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A COMBINATORIAL PROOF OF KNESER'S CONJECTURE*

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Kneser's conjecture, first proved by Lovász in 1978, states that the graph with all k-element subsets of $\{1,2,\dots,n\}$ as vertices and with edges connecting disjoint sets has chromatic number $n-2k+2$. We derive this result from Tucker's combinatorial lemma on labeling the vertices of special triangulations of the octahedral ball. By specializing a proof of Tucker's lemma, we obtain self-contained purely combinatorial proof of Kneser's conjecture.

1. Introduction

Let $\binom{[n]}{k}$ denote the system of all k-element subsets of the set $[n] = \{1,2,\dots,n\}$. The *Kneser graph* $\mathrm{KG}(n,k)$ has vertex set $\binom{[n]}{k}$ and edge set $\{\{S,S'\} : S,S' \in \binom{[n]}{k}, S\cap S' = \emptyset\}$. Kneser [7] conjectured in 1955 that $\chi(\mathrm{KG}(n,k)) \geq n-2k+2$, $n \geq 2k \geq 2$, where χ denotes the chromatic number. This was proved in 1978 by Lovász [11], as one of the earliest and most spectacular applications of topological methods in combinatorics. Several other proofs have been published since then, all of them topological; among them, at least those of Bárány [2], Dol'nikov [4] (also see e.g. [3]), and Sarkaria [15] can be regarded as substantially different from each other and from Lovász' original proof. These proofs are also presented in [13].

In this paper, we first we show how Kneser's conjecture can be derived directly from Tucker's lemma, which is a combinatorial statement concern-

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ing labelings of vertices of certain triangulations (somewhat resembling the well-known Sperner's lemma). Then we give a proof of Kneser's conjecture that avoids any mentioning of topology or triangulations, by reformulating and specializing a known proof of Tucker's lemma (namely one due to Freund and Todd [6]) for our particular setting. The proof is self-contained and can be read without reference to the part with Tucker's lemma. Other known proofs of Tucker's lemma, such as the one in [10], seem to lead to similar considerations when traced down to the "combinatorial core", although many details can be varied.

Currently the topological proofs appear preferable in all respects, except possibly for a short presentation to an audience with no topological background. On the other hand, it is interesting that a reasonable combinatorial proof can be given, and perhaps further investigation might lead to combinatorial methods of real interest.

The proof from Tucker's lemma is inspired by previous author's simplified proof [12] of a theorem of Kříž [8], [9] (a generalization of Kneser's conjecture). The approach of Kříž is in turn based on some ideas from Alon, Frankl, and Lovász [1], who established a generalized Kneser's conjecture for hypergraphs.

2. A proof from Tucker's lemma

Tucker [16] in his lecture at the First Canadian Math. Congress gave an elementary proof of the 2-dimensional Borsuk–Ulam theorem using a combinatorial lemma about labelings of the vertices of a particular triangulation of the square. A slightly different version, concerning triangulations of the n-dimensional octahedral sphere, was used in Lefschetz's textbook [10]. Later, this lemma has been re-proved and generalized in several papers, mainly because it provides an "effective" proof of the Borsuk–Ulam theorem; see, for example, Freund [5] or Freund and Todd [6]. We use the version of Tucker's lemma given in the latter paper.

Let B^n denote octahedral ball in $\mathbb{R}^n$ (the unit ball of the ℓ_1-norm), let S^{n-1} denote its boundary (the octahedral sphere), and let K_0 be the natural triangulation of B^n induced by the coordinate hyperplanes (the n-dimensional simplices are in one-to-one correspondence with the orthants in $\mathbb{R}^n$). Call a triangulation K of B^n a *special triangulation* if it refines K_0 and is antipodally symmetric around the origin.

Lemma 2.1 (Tucker's lemma). *Let K be a special triangulation of B^n, and suppose that each vertex v of K is assigned a label $\lambda(u) \in$*

$\{+1, -1, +2, -2, \ldots, +n, -n\}$ *in such a way that for the vertices of K lying in S^{n-1} the labeling satisfies $\lambda(-u) = -\lambda(u)$. Then there exists a 1-simplex (edge) of K which is* complementary*, i.e. its two vertices are labeled by opposite numbers.*

Proof of Kneser's conjecture from Tucker's lemma. First we define the appropriate triangulation K. Let L_0 be the subcomplex of K_0 consisting of the simplices lying on S^{n-1}. We note that the nonempty simplices of L_0 are in one-to-one correspondence with nonzero vectors from $V = \{-1, 0, 1\}^n$; see Fig. 1(a) for a 2-dimensional illustration. The inclusion relation on the

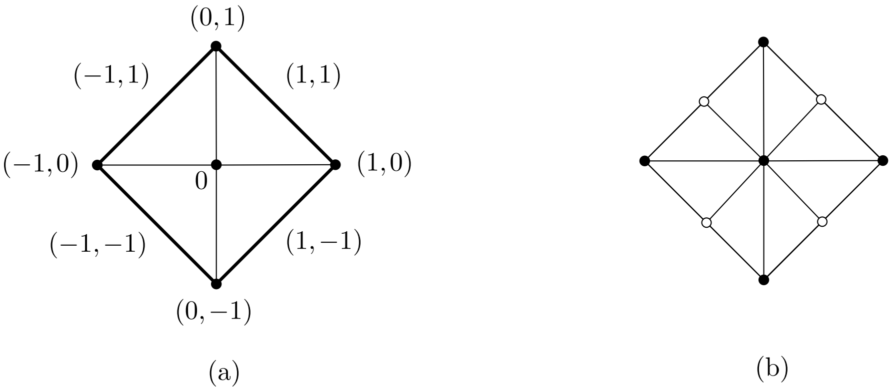

Figure 1. The triangulations K_0 and L_0 (a), and K (b).

simplices of L_0 corresponds to the relation $\preceq$ on V, where $u \preceq v$ if $u_i \preceq v_i$ for all $i = 1, 2, \ldots, n$ and where $0 \preceq 1$ and $0 \preceq -1$.

Let L_0' be the first barycentric subdivision of L_0. Thus, the vertices of L_0' are centers of gravity of the simplices of L_0 and the simplices of L_0' correspond to chains of simplices of L_0 under inclusion. A simplex of L_0' can be identified with a chain in the set $V \setminus \{(0, \ldots, 0)\}$ under $\preceq$. Finally, we define the triangulation K: it consists of the simplices of L_0' and of the cones with apex 0 over such simplices. This is a special triangulation of B^n as in Tucker's lemma.

Let $n > 2k \geq 2$. Suppose that c is a coloring of the vertices of the Kneser graph $\mathrm{KG}(n, k)$ with $n - 2k + 1$ colors (as we will show, it cannot be a proper coloring, so we regard it just as an arbitrary assignments of colors to the k-tuples in $\binom{[n]}{k}$). For technical convenience, we suppose that the colors are numbered $2k, 2k+1, \ldots, n$. We are going to define a labeling of the vertices of K as in Tucker's lemma. These vertices include 0 and so they can be identified with the vectors of V,[1] and we want to define a labeling $\lambda: V \to \{\pm 1, \ldots, \pm n\}$.

Let us fix some arbitrary linear ordering $\leq$ on $2^{[n]}$ that refines the partial ordering according to size, i.e. such that if $|A| < |B|$ then $A < B$. Let $v \in V$ be a sign vector. To define $\lambda(v)$, we consider the ordered pair (A, B) of disjoint subsets of $[n]$ defined by

$$A = \{i \in [n] : v_i = 1\} \text{ and } B = \{i \in [n] : v_i = -1\}.$$

We distinguish two main cases. If $|A| + |B| \leq 2k - 2$ (Case I) then we put

$$\lambda(v) = \begin{cases} |A| + |B| + 1 & \text{if } A \geq B \\ -(|A| + |B| + 1) & \text{if } A < B. \end{cases}$$

If $|A| + |B| \geq 2k - 1$ (Case II) then at least one of A and B has size at least k. If, say, $|A| \geq k$ we define $c(A)$ as $c(A')$, where A' consists of the first k elements of A, and for $|B| \geq k$, $c(B)$ is defined similarly. We set

$$\lambda(v) = \begin{cases} c(A) & \text{if } A > B \\ -c(B) & \text{if } B > A. \end{cases}$$

So labels assigned in Case I are in $\{\pm 1, \ldots, \pm(2k-1)\}$ while labels assigned in Case II in $\{\pm 2k, \ldots, \pm n\}$.

One can check that this is a well-defined mapping from V to $\{\pm 1, \pm 2, \ldots, \pm n\}$ and that it has the antipodal property, i.e. for every nonzero $v \in V$ we have $\lambda(-v) = -\lambda(v)$. Therefore, by Tucker's lemma, there is a complementary edge of K.

Suppose that the complementary edge connects vertices $v_1, v_2 \in V$, so $\lambda(v_1) = -\lambda(v_2)$. Since $\{v_1, v_2\}$ is an edge of K, we have $v_1 \preceq v_2$ (after a possible renaming), and so if (A_1, B_1) and (A_2, B_2) are the corresponding set pairs, we have $A_1 \subseteq A_2$ and $B_1 \subseteq B_2$, with at least one of the inclusions being proper. From this it is seen that neither of the labels $\lambda(v_1)$ and $\lambda(v_2)$ could have been assigned in Case I, since $|A_1| + |A_2| < |B_1| + |B_2|$.

Suppose, for instance, that $\lambda(v_1) > 0$ (the other case is symmetric). Then, by the definition in Case II, $\lambda(v_1)$ is the color of a k-tuple contained in A_1 and $-\lambda(v_2) = \lambda(v_1)$ is the color of a k-tuple contained in B_2. We have $B_2 \cap A_2 = \emptyset$ and $A_1 \subseteq A_2$, and so $A_1 \cap B_2 = \emptyset$. We have found two disjoint k-tuples with the same color and c is not a proper coloring of the Kneser graph $\mathrm{KG}(n, k)$. This proves Kneser's conjecture. ∎

[1] The coordinate vector of the vertex corresponding to a nonzero $v \in V$ is $\frac{v}{\|v\|_1}$, where $\|v\|_1 = |v_1| + |v_2| + \cdots + |v_n|$.

3. A direct combinatorial proof

The basic objects in the proof are *disjoint ordered pairs* (A,B), where $A,B\subseteq [n]$ and $A\cap B=\emptyset$. The forthcoming definition of a labeling function on disjoint ordered pairs is identical to that in the preceding section but we repeat it for reader's convenience.

For contradiction, we suppose that c is a proper coloring of the Kneser graph $\mathrm{KG}(n,k)$ by $n-2k+1$ colors, which for convenience we assume to be the integers $2k, 2k{+}1, \ldots, n$. We let $\leq$ be a linear ordering of the subsets of $[n]$ such that if $|A|<|B|$ then $A<B$. For each disjoint ordered pair (A,B), we define the label $\lambda(A,B)$. We distinguish two main cases. If $|A|+|B|\leq 2k-2$ (Case I) then we put

$$\lambda(A,B)=\begin{cases}|A|+|B|+1 & \text{if } A\geq B\\ -(|A|+|B|+1) & \text{if } A<B.\end{cases}$$

If $|A|+|B|\geq 2k-1$ (Case II) then

$$\lambda(A,B)=\begin{cases}c(A) & \text{if } A>B\\ -c(B) & \text{if } B>A,\end{cases}$$

where $c(A)$ is the color of the first k elements of A and $c(B)$ is the color of the first k elements of B. This is a valid definition since if, for instance, $A>B$ and $|A|+|B|\geq 2k-1$ then $|A|\geq k$. The labels assigned in Case I are in $\{\pm 1,\ldots,\pm(2k-1)\}$ while labels assigned in Case II in $\{\pm 2k,\ldots,\pm n\}$.

Next, we consider labels of signed sequences. A *signed sequence* is a sequence $(s_1,s_2,\ldots,s_m)$, where $0\leq m\leq n$, $s_1,\ldots,s_m\in\{\pm 1,\pm 2,\ldots,\pm n\}$, and $|s_i|\neq|s_j|$ for $i\neq j$. A signed sequence $(s_1,\ldots,s_m)$ defines a sequence of $m{+}1$ disjoint ordered pairs (A_0,B_0), (A_1,B_1),…, (A_m,B_m), where

$$A_i=\{s_j : j\in[i],\ s_j>0\},\quad B_i=\{-s_j : j\in[i],\ s_j<0\}.$$

In other words, $A_0=B_0=\emptyset$ and (A_i,B_i) is obtained from (A_{i-1},B_{i-1}) by adding s_i to A_{i-1} if $s_i>0$ and by adding $-s_i$ to B_{i-1} if $s_i<0$. The *label sequence* associated to $(s_1,\ldots,s_m)$ is $(\lambda_0,\lambda_1,\ldots,\lambda_m)$, where $\lambda_i=\lambda(A_i,B_i)$. By the definition of λ, each label sequence begins with $(1,\pm 2,\pm 3,\ldots)$, up until $\pm(2k-1)$, and then it has terms $\pm i$ with $2k\leq i\leq n$ (but of course it may have fewer than $2k-1$ terms).

We claim that if c is a proper coloring of the Kneser graph then the label seuqence of any signed sequence never contains two complementary labels, i.e. there are no $i\neq j$ with $\lambda_i=-\lambda_j$. This is exactly as in the preceding section: supposing $\lambda_i=-\lambda_j$, we first observe that $i,j\geq 2k-1$, i.e. the labels must have been assigned according to Case II. Supposing, for example, that

$i < j$ and $\lambda_i > 0$, we get that A_i and B_j are two disjoint subsets of $[n]$ with $c(A_i) = c(B_j)$, which means that two disjoint k-tuples received the same color under c.

We prove that there exists a signed sequence whose label sequence contains complementary labels. We proceed again by contradiction. (But the proof actually provides an algorithm for finding such a signed sequence and, consequently, exhibiting two k-tuples given the same color by c.)

Let us call a signed sequence $s = (s_1, \ldots, s_m)$ a *permissible sequence* if $s_i \in \{\lambda_0, \lambda_1, \ldots, \lambda_m\}$ for all $i \in [m]$, where $(\lambda_0, \ldots, \lambda_m)$ is the label sequence of s. So, for example, a permissible sequence with t or more terms, where $t \leq 2k-1$, contains $+1$, one of -2 and $+2$, ..., one of $-t$ and $+t$.

Assuming that no label sequence contains complementary labels, we now define one or two *neighbor* permissible sequences for each permissible sequence, where the relation of being neighbor is symmetric. The only permissible sequence with a single neighbor is the empty one; all others have exactly two neighbors. This, of course, is impossible since there are only finitely many permissible sequences, and it provides the desired contradiction.

The single neighbor of the empty sequence is the (permissible) sequence $(+1)$.

Let $s = (s_1, \ldots, s_m)$ be a nonempty permissible sequence and let $(\lambda_0 = 1, \lambda_1, \ldots, \lambda_m)$ be its label sequence. There are at least m distinct numbers among the $m+1$ labels, namely $s_1, \ldots, s_m$ (in some arbitrary order). Depending on the single remaining label, two cases can occur:

(i) Two labels coincide: $\lambda_i = \lambda_j$, $i < j$.
(ii) There is one extra label $\lambda_i \not\in \{s_1, s_2, \ldots, s_m\}$.

In case (i), one of the two neighbors of s is obtained by *transposition at* $(i, i+1)$, by which we mean that the neighbor sequence is $(s_1, s_2, \ldots, s_{i-1}, s_{i+1}, s_i, s_{i+2}, \ldots, s_m)$. (Note that $i = 0$ is not possible since the label $\lambda_0 = 1$ cannot occur at any other position of the label sequence.) Such a transposition preserves all terms of the label sequence except possibly for λ_i, and so the just defined neighbor is a permissible sequence. The second neighbor of s is defined similarly, by transposition at $(j, j+1)$, unless $j = m$. For $j = m$, the second neighbor of s is the contracted sequence $(s_1, \ldots, s_{m-1})$.

In case (ii), one of the two neighbors of s is the expanded sequence $(s_1, \ldots, s_m, \lambda_i)$. This sequence is permissible because we assume that there are no two complementary labels, and therefore $|\lambda_i| \not\in \{|s_1|, \ldots, |s_m|\}$. To define the second neighbor, we must distinguish a few cases. If λ_i is neither the first nor the last term of the label sequence, i.e. $1 \leq i \leq m-1$, the second neighbor is obtained from s by transposition at $(i, i+1)$. If $i = m$ then the second neighbor arises by contraction, i.e. it is $(s_1, \ldots, s_{m-1})$. Finally if $i = 0$ then the second neighbor is obtained by *sign change*: it is $(-s_1, -s_2, \ldots, -s_m)$.

In this way, two distinct neighbors are assigned to each nonempty permissible sequence. It remains to check that the neighbor relation is symmetric, which is done by discussing the few possible cases. This concludes the proof.

Added in proof. Methods of this paper were further developed and applied to generelizations of Kneser's conjecture by Ziegler [17]. Relations of the various topological proofs of Kneser's conjecture were investigated in [14].

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