Vertex Splitting, Coincident Realisations and Global Rigidity of Braced Triangulations

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Abstract

We give a relatively short graph theoretic proof of a result of Jordán and Tanigawa that a 4-connected graph which has a spanning plane triangulation as a proper subgraph is generically globally rigid in $\mathbb{R}^3$. Our proof is based on a new sufficient condition for the so called vertex splitting operation to preserve generic global rigidity in $\mathbb{R}^d$.

Keywords Bar-joint framework, global rigidity, vertex splitting, plane triangulation.

Mathematics Subject Classification 52C25, 05C10, 05C75

1 Introduction

We consider the problem of determining when a configuration consisting of a finite set of points in d-dimensional Euclidean space $\mathbb{R}^d$ is uniquely defined up to congruence by a given set of constraints which fix the Euclidean distance between certain pairs of points.

More formally, we define a *d-dimensional framework* to be a pair (G, p) where $G = (V, E)$ is a graph and $p : V \to \mathbb{R}^d$ is a point configuration. We will also refer to the framework (G, p) and the configuration p as *realisations* of G in $\mathbb{R}^d$. The length of an edge of G in (G, p) is given by the Euclidean distance between its endvertices. Two realisations (G, p) and (G, q) of G in $\mathbb{R}^d$ are *congruent* if (G, p) can be obtained from (G, q) by an isometry of $\mathbb{R}^d$ i.e. a combination of translations, rotations and reflections. The framework (G, p) is *globally rigid* if every framework which has the same edge lengths as (G, p) is congruent to (G, p). It is *rigid* if every continuous motion of the vertices of (G, p) in $\mathbb{R}^d$ which preserves the edge lengths results in a framework which is congruent to (G, p). It is *infinitesimally rigid* if it satisfies the stronger property that every infinitesimal motion of the points which preserves the edge lengths is induced by an infinitesimal isometry of $\mathbb{R}^d$ i.e. a combination of infinitesimal translations and rotations. (A formal definition of infinitesimal rigidity will be given in Section 2.)

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Saxe [22] showed that it is NP-hard to determine when a realisation of G in $\mathbb{R}^d$ is globally rigid for all $d \geq 1$. Abbot [1] showed that the same holds for rigidity for all $d \geq 2$. (It is straightforward to show that a 1-dimensional framework is rigid if and only if its underlying graph is connected.) These decision problems become more tractable, however, if we restrict our attention to 'generic realisations'. A configuration p or framework (G, p) is said to be *generic* if the set of coordinates of the points $p(v)$, $v \in V(G)$, is algebraically independent over $\mathbb{Q}$. Gluck [7] showed that rigidity and infinitesimal rigidity are equivalent properties of generic frameworks and depend only on the underlying graph. This enables us to define a graph G as being *rigid in* $\mathbb{R}^d$ if some (or equivalently every) generic realisation of G in $\mathbb{R}^d$ is rigid (or equivalently infinitesimally rigid). Analogous, but much deeper, results of Connelly [3] and Gortler, Healy and Thurston [9] imply that the global rigidity of a generic framework depends only on its underlying graph and allows us to define a graph G as being *globally rigid in* $\mathbb{R}^d$ if some (or equivalently every) generic realisation of G in $\mathbb{R}^d$ is globally rigid. It is known that a graph is globally rigid in $\mathbb{R}$ if and only if it is 2-connected. Characterisations of graphs which are rigid or globally rigid in $\mathbb{R}^2$ are given in [20, 21] and [14], respectively. It is a major open problem in distance geometry to characterise rigid or globally rigid graphs when $d \geq 3$, although many partial results exist for particular families of graphs. We refer the reader to [23, 19] for recent survey articles on framework rigidity.

Gluck [7] showed that every plane triangulation, i.e. maximal planar graph on at least three vertices, is rigid in $\mathbb{R}^3$. Plane triangulations other than K_3 and K_4 do not have enough edges to be globally rigid but a recent result of Jordán and Tanigawa [18] characterises when *braced plane triangulations*, i.e. graphs constructed from plane triangulations by adding additional edges called *braces*, are globally rigid in $\mathbb{R}^3$.

Theorem 1.1. *Suppose that G is a braced plane triangulation with at least five vertices. Then G is globally rigid in $\mathbb{R}^3$ if and only if G is 4-connected and has at least one brace.*

We will give a relatively short proof of this result.

1.1 Vertex splitting

Our proof of Theorem 1.1 is inductive and is based on (the 3-dimensional version of) Theorem 1.2 below, which verifies a special case of the so-called vertex splitting conjecture. Before stating this theorem we need to introduce some new terminology.

Given a graph $G = (V, E)$ and $v \in V$ with neighbour set $N_G(v)$, the *d-dimensional vertex splitting operation* constructs a new graph G' by choosing pairwise disjoint sets U_1, U_2, U_3 with $U_1 \cup U_2 \cup U_3 = N_G(v)$ and $|U_2| \geq d - 1$, deleting all edges from v to U_3, and then adding a new vertex v' and new edges from v' to every vertex in $U_2 \cup U_3 \cup \{v\}$. Whiteley [25] showed that the vertex splitting operation preserves rigidity in $\mathbb{R}^d$ and conjectured that it also preserves global rigidity in $\mathbb{R}^d$ whenever v and v' both have degree at least $d + 1$ in G', see [5, 19]. We will verify a special case of this conjecture.

Theorem 1.2. *Let $G = (V, E)$ be a graph which is globally rigid in $\mathbb{R}^d$ and $v \in V$. Suppose that G' is obtained from G by a d-dimensional vertex splitting operation which splits v into two vertices v and v', and that G' can be realised as an infinitesimally rigid framework in $\mathbb{R}^d$ in which v and v' are coincident. Then G' is generically globally rigid in $\mathbb{R}^d$.*

The assertion that Theorem 1.2 is a special case of Whiteley's conjecture follows from Lemma 1.3 below (which implies that, if G' can be realised as a vv'-coincident infinitesimally rigid framework in $\mathbb{R}^d$, then $G' - vv'$ is rigid so v and v' must both have degree at least $d + 1$ in G').

Theorem 1.2 can be deduced from the results of Connelly[4] using the stress matrix characterisation of global rigidity in [9]. We give a shorter direct proof in Section 3. Theorem 1.2 has already been used by Jordán, Kiraly and Tanigawa in [17] to repair a gap in the proof of their characterisation of generic global rigidity for body-hinge frameworks given in [16].[1] An analogous result to Theorem 1.2 was used in [13, 15] to obtain a characterisation of generic global rigidity for cylindrical frameworks.

1.2 Coincident rigidity

In order to use Theorem 1.2, we need to have sufficient conditions for a graph to have an infinitesimally rigid realisation with two coincident points. The analysis of such graphs was initiated by Fekete, Jordán and Kaszanitzky [8], who gave a complete combinatorial characterization in the two dimensional case.

Given two vertices u, v in a graph $G = (V, E)$ we use $G - uv$ to denote the graph obtained from G by deleting the edge uv if it exists in G (and putting $G - uv = G$ if it does not exist). We also use G/uv to denote the graph obtained from G by replacing u and v by a single vertex w which is adjacent to every neighbour of u and v in $V \backslash \{u, v\}$. We say that G is *uv-coincident rigid* in $\mathbb{R}^d$ if G can be realised as an infinitesimally rigid framework (G, p) in $\mathbb{R}^d$ with $p(u) = p(v)$. Fekete, Jordán and Kaszanitzky [8] showed that G is *uv*-coincident rigid in $\mathbb{R}^2$ if and only if $G - uv$ and G/uv are both rigid in $\mathbb{R}^2$. Their proof of necessity extends immediately to $\mathbb{R}^d$.

Lemma 1.3. *Suppose u, v are two vertices of a graph G. If G is uv-coincident rigid in $\mathbb{R}^d$ then $G - uv$ and G/uv are both rigid in $\mathbb{R}^d$.*

The converse direction does not hold in $\mathbb{R}^3$ (see [10, Section 5.2]), and it is an open problem to characterize *uv*-coincident rigidity in terms of rigidity in $\mathbb{R}^d$.

In order to link Theorems 1.1 and 1.2, we will obtain sufficient conditions for the *uv*-coincident rigidity of braced plane triangulations in $\mathbb{R}^3$. Lemma 1.3 implies that no plane triangulation T can be *uv*-coincident rigid in $\mathbb{R}^3$ for two adjacent vertices u, v since $T - uv$ has too few edges to be rigid. Our third main result gives a sufficient condition for T to become *uv*-coincident rigid after the addition of at least one brace.

Theorem 1.4. *Let G be a 4-connected braced plane triangulation which is obtained from a plane triangulation T by adding at least one brace. Suppose that $e = uv$ is an edge of T which does not belong to any separating 3-cycle of T. Then G is uv-coincident rigid in $\mathbb{R}^3$*

In the forthcoming paper [6], we shall verify a conjecture of Connelly concerning the global rigidity of triangulated surfaces in $\mathbb{R}^3$. Theorems 1.2 and 1.4 will be key ingredients in our proof.

2 Infinitesimally rigid realisations

We can determine whether a given d-dimensional framework (G, p) is infinitesimally rigid by calculating the rank of its *rigidity matrix*. This is the matrix of size $|E| \times d|V|$ in which

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[1]They mistakenly state in [16] that a stronger form of Theorem 1.2 (in which the hypothesis that G' can be realised as a vv'-coincident infinitesimally rigid framework in $\mathbb{R}^d$ is replaced by the hypothesis that $G' - vv'$ is rigid) is implied by [4].

each row is indexed by an edge, sets of d consecutive columns are indexed by the vertices, and the row indexed by the edge $e = uv$ has the form:

$$e = uv \quad \begin{matrix} & \overset{u}{} & & & \overset{v}{} & \\ [0 \ldots 0 & p(u) - p(v) & 0 \ldots 0 & p(v) - p(u) & 0 \ldots 0] . \end{matrix}$$

The space of *infinitesimal motions* of (G, p) is given by the right kernel of $R(G, p)$. The framework (G, p) is *infinitesimally rigid* if rank $R(G, p) = d|V| - \binom{d+1}{2}$ when $|V| \geq d$ and rank $R(G, p) = \binom{|V|}{2}$ when $|V| < d$, since this will imply that every vector in $\ker R(G, p)$ is an infinitesimal isometry of $\mathbb{R}^d$. Since the rank of $R(G, p)$ will be maximised whenever (G, p) is generic, the infinitesimal rigidity of a generic framework (G, p) in $\mathbb{R}^d$ depends only on the underlying graph G. We say that (G, p) is *independent* if $R(G, p)$ is row independent, and is *minimally rigid* if it is both infinitesimally rigid and independent.

The fact that the entries in the rigidity matrix of (G, p) are linear in the coordinates of p implies that, if G is uv-coincident rigid and (G, p) is a uv-coincident realisation of G in which $p|_{V-v}$ is generic, then (G, p) will be infinitesimally rigid. We will refer to such a realisation as a *generic uv-coincident realisation* of G.

In order to apply Theorem 1.2, we need to construct an infinitesimally rigid realisation of a graph in which two given vertices are coincident. We will use the following results on infinitesimal rigidity to do this.

We first give a precise statement of Whiteley's above mentioned vertex splitting theorem [25, Corollary 11].

Lemma 2.1. *Suppose that (G, p) is an infinitesimally rigid framework in $\mathbb{R}^d$ and that G' is obtained from G by a vertex splitting operation which splits a vertex v of G into two vertices v', v''. Suppose further that, for some $X \subseteq N_{G'}(v') \cap N_{G'}(v'')$ with $|X| = d-1$, the points in $\{p(w) : w \in X + v\}$ are in general position in $\mathbb{R}^d$. Then (G', p') is infinitesimally rigid for some p' with $p'(v') = p(v)$ and $p'(w) = p(w)$ for all $w \in V(G) \setminus \{v\}$.*

Our next two lemmas concern the so called 1-extension and gluing operations. The *d-dimensional 1-extension operation* on a graph G constructs a new graph G' by deleting an edge uv and then adding a new vertex w with $u, v \in N_{G'}(w)$ and $|N_{G'}(w)| = d + 1$. The *d-dimensional gluing operation* constructs a new graph by taking the union of two graphs with at least d vertices in common. These lemmas can be proved using standard techniques: for example, Lemma 2.2 follows from the first part of the proof of [26, Theorem 11.1.7], and the proof of [26, Lemma 3.1.4] easily extends to give Lemma 2.3.

Lemma 2.2. *Suppose that (G, p) is an infinitesimally rigid framework in $\mathbb{R}^d$ and that G' is obtained from G by a d-dimensional 1-extension operation which adds a new vertex w. Suppose further that the points $\{p(x) : x \in N_{G'}(w)\}$ are in general position in $\mathbb{R}^d$. Then there is an extension $p' : V(G') \to \mathbb{R}^d$ of p such that (G', p') is infinitesimally rigid.*

Lemma 2.3. *Let (G, p) be a framework in $\mathbb{R}^d$, G_1 and G_2 be subgraphs of G with $G = G_1 \cup G_2$, and $U \subseteq V(G_1) \cap V(G_2)$ with $|U| = d$. Suppose that $(G_1, p|_{G_1})$ and $(G_2, p|_{G_2})$ are infinitesimally rigid and $p(U)$ is in general position in $\mathbb{R}^d$. Then (G, p) is infinitesimally rigid.*

To prove Theorem 1.1, we also need the following combination of Lemmas 2.2 and 2.3.

Lemma 2.4. *Let G_1, G_2 be graphs which are rigid in $\mathbb{R}^d$ and satisfy $|V(G_1) \cap V(G_2)| \geq d$, $x \in V(G_1) \backslash V(G_2)$, $y \in V(G_2) \backslash V(G_1)$, $z \in V(G_1) \cap V(G_2)$ and $xz \in E(G_1)$. Put $G = (G_1 \cup G_2) - xz + xy$. Suppose that (G_1, p_1) is an infinitesimally rigid realisation of G_1*

and that p_1 is generic on $(V(G_1) \cap V(G_2)) \cup \{x\}$. Then (G, p) is infinitesimally rigid for some p with $p(v) = p_1(v)$ for all $v \in V(G_1)$.

Proof. Let (G_1', p_1') be obtained from $(G_1 - xz, p_1)$ by adding the vertex y at a point $p_1'(y)$ whose coordinates are algebraically independent over the field obtained by extending $\mathbb{Q}$ by the coordintes of p_1, and then adding an edge from y to x and all vertices in $V(G_1) \cap V(G_2)$. Then (G_1', p_1') is infinitesimally rigid by Lemma 2.2 (since it can be obtained from (G_1, p_1) by a 1-extension and a possibly empty sequence of edge additions). Lemma 2.3 now implies that $(G_1' \cup G_2, p)$ is infinitesimally rigid for any generic extension p of p_1'. We can now deduce that that (G, p) is infinitesimally rigid (since $(G_2, p|_{G_2})$ is infinitesimally rigid, we can delete any edges of G_1' from y to $V(G_1) \cap V(G_2)$ which do not belong to G_2 without destroying the infinitesimal rigidity of $(G_1' \cup G_2, p)$). $\qquad\square$

3 Global rigidity and vertex splitting

In this section we prove Theorem 1.2. We need the following result of Connelly and Whiteley [5, Theorem 13], which shows that global rigidity is a stable property for infinitesimally rigid frameworks, to prove Theorem 1.2.

Lemma 3.1. *Suppose that (G, p) is an infinitesimally rigid, globally rigid framework on n vertices in $\mathbb{R}^d$. Then there exists an open neighbourhood N_p of p in $\mathbb{R}^{dn}$ such that (G, q) is infinitesimally rigid and globally rigid for all $q \in N_p$.*

Note that, although the definition of a framework (G, p) in [5] requires $p(u) \neq p(v)$ for all $uv \in E(G)$, this simplifying assumption is not needed in the proof of [5, Theorem 13].

Proof of Theorem 1.2. Let (G, p) be a generic realisation of G in $\mathbb{R}^d$ and let (G', p') be the vv'-coincident realisation of G' obtained by putting $p'(u) = p(u)$ for all $u \in V$ and $p'(v') = p(v)$. The genericity of p implies that the rank of the rigidity matrix of any vv'-coincident realisation of G' will be maximised at (G', p') and hence (G', p') is infinitesimally rigid. The genericity of p also implies that (G, p) is globally rigid, and this in turn implies that (G', p') is globally rigid. We can now use Lemma 3.1 to deduce that (G', q) is globally rigid for any generic q sufficiently close to p'. Hence G' is globally rigid. $\qquad\square$

We will be exclusively concerned with 3-dimensional frameworks in the remainder of the paper so will henceforth suppress reference to the ambient space $\mathbb{R}^3$ and say, for example, that a graph is rigid to mean it is rigid in $\mathbb{R}^3$.

4 Coincident rigidity of plane triangulations

A graph T is a *plane triangulation* if it has a 2-cell embedding in the plane in which every face has three edges on its boundary. The infinitesimal rigidity of realisations of plane triangulations in $\mathbb{R}^3$ is one of the fundamental topics in graph rigidity, see, e.g., [24]. As a warm up to the analysis of uv-coincident rigidity of braced plane triangulations, we shall analyze the uv-coincident rigidity of plane triangulations. As remarked in the introduction, no plane triangulation T can be uv-coincident rigid in $\mathbb{R}^3$ for two adjacent vertices u, v in T. We will give a sufficient condition for the uv-coincident rigidity of T when u, v are not adjacent. This result will be used to obtain our characterisation of globally rigid triangulated surfaces in [6].

We will need the following notation and elementary results for (a particular embedding of) a plane triangulation T. Every cycle C of T divides the plane into two open regions exactly one of which is bounded. We refer to the bounded region as the *inside of C* and the unbounded region as the *outside of C*. We say that C is a *separating cycle* of T if both regions contain vertices of T. If S is a minimal vertex cut-set of T then S induces a separating cycle C. It follows that every plane triangulation with at least four vertices is 3-connected and that a plane triangulation with at least five vertices is 4-connected if and only if it contains no separating 3-cycles. Given an edge e of T which belongs to no separating 3-cycle of T, we can obtain a new plane triangulation T/e by contracting the edge e and its end-vertices to a single vertex (which is located at the same point as one of the two end-vertices of e), and replacing the multiple edges created by this contraction by single edges. For $X \subseteq V(T)$, let $T[X]$ be the subgraph of T induced by X. Given two vertices u, v of T, we say that an edge $f \in E(T)$ is *uv-admissible* if $\{u, v\}$ is not contained in the unique face of $T - f$ of size four. Our motivation for considering such edges is that, if f is uv-admissible and T/f is uv-coincident rigid, then we can apply Lemma 2.1 to deduce that T is uv-coincident rigid.

We can now give our sufficient condition for uv-coincident rigidity of plane triangulations in $\mathbb{R}^3$. Its proof also illustrates our strategy for proving Theorem 1.4.

Theorem 4.1. *Let $T = (V, E)$ be a plane triangulation and $u, v \in V$. Suppose that $uv \notin E$ and that no separating 4-cycle of T contains both u and v. Then T is uv-coincident rigid.*

Proof. Suppose the statement does not hold, and let $T = (V, E)$ be a counterexample such that $|V|$ is minimal. Then T is not uv-coincident rigid for two non-adjacent vertices u and v which are not contained in any separating 4-cycle of T. We first consider the case when T is not 4-connected. Then T has a separating 3-cycle C and we have $T = T_1 \cup T_2$ for two subtriangulations T_1, T_2 of T with $T_1 \cap T_2 = C$. Note that $\{u, v\} \not\subseteq V(C)$ since $uv \notin E$. If $u \in V(T_1) \setminus V(C)$ and $v \in V(T_2) \setminus V(C)$ then we can choose generic realisations p_i of T_i for $i = 1, 2$ with $p_1(w) = p_2(w)$ for $w \in V(C)$ and $p_1(u) = p_2(v)$. Then each (T_i, p_i) is infinitesimally rigid and we can now use Lemma 2.3 to deduce that the realisation p of T with $p(x) = p_i(x)$ for all $x \in V(T_i)$ is a uv-coincident infinitesimally rigid realisation of T. Hence we may assume that $u, v \in V(T_1)$. Then we may apply the minimality of T to deduce that there exists a generic uv-coincident infinitesimally rigid realisation p_1 of T_1. Since $\{u, v\} \not\subseteq V(C)$, we may now choose a generic realisation p_2 of T_2 with $p_2(w) = p_1(w)$ for all $w \in V(C)$ and proceed as in the previous subcase.

It remains to consider the case when T is 4-connected. Then T/f is a plane triangulation for all $f \in E$. Let S be the (possibly empty) set of all edges of T which lie on a uv-path in T of length two.

Claim 4.2. *T has a uv-admissible edge f with $f \notin S$.*

Proof. Recall that u and v are not adjacent in T. If there is an edge f incident to u or v and not contained in a uv-path of length two, then f is uv-admissible with $f \notin S$. Hence, we may assume that every edge incident to u or v is contained in a uv-path of length two. By 4-connectivity, there are at least four internally disjoint uv-paths of length two. By planarity, at least one pair of these uv-paths forms a separating 4-cycle in T. This contradicts the hypothesis that $\{u, v\}$ is not contained in a separating 4-cycle. $\square$

We also have the following.

Claim 4.3. *For every uv-admissible edge f with $f \notin S$, T has a separating 5-cycle which contains $\{u, v\}$ and f.*

Proof. Let w_1 and w_2 be the vertices of T which lie on the same face as f and are not incident to f. Let z be the new vertex we obtain when we contract f to form T/f. We modify this notation when f is incident to u or v by putting $z = u$ or $z = v$, respectively, to ensure that $\{u, v\} \subseteq V(T/f)$. Note that $uv \notin E(T/f)$ since f does not lie on a uv-path of length two. If $\{u, v, f\}$ is not contained in a separating 5-cycle of T, then no separating 4-cycle of T/f contains both u and v. So by the minimality of T, there is a generic uv-coincident infinitesimally rigid framework $(T/f, p)$. Since f is uv-admissible, $\{u, v\} \not\subseteq \{w_1, w_2, z\}$. So the points in $p(w_1), p(w_2), p(z)$ are in general position and we may apply Lemma 2.1 to deduce that T is uv-coincident rigid, which is a contradiction. $\qquad\square$

By Claim 4.2 and Claim 4.3, T has a separating 5-cycle containing $\{u, v\}$. Choose a separating 5-cycle C of T which contains $\{u, v\}$ and is such that some component H of $T - V(C)$ has the minimum number of vertices. Let $C = uwvxyu$. By symmetry we may assume that H is contained inside C. By the 4-connectivity of T and the hypothesis that T has no separating 4-cycle containing u and v,

$$C \text{ has no chord incident to } u \text{ or } v. \qquad (1)$$

Since T is 4-connected, at least one of x or y is incident to a vertex inside C. Without loss of generality, we assume T has an edge $f = xz$ for some z inside C. Then f is uv-admissible by (1) and clearly $f \notin S$. Hence, by Claim 4.3, T has a separating 5-cycle C' that contains $\{u, v\}$ and f. By (1), $C' = uzxvz'u$ holds for some $z' \in V(T)$. The minimum choice of C and H implies that z' is not inside C.

Claim 4.4. $N_T(w) = \{u, z, x, v, z'\}$.

Proof. Since T has the 5-cycle $C' = uzxvz'u$, it suffices to show that wz', wx, wz exist in T. Observe first that $uwvz'$ forms a 4-cycle. Since there is no separating 4-cycle containing u and v, we have $wz' \in E(T)$.

To see $wx, wz \in E(T)$, consider the 5-cycle $C'' = uzxvwu$. The minimum choice of C and H implies that there are no vertices inside C''. Moreover, by (1), wx and wz are the only possible chords of C''. This gives $wx, wz \in E(T)$. $\qquad\square$

By Claim 4.4, C' and the set of edges incident to w forms a wheel on six vertices, and the 4-connectivity of T implies that C' has no chord in T. Then wx is uv-admissible with $wx \notin S$, and hence T has a separating 5-cycle C'' which contains $\{u, v\}$ and wx by Claim 4.3. By (1), we have $C'' = uwxvz''u$ for some vertex $z'' \in V(T) \backslash N_T(w)$. This implies that $uwvz''u$ is a separating 4-cycle in T which contains $\{u, v\}$, a contradiction. $\qquad\square$

The plane triangulation T obtained by joining two nonadjacent vertices u, v to all vertices of a cycle C of length at least four shows that the conclusion of Theorem 4.1 may not hold if we remove the hypothesis that no separating 4-cycle contains both u and v: Lemma 1.3 implies that T is not uv-coincident rigid since T/uv has too few edges to be rigid.

5 Contractible edges in plane triangulations

Hama and Nakamoto [11], see also Brinkmann et al [2], showed that every 4-connected plane triangulation T other than the octahedron has an edge e such that T/e is a 4-connected plane triangulation. For the analysis of braced triangulations, we will need more detailed information on the distribution of such contractible edges. We will frequently use the following well known properties of a 4-connected plane triangulation T.

- T/e is 4-connected if and only if e belongs to no separating 4-cycle of T.

- No separating 4-cycle in T can have a chord.

- No proper subgraph of T can be a plane triangulation.

- The octahedron is the unique 4-connected plane triangulation with at most six vertices.

Our first lemma is statement (b) in the proof of [2, Theorem 0.1]. We include a proof for the sake of completeness.

Lemma 5.1. *Let T be a 4-connected plane triangulation with at least seven vertices, u be a vertex of T of degree four and $e_1 = uv_1, e_2 = uv_2$ be two cofacial edges of T. Then T/e_i is 4-connected for some $i = 1, 2$.*

Proof. Suppose, for a contradiction, that T/e_i is not 4-connected for both $i = 1, 2$. Let $C_1 = v_1 v_2 v_3 v_4 v_1$ be the separating 4-cycle of T which contains the neighbours of u. Since T/e_1 is not 4-connected, T has a separating 4-cycle C_2 containing e_1. Since no separating 4-cycle of T can have a chord, $C_2 = wv_1 uv_3 w$ for some vertex $w \in V(T)\backslash(V(C_1) \cup \{u\})$. Similarly, since T/e_2 is not 4-connected, T has a separating 4-cycle $C_3 = w' v_2 u v_4 w'$ for some $w' \in V(T)\backslash(V(C_1) \cup \{u\})$. If $w' \neq w$ then $T[V(C_1) \cup \{u, w, w'\}]$ contains a subgraph homeomorphic to K_5 contradicting the planarity of T. On the other hand, if $w = w'$, then $T[V(C_1) \cup \{u, w\}]$ is a proper subtriangulation of T and this contradicts the hypothesis that T is a 4-connected triangulation. $\square$

Lemma 5.2. *Suppose that T is a 4-connected plane triangulation with at least seven vertices and F is a face of T. Then T/e is 4-connected for some edge e of $T - V(F)$.*

Proof. Fix a plane embedding of T with F as the unbounded face. Since T is 4-connected, some edge e of T is not incident with $V(F)$. If T has no separating 4-cycle then T/e would be 4-connected so we may assume that T has a separating 4-cycle. Let $C = v_1 v_2 v_3 v_4 v_1$ be a separating 4-cycle such that the set of vertices inside C is minimal with respect to inclusion. Since T is 4-connected, C has no chords and hence, relabelling $V(C)$ if necessary, we may assume that $v_1, v_2 \notin V(F)$. Let uv_1 be an edge from a vertex u inside C to v_1. If T/uv_1 is 4-connected then we are done, so we may also assume that $v_1 u$ belongs to a separating 4-cycle C_1 of T. The minimality of C implies that $C_1 = wv_1 uv_3 w$ for some vertex w outside C, and hence that u is the only vertex inside C (otherwise $C_2 = v_1 uv_3 v_2 v_1$ or $C_3 = v_1 uv_3 v_4 v_1$ would contradict the minimality of C). This in turn implies that u has degree 4 in T. We can now use Lemma 5.1 and the fact that T/uv_1 is not 4-connected to deduce that T/uv_2 is 4-connected. $\square$

Let C be a separating cycle in a plane triangulation $T = (V, E)$, H_1 and H_2 be the components of $G - C$ such that H_1 is inside C, and $R \subset V$. We say that R *crosses* C if both H_1 and H_2 contain a vertex in R. For $i = 1, 2$, let H_i^+ be the subgraph of T consisting of $C \cup H_i$ and all edges of T joining C to H_i. A separating 4-cycle C is said to be *inner minimal* if no other separating 4-cycle of T is contained in H_1^+. We say that C is *outer minimal* if it satisfies the same condition with respect to H_2^+, and that C is *minimal* if it is either inner minimal or outer minimal.

Lemma 5.3. *Let $T = (V, E)$ be a 4-connected plane triangulation, and $R \subset V$ with $4 \leq |R| < |V|$. Then there exists an edge $f \in E$ such that f is not induced by R and R crosses every separating 3-cycle of T/f.*

Proof. Let $f = xy$ be an edge of G incident to a vertex $x \in V \backslash R$. Then T/f is a plane triangulation by the 4-connectivity of T. Furthermore, R crosses every separating 3-cycle of T/f unless

(†) f belongs to a separating 4-cycle C of T and some component H of $T - C$ contains no vertex of R.

Suppose, for a contradiction, that the lemma is false and that (T, R) is a counterexample. Then, for every $x \in V \backslash R$, each edge f which is incident to x belongs to a separating 4-cycle $C = x_1 x_2 x_3 x_4 x_1$ which satisfies condition (†). We may assume that x, f and C have been chosen such that the specified component H of $T - C$ has a minimal number of vertices. By symmetry, we may also assume that H is contained inside C. Then H^+ is a plane graph and all interior faces of H^+ are triangles.

Supoose $E(H) \neq \emptyset$, and let $f' = x'x'' \in E(H)$. Since T is a counterexample, f' belongs to a separating 4-cycle C' of T which satisfies (†). The choice of (x, f, C) now implies that C' is not contained in H^+. This in turn implies that $C' \cap H^+$ is a path of length three joining two non-adjacent vertices, say x_1, x_3, of C, and $x_1 x_3$ is a chord of C. This contradicts the hypothesis that T is a 4-connected plane triangulation.

It remains to consider the case when $E(H) = \emptyset$. Then $|V(H)| = 1$ and H^+ is a wheel on five vertices. Let x_5 be the unique vertex of H. Then $x_5 \notin R$, and each of the four triangular faces of H^+ incident to x_5 is a face of T. If T has more than six vertices, (†) would be violated by Lemma 5.1, which is a contradiction. Hence T has at most six vertices, and T is the octahedron. Let x_6 be the unique vertex of $T - H^+$. Then the unique separating 4-cycle of T which contains $x_3 x_5$, respectively $x_4 x_5$, is $C_1 = x_3 x_5 x_1 x_6 x_3$, respectively $C_2 = x_4 x_5 x_2 x_6 x_4$. Since the edges $x_3 x_5$ and $x_4 x_5$ must both satisfy (†), each of $T - C$, $T - C_1$ and $T - C_2$ has a component with no vertices in R. This gives $|V \backslash R| \geq 3$ and contradicts the hypothesis that $|R| \geq 4$. $\square$

6 Braced triangulations

Recall that a *braced plane triangulation* is a graph $G = (V, E \cup B)$ which is the union of a plane triangulation $T = (V, E)$ and a (possibly empty) set B of additional edges which we refer to as *braces*. Given a braced plane triangulation $G = (T, B)$ and an edge f of T which belongs to no separating 3-cycle of T, we denote the braced plane triangulation obtained by contracting the edge f by $G/f = (T/f, B_f)$ where the set of bracing edges B_f is obtained from B by replacing any multiple edges in G/f by single edges (in particular any edge of B which becomes parallel to an edge of T/f is deleted).

Our general strategy to prove Theorem 1.4 is to find a *uv-admissible* edge $f \in E(T)$ such that G/f satisfies the hypotheses of the theorem, apply induction to G/f, and then apply the vertex splitting lemma (Lemma 2.1) to return to G. In order to apply induction to G/f when f is incident to one of u or v, say u, we will adopt the convention that the new vertex obtained by contracting f is also labeled as u whenever this occurs.

6.1 Admissible edges

Let T be a plane triangulation. Our first result shows how Lemma 2.1 can be used to complete the inductive step in our proof of Theorem 1.4.

Lemma 6.1. *Let $G = (T, B)$ be a braced triangulation, and $e = uv \in E(T)$. Suppose $f \in E(T)$ is uv-admissible and G/f is uv-coincident rigid in $\mathbb{R}^3$. Then G is uv-coincident rigid.*

Proof. Let $(G/f, p)$ be a generic uv-coincident rigid realization of G/f. Let $f = ab$. Let w be the new vertex of G/f created by contracting f, and B_f be the set of bracing edges in G/f. We may assume B_f is a subset of B. (If an edge e' in B_f corresponds to more than one edge in B, then we choose one of these edges arbitrarily and identify it with e'.) Since p is a generic uv-coincident realization and f is uv-admissible, $\{p(x) : x \in \{w\} \cup (N_H(a) \cap N_H(b))\}$ is in general position. Hence, the uv-coincident rigidity of G follows from Lemma 2.1. $\square$

We next give a lemma which will be used to solve the most difficult case in our proof of Theorem 1.4: when G is obtained from a 4-connected plane triangulation T by adding exactly one brace xy. Specifically, for $e = uv \in E(T)$, and $xy \notin E(T)$, we need to find a uv-admissible edge $f \in E(T)$ such that

(i) $(T + xy)/f$ is a braced planar triangulation (with at least one brace),

(ii) $(T + xy)/f$ is 4-connected, and

(iii) e does not belong to any separating 3-cycle of T/f.

This can be guaranteed if f satisfies the following conditions in T:

(i') f does not belong to an xy-path of length two in T,

(ii') for every separating 4-cycle C of T which contains f, $\{x, y\}$ crosses C, and

(iii') for every separating 4-cycle C of T which contains f, $e \notin E(C)$.

The following lemma shows that we can always find a uv-admissible edge $f \in E(T)$ satisfying (i'), (ii'), and (iii').

Lemma 6.2. *Let $T = (V, E)$ be a 4-connected plane triangulation on at least seven vertices, $e = uv \in E$ and x, y be two non-adjacent vertices of T. Then there exists a uv-admissible edge $f \in E$ such that f does not belong to an xy-path of length two and, for every separating 4-cycle C of T which contains f, we have $e \notin E(C)$ and $\{x, y\}$ crosses C.*

Proof. Suppose, for a contradiction, that the lemma is false and T is a counterexample. Let F, F' be the faces of T which contain e and let w, respectively w', be the vertex of F, respectively F', that is different from u, v. Let S be the (possibly empty) set of all edges of T which lie on an xy-path in T of length two. The assumption that T is a counterexample implies that

(∗) every edge of $E(T) \setminus (E(F) \cup E(F') \cup S)$ belongs to a separating 4-cycle C of T such that either $e \in E(C)$ or C is not crossed by $\{x, y\}$.

Since T is a triangulation with more than five vertices, $T - (E(F) \cup E(F'))$ contains a triangle. Since S induces a bipartite subgraph, this implies that $E(T) \setminus (E(F) \cup E(F') \cup S)$ is nonempty, so T has a separating 4-cycle by (∗).

Claim 6.3. *Let C be a minimal separating 4-cycle of T and H be a component of $T - C$ such that no separating 4-cycle of T is contained in H^+. Suppose that $e \notin E(H^+) \setminus E(C)$. Then H^+ is a wheel on five vertices. Furthermore:*
(a) if $e \in E(C)$, then the unique vertex z of H satisfies $z \in \{w, w'\} \cap \{x, y\}$;
(b) if $e \notin E(C)$ then $\{x, y\} \subset V(C)$, exactly one of $\{u, v\}$, say u, belongs to $V(C) \setminus \{x, y\}$ and v is adjacent to the unique vertex of $V(C) \setminus \{u, x, y\}$.

Proof of Claim. By symmetry, it will suffice to prove the claim when C is inner minimal. The 4-connectivity of T and minimality of C imply that H^+ is either a wheel on five vertices or is 4-connected.

Suppose H^+ is 4-connected. We first show that we can choose an edge $f \in E(H^+) \setminus E(C)$ such that $f \notin E(F) \cup E(F') \cup S$. If there exists a triangular face F_1 of H^+ such that $E(F_1) \subseteq E(H^+)\setminus(E(C) \cup E(F) \cup E(F'))$, then we can choose an edge $f \in E(F_1)\setminus S$ since S induces a bipartite subgraph of T. This edge f satisfies $f \notin E(F) \cup E(F') \cup S$ as required. On the other hand, if H^+ has no triangular face F_1 such that $E(F_1) \subseteq E(H^+)\setminus(E(C) \cup E(F) \cup E(F'))$, then, since $e \notin E(H^+)\setminus E(C)$, $e \in E(C)$ holds and $H^+ - e$ forms the wheel on six vertices. In this case H^+ would have a vertex of degree three, contradicting the 4-connectivity of H^+.

Thus, we can choose an edge $f \in E(H^+) \setminus E(C)$ such that $f \notin E(F) \cup E(F') \cup S$, and f is contained in a separating 4-cycle C' of T by $(*)$. The minimality of C implies that $C' \not\subseteq H^+$ and the fact that $|V(H)| \geq 2$ now implies that either C' or C has a chord. This contradicts the 4-connectivity of T. Hence H^+ is a wheel on five vertices. It remains to verify (a) and (b).

To see that (a) holds, we suppose that $e \in E(C)$. Then exactly one of w, w' is contained in H^+. Relabelling if necessary, we may assume that $V(H) = \{w'\}$. Let $C = uvstu$. By Lemma 5.1, at least one of the cofacial edges $w's, w't$, say $w's$, does not belong to a separating 4-cycle of T. We can now apply $(*)$ to deduce that $w's \in S$ so $\{w', s\} \cap \{x, y\} \neq \emptyset$. If $s \in \{x, y\}$, say $s = x$, then the facts that $w's \in S$ and $xy \notin E$ give $y = u$. Then $w't$ satisfies $w't \in E(T) \setminus (E(F) \cup E(F') \cup S)$. Moreover, since every separating 4-cycle C' which contains $w't$ is of the form $tw'vu't$ for some $u' \in V$, C' does not contain uv and is crossed by $\{x, y\}$. This contradicts $(*)$. Hence $w' \in \{x, y\}$ and (a) holds.

To see that (b) holds, we suppose that $e \notin E(C)$. Since $e \notin E(H^+) \setminus E(C)$, we have $e \notin E(H^+)$. Let $C = v_1v_2v_3v_4v_1$. Since H^+ is a wheel on five vertices, the unique vertex z of H has degree four in T and we can apply Lemma 5.1 to T to deduce that, after a possible relabelling of $V(C)$, neither zv_1 nor zv_3 belong to a separating 4-cycle of T. By $(*)$ and $e \notin E(H^+)$, we have $\{zv_1, zv_3\} \subseteq S$ and $\{x, y\} = \{v_1, v_3\}$. Then $zv_2, zv_4 \notin S$ and $(*)$ implies that there exists a separating 4-cycle $C' = v_2zv_4z'v_2$ of T such that either $e \in E(C')$ or $\{x, y\}$ does not cross C'. Since $\{x, y\}$ crosses C', we have $e \in E(C')$. Hence $z' \in \{u, v\}$, $|\{u, v\} \cap \{v_2, v_4\}| = 1$ and (b) holds. $\qquad\square$

Claim 6.4. *Some separating 4-cycle of T does not contain e.*

Proof of Claim. Suppose for a contradiction that every separating 4-cycle of T contains e. Choose two separating 4-cycles $C_0 = uvu_1u_2$ and $C_0' = uvu_1'u_2'$, such that C_0 is outer minimal and C_0' is inner minimal. Relabelling w, w' and x, y if necessary, Claim 6.3 implies that $w = x$ is the unique vertex outside C_0 and $w' = y$ is the unique vertex inside C_0'. If $xu_1 \notin S$ then $(*)$ would give us a separating 4-cycle C' which contains xu_1. Then C' cannot contain e (otherwise, since C' is separating, $C' = xu_1uvx$ holds and C_0 has the chord uu_1, contradicting the 4-connectivity of T) so C' would be the required separating 4-cycle in T. Hence $xu_1 \in S$. By symmetry, we also have $xu_2, yu_1', yu_2' \in S$. This implies that $C_0 = C_0'$ and $\{x, y\} \cup V(C_0)$ induces a copy of the octahedron in T. This contradicts the assumption that T is 4-connected and has at least seven vertices. $\qquad\square$

We can now complete the proof of the lemma. By Claim 6.4, T has a separating 4-cycle C which does not contain e. By symmetry we may assume that e lies outside of C. Replacing C by a minimal separating 4-cycle which is contained inside C if necessary, we

may assume that C is inner minimal. Relabelling u, v if necessary, Claim 6.3 now implies that $C = xuytx$ for some vertex $t \neq v$ which is adjacent to v and there is exactly one vertex z inside C.

Consider the separating 4-cycle $C' = uvtzu$ of T. By symmetry we may assume that x, w lie outside C' and y, w' lie inside C'. If $x = w$ and $y = w'$ then T would contain a copy of the octahedron. This would contradict the hypothesis that T is 4-connected plane triangulation with at least seven vertices so we may also assume by symmetry that $x \neq w$. Then $C'' = uxtvu$ is another separating 4-cycle of T. We may now choose an outer minimal separating 4-cycle C_1 of T which is contained in the closed region outside of C''. Let H_1 be the component of $T - C_1$ outside C_1.

We will show that the pair (C_1, H_1^+) contradicts Claim 6.3. We have $e \notin E(H_1^+) \backslash E(C_1)$ since $e \in E(C'')$ so (C_1, H_1^+) satisfies the hypotheses of Claim 6.3. We also have $y \notin V(H_1^+)$ since y is inside C'' and $x \notin V(H_1^+) \backslash V(C_1)$ since $x \in V(C'')$. Hence $(V(H_1^+) \backslash V(C_1)) \cap \{x, y\} = \emptyset$ so part (a) of Claim 6.3 does not hold, and $y \notin V(C_1)$ so part (b) of Claim 6.3 does not hold. $\qquad \square$

6.2   Coincident rigidity: proof of Theorem 1.4

We will use the well known result that the maximal 4-connected subgraphs, or 4-*blocks*, of a plane triangulation T form a tree like structure. More precisely we can define the 4-*block tree* of T to be the tree $\mathcal{T}$ whose vertex set is the union of the set of 4-blocks and the set of separating 3-cycles of T, in which a 4-block D is adjacent to a separating 3-cycle C if $C \subseteq D$.

Proof of Theorem 1.4.   We proceed by contradiction. Suppose the theorem is false and let $G = (T, B)$ be a counterexample such that $|E(G)|$ is as small as possible. If $|V| = 5$ then $G \cong K_5$ and the theorem holds since K_5 is uv-coincident rigid for every edge $uv \in E(K_5)$. Hence we may assume that $|V| \geq 6$. Consider the following two cases.

Case 1: T is 4-connected.   Choose $b = xy \in B$. If $T + b \neq G$ then we can apply induction to $G - b$. Hence we have $G = T + b$. Let F and F' be the faces of T which contain $e = uv$ and let S be the set of edges of T which lie on an xy-path of length two. Since T is 4-connected, we have $|V| \geq 6$ with equality only if T is the octahedron.

Suppose T is the octahedron. Then $T - V(F \cup F') \cong K_2$. Let f be the unique edge in $T - V(F \cup F')$. Then f is uv-admissible. If $f \in S$ then b is incident with an end-vertex of both e and f and, up to symmetry, there is a unique choice for e and b. Taking $e = uv$, $f = wz$ and $b = uz$ we have $G - w + g \cong K_5$ for a unique edge g joining two neighbours of w and we can now use Lemma 2.2 to obtain an infinitesimally rigid uv-coincident realisation of G from an infinitesimally rigid uv-coincident realisation of K_5. Hence $f \notin S$. Then $G/f \cong K_5$ and G/f is uv-coincident rigid. Since f is uv-admissible, Lemma 6.1 now implies that G is uv-coincident rigid.

Hence $|V(T)| \geq 7$. Lemma 6.2 now implies that there exists an edge $f \in E \backslash (E(F) \cup E(F') \cup S)$ such that $\{x, y\}$ crosses every separating 3-cycle of T/f that contains e. Then $G/f = T/f + b$ is a 4-connected braced triangulation and b is a brace of G/f. Hence G/f is uv-coincident rigid by the minimality of G. Since f is uv-admissible, G is uv-coincident rigid by Lemma 6.1.

Case 2: T is not 4-connected.   Since e does not belong to any separating 3-cycle of T, e belongs to a unique 4-block D_e of T. Since $T + B$ is 4-connected we can choose a path

$P = D_0 X_1 D_1 \ldots X_k D_k$ in the 4-block tree $\mathcal{T}$ of T such that $D_i = D_e$ for some $0 \leq i \leq k$ and some brace $b = xy \in B$ has $x \in V(D_0) \backslash X_1$ and $y \in V(D_k) \backslash X_k$. Let $T' = \bigcup_{i=0}^{k} D_i$. If $T' + b \neq G$ then the minimality of $|E(G)|$ implies that $T' + b$ is uv-coincident rigid and we can now use Lemma 2.3 to deduce that G is uv-coincident rigid. Hence $T' + b = G$. By symmetry, we may assume that $D_e \neq D_k$.

We claim that D_k is isomorphic to K_4. To see this, suppose $D_k \not\cong K_4$. Then D_k is 4-connected. Let $R = X_k + y$. Then Lemma 5.3 gives an edge $f \in E(D_k)$ such that f is not induced by R and R crosses every separating 3-cycle of D_k/f. Then $(T/f) + b$ is 4-connected so the minimality of $|E(G)|$ implies that $T/f + b$ is uv-coincident rigid. Since $e \notin E(D_k)$, f is uv-admissible and Lemma 6.1 implies that G is uv-coincident rigid, a contradiction.

Hence D_k is the complete graph on $X_k \cup \{y\}$. Denote $X_k = \{a_1, a_2, a_3\}$. We next show that there is an edge $f' = xa_i$ such that $T - y + f'$ is 4-connected for some $i \in \{1, 2, 3\}$. If $a_i \in X_k \setminus X_{k-1}$, then $T - y + f'$ is 4-connected for $f' = xa_i$. Hence, assume $k = 1$ and $X_1 = \{a_1, a_2, a_3\}$. Then $T - y = D_0$, which is a 4-connected triangulation. Also, $xa_i \notin E(D_0)$ for some i since otherwise $D_0 \cong K_4$ and $|V| = 5$, contradicting $|V| \geq 6$.

Thus, without loss of generality, we have that $T - y + xa_1$ is a 4-connected braced plane triangulation. Since e does not belong to any separating 3-cycle in $T - y$, by induction $T - y + xa_1$ is uv-coincident rigid. Observe further that G can be obtained from $T - y + xa_1$ by applying the vertex splitting operation to a_1. Since e is not induced by $\{a_1, a_2, a_3\}$ (as e belongs to no separating 3-cycle in T), we can apply Lemma 2.1 to conclude that G is uv-coincident rigid. $\qquad\square$

7 Global rigidity: proof of Theorem 1.1

In this section, we use Theorems 1.2 and 1.4 to prove Theorem 1.1.

Proof of Theorem 1.1. Let $G = (T, B)$ be a braced triangulation with at least five vertices, where $T = (V, E)$ is a plane triangulation and B is a set of braces. Necessity follows from (the 3-dimensional version of) a result of Hendrickson [12, Theorem 5.9] which implies that every globally rigid graph on at least five vertices is 4-connected and remains rigid after the removal of any edge. Hence, if G is globally rigid, then G is 4-connected, and $B \neq \emptyset$ since otherwise $G = T$ would have $3|V| - 6$ edges so $G - e$ would have too few edges to be rigid for all $e \in E(G)$.

To prove sufficiency we assume G is 4-connected and $B \neq \emptyset$. We prove that G is globally rigid in $\mathbb{R}^3$ by induction on $|V|$. If $|V| = 5$ then $G \cong K_5$ and we are done since K_5 is globally rigid. Hence we may suppose that $|V| \geq 6$.

We first consider the case when T is a 4-connected plane triangulation. Choose a brace $xy \in B$ and let S be the set of edges of T which lie on an xy-path of length two. If $|V(T)| = 6$, then T is the octahedron and $G/f \cong K_5$ for all $f \in E(T) \backslash S$, so G/f is globally rigid. This implies that G is globally rigid by Theorems 1.4 and 1.2. Hence we may assume that $|V(T)| \geq 7$. We can now apply Lemma 6.2 (taking e to be an arbitrary edge of T) to deduce that there exists an edge $f \in E(T) \setminus S$ such that $T/f + xy$ is 4-connected. Then $T/f + xy$ is a 4-connected braced triangulation so is globally rigid by induction, and we can again use Theorems 1.4 and 1.2 to deduce that G is globally rigid.

Hence we may assume that T is not 4-connected. Choose a fixed embedding of T in the plane and let C be a separating 3-cycle in T such that the component H of $G - C$ which lies inside C is minimal with respect to inclusion. Let H^+ be the subtriangulation

of T obtained from $C \cup H$ by adding all edges of T from H to C. Since G is 4-connected there is a brace $xy \in B$ with $x \in V(H)$ and $y \in V(T) \backslash V(H^+)$. The minimality of H implies that H^+ is 4-connected or is isomorphic to K_4.

Case 1: H^+ is isomorphic to K_4. We first consider the subcase when there exists a vertex $z \in V(C)$ which is not adjacent to y in T. Then G/xz is a 4-connected braced triangulation with at least one brace so is globally rigid by induction. In addition, $T - x$ is a plane triangulation so is rigid by Gluck's Theorem. This allows us to construct an xz-coincident infinitesimally rigid realisation (G, p) from a generic infinitesimally rigid realisation $(G - x, p')$ by putting $p(x) = p'(z)$ and using the fact that x has at least three neighbours other than z in G. Theorem 1.2 now implies that G is globally rigid.

It remains to consider the subcase when, for every brace xy incident to x in G, y is adjacent to every vertex of C in T. Planarity now implies that xy is the unique brace incident to x and $V(C) \cup \{y\}$ induces a copy of K_4 in T. The fact that $|V(T)| \geq 6$ now implies that $T - x$ is not 4-connected. In addition, $G - x = (T - x, B - xy)$ is a 4-connected braced plane triangulation, and has at least one brace since $T - x$ is not 4-connected. Then $G - x$ is globally rigid, by induction, and the fact that x has degree four in G now implies that G is globally rigid.

Case 2: H^+ is 4-connected. Then $|V(H)| \geq 2$ and the minimality of H now implies that some vertex $z \in V(C)$ is not adjacent to x in H^+. Then $G' = H^+ + xz$ is a braced 4-connected plane triangulation with exactly one brace. By Theorem 1.4, G' has an infinitesimally rigid uv-coincident realisation for all edges uv of H^+. We can now use Lemma 2.4, taking $G_1 := G'$ and $G_2 := G - V(H)$, to deduce:

($\diamond$) G has an infinitesimally rigid uv-coincident realisation for all edges uv of H^+ with $\{u, v\} \not\subset V(C) \cup \{x\}$.

Suppose H^+ is isomorphic to the octahedron. Let uv be the unique edge of H^+ which is not incident to a vertex in $V(C) \cup \{x\}$. Then $G/uv = T/uv + xy$ is a 4-connected braced triangulation with at least one brace so is globally rigid by induction. We can now use Theorem 1.2 and ($\diamond$) to deduce that G is globally rigid.

It remains to consider the subcase when $|V(H^+)| \geq 7$. By Lemma 5.2, there is an edge $uv \in E(H)$ such that H^+/uv is 4-connected. Then G/uv is a 4-connected braced triangulation with at least one brace which, by induction, is globally rigid. Theorem 1.2 and ($\diamond$) now imply that G is globally rigid. $\qquad\square$

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