# On the chromatic number of 2-dimensional spheres 

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#### Abstract

In 1976 Simmons conjectured that every coloring of a 2 -dimensional sphere of radius strictly greater than $1 / 2$ in three colors has a pair of monochromatic points at the distance 1 apart. We prove this conjecture.


## Introduction

A coloring of a given set $M$ is a map from $M$ to the set of colors. A coloring of a subset $M$ of a metric space is proper if no pair of monochromatic points lie at distance 1 apart. The minimum number of colors that admits a proper coloring of $M$ s called the chromatic number of $M$; we denote it by $\chi(M)$. In the case of $M \subset \mathbb{R}^{n}$, the distance typically comes from the induced Euclidean metric on $M$.

A slightly different point of view is to consider a unit distance graph $G(M)$ : the points of $M$ are the vertices of $G(M)$ and edges connect points at unit distance apart. By definition, $\chi(M)=\chi(G(M))$. The de Bruijn-Erdôs theorem states that If $\chi(M)$ is finite then there is a finite subgraph $H$ of $G(M)$ such that $\chi(H)=\chi(G(M))$.
Denote by $S^{2}(r)$ the two-dimensional sphere of radius $r$ in $\mathbb{R}^{3}$ centered at the origin. Let $\chi\left(S^{2}(r)\right)$ be the chromatic number of $S^{2}(r)$ with respect to the Euclidean metric. Obviously if $r<1 / 2$ and $r=1 / 2$ then the chromatic number is equal ' 1 and 2 , respectively. Note that for any $r>\frac{1}{2}$ there is $r_{1}<r$ such that $S^{1}\left(r_{1}\right)$ contains an odd cycle. Since $S^{1}\left(r_{1}\right) \subset S^{2}(r)$, e obtain that $\chi\left(S^{2}(r)\right) \geqslant 3$. G. Simmons [15] proved that

$$
\chi\left(S^{2}(r)\right) \geqslant 4 \text { for } r \geqslant \frac{\sqrt{3}}{3} .
$$

n the proof, Simmons constructs certain subgraphs of $G\left(S^{2}(r)\right)$ that contain triangles. Obviously, for smaller values of the didius $G\left(S^{2}(r)\right)$ is triangle-free, and so other ideas are needed.
Then L. Lovász [10] generalized the odd cycle construction to an arbitrary dimension, showing that for every $n \geqslant 3$ there xists a family of strongly self-dual polytopes inscribed in $S^{n-1}(r)$ whose graphs of diameters have chromatic number $n+1$ and that $r$ can be arbitrarily close to $\frac{1}{2}$. In our notation this result can be formulated as follows:
heorem 1 (Lovász, [10]). For every $n \geqslant 2$ there exists a monotonically decreasing sequence $r_{k}^{(n)}, k=1,2, \ldots$, such that

$$
\lim _{k \rightarrow \infty} r_{k}^{(n)}=\frac{1}{2} \quad \text { and } \quad \chi\left(S^{n-1}\left(r_{k}^{(n)}\right)\right) \geqslant n+1
$$

Since $S^{n-1}\left(r_{1}\right) \subset S^{n}(r)$ for $r_{1} \leqslant r$, we get the following inequality.
Corollary 1.

$$
\chi\left(S^{n-1}(r)\right) \geqslant n \quad \text { for } \quad r>\frac{1}{2} .
$$

Some sources state that the chromatic number of a two-dimensional sphere $S^{2}(r)$ is known only for $r \leqslant \frac{1}{2}$ and for $r=\frac{\sqrt{2}}{2}$ [5, (11]. But it should be clarified that the equality $\chi\left(S^{2}(r)\right)=n+1=4$ is true for $r \in\left\{r_{k}^{(3)}\right\} \cap\left(\frac{1}{2}, \frac{\sqrt{3-\sqrt{3}}}{2}\right]$. Explicit formulas for algebraic numbers $r_{k}^{(3)}$, if such exist, seem to be too complicated, but it is not difficult to compute $r_{k}^{(3)}$ for a given $k$ with an arbitrary precision by approximately solving a certain optimization problem. For example, the first non-trivial construction in the case of a two-dimensional sphere corresponds to a unit distance embedding of the Grötzsch graph at $r=0.54003829 \ldots$

It is worth noting that chromatic numbers in high dimensions were studied using algebraic, topological and combinatorial methods. A.M. Raigorodskii 14 showed that for every fixed $r>1 / 2$ the chromatic number of an $n$-dimensional sphere grows exponentially with $n$. O. Kostina 7 refined asymptotic lower bounds. R. Prosanov [12] gave a new asymptotic upper bound. The paper of A. Kupavskii 9$]$ contains several results on the number of different colors on a sphere of given radius in every proper coloring of $\mathbb{R}^{n}$.

A lot of results on colorings of 2-dimensional spheres were obtained by Simmons [15. Recent discovery of a 5 -chromatic unit distance subgraph of the Euclidean plane [2] spurred interest to the topic and in particular to the chromatic number of a 2 -dimensional sphere.

Among the other results, in 18 the authors constructed several 5 -chromatic subgraphs of 2-dimensional spheres, which lead to the bounds

$$
\begin{aligned}
& \chi\left(S^{2}\left(r_{1}\right)\right) \geqslant 5 \quad \text { where } \quad r_{1}=\cos \frac{3 \pi}{10}=\frac{\sqrt{5-\sqrt{5}}}{2 \sqrt{2}}=0.58778 \ldots \\
& \chi\left(S^{2}\left(r_{2}\right)\right) \geqslant 5 \quad \text { where } \quad r_{2}=\cos \frac{\pi}{10}=\frac{\sqrt{5+\sqrt{5}}}{2 \sqrt{2}}=0.95105 \ldots
\end{aligned}
$$

The paper [16] contains a family of proper colorings of $S^{2}(r)$ spheres in 7 colors, provided $r$ is large enough.
The following statement was formulated by Simmons as a conjecture [15]. The proof of Simmons' conjecture is the main result of the present paper.
Theorem 2. For every $r>\frac{1}{2}$ we have

$$
\chi\left(S^{2}(r)\right) \geqslant 4
$$

We note that for $\frac{1}{2}<r \leqslant \frac{\sqrt{3-\sqrt{3}}}{2}=0.563 \ldots$ a proper 4-coloring of $S^{2}(r)$ can be obtained from a partition of the sphere into four equal spherical triangles [15]. It implies the following corollary.

Corollary 2. $\chi\left(S^{2}(r)\right)=4$ for $\frac{1}{2}<r \leqslant \frac{\sqrt{3-\sqrt{3}}}{2}=0.563 \ldots$.
Structure of the paper. Section 2 contains the proof of Theorem 2. In Section 3 we summarize the results and discuss some further questions.

## 2 Proof of Theorem 2

Recall that for $r \geqslant \frac{\sqrt{3}}{3}$ the statement was proved in [15].
Here is the sketch of the proof. Fix $r \in\left(\frac{1}{2}, \frac{\sqrt{3}}{3}\right)$. The proof consists of two steps. Suppose that there is a proper 3 -coloring of the sphere $S^{2}(r)$. In the first step we use the Borsuk-Ulam theorem to show that every color is dense in the sphere. Consider a graph $G_{k}$ with vertices $x_{1}, \ldots x_{2 k+1}, y_{1}, \ldots, y_{2 k+1}$ and edges $\left\{\left(y_{i}, y_{i+1}\right),\left(x_{i}, y_{i}\right): 1 \leqslant i \leqslant 2 k+1\right\}$ (where indices are modulo $2 k+1$ ). We provide an explicit representation of $G_{k}$ as a unit distance subgraph of the sphere. The second step is to show that this embedding is stable under small perturbations of $x_{i}$. Then one can move every $x_{i}$ at a red point, which forces the odd cycle on vertices $y_{i}$ to be colored in the remaining two colors. The contradiction proves the theorem.

Note that the idea of attaching an odd cycle to a finite set $A$ in order to exclude the possibility of $A$ to be monochromatic was used in a series of papers devoted to the existence of planar unit distance graphs with chromatic number 4 and arbitrarily large girth [4, 17, 19]. The key twist in step 2 is to find the required embedding of $G_{k}$ implicitly, i.e. the corresponding $A$ is not a constructive set. Similar ideas were used by the authors in [6].

### 2.1 Step 1. Each color is a dense set

All the distances are considered in the metrics induced from Euclidean space $\mathbb{R}^{3}$, the distance between $x$ and $y$ is denoted by $\|x-y\|$.

Fix $r \in\left(\frac{1}{2}, \frac{\sqrt{3}}{3}\right)$ and consider $S^{2}(r)$. Suppose that there is a proper coloring of $S^{2}(r)$ in three colors. Consider the unit distance graph $G=G\left(S^{2}(r)\right)$. Then neighborhood of a vertex in $G$ forms a circle of diameter $d=\frac{\sqrt{4 r^{2}-1}}{r}$ and radius $\rho=\sqrt{4 r^{2}-1}$ in the induced metric, centered at the opposite point of the sphere. Vice versa, any circle of such radius is a graph-neighborhood of some vertex, and hence contains points of at most two colors. We need the following technical statement.

Lemma 1. Let $D \subset S^{2}(r) \times S^{2}(r)$ be a set of pairs $(x, y)$ such that $0<\|x-y\|<d$. Then

- for every $(x, y) \in D$ there are two circles of radius $\rho$ containing $x$ and $y$. One may denote their centers by $c_{r}$ and $c_{l}$ in such a way that the triple of radius-vectors $\left(x, y, c_{r}\right)$ is right-handed and the triple ( $x, y, c_{l}$ ) is left-handed.
- The functions $c_{r}(x, y)$ and $c_{l}(x, y)$ from $D$ to $S^{2}(r)$ are continuous.

In what follows, we will call a circle passing through the points $x, y$ with center $c$ right-handed if the triple $(x, y, c)$ is right-handed, and left-handed otherwise.

Let $\underline{C_{\text {red }}}, \underline{C_{\text {blue }}}, \underline{C_{\text {green }}}$ be the sets of red, blue and green points, respectively. A chromaticity of a point $x$ is the number of sets $\overline{C_{\text {red }}}, \overline{C_{\text {blue }}}, \overline{C_{\text {green }}}$ containing $x$ (as usual, $\bar{T}$ stands for the closure of a set $\left.T\right)$. A set $T \subset S^{2}(r)$ is called dense if $\bar{T}=S^{2}(r)$. Let $B_{\rho}(x)$ denote the set of points $y \in S^{2}(r)$ such that $\|x-y\|<\rho$, i.e. an open ball of radius $\rho$ and diameter $d$.

Lemma 2. If some open ball of diameter d contains points of all three colors then each of $C_{r e d}, C_{b l u e}, C_{\text {green }}$ is dense in the sphere.

Proof. Consider points $x \in C_{r e d}, y \in C_{\text {blue }}$ and $z \in C_{\text {green }}$ inside a ball $K_{0}$ of diameter $d$. Then one can continuously move $K_{0}$ to a ball $K$ containing two points (say, $x$ and $y$ ) on the boundary; at the first such moment the point $z$ lies inside $K$. The circle $\partial K$ contains blue and red points and so it is colored in blue and red only. Hence, it contains a point $u$ lying in the closures of $C_{r e d}$ and $C_{b l u e}$; without loss of generality, assume that point $u$ is red. A red-green circle (right-handed, see Lemma 1) of diameter $d$ containing $z$ and $u$ and a blue-green circle (left-handed) with the diameter $d$ containing $z$ and blue point $u^{\prime}$ in a small neighborhood of $u$ intersect in a green point $v$. Note that if $u=u^{\prime}$ then $v=u=u^{\prime}$. Hence, due to the continuity of circles in Lemma 10 may be arbitrarily close to $u$ with a proper choice of $u^{\prime}$ (see Fig. (1). It implies that the chromaticity of $u$ is three.


Figure 1: Finding a point with chromaticity 3 in Lemma 2
Since $u$ has chromaticity 3 , a small neighborhood of $u$ contains a point $a \neq u$ with the chromaticity at least 2 . Suppose that $a$ has chromaticity 2 (say, $a$ does not lie in $\overline{C_{\text {green }}}$ ) and $\|a-u\|<d$. Consider a green point $b$ in a small neighborhood of $u$. Consider a red point $e$ and a blue point $f$ in a small neighborhood of $a$. Then the right-handed circle containing $b$ and $e$ is red-green and the left-handed circle containing $b$ and $f$ is blue-green, so they intersect in a green point $g$. Since the neighborhoods can be chosen arbitrarily small, $g$ can be arbitrarily close to $a$. Hence $a$ has chromaticity 3 , a contradiction.

Thus we have shown that if a point with the chromaticity 3 and a point with the chromaticity at least 2 lie at a distance smaller than $d$, then they both have chromaticity 3 .


Figure 2: Propagation of 3-chromaticity along a circle in Lemma 2
Now let $x_{1}$ and $x_{2}$ be points of chromaticity 3 such that $\left\|x_{1}-x_{2}\right\|<d$. We claim that any point on a circle $L$ of diameter $d$ containing $x_{1}$ and $x_{2}$ has chromaticity three. By the previous argument it is enough to show that the chromaticity is at least 2. Without loss of generality, a triple $\left(x_{1}, x_{2}, c\right)$ is left-handed, where $c$ is the center of $L$ on the sphere. Arguing indirectly, assume that a point $y_{1} \in L$ has a small red neighborhood $U_{y_{1}}$. Choose a blue point $u_{1}$ in a small neighborhood of $x_{1}$ and a green point $v_{1}$ in a small neighborhood of $x_{2}$ (see Fig. 24). By Lemma 1 the left-handed circle of diameter $d$ passing through blue point $u_{1}$, green point $v_{1}$ is close to $L$ so it intersects red set $U_{y_{1}}$; this contradiction shows that every point on $L$ has chromaticity 3 .

Let $q$ be an arbitrary point of $S^{2}(r)$. Consider a path $q_{0}, q_{1} \ldots q_{t}=q$ such that $q_{0} \in L$ and $\left\|q_{i+1}-q_{i}\right\|<\rho$ for $0 \leqslant i \leqslant t-1$. A circle $L_{1}$ of diameter $d$ that passes through $q_{1}$ and $q_{0}$ intersects $L$ in two points, so by the previous argument every point (in particular, $q_{1}$ ) of $L_{1}$ has chromaticity 3 . By induction, a circle $L_{i+1}$ of diameter $d$ that passes through $q_{i+1}$ and $q_{i}$ intersects $L_{i}$ in two points, so every point in $L_{i+1}$ (in particular $q_{i+1}$ ) has chromaticity 3 . So $q=q_{t}$ also has chromaticity 3 . Since $q \in S^{2}(r)$ was arbitrary, every point of $S^{2}(r)$ has chromaticity 3.

Suppose that the condition of Lemma 2 does not hold, i.e.
every open ball of diameter d contains points of at most two colors.
Consider a continuous function

$$
f: S^{2}(r) \rightarrow \mathbb{R}^{2}, \quad f(x)=\left(\operatorname{dist}\left(x, \overline{C_{\text {red }}}\right), \operatorname{dist}\left(x, \overline{C_{\text {blue }}}\right)\right)
$$

where $\operatorname{dist}(\cdot)$ stands for the distance between a point and a set in $\mathbb{R}^{3}$. By the Borsuk-Ulam theorem there exists $x^{*} \in S^{2}(r)$ such that $f\left(x^{*}\right)=f\left(-x^{*}\right)$. We have to deal with three cases.


Figure 3: Case 1

Case 1: $f\left(x^{*}\right)=(0,0)$. Without loss of generality, the point $x^{*}$ is blue. One may pick a red point $z$, which is arbitrarily close to $x^{*}$. If $\|x-z\|<\rho$, then the intersection of circles of unit Euclidean radius with centers $x^{*}$ and $z$ consists of two green points $y_{1}, y_{2}$ belonging to the circle of radius $\rho$ centered at $-x^{*}$. Hence, one can cover a small neighborhood of $-x^{*}$ and $y_{1}$ by a ball of diameter $d$. Every neighborhood of $-x^{*}$ contains red and blue points; point $y_{1}$ is green (see Fig. (3). We have a contradiction with assumption ( $\star$ ).

Case 2: $f\left(x^{*}\right)=(a, b), a, b>0$. Then both points $x^{*},-x^{*}$ are green. We may swap blue and green colors to reduce the situation to the next case with the same $x^{*}$.

Case 3: $f\left(x^{*}\right)=(a, 0), a>0$. We claim that $a>\rho$. Assume the contrary, i.e. $x^{*} \in \overline{C_{b l u e}}$ and for every $\eta>0$ there is a red point $z=z_{\eta}$ such that $\left\|x^{*}-z\right\| \leqslant \rho+\eta$. Note that if $x^{*}$ is green, then it contradicts $(\star)$, so $x^{*}$ is blue. There are distinct points $y_{1}, y_{2} \in \overline{B_{\rho}\left(-x^{*}\right)}$ such that $\left\|x^{*}-y_{1}\right\|=\left\|x^{*}-y_{2}\right\|=\left\|z-y_{1}\right\|=\left\|z-y_{2}\right\|=1$. Since $x^{*}$ is blue and $z$ is red $y_{1}, y_{2} \in \overline{C_{\text {green }}}$. Recall that $f\left(-x^{*}\right)=f\left(x^{*}\right)$, so there is a point $z^{\prime} \in \overline{C_{r e d}} \cap \overline{B_{\rho}\left(-x^{*}\right)}$. Let $y^{\prime} \in\left\{y_{1}, y_{2}\right\}$ be such that $z^{\prime}$, $-x^{*}$ and $y^{\prime}$ do not lie on a great circle of $S^{2}(r)$. Then for a small enough $\eta$ the neighborhoods of $-x^{*}, y^{\prime}$ and $z^{\prime}$ can be covered by a ball of diameter $d$. This is a contradiction with ( $\star$ ).

So the set $\overline{B_{\rho}\left(x^{*}\right)} \cup \overline{B_{\rho}\left(-x^{*}\right)}$ is colored with blue and green.
Lemma 3. The bipartite subgraph of $S^{2}(r)$ with parts $\overline{B_{\rho}\left(x^{*}\right)}$ and $\overline{B_{\rho}\left(-x^{*}\right)}$ is connected.
Proof. Any point $x \in \overline{B_{\rho}\left(x^{*}\right)}$ has a common neighbor with $x^{*}$ since the corresponding unit circles intersect. So $\overline{B_{\rho}\left(x^{*}\right)}$ belong to the same connected component; the same holds for $\overline{B_{\rho}\left(-x^{*}\right)}$. There is an edge between $\overline{B_{\rho}\left(x^{*}\right)}$ and $\overline{B_{\rho}\left(-x^{*}\right)}$, and so the subgraph is connected.

By Lemma 3, one can color $\overline{B_{\rho}\left(x^{*}\right)} \cup \overline{B_{\rho}\left(-x^{*}\right)}$ in two colors in the unique way (up to symmetry): the first part is blue and the second one is green. Then the distance from $x^{*}$ and $-x^{*}$ to $\overline{C_{b l u e}}$ is zero and nonzero simultaneously.

This contradiction implies that each color is dense in the sphere.

### 2.2 Step 2. Stability of embedding

In this section we will need the implicit function theorem [8] in the following weakened formulation.
Theorem 3. Let $F: \mathbb{R}^{2 s} \rightarrow \mathbb{R}^{s}$ be a continuously differentiable function,

$$
F=F(X, Y)=F\left(x_{1}, \ldots, x_{s} ; y_{1}, \ldots, y_{s}\right)
$$

and at some point $X=a, Y=b$ the following conditions are satisfied

$$
F(a, b)=0, \quad \operatorname{det}\left(\frac{\partial F(X, Y)}{\partial Y}\right)_{X=a, Y=b} \neq 0
$$

Then there exists $\eta>0$ such that the system of equations $F(X, Y)=0$ is solvable in $Y$ for any $X$ satisfying the condition $\|X-a\|<\eta$.

Recall that $G_{k}$ is an odd cycle of length $m=2 k+1$ with an extra pendant (leaf) vertex attached to each vertex of the cycle. In particular, $G_{k}$ has $2 m$ vertices and $2 m$ edges.

Denote by $y_{1}, \ldots, y_{m}$ the points of $S^{2}(r)$ that correspond to the cycle vertices and by $x_{1}, \ldots, x_{m}$ the points of $S^{2}(r)$ that correspond to the pendant vertices. For convenience, let us put $X=\left(x_{1}, \ldots, x_{m}\right)$ and $Y=\left(y_{1}, \ldots, y_{m}\right)$ the vectors of dimension $s=3 m$ containing all coordinates. Then the embedding of $G_{k}$ can be given by the pair $(X, Y)$.
Lemma 4. Fix the radius $r \in\left(\frac{1}{2}, \frac{\sqrt{3}}{3}\right)$. Then if $k$ is large enough, there exists a unit distance embedding $(X, Y)$ of $G_{k}$ into $S^{2}(r)$ and a constant $\eta>0$ such that for any $\tilde{X}$ satisfying $\|\tilde{X}-X\|<\eta$ there exists $Y$ such that $(\tilde{X}, \tilde{Y})$ is a "perturbed" unit distance embedding of $G_{k}$.

In other words, for any sufficiently small perturbation of pendant vertices, it is possible to find the embedding of the cycle vertices.

Proof. We provide the desired unit distance embedding explicitly. In what follows we slightly abuse the notation and write $x_{i}$ and $y_{i}$ for a vertex of the graph, the corresponding point on $S^{2}(r)$, and its 3-dimensional vector representation. Consider the system of equations defining the embedding $G_{k}$ in $S^{2}(r)$ :

$$
\left\{\begin{array}{l}
f_{i}=\left\|y_{i}\right\|^{2}-r^{2}=0, \quad 1 \leqslant i \leqslant m  \tag{1}\\
f_{i+m}=\left\|y_{i}-y_{i+1}\right\|^{2}-1=0, \quad 1 \leqslant i \leqslant m-1 \\
f_{2 m}=\left\|y_{m}-y_{1}\right\|^{2}-1=0 ; \\
f_{i+2 m}=\left\|x_{i}-y_{i}\right\|^{2}-1=0, \quad 1 \leqslant i \leqslant m
\end{array}\right.
$$

Next, we will be interested in the family of embeddings, the $k=2$ case of which is depicted on Fig. (4)


Figure 4: Unit distance embedding of $G_{k}$, the $k=2$ case
Note that (1) allows $x_{i}$ to lie in $\mathbb{R}^{3}$, not only $S^{2}(r)$, but the cycle $y_{1}, \ldots, y_{m}$ must lie on the sphere.
One can consider the function corresponding to the left-hand side of the system (1).

$$
F=\left(f_{1}, \ldots, f_{3 m}\right)=F\left(x_{11}, x_{12}, x_{13}, \ldots, x_{m 3} ; y_{11}, \ldots, y_{m 3}\right)
$$

Suppose that the Jacobian matrix $J=\left(\frac{\partial F}{\partial Y}\right)$ is nondegenerate,

$$
\operatorname{det} J=\operatorname{det}\left(\frac{\partial F}{\partial Y}\right) \neq 0
$$

then the statement of the lemma follows from Theorem 3. The rest of the proof is devoted to the calculation of this determinant.

The matrix $J$ has the following form (recall that $x_{i}$ and $y_{i}$ are $1 \times 3$ vectors):

$$
J(X, Y)=2\left(\begin{array}{cccccc}
y_{1} & 0 & 0 & 0 & \ldots & 0 \\
0 & y_{2} & 0 & 0 & \ldots & 0 \\
0 & 0 & y_{3} & 0 & \ldots & 0 \\
\vdots & & & \ddots & & \\
0 & 0 & 0 & 0 & \ldots & y_{m} \\
y_{1}-y_{2} & y_{2}-y_{1} & \ldots & 0 & \ldots & 0 \\
0 & y_{2}-y_{3} & y_{3}-y_{2} & 0 & \ldots & 0 \\
\vdots & & & & & \\
y_{1}-y_{m} & 0 & \ldots & \ldots & 0 & y_{m}-y_{1} \\
y_{1}-x_{1} & 0 & \ldots & \ldots & 0 & 0 \\
0 & y_{2}-x_{2} & 0 & \ldots & 0 & 0 \\
\vdots & & \vdots & & \vdots & \\
0 & \ldots & 0 & \ldots & 0 & y_{m}-x_{m}
\end{array}\right) .
$$

Subtracting some rows from each other, we get

$$
\operatorname{det} J=2^{3 m} \operatorname{det}\left(\begin{array}{cccccc}
y_{1} & 0 & 0 & 0 & \ldots & 0 \\
0 & y_{2} & 0 & 0 & \ldots & 0 \\
0 & 0 & y_{3} & 0 & \ldots & 0 \\
\vdots & & & \ddots & & \\
0 & 0 & 0 & 0 & \ldots & y_{m} \\
y_{2} & y_{1} & \ldots & 0 & \ldots & 0 \\
0 & y_{3} & y_{2} & 0 & \ldots & 0 \\
\vdots & & & \ddots & & \\
y_{m} & \ldots & 0 & \ldots & 0 & y_{1} \\
x_{1} & 0 & 0 & 0 & \ldots & 0 \\
\vdots & & & \ddots & & \\
0 & 0 & 0 & 0 & \ldots & x_{m}
\end{array}\right)=2^{3 m} \operatorname{det}\left(\begin{array}{cccccc}
y_{1} & 0 & 0 & 0 & \ldots & 0 \\
x_{1} & 0 & 0 & 0 & \ldots & 0 \\
y_{2} & y_{1} & 0 & \ldots & \ldots & 0 \\
0 & y_{2} & 0 & 0 & \ldots & 0 \\
0 & x_{2} & 0 & 0 & \ldots & 0 \\
0 & y_{3} & y_{2} & 0 & \ldots & 0 \\
\vdots & & & \ddots & & \\
0 & 0 & 0 & 0 & \ldots & y_{m} \\
0 & 0 & 0 & 0 & \ldots & x_{m} \\
y_{m} & 0 & 0 & \ldots & 0 & y_{1}
\end{array}\right)=
$$

where

$$
V_{i}=-\operatorname{det}\left(\begin{array}{c}
y_{i} \\
y_{i+1} \\
x_{i}
\end{array}\right), \quad V_{i}^{\prime}=\operatorname{det}\left(\begin{array}{c}
y_{i} \\
y_{i+1} \\
x_{i+1}
\end{array}\right)
$$

Now we fix the following embedding (Fig. (4). Let vertices $y_{i}$ lie in the plane $z=h$ (and form a regular $m$-gon), and vertices $x_{i}$ lie in the plane $z=-h$ (and also form a regular $m$-gon). Note that the radius of the circumcircle of the $m$-gon is greater than $1 / 2$, hence

$$
\begin{equation*}
h<\left(\frac{1}{3}-\frac{1}{4}\right)^{1 / 2}=\frac{1}{2 \sqrt{3}}<\frac{1}{2} \tag{2}
\end{equation*}
$$

Denote by $U_{m}$ the rotation matrix by an angle $2 \pi / m$ counterclockwise around $z$-axis. Then $y_{i+1}=U_{m} y_{i}, x_{i+1}=U_{m} x_{i}$. Hence, all $V_{i}$ coincide and all $V_{i}^{\prime}$ also coincide; put $V=V_{i}$ and $V^{\prime}=V_{i}^{\prime}$. Hence

$$
\operatorname{det} J=V^{m}+\left(V^{\prime}\right)^{m}
$$

We claim that

$$
V+V^{\prime}=\operatorname{det}\left(\begin{array}{c}
y_{1} \\
y_{2} \\
x_{2}-x_{1}
\end{array}\right) \neq 0
$$

Indeed, since $y_{13}=y_{23}=h, x_{13}=x_{23}=-h$, the equality

$$
\alpha y_{1}+\beta y_{2}+\gamma\left(x_{2}-x_{1}\right)=0
$$

implies $\alpha=-\beta$, i.e.

$$
\begin{equation*}
\alpha\left(y_{1}-y_{2}\right)=\gamma\left(x_{1}-x_{2}\right) \tag{3}
\end{equation*}
$$

Recall that $\left\|y_{1}-y_{2}\right\|=\left\|x_{1}-x_{2}\right\|=1$, so $\alpha= \pm \gamma$.
Since both sets of points $\mathcal{X}=\left\{x_{1}, \ldots, x_{m}\right\}, \mathcal{Y}=\left\{y_{1}, \ldots, y_{m}\right\}$ form vertices of congruent regular $m$-gons, in the case $\alpha=\gamma$, we have $x_{1}-x_{2}=y_{1}-y_{2}$ and the projections of $x_{i}$ and $y_{i}$ on the plane $z=0$ coincide, $i=1,2, \ldots, m$, and taking into account (2), we have

$$
\left\|x_{1}-y_{1}\right\|=2 h<1
$$

In the case $\alpha=-\gamma$, we have $x_{1}-x_{2}=y_{2}-y_{1}$ and the sets $\mathcal{X}$ and $\mathcal{Y}$ are symmetric about the origin. Then $x_{1} x_{2} y_{1} y_{2}$ is a rectangle, and

$$
\left\|x_{1}-y_{1}\right\|^{2}>\left\|x_{1}-x_{2}\right\|^{2}+4 h^{2}>1 .
$$

In both cases we got a contradiction. Then the equation (3) does not hold and so $V+V^{\prime} \neq 0$. Hence

$$
\operatorname{det} J=V^{m}+\left(V^{\prime}\right)^{m} \neq 0
$$

as required.

## 3 Open questions

Does the chromatic number of $S^{2}(r)$ «almost» grow with $r$ ? Id est is the chromatic number monotonic except for at most countable set of values $r$ ? Recall that the known results (see Table 1) allow for such possibility.

| $r$ | Estimate for $\chi(r)=\chi\left(S^{2}(r)\right)$ | Source |
| :---: | :---: | :---: |
| $r<1 / 2$ | $\chi(r)=1$ |  |
| $r=1 / 2$ | $\chi(r)=2$ |  |
| $\frac{1}{2}<r \leqslant \frac{\sqrt{3-\sqrt{3}}}{2}$ | $\chi(r)=4$ | Corollary 1 |
| $r>\frac{\sqrt{3-\sqrt{3}}}{2}$ | $\chi(r) \geqslant 4$ | Theorem 2 |
| $r=\frac{\sqrt{5-\sqrt{5}}}{2 \sqrt{2}}$ | $\chi(r) \geqslant 5$ | $[18]$ |
| $r=\frac{1}{\sqrt{2}}$ | $\chi(r)=4$ | $[15,3]$ |
| $r=\frac{\sqrt{5+\sqrt{5}}}{2 \sqrt{2}}$ | $\chi(r) \geqslant 5$ | $[18]$ |
| $r \leqslant \frac{1}{\sqrt{3}}$ | $\chi(r) \leqslant 5$ | $[15,[1]$ |
| $r \leqslant \sqrt{3} / 2$ | $\chi(r) \leqslant 6$ | $[11]$ |
| $r \geqslant 12.44$ | $\chi(r) \leqslant 7$ | $[16]$ |
| $r>1 / 2$ | $\chi(r) \leqslant 15$ | $[1,[13]$ |

Table 1: Lower and upper estimates for $\chi\left(S^{2}(r)\right)$.

Is there a proper coloring of $S^{2}(r)$ in $\chi\left(S^{2}(r)\right)$ colors such that every color is dense? It is interesting that all known upper bounds are given by explicit colorings in which every color is a finite union of regions bounded by piecewise-continuous curves.

What is the minimal number of vertices in a subgraph $G$ of a sphere $S^{2}(r)$ with $\chi(G)=\chi\left(S^{2}(r)\right)$ ? By the de Bruijn-Erdős theorem this number is finite. Note that the proof of Theorem 2 does not give any finite 4 -chromatic unit distance graph.


Figure 5: 4-coloring of the sphere. Here $s_{0} \rightarrow 0$ as $r \rightarrow 1 / 2$
Let us focus on the case $r=1 / 2+\varepsilon, \varepsilon \rightarrow 0$. Then the sphere can be colored in 4 colors in the way shown in Figure 5, 5 Let us denote by $s_{0}$ the area of the spherical cap of color 0 . Observe that $s_{0}=4 \pi \varepsilon+o(\varepsilon)$, and thus, via averaging, we have the lower bound $n_{4}(r) \geqslant c \varepsilon^{-1}$ for some $c>0$, where $n_{4}(r)$ is the minimal number of vertices in a 4 -chromatic unit distance graph. Can this obvious bound be refined?

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