# Oriented matroids and Ky Fan's theorem 

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#### Abstract

L. Lovász has shown in 9 that Sperner's combinatorial lemma admits a generalization involving a matroid defined on the set of vertices of the associated triangulation. We prove that Ky Fan's theorem admits an oriented matroid generalization of similar nature (Theorem 3.1). Classical Ky Fan's theorem is obtained as a corollary if the underlying oriented matroid is chosen to be the alternating matroid $C^{m, r}$.


## 1 Introduction

The following extension of Sperner's combinatorial lemma was proposed by Lászlo Lovász in (9].

Theorem 1.1. ([9]) Let $K$ be a simplicial complex which triangulates a ddimensional manifold. Suppose that a matroid $\mathcal{N}$ of rank $d+1$ is defined on the set $\operatorname{vert}(K)$ of vertices of $K$. If $K$ has a simplex whose vertices form a basis of the matroid $\mathcal{N}$, than there exist at least two such simplices.

Homological nature of this result was subsequently emphasized by Bernt Lindström who demonstrated that it suffices to assume that the complex $K$ (in Theorem 1.1) supports a $d$-cycle.

Theorem 1.2. ([7]) Let $C=\sum_{i \in I} \alpha_{i} \sigma_{i}$ be a d-cycle in a complex K. Assume that the vertices of simplices in $C$ are labelled by elements of a matroid $\mathcal{N}$ of rank $(d+1)$. If some simplex is labelled by the elements of a base of $\mathcal{N}$, then there are at least two simplices in $C$ with this property.

As a corollary of Theorem 1.1 Lovász deduced the following result which reduces to the classical version of Sperner's lemma if $\mathcal{N}$ is the matroid such that $S \subset \operatorname{vert}(K)$ is an independent set if and only if its elements are labelled (colored) by different labels.

Corollary 1.3. Suppose that $K$ is a simplicial subdivision of a d-dimensional simplex $\sigma$. Suppose that a matroid $\mathcal{N}$ of rank $d+1$ is defined on the set $\operatorname{vert}(K)$ of vertices of $K$ and let $\operatorname{vert}(\sigma)$, the set of vertices of $\sigma$, be a basis of $\mathcal{N}$. Assume that for each face $F$ of $\sigma$, the set $F \cap \operatorname{vert}(K)$ is in the flat of the matroid $\mathcal{N}$ spanned by $\operatorname{vert}(F) \subset \operatorname{vert}(\sigma)$. Then $K$ has a simplex whose vertices form a basis of $\mathcal{N}$.

Well known $\mathbb{Z}_{2}$-counterparts of Sperner's lema are Tucker's lemma [17], [10] and its generalization due to Ky Fan [5].

Theorem 1.4. (5]) Suppose that $K$ is a $\mathbb{Z}_{2}$-invariant triangulation of the sphere $S^{n}$. Let $\diamond^{m}:=\operatorname{conv}\left\{ \pm e_{1}, \ldots, \pm e_{m}\right\}$ be the $m$-dimensional crosspolytope and $\partial\left(\diamond^{m}\right) \cong S^{m-1}$ its boundary with inherited $\left(\mathbb{Z}_{2}\right.$-invariant) triangulation. If $f: K \rightarrow \partial\left(\diamond^{m}\right)$ is a simplicial, $\mathbb{Z}_{2}$-equivariant map, then $n<m$ and

$$
\sum_{1 \leq k_{1}<k_{2}<\ldots<k_{n+1} \leq m} \alpha\left(k_{1},-k_{2}, k_{3},-k_{4}, \ldots,(-1)^{n} k_{n+1}\right) \cong 1 \quad(\bmod 2)
$$

where $\alpha\left(j_{1}, j_{2}, \ldots, j_{n+1}\right)$ is the number of $n$-simplices in $K$ mapped to the simplex spanned by vectors $e_{j_{1}}, e_{j_{2}}, \ldots, e_{j_{n+1}}$ and by definition $e_{-j}:=-e_{j}$.
A natural question is whether there exists a counterpart of Lovász' theorem which extends Ky Fan's theorem (Tucker's theorem) in the manner Theorem 1.1 extends Sperner's lemma. Our objective is to prove such a result, Theorem 3.1. It is not a surprise that oriented matroids appear in this extension and play a role similar to the role matroids play in Theorem 1.1.

## 2 Oriented matroids in a nutshell

Oriented matroids provide combinatorial models for important geometric objects, configurations and structures including the following:

- linear subspaces $L \subset \mathbb{R}^{n}$,
- configurations of points (vectors) in $\mathbb{R}^{n}$,
- matrices,
- directed graphs,
- convex polytopes,
- linear programs,
- hyperplane arrangements etc.

Although they appear in many incarnations and disguises, oriented matroids always provide essentially the same amount of information about the object they discretize (cryptomorphism). The reader is referred to [19] (Section 6) for a quick introduction and initial motivation and to [1] for illuminating orientation sessions (Sections 1 and $2)$ and thorough treatment of the general theory with many interesting applications. More recent reference [14] offers both an outline of the theory and a guide to the papers published after the appearance of [1].

## $2.1 \nabla^{m}$-oriented matroid of a linear subspace

Let us briefly review how an oriented matroid can be associated to a d-dimensional linear subspace $L \subset \mathbb{R}^{m}$. Let $\nabla^{m}:=\operatorname{conv}\left\{ \pm e_{1}, \ldots, \pm e_{m}\right\}$ be the crosspolytope in $\mathbb{R}^{m}$ and let $P\left(\diamond^{m}\right)$ be the associated face poset. Define $\diamond^{m}$-oriented matroid $L \diamond^{m}$ of $L$ as the collection (poset) $L_{\diamond^{m}}=\left\{F \in P\left(\diamond^{m}\right) \mid L \cap F \neq \emptyset\right\}$ of all faces of the crosspolytope intersected by $L$. The face poset $P\left(\diamond^{m}\right)$ is isomorphic to the poset $\{0,+,-\}^{m}$ of all sign-vectors of length $m$. This isomorphism associates to each face $F \in P\left(\diamond^{m}\right)$ the $\operatorname{sign}$ vector $\operatorname{sign}(F):=\operatorname{sign}(v) \in\{0,+,-\}^{m}$ for some (any) $v \in \operatorname{relint}(F)$. It follows that $L_{\diamond^{m}}$ is essentially the set $\mathcal{V}^{*}$ of all covectors of an oriented matroid $\mathcal{M}=\mathcal{M}(L)$ which captures the combinatorial information about how the subspace $L$ is placed in the ambient space $\mathbb{R}^{m}$. Similarly, the set $\mathcal{C}^{*}$ of all cocircuits of $\mathcal{M}(L)$ can be described as the set of all $\subseteq$-minimal elements in $L_{\diamond^{m}}$ or alternatively as the collection of faces $F \in P\left(\diamond^{m}\right)$ such that the subspace $L$ intersects $\operatorname{relint}(F)$ in a single point.

### 2.2 Topological representation theorems

The original Topological Representation Theorem for oriented matroids was proved by Folkman and Lawrence [1]. The following strengthening of this result, due to Brylavski and Ziegler [3], provides a simultaneous representation for both the oriented matroid $\mathcal{M}$ and its dual $\mathcal{M}^{*}$.

Theorem 2.1. ([3]) For each oriented matroid $\mathcal{M}$ of rank $r$ on $\{1, \ldots, m\}$ there exists a (signed) pseudosphere arrangement $\mathcal{A}=\left(S_{i}\right)_{1 \leq i \leq 2 m}$ in $S^{m-1}$ such that:
(1) $S_{i}=\left\{x \in S^{m-1} \mid x_{i}=0\right\}$ for $1 \leq i \leq m$.
(2) The $(r-1)$-subsphere $S_{B}:=S_{m+r+1} \cap \ldots \cap S_{2 m}$ and $(m-r-1)$-subsphere $S_{A}:=S_{m+1} \cap \ldots \cap S_{m+r}$ are disjoint.
(3) The arrangement $\left(S_{i} \cap S_{B}\right)_{1 \leq i \leq m}$ is a topological representation of the oriented matroid $\mathcal{M}$ in $S_{B}$.
(4) The arrangement $\left(S_{j} \cap S_{A}\right)_{1 \leq j \leq m}$ is a topological representation of the oriented matroid $\mathcal{M}^{*}$ in $S_{A}$.
Corollary 2.2. Let $\mathcal{M}$ be an oriented matroid of rankr on $[m]=\{1,2, \ldots, m\}$. Let $S_{i}, i=1, \ldots, m$ be the coordinate sphere $S_{i}=\left\{x \in S^{m-1} \mid x_{i}=0\right\}$. Then there exists a $(r-1)$-dimensional pseudosphere $S_{B}$ in $S^{m-1}$ such that the arrangement $\left\{S_{i} \cap S_{B}\right\}_{i=1}^{m}$ is a topological representation of $\mathcal{M}$ in $S_{B}$. Moreover $S_{B}$ can be chosen to be centrally symmetric and transverse to each of the spheres $S_{I}:=S_{i_{1}} \cap \ldots \cap S_{i_{r}}$ for any r-element subset

$$
I=\left\{i_{1}, \ldots, i_{r}\right\} \subset[m] .
$$

Proof: According to Section 5.2 in [1] (Theorem 5.2.1), the condition that all pseudospheres are centrally symmetric can be always satisfied. In particular all pseudospheres in Theorem 2.1 can be assumed to have this property. Also, the transversality condition from Corollary [2.2 is "built in" the Topological Representation Theorem for oriented matroids.

## 3 Generalization of Ky Fan's theorem

Theorem 3.1. Suppose that $M$ is a connected, $n$-dimensional, triangulated $\mathbb{Z}_{2}$-manifold. Moreover, it is assumed that the involution $\nu: M \rightarrow M$ defining the $\mathbb{Z}_{2}$-action is fixed-point-free. Given a positive integer $m$, let

$$
\lambda: \operatorname{vert}(M) \rightarrow\{ \pm 1, \pm 2, \ldots, \pm m\}
$$

be a labelling of the vertices of $M$ which satisfies the conditions:
(a) $\lambda(\nu(v))=-\lambda(v)$ for each $v \in \operatorname{vert}(M)$,
(b) $\lambda(v)+\lambda\left(v^{\prime}\right) \neq 0$ for each 1-simplex $\tau=\left\{v, v^{\prime}\right\}$ in $M$.

Let $\mathcal{M}=\mathcal{M}\left([m], \mathcal{C}^{*}\right)$ be an oriented matroid of rank $r=m-n$ on the set $E=[m]=\{1, \ldots, m\}$ where $\mathcal{C}^{*}$ is the set of associated cocircuits. Moreover, $\mathcal{M}$ is assumed to be uniform in the sense that all cocircuits in $\mathcal{C}^{*}$ have the same cardinality $n=m-r$. Let $w_{1} \in H_{\mathbb{Z}_{2}}^{1}\left(M, \mathbb{Z}_{2}\right) \cong H^{1}\left(M / \mathbb{Z}_{2}, \mathbb{Z}_{2}\right)$ be the first Stiefel-Whitney class of the $\mathbb{Z}_{2}$-manifold $M$ and let $w(M)=w_{1}^{n}\left(\left[M / \mathbb{Z}_{2}\right]\right)$ be the associated Stiefel-Whitney number where $\left[M / \mathbb{Z}_{2}\right]$ is the $\mathbb{Z}_{2}$-fundamental class of $M / \mathbb{Z}_{2}$. Then,

$$
\begin{equation*}
w(M)=\frac{1}{2} \sum_{\tau \in \mathcal{C}^{*}} \alpha(\tau)=\sum_{[\tau] \in \mathcal{C}^{*} / \mathbb{Z}_{2}} \alpha(\tau) \quad(\bmod 2) \tag{1}
\end{equation*}
$$

where $\alpha(\tau)$ is the number of $n$-simplices $\sigma \in M$ whose vertices receive labels from $\tau$, i.e. such that $\tau=\{\lambda(v) \mid v \in \operatorname{vert}(\sigma)\}$.

Proof: Let $\partial \diamond^{m}$ be the boundary of the crosspolytope $\diamond^{m}=\operatorname{conv}\left\{ \pm e_{1}, \ldots, \pm e_{m}\right\}$. As a $\mathbb{Z}_{2}$-space, $\partial \diamond^{m}$ is isomorphic to the join [2]*. . $*[2]$ of $m$ copies of $[2]=\{1,2\}$. We may therefore see $\partial \diamond^{m}$ as a subcomplex of (the boundary of) an "infinite crosspolytope" $E \mathbb{Z}_{2}=[2] * \ldots *[2] * \ldots$, which is nothing but the well known Milnor's model for the classifying space of the group $\mathbb{Z}_{2}$.

Any labelling $\lambda: \operatorname{vert}(M) \rightarrow\{ \pm 1, \ldots, \pm m\}$ can be in an unique way extended to a $\mathbb{Z}_{2}$-equivariant map

$$
\Lambda: M \rightarrow \diamond^{m}
$$

which is affine on each simplex $\sigma \in M$. Conditions (a) and (b) on the labelling guarantee that $\operatorname{Im}(\Lambda) \subset \partial \diamond^{m}$ and that $\Lambda$ is a simplicial $\mathbb{Z}_{2}$-map. Consequently, $\Lambda$ is essentially a classifying $\operatorname{map} \Lambda: M \rightarrow \partial \diamond^{m} \hookrightarrow E \mathbb{Z}_{2}$, that is the unique (up to a $\mathbb{Z}_{2}$-homotopy) $\mathbb{Z}_{2}$-equivariant map $\Lambda: M \rightarrow E \mathbb{Z}_{2}$. Let $\xi=\Lambda / \mathbb{Z}_{2}: M / \mathbb{Z}_{2} \rightarrow \partial \diamond^{m} / \mathbb{Z}_{2}$ be the induced map. Let $\gamma \in H_{\mathbb{Z}_{2}}^{1}\left(\partial \nabla^{m} / \mathbb{Z}_{2} ; \mathbb{Z}_{2}\right) \cong H^{1}\left(\partial \nabla^{m} / \mathbb{Z}_{2} ; \mathbb{Z}_{2}\right)$ be the first StiefelWhitney class of the $\mathbb{Z}_{2}$-space $\partial \nabla^{m}$, or equivalently the first S-W-class of the line bundle

$$
\mathbb{R}^{1} \rightarrow \partial \diamond^{m} \times_{\mathbb{Z}_{2}} \mathbb{R}^{1} \rightarrow \partial \diamond^{m} / \mathbb{Z}_{2}
$$

Let $S_{B}$ be the $(r-1)$-dimensional pseudosphere in $S^{m-1} \subset \mathbb{R}^{m}$ which represents the oriented matroid $\mathcal{M}$ in the sense of Corollary 2.2. The fundamental class $y=\left[S_{B} / \mathbb{Z}_{2}\right]$
of the quotient projective space $S_{B} / \mathbb{Z}_{2} \subset R P^{m-1}$ represents the generator of the group $H_{r-1}\left(R P^{m-1} ; \mathbb{Z}_{2}\right) \cong \mathbb{Z}_{2}$. This follows from the fact that, according to the definition of a (centrally symmetric) pseudosphere, there exists a $\mathbb{Z}_{2}$-homeomorphisms $h: S^{m-1} \rightarrow$ $S^{m-1}$ which maps the $(r-1)$-dimensional equator of $S^{m-1}$ to the pseudosphere $S_{B}$. By the well-known formula, see e.g. [4], Section VII.12, or [2], Theorem VI.5.2 part (4), and standard (loc. cit.) properties of the products of (co)homology classes and the Poincaré duality map $D$,

$$
\begin{align*}
& w(M)=w_{1}^{n}\left(\left[M / \mathbb{Z}_{2}\right]\right)=w_{1}^{n} \cap\left[M / \mathbb{Z}_{2}\right]=\xi_{*}\left(\xi^{*}\left(\gamma^{n}\right) \cap\left[M / \mathbb{Z}_{2}\right]\right)=  \tag{2}\\
& \quad=\gamma^{n} \cap \xi_{*}\left[M / \mathbb{Z}_{2}\right]=D\left(\gamma^{n}\right) \bullet \xi_{*}\left[M / \mathbb{Z}_{2}\right]=\left[S_{B}\right] \bullet \xi_{*}\left[M / \mathbb{Z}_{2}\right] . \tag{3}
\end{align*}
$$

For the completion of the proof it is sufficient to observe that the intersection product $\left[S_{B}\right] \bullet \xi_{*}\left[M / \mathbb{Z}_{2}\right]$ is precisely the right hand side of the equation (1). Indeed, the intersection $\tau \cap S_{B}$ is transverse for each of the simplices (cocircuits) $\tau \in \mathcal{C}^{*}$ and each of them is counted with the multiplicity $\alpha(\tau)$.

Corollary 3.2. If $w_{1}^{n}$ is non-trivial, i.e. if $w(M)=w_{1}^{n}\left(\left[M / \mathbb{Z}_{2}\right]\right)=1$, for example if $M=S^{n}$ is the $n$-sphere, then

$$
\begin{equation*}
1=\sum_{[\tau] \in \mathcal{C}^{*} / \mathbb{Z}_{2}} \alpha(\tau) \quad(\bmod 2) \tag{4}
\end{equation*}
$$

where the summation is over the representatives of classes $[\tau]=\{\tau,-\tau\}$ of antipodal simplices (cocircuits).

Corollary 3.3. The usual Ky Fan's theorem (Theorem (1.4) follows from Theorem 3.1 (Corollary 3.2) if $\mathcal{M}$ is chosen as (the dual of) the alternating oriented matroid $C^{m, r}$.

Proof: The (dual of) the alternating matroid $C^{m, r}$ is defined, [1] Section 9.4, as the oriented matroid of the vector configuration $\mathcal{W}=\left\{v_{1}, \ldots, v_{m}\right\}$ where $v_{i}:=\left(1, t_{i}, \ldots, t_{i}^{r}\right)$ are the points on the moment curve corresponding to a sequence $0<t_{1}<\ldots<t_{m}$. The associated set of cocircuits is, following the description given in Section 2.1, obtained as the $\diamond^{m}$-oriented matroid associated to the subspace $L:=\operatorname{Im}(W)$ where $W$ is the matrix

$$
W=\left[v_{1}^{t}, \ldots, v_{m}^{t}\right]=\left[\begin{array}{cccc}
1 & 1 & \ldots & 1 \\
t_{1} & t_{2} & \ldots & t_{m} \\
\vdots & \vdots & \ddots & \vdots \\
t_{1}^{r-1} & t_{2}^{r-1} & \ldots & t_{m}^{r-1}
\end{array}\right]
$$

It is well known and not difficult to prove that the associated cocircuits are precisely the alternating sequences (and their antipodes) that appear in the formulation of Theorem 1.4 .

## 4 Homological reformulation

Theorem 3.1 admits a reformulation which emphasizes its homological nature. It is parallel to and, to some extent, inspired by Lindström's extension of the result of Lovász. Theorem4.1 can be interpreted as a combinatorial formula (involving oriented matroids) for a power of the first Stiefel-Whitney characteristic cohomological class of the $\mathbb{Z}_{2}$-complex $\partial \diamond^{m} \cong S^{m-1}$. Moreover, it points in the direction of a hypothetical "homological representation theorem" for oriented matroids.

As before, each cocircuit (covector) of an oriented matroid $\mathcal{M}$ is identified with the corresponding face $\tau$ of the cross-polytope $\diamond^{m}$ (Section 2.1).

Theorem 4.1. Let $\mathcal{M}=\mathcal{M}\left([m], \mathcal{C}^{*}\right)$ be an uniform oriented matroid of rank $r=m-n$ on the set $E=[m]=\{1, \ldots, m\}$ where $\mathcal{C}^{*}$ is the set of associated cocircuits. For each cocircuit $\tau \in \mathcal{C}^{*} \subset P\left(\diamond^{m}\right)$ let $\hat{\tau}$ be the $\mathbb{Z}_{2}$-cochain dual to $\tau$ where $\hat{\tau}(\tau)=1$ and $\hat{\tau}(\theta)=0$ if $\theta \neq \tau$. Then the $n$-dimensional, $\mathbb{Z}_{2}$-equivariant cochain

$$
\begin{equation*}
C_{\mathcal{M}}:=\sum_{\tau \in \mathcal{C}^{*}} \hat{\tau} \in C_{\mathbb{Z}_{2}}^{n}\left(\partial \diamond^{m}\right) \tag{5}
\end{equation*}
$$

represents the Stiefel-Whitney characteristic class

$$
w_{1}^{n} \in H_{\mathbb{Z}_{2}}^{n}\left(\partial \diamond^{m}\right) \cong H^{n}\left(R P^{m-1}\right)
$$

Proof: The proof uses similar ideas as the proof of Theorem 3.1 so we omit the details. The key observation is that the cochain $C_{\mathcal{M}}$ is a Poincaré dual (on the level of chains) of the class $\left[S_{B}\right]$. For this it is sufficient to check that $C_{\mathcal{M}}$ is a cochain which can be deduced from the fact that the pseudosphere $S_{B}$ is transverse to all the simplices in the triangulation of $\partial \diamond^{m}$.

Example: Choose a rank 2 oriented matroid $\mathcal{M}$ associated to a nonstretchable arrangement $\mathcal{A}$ of pseudolines. For example let $\mathcal{A}$ be the arrangement of nine pseudolines described by Ringel, see [14] Figure 6.1.2. In this case $m=9, r=3$ and $n=9-3=6$. According to formula (5) this yields a cochain representative for the class $w_{1}^{6}$ which has a 6-dimensional simplex (and its antipode) for each intersection of two pseudolines in $\mathcal{A}$.

## 5 Concluding remarks

The proof of Theorem 3.1 appears to be new already in the realizable case where we don't need the full power of the Topological representation theorem. In particular this approach yields a short and conceptual, albeit non-constructive, proof of the classical Ky Fan's theorem.

It is natural to ask if the condition (Theorem 3.1) that $\mathcal{M}$ is a uniform oriented matroid can be relaxed. Indeed, it would be desirable and hopefully not too difficult
to come up with analogues of the formula (5) for the case of non-uniform matroids. The critical step is to express the intersection product $\left[S_{B}\right] \bullet \xi_{*}\left[M / \mathbb{Z}_{2}\right]$ in terms of the underlying oriented matroid. One way around this difficulty is to analyze perturbations of the associated pseudosphere arrangement.

Aside from the conceptual interest, more general formulas could lead to new inductive and constructive proofs based on standard oriented matroid technique. This may prove useful in finding new systematic ways of producing combinatorial proofs for combinatorial statements which originally required topological methods, cf. [8] [11] [13] [20] for some of the more recent related developments.

Considering some recent advances [12] [20] in understanding $Z_{q}$-analogues of Tucker's and Ky Fan's theorem, it would be interesting to know if such analogs exist for Theorem 3.1. This may involve a development of an analogue (or replacement) for the concept of a ( $\mathbb{Z}_{q}$-oriented) matroid, see [18] for a related development.

Formula (1) seems to indicate that, at least in principle, all the formulas involving algebraic count of alternating simplices, could be instances of more general statements involving oriented matroids. For example it is plausible that Sarkaria's "Generalized Tucker-Ky Fan theorem" [15] admits such a generalization.

Finally, in light of the fact that Ky Fan's theorem and its consequences have found numerous applications in combinatorics and discrete geometry, [16] being one of the latest examples, it remains to be seen if Theorem 3.1 can be used for a similar purpose.

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