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EVERY NON-EUCLIDEAN ORIENTED MATROID ADMITS A BIQUADRATIC FINAL POLYNOMIAL

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Richter-Gebert proved that every non-Euclidean uniform oriented matroid admits a biquadratic final polynomial. We extend this result to the non-uniform case.

1. Introduction

In this paper, we identify a *chirotope* χ with an oriented matroid $\mathcal{M}=(E,\chi)$, which we abbreviate by OM. A standard reference for the theory of oriented matroids is [2]. The set $E=\{1,\dots,n\}$ is called *ground set* and $\chi\colon E^r \to \{+1,-1,0\}$ satisfies chirotope axioms, where r is a rank of an OM and n is a number of elements of the ground set.

Let $X=(x_1,\dots,x_n)\in\mathbb{R}^{r\times n}$ be a configuration of n points in $\mathbb{R}^r$. Let $[i_1\cdots i_r]$ denote the determinant $\det(x_{i_1}\cdots x_{i_r})$. By setting $\chi_X(i_1,\dots,i_r)=\operatorname{sgn}[i_1\cdots i_r]$, the function χ_X satisfies the chirotope axioms. A chirotope arising this way is called *representable* or *realizable*. It is well known that not all chirotopes are realizable.

In the sequel, we regard a bracket $[i_1\cdots i_r]$ as a bracket variable. For any given ordered sequences of indices $\tau=(\tau_1\cdots\tau_{r-2})$ and $\lambda=(\lambda_1\cdots\lambda_4)$, we call a bracket polynomial

$$[\tau\lambda_1\lambda_2][\tau\lambda_3\lambda_4]-[\tau\lambda_1\lambda_3][\tau\lambda_2\lambda_4]+[\tau\lambda_1\lambda_4][\tau\lambda_2\lambda_3] \quad (1)$$

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a *3-term Grassmann–Plücker polynomial*. If the chirotope is realizable, the value of (1) is always 0. Now we introduce *biqudratic inequalities (equations)* and define *biquadratic final polynomials*.

Definition 1.1. Let χ be an OM of rank r, let $\tau \in E^{r-2}, \lambda \in E^4$ be index sequences, and let $A = (\tau\lambda_1\lambda_2)$, $B = (\tau\lambda_3\lambda_4)$, $C = (\tau\lambda_1\lambda_3)$, $D = (\tau\lambda_2\lambda_4)$, $E = (\tau\lambda_1\lambda_4)$ and $F = (\tau\lambda_2\lambda_3)$. Then

1. A pair (τ,λ) is called *χ-normalized* if $\chi(A)\cdot\chi(B) \geq 0$, $\chi(C)\cdot\chi(D) \geq 0$ and $\chi(E)\cdot\chi(F) \geq 0$.
2. For a χ-normalized pair (τ,λ), we call
   $[A][B] < [C][D]$ and $[E][F] < [C][D]$ *biquadratic inequalities*,
   $[A][B] = [C][D]$ or $[E][F] = [C][D]$ a *biquadratic equation*.

We remark that for any pair (τ,λ), by permutating $(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ appropriately, (τ,λ) becomes χ-normalized. We denote the set of biquadratic inequalities and biquadratic equations by $\mathcal{A}_\chi$ and $\mathcal{B}_\chi$, respectively. If χ is uniform, $\mathcal{B}_\chi = \emptyset$.

Definition 1.2. An OM χ is said to *admit a biquadratic final polynomial* if there are a non-empty subset of $\mathcal{A}\chi$: $\{[A_i][B_i] < [C_i][D_i] \mid 1 \leq i \leq k\}$ and a (possibly empty) subset of $\mathcal{B}_\chi$: $\{[A_j][B_j] = [C_j][D_j] \mid 1 \leq j \leq l\}$ such that the following equality holds

$$\prod_{i=1}^{k}[A_i][B_i] \cdot \prod_{j=1}^{l}[A_j][B_j] = \prod_{i=1}^{k}[C_i][D_i] \cdot \prod_{j=1}^{l}[C_j][D_j].$$

The following is a direct consequence of the definition above.

Lemma 1.3. *If χ admits a biquadratic final polynomial, χ is non-realizable.*

Richter-Gebert [6] proved that every non-Euclidean uniform oriented matroid admits a biquadratic final polynomial. Our main theorem extends this result to the non-uniform case.

Theorem 1.4. *Every non-Euclidean oriented matroid admits a biquadratic final polynomial.*

2. Oriented Matroid Programming

Oriented matroid programming is formulated as a combinatorial abstraction of linear programming [1]. The simplex method in linear programming has a natural extension in the setting of oriented matroids. Edmonds and

Fukuda [3] showed that there exist OMs allowing the simplex method to generate a cycle of *non-degenerate* pivots, which cannot[1] occur in linear programming. Consequently, one can show the non-realizability of an OM by exhibiting a non-degenerate cycle of simplex pivots if exists.

Let χ be an OM of rank r on an $(n+2)$ element set $E=\{1,\dots,n,f,g\}$. Here, the last two elements f and g of E are distinguished. The triple (χ,f,g) is called an *oriented matroid program* (abbreviated by OMP). The element g represents a hyperplane at infinity and f represents an objective function.

Definition 2.1. Let (χ,f,g) be an OMP and $\mathcal{A}$ ($\mathcal{A}^\infty$, respectively) be the affine (infinite) space with respect to g, i.e., the set of covectors with positive (zero) g-component.

1. A set $B=(\lambda_1,\dots,\lambda_{r-1})\in E-\{f,g\}$, such that $B\cup\{g\}$ is independent, is called an *affine basis*. The unique vertex (i.e., a covector with minimal support, or equivalently a cocircuit) X with $X_B=0$ and $X_g=+$ is denoted by $v(B)$.
2. $B_1\to B_2$ is called a *pivot operation* if B_1,B_2 are affine bases and $L=B_2-\{b\}=B_1-\{a\}$ where $a,b\in E-\{f,g\}$ and $a\neq b$. L is called the *edge* of $B_1\to B_2$.
3. The *direction* of a pivot $L\cup\{a\}=B_1\to B_2=L\cup\{b\}$ where $L\cup\{a,b\}$ is assumed to be independent, is the unique vertex $d=d(B_1\to B_2)\in\mathcal{A}^\infty$ with $d_L=0$ and $d_a=v(B_2)_a$.
4. A pivot operation $L\cup\{a\}=B_1\to B_2=L\cup\{b\}$ where $a\neq b$ is called *degenerate* if $v(B_1)=v(B_2)$, *horizontal* if $L\cup\{f,g\}$ is dependent, *strictly increasing* if $d(B_1\to B_2)_f>0$ and $B_1\to B_2$ is not degenerate.

We remark that neither degenerate nor horizontal pivot operation occurs when an OM χ is uniform.

Definition 2.2. A sequence of pivot operations $B_1\to B_2\to\cdots\to B_k$ is called a *non-degenerate cycle* on χ if $B_1=B_k$ and all pivot operations are either degenerate, horizontal or strictly increasing and at least one pivot is strictly increasing.

Since no non-degenerate cycling occurs in linear programming, the following proposition holds.

Proposition 2.3. *If an OMP (χ,f,g) admits a non-degenerate cycle, then the oriented matroid χ is non-realizable.*

[1] Note that in linear programming, the simplex method can generate a cycle of *degenerate* pivots, known as cycling.

The following characterization of Euclidean OMs is fundamental.

Proposition 2.4 **([3]).** *An OMP (χ, f, g) on E admits a non-degenerate cycle for some choice of two distinguished elements f and g from E if and only if the oriented matroid χ is non-Euclidean.*

3. From Cycling to Biquadratic Final Polynomial

In the case of uniform OMs, Richter-Gebert [6] gave a method to obtain a biquadratic final polynomial from a non-degenerate cycle. Now we extend this method to the non-uniform case. In the following proof, we translate each pivot operation to one Grassmann–Plücker polynomial.

Lemma 3.1. *Let (χ, f, g) be an OMP and $L = \{\lambda_1, \ldots, \lambda_{r-2}\} \subset E - \{f, g\}$, $a, b \in E - \{f, g\}$ such that $L \cup \{a\} = B_1 \to B_2 = L \cup \{b\}$ is a pivot operation along edge L. Then*

- *if $B_1 \to B_2$ is strictly increasing, $\chi(\lambda_1, \ldots, \lambda_{r-2}, g, f) \cdot \chi(\lambda_1, \ldots, \lambda_{r-2}, a, b) \cdot \chi(\lambda_1, \ldots, \lambda_{r-2}, g, a) \cdot \chi(\lambda_1, \ldots, \lambda_{r-2}, g, b) = +1$,*
- *if $B_1 \to B_2$ is either degenerate or horizontal, $\chi(\lambda_1, \ldots, \lambda_{r-2}, g, f) \cdot \chi(\lambda_1, \ldots, \lambda_{r-2}, a, b) \cdot \chi(\lambda_1, \ldots, \lambda_{r-2}, g, a) \cdot \chi(\lambda_1, \ldots, \lambda_{r-2}, g, b) = 0$.*

Proof. For the first case, see [6]. If the pivot operation is degenerate, which means two affine vertices $v(B_1)$ and $v(B_2)$ are at the same point, $\chi(\lambda_1, \ldots, \lambda_{r-2}, a, b) = 0$. Similary, if the pivot operation is horizontal, that is $L \cup \{f, g\}$ is dependent, $\chi(\lambda_1, \ldots, \lambda_{r-2}, f, g) = 0$ is satisfied. For both two cases, the values become 0. ∎

We are now ready to prove the main theorem.

Proof of Theorem 1.4. Let χ be a non-Euclidean OM on E. By Proposition 2.4, there exist f and g in E such that the OMP (χ, f, g) admits a non-degenerate cycle, say, $B_1 \to B_2 \to \cdots \to B_k$ where $B_1 = B_k$. We shall construct a suitable biquadratic final polynomial. We define L^i, a^i, b^i by the relations:

$$L^i \cup \{a^i\} = B_i \to B_{i+1} = L^i \cup \{b^i\} \quad \text{for all } 1 \leq i \leq k. \tag{2}$$

In (2), we set $B_{k+1} = B_2$. $L^i = \{\lambda_1^i, \ldots, \lambda_{r-2}^i\}$ is the edge of the pivot operation $B_i \to B_{i+1}$. We denote $\lambda^i = (\lambda_1^i, \ldots, \lambda_{r-2}^i)$. Consider the following sequence of Grassmann–Plücker polynomials:

$$GP^i = [\lambda^i, g, f][\lambda^i, a^i, b^i] - [\lambda^i, g, a^i][\lambda^i, f, b^i] + [\lambda^i, g, b^i][\lambda^i, f, a^i].$$

Note that $GP^1 = GP^k$. As in Definition 1.1, we set $A^i = (\lambda^i, g, f)$, $B^i = (\lambda^i, a^i, b^i)$, $C^i = (\lambda^i, g, a^i)$, $D^i = (\lambda^i, f, b^i)$, $E^i = (\lambda^i, g, b^i)$ and $F^i = (\lambda^i, f, a^i)$. Then, we have

$$GP^i = A^i \cdot B^i - C^i \cdot D^i + E^i \cdot F^i.$$

Now we consider the signs of terms appearing in GP^i. If the pivot operation $B_i \to B_{i+1}$ is strictly increasing, $\chi(A^i) \cdot \chi(B^i) \cdot \chi(C^i) \cdot \chi(E^i) = +1$ is satisfied. Using OM axioms, the following 12 types of signs are possible:

$$\begin{array}{ccccccl}
A^i & \cdot\, B^i - & C^i & \cdot\, D^i + & E^i & \cdot\, F^i & \\
+ & & + & + & + & + & \text{type } 1 \\
+ & & + & + & + & - & \text{type } 2 \\
+ & & + & - & + & - & \text{type } 3 \\
+ & & - & - & - & - & \text{type } 4 \\
+ & & - & - & - & + & \text{type } 5 \\
+ & & - & + & - & + & \text{type } 6 \\
- & & + & - & - & + & \text{type } 7 \\
- & & + & - & - & - & \text{type } 8 \\
- & & + & + & - & - & \text{type } 9 \\
- & & - & + & + & - & \text{type } 10 \\
- & & - & + & + & + & \text{type } 11 \\
- & & - & - & + & + & \text{type } 12
\end{array} \tag{3}$$

After normalization, type 1, 4, 7 or 10 generates a biquadratic inequality $[E^i][F^i] < [C^i][D^i]$ and type 3, 6, 9 or 12 generates a biquadratic inequality $[A^i][B^i] < [C^i][D^i]$. Remaining type 2, 5, 8 or 11 does not determine either of the two inequalities, but these four types cannot appear. We explain the fact by using a transition diagram in Figure 1 later.

If the pivot operation $B_i \to B_{i+1}$ is degenerate or horizontal, $\chi(A^i) \cdot \chi(B^i) = 0$ is satisfied. Using OM axioms, the following 8 types of signs are possible:

$$\begin{array}{ccccccl}
A^i & \cdot\, B^i - & C^i & \cdot\, D^i + & E^i & \cdot\, F^i & \\
0 & & + & + & + & + & \text{type } 1' \\
0 & & + & + & - & - & \text{type } 2' \\
0 & & + & - & + & - & \text{type } 3' \\
0 & & + & - & - & + & \text{type } 4' \\
0 & & - & + & + & - & \text{type } 5' \\
0 & & - & + & - & + & \text{type } 6' \\
0 & & - & - & + & + & \text{type } 7' \\
0 & & - & - & - & - & \text{type } 8'
\end{array} \tag{4}$$

Clearly, each one of the eight types implies a biquadratic equation $[E^i][F^i]=[C^i][D^i]$.

In both cases above, the fact $L^i\cup\{b^i\}=B_{i+1}=L^{i+1}\cup\{a^{i+1}\}$ implies the following relation:

$$\chi(D^i)\cdot\chi(E^i)\cdot\chi(C^{i+1})\cdot\chi(F^{i+1}) = \\ \chi(\lambda^i,f,b^i)\cdot\chi(\lambda^i,g,b^i)\cdot\chi(\lambda^{i+1},g,a^{i+1})\cdot\chi(\lambda^{i+1},f,a^{i+1})=1,$$

which restricts the types of possible successors GP^{i+1} of a Grassmann–Plücker relation GP^i of certain type. The transition diagram is given in the following Figure 1. For example, if a Grassmann–Plücker polynomial GP^i is type 3, the type of GP^{i+1} is either 3, 6, 9, 12, $2'$, $3'$, $6'$, $7'$, 2, 5, 8 or 11.

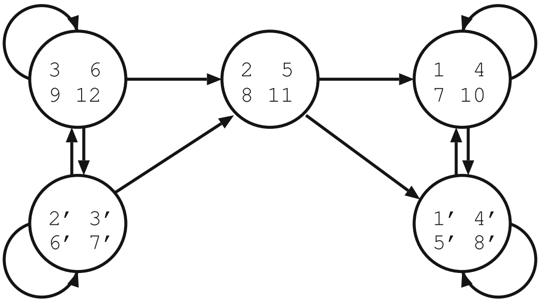

Figure 1. transition diagram among types

A Grassmann–Plücker relation of type t can be succeeded by a Grassmann–Plücker relation of type s if and only if there is an arrow from the circle containing t to the circle containing s. We have $GP^1=GP^k$ and $B_1\to B_2$ is strictly increasing, that is, the type of GP^1 and GP^k is same and either of (3). Then a sequence of transition either

- contains only two states $(1,4,7,10)$ and $(1',4',5',8')$, and its initial state is $(1,4,7,10)$, or
- contains only two states $(3,6,9,12)$ and $(2',3',6',7')$, and its initial state is $(3,6,9,12)$.

In both cases, the state 2, 5, 8 or 11 cannot appear in the transition sequence and the resulting set of biquadratic inequalities and biquadratic equations yields a biquadratic final polynomial. ∎

The converse of Theorem 1.4 does not hold: that is, there exist non-realizable OMs which have a BFP but are Euclidean. For example, below the non-Pappus OM has a BFP, but is Euclidean since the rank is three.

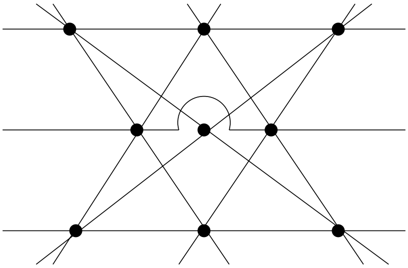

Figure 2. non-Pappus OM

Historical Notes

The main theorem, Theorem 1.4, was independently proved by the last author and by the first three authors. Although the last author's work predates the other, it was published only in the form of doctoral dissertation [5] and left unknown for many years. This led to an independent proof by the first three authors. The present paper is a natural synthesis of the two proofs.

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