# A NEW APPROACH TO THE RAMSEY-TYPE GAMES AND THE GOWERS DICHOTOMY IN F-SPACES 

G. ANDROULAKIS, S. J. DILWORTH, AND N. J. KALTON


#### Abstract

We give a new approach to the Ramsey-type results of Gowers on block bases in Banach spaces and apply our results to prove the Gowers dichotomy in F-spaces.


## 1. Introduction

Our aim in this note is to establish the Gowers dichotomy [4 in a general Fspace (complete metric linear space). We say that an F-space $X$ is hereditarily indecomposable if it is impossible to find two separated infinite-dimensional closed subspaces $V, W$, i.e. such that $V \cap W=\{0\}$ and $V+W$ is closed (or equivalently that the natural projection from $V+W$ onto $V$ is continuous). Our main result is that an F-space either contains an unconditional basic sequence or an infinitedimensional HI subspace. In order to prove such a result we give a new and, we hope, interesting approach to the Gowers Ramsey-type result about block bases in a Banach space. We now state this result (terminology is explained in 92 and in [5], [6]):

Theorem 1.1 ([4],5], [6]). Let $X$ be a Banach space with a basis. Let $\partial B_{X}$ denote the unit sphere of $X$ i.e. $\partial B_{X}=\{x \in X:\|x\|=1\}$. Let $\sigma \subseteq \Sigma_{<\infty}\left(\partial B_{X}\right)$. Let $\Theta=\left(\theta_{i}\right)_{i}$ be a sequence of positive numbers. If $\sigma$ is large then there exists a block subspace $Y$ of $X$ such that $\sigma_{\Theta}$ is strategically large for $Y$, where $\sigma_{\Theta}$ is the set of all finite block bases $\left\{u_{1}, \ldots, u_{n}\right\}$ such that for some $\left\{v_{1}, \ldots, v_{n}\right\} \in \sigma$ we have $\left\|u_{i}-v_{i}\right\|<\theta_{i}$.

In [5] and [6] the statement of Theorem 1.1 is announced for $\partial B_{X}$ replaced by the unit ball except the origin, i.e. $B=B_{X} \backslash\{0\}=\{x \in X: 0<\|x\| \leq 1\}$. There appears to be a slight problem in the non-normalized case in [5, page 805, line -9] and [6, page 1092, line -2]. Theorem 1.1 (including the non-normalized case) follows from our Theorem 3.8.

Gowers also considers an infinite version of the same result (Theorem 4.1 of [5]):
Theorem 1.2 ([5], [6]). Let $X$ be a Banach space with a basis. Let $\sigma \subseteq \Sigma_{\infty}\left(\partial B_{X}\right)$. Let $\Theta=\left(\theta_{i}\right)_{i}$ be a sequence of positive numbers. If $\sigma$ is analytic and large then there exists a block subspace $Y$ of $X$ such that $\sigma_{\Theta}$ is strategically large for $Y$,

[^0]where $\sigma_{\Theta}$ is the set of all infinite block bases $\left\{u_{1}, \ldots, u_{n}, \ldots\right\}$ such that for some $\left\{v_{1}, \ldots, v_{n}, \ldots\right\} \in \sigma$ we have $\left\|u_{i}-v_{i}\right\|<\theta_{i}$.

Other proofs of these results can be found in the work of Bagaria and Lopez-Abad [1], 2]. Direct proofs of the dichotomy result without these theorems can be found in [13] and [3] see also [14].

Our main objective is to prove Theorem 1.2 in a form that is suitable for our intended applications. We take a somewhat different viewpoint (see Theorem 4.4 below) by treating this theorem as a result about block bases in a countable dimensional space $E$ with no topology assumed. We consider in fact only the intrinsic topology on $E$ i.e. the finest vector space topology. We then give a proof which is rather distinct from that given by Gowers, and we feel has some advantages. A benefit of this approach is that we are able to apply the result very easily to the setting of a general F-space.

In §5we prove that the Gowers dichotomy extends to general F-spaces and discuss connections with similar (but easier) dichotomies for the existence of basic sequence. In the final section, $\sqrt{6}$ we prove the result of Gowers and Maurey [7] that on a complex HI-space every operator is the sum of a scalar and a strictly singular operator in the context of quasi-Banach spaces. This generalization is not entirely trivial and requires a few new tricks, although we broadly follow the same ideas as Gowers and Maurey.

## 2. Countable dimensional vector spaces

Let $E$ be a real or complex vector space of countable algebraic dimension. There is a natural intrinsic topology $\mathcal{T}=\mathcal{T}_{E}$ on $E$ defined as follows: a set $U$ is $\mathcal{T}$-open if $U \cap F$ is open relative to $F$ for every finite-dimensional subspace $F$. The topology $\mathcal{T}$ is a vector topology on $E$ and is, indeed, the finest vector topology on $E$. It is known that $(E, \mathcal{T})$ is in fact locally convex. More precisely if $\left(e_{j}\right)_{j=1}^{\infty}$ is any fixed Hamel basis then the topology is induced by the family of norms

$$
\left\|\sum_{j=1}^{m} a_{j} e_{j}\right\|_{\lambda}=\max _{1 \leq j \leq m} \lambda_{j}\left|a_{j}\right|
$$

where $\lambda=\left(\lambda_{j}\right)_{j=1}^{\infty}$ is any sequence of positive numbers. In the case when $\lambda_{j}=1$ for all $j$ we denote the resulting norm by $\|\cdot\|_{\infty}$.

We will also be concerned with the product $E^{\mathbb{N}}$. On this there are two natural topologies: the product topology $\mathcal{T}_{p}$ and the box topology. The box topology $\mathcal{T}_{b x}$ is a topology which makes $E^{\mathbb{N}}$ a topological group but not a topological vector space. A base of neighborhoods of the origin is given by sets of the form $\prod_{n=1}^{\infty} U_{n}$ where each $U_{n}$ is a $\mathcal{T}$-neighborhood of zero in $E$. A base can also be given by sets of the form $\prod_{n=1}^{\infty}\left\{x:\|x\|_{\lambda}<\delta_{n}\right\}$ for some fixed norm $\|\cdot\|_{\lambda}$ and a sequence $\delta_{n}>0$. We observe the obvious fact that if $V$ is an infinite-dimensional subspace of $E$ then $\mathcal{T}_{E} \mid V=\mathcal{T}_{V}$ and $\mathcal{T}_{E, b x} \mid V^{\mathbb{N}}=\mathcal{T}_{V, b x}$.

Now let us suppose that $E$ has a given fixed Hamel basis $\left(e_{n}\right)_{n=1}^{\infty}$. Let $E_{n}=$ $\left[e_{1}, \ldots, e_{n}\right]$ and $E^{(n)}=\left[e_{n+1}, e_{n+2}, \ldots\right]$. A sequence $\left(v_{k}\right)_{k=1}^{n}$ where $1 \leq n \leq \infty$ is
called a block basis of $\left(e_{k}\right)_{k=1}^{\infty}$ if each $v_{k} \neq 0$ and

$$
v_{k}=\sum_{j=p_{k-1}+1}^{p_{k}} a_{j} e_{j}
$$

for some increasing sequence $p_{0}=0<p_{1}<p_{2}<\cdots$. A subspace $V$ of $E$ is called a block subspace if $V$ is the linear span of a block basis.

We let $\Sigma_{\infty}(E)$ be the subset of $E^{\mathbb{N}}$ consisting of all infinite block bases. For each $n \in \mathbb{N}$ we let $\Sigma_{n}(E)$ be the subset of $E^{\mathbb{N}}$ of all block bases of length $n$. We also let $\Sigma_{0}(E)$ be the one-point set with a single member $\emptyset$. Let $\Sigma_{<\infty}(E)$ denote the union of all $\Sigma_{n}(E)$ for $0 \leq n<\infty$. If $A$ is a subset of $E$ we denote by $\Sigma_{n}(A)$, etc., the subset of $\Sigma_{n}(E)$ with each element in $A$. In particular we will be interested in the sets

$$
A_{\infty}=\left\{x \in E: 0<\|x\|_{\infty} \leq 1\right\}, S_{\infty}=\left\{x \in E:\|x\|_{\infty}=1\right\} .
$$

Lemma 2.1. Let $\|\cdot\|$ be any norm on $E$ so that $\left(e_{n}\right)_{n=1}^{\infty}$ is a Schauder basis of the completion $\tilde{E}$ of $(E,\|\cdot\|)$. Then, on the space $\Sigma_{\infty}(E)$ the product topology $\mathcal{T}_{p}$ coincides with the product topology induced by $\|\cdot\|$. In particular $\left(\Sigma_{\infty}, \mathcal{T}_{p}\right)$ is a Polish space.

Proof. Let $\xi_{n}=\left(\xi_{n k}\right)_{k=1}^{\infty}$ be a sequence in $\Sigma_{\infty}(E)$ so that for some $\xi=\left(\xi_{k}\right)_{k=1}^{\infty} \in$ $\Sigma_{\infty}(E), \lim _{n \rightarrow \infty}\left\|\xi_{n k}-\xi_{k}\right\|=0$ for each $k$. Let us suppose $\xi_{n k} \in\left[e_{p_{n, k-1}+1}, \ldots, e_{p_{n k}}\right]$ where $p_{n 0}<p_{n 1}<\cdots$ and that $\xi_{k} \in\left[e_{p_{k-1}+1}, \ldots, e_{p_{k}}\right]$. Then it is clear that

$$
\limsup _{n \rightarrow \infty} p_{n, k-1} \leq p_{k} \quad k=1,2, \ldots
$$

using the fact that $\left(e_{n}\right)_{n=1}^{\infty}$ is a Schauder basis. It follows that each sequence $\left(\xi_{n k}\right)_{n=1}^{\infty}$ is contained in some fixed finite-dimensional space and so the convergence is also in $\mathcal{T}_{p}$.

For the product-norm topology it is also easy to see that $\Sigma_{\infty}(E)$ is a closed subset of $(\tilde{E} \backslash\{0\})^{\mathbb{N}}$ and hence is Polish.

Let $\mathcal{B}=\mathcal{B}(E)$ be the collection of all infinite-dimensional block subspaces of $\left(e_{k}\right)_{k=1}^{\infty}$. If $V \in \mathcal{B}$ then $V$ is the span of a block basis $\left(v_{n}\right)_{n=1}^{\infty}$ and we write $\mathcal{B}(V)$ for the collection of infinite-dimensional block subspaces of $V$ with respect to $\left(v_{n}\right)_{n=1}^{\infty}$ (this is clearly independent of the choice of the block basis). We will use the notation $\left(v_{1}, \ldots, v_{r}\right) \prec\left(u_{1}, \ldots, u_{s}\right)$ to mean that $\left(v_{1}, \ldots, v_{r}\right)$ is a block basis of $\left(u_{1}, \ldots, u_{s}\right)$.

Let $\sigma$ be a subset of $\Sigma_{\infty}(E)$. We shall say that $\sigma$ is large if for every $V \in \mathcal{B}(E)$ we have $\sigma \cap \Sigma_{\infty}(V) \neq \emptyset$.

A strategy is a map $\Phi: \Sigma_{<\infty}(E) \times \mathcal{B}(E) \rightarrow \Sigma_{<\infty}(E)$ if for all $\left(u_{1}, \ldots, u_{n}\right) \in \Sigma_{n}(E)$ we have $\Phi\left(u_{1}, u_{2}, \ldots, u_{n} ; V\right)=\left(u_{1}, \ldots, u_{n}, u_{n+1}\right)$ with $u_{n+1} \in V$.

If $\left(V_{j}\right)_{j=1}^{\infty}$ is a sequence of block subspaces then we will write

$$
\Phi\left(u_{1}, \ldots, u_{n} ; V_{1}, \ldots, V_{m}\right)=\left(u_{1}, \ldots, u_{m+n}\right)
$$

and

$$
\Phi\left(u_{1}, \ldots, u_{n} ; V_{1}, \ldots, V_{m}, \ldots\right)=\left(u_{1}, \ldots, u_{m+n}, \ldots\right)
$$

where $u_{n+k}=\Phi\left(u_{1}, \ldots, u_{n+k-1} ; V_{k}\right)$ for $k \geq 1$. In the case when $n=0$ we write $\Phi\left(V_{1}, \ldots, V_{m}\right)$ or $\Phi\left(V_{1}, \ldots, V_{m}, \ldots\right)$ for $\Phi\left(\emptyset ; V_{1}, \ldots, V_{m}\right)$ or $\Phi\left(\emptyset ; V_{1}, \ldots, V_{m}, \ldots\right)$

A subset $\sigma$ of $\Sigma_{\infty}(E)$ is called strategically large for $V \in \mathcal{B}(E)$ and $\left(u_{1}, \ldots, u_{n}\right) \in$ $\Sigma_{<\infty}(E)$ if there is a strategy $\Phi$ with the property that for every sequence $\left(V_{j}\right)_{j=1}^{\infty}$ with $V_{j} \subset V$ we have

$$
\Phi\left(u_{1}, \ldots, u_{n} ; V_{1}, \ldots, V_{m}, \ldots\right) \in \sigma
$$

$\sigma$ is strategically large for $V \in \mathcal{B}(E)$ if it is strategically large for $V \in \mathcal{B}(E)$ and $\emptyset$.

## 3. Functions on subsets of $\Sigma_{<\infty}(E)$

If $V, W$ are subspaces of $E$ let us write $V \subset_{a} W$ to mean that there exists a finite dimensional subspace $F$ so that $V \subset W+F$.

Lemma 3.1 (Stabilization Lemma). Let $E$ be a countable dimensional space with fixed Hamel basis $\left(e_{k}\right)_{k=1}^{\infty}$. Let $X$ be a separable topological space and suppose that, for each $V \in \mathcal{B}(E), f_{V}: X \rightarrow \mathbb{R}$ is a continuous function. Suppose further that

$$
f_{V_{1}}(x) \geq f_{V_{2}}(x) \quad x \in X
$$

whenever $V_{1} \subset_{a} V_{2}$. Then there is a block subspace $W$ of $E$ so that $f_{V}=f_{W}$ whenever $V \subset W$.

More generally suppose $\left(X_{n}\right)_{n=1}^{\infty}$ is a sequence of separable topological spaces and for each $V \in \mathcal{B}$ and $n \in \mathbb{N}, f_{V}^{(n)}: X_{n} \rightarrow \mathbb{R}$ is a continuous function. Suppose further that

$$
f_{V_{1}}^{(n)}(x) \geq f_{V_{2}}^{(n)}(x) \quad x \in X_{n}
$$

whenever $V_{1} \subset_{a} V_{2}$. Then there is a block subspace $W$ of $E$ so that $f_{V}^{(n)}=f_{W}^{(n)}$ whenever $V \subset_{a} W$ and $n \in \mathbb{N}$.

Proof. We prove the first part. We define block subspaces $V_{\alpha}$ for every countable ordinal $\alpha$ by transfinite induction, so that $\alpha \leq \beta \Longrightarrow V_{\beta} \subset_{a} V_{\alpha}$. Set $V_{1}=E$. For each $\alpha$ which is not a limit ordinal, say $\alpha=\beta+1$ define $V_{\alpha} \subset V_{\beta}$ so that $f_{V_{\alpha}} \neq f_{V_{\beta}}$ if possible; otherwise let $V_{\alpha}=V_{\beta}$. If $\alpha$ is a limit ordinal then $\alpha=\sup _{n} \beta_{n}$ for some increasing sequence $\left(\beta_{n}\right)_{n=1}^{\infty}$ with $\beta_{n}<\alpha$. Thus $V_{\beta_{m}} \subset_{a} V_{\beta_{n}}$ if $m>n$. In this case we may by a diagonal argument find $V_{\alpha}$ so that $V_{\alpha} \subset_{a} V_{\beta_{n}}$ for every $n$ (simply choose a block basis $v_{n}$ with $v_{n} \in V_{\beta_{1}} \cap \cdots \cap V_{\beta_{n}}$ ). Now it follows that the functions $f_{V_{\alpha}}$ are increasing in $\alpha$ for $1 \leq \alpha<\omega_{1}$. If $D$ is a countable dense set in $X$ there must therefore exist a countable ordinal $\beta$ so that

$$
f_{V_{\beta}}(x)=f_{V_{\alpha}}(x) \quad x \in D, \beta \leq \alpha
$$

Thus $f_{V_{\beta+1}}=f_{V_{\beta}}$ so that $W=V_{\beta}$ satisfies the conclusion.
The second part reduces to the first if we consider $X=\cup_{n=1}^{\infty} X_{n}$ topologized as a disjoint union and $f_{V}: X \rightarrow \mathbb{R}$ given by $f_{V}(x)=f_{V}^{(n)}(x)$ when $x \in X_{n}$.

Consider a function $f: \Sigma_{<\infty}(A) \rightarrow[0, \infty)$ where $A=S_{\infty}$ or $A=A_{\infty}$. We shall say that $f$ is uniformly $\mathcal{T}_{b x}$-continuous if given $\epsilon>0$ there is a sequence $\left(U_{n}\right)_{n=1}^{\infty}$ of $\mathcal{T}$-neighborhoods of 0 such that if $\left(u_{1}, \ldots, u_{r}\right),\left(v_{1}, \ldots, v_{r}\right) \in \Sigma_{<\infty}(A)$ and $u_{j}-v_{j} \in U_{j}$ for $1 \leq j \leq r$ then

$$
\left|f\left(u_{1}, \ldots, u_{r}\right)-f\left(v_{1}, \ldots, v_{r}\right)\right|<\epsilon
$$

In effect if we induce maps $f^{[n]}$ on $\Sigma_{\infty}(A)$ by

$$
f^{[n]}\left(u_{1}, \ldots, u_{k}, \ldots\right)=f\left(u_{1}, \ldots, u_{n}\right)
$$

this requires that the family of functions $\left(f^{[n]}\right)_{n=1}^{\infty}$ is equi-uniformly continuous for the box topology $\mathcal{T}_{b x}$.

We will need a slightly weaker notion for maps $f: \Sigma_{<\infty}\left(A_{\infty}\right) \rightarrow[0, \infty)$. We will say that $f$ is admissible if it is bounded and
(i) Given $\epsilon>0$, there is a sequence $\left(U_{n}\right)_{n=1}^{\infty}$ of $\mathcal{T}$-neighborhoods of 0 such that if $\left(u_{1}, \ldots, u_{r}\right),\left(v_{1}, \ldots, v_{r}\right) \in \Sigma_{<\infty}\left(S_{\infty}\right)$ and $u_{j}-v_{j} \in U_{j}$ for $1 \leq j \leq r$ then

$$
\left|f\left(\lambda_{1} u_{1}, \ldots, \lambda_{r} u_{r}\right)-f\left(\lambda_{1} v_{1}, \ldots, \lambda_{r} v_{r}\right)\right|<\epsilon \quad\left(\lambda_{1}, \ldots, \lambda_{r}\right) \in(0,1]^{r}
$$

and
(ii) Given $\epsilon>0$ and $\left(u_{1}, \ldots, u_{r}\right) \in \Sigma_{<\infty}\left(S_{\infty}\right)$ there exists $\delta=\delta\left(u_{1}, \ldots, u_{r}, \epsilon\right)>0$ so that if $0<\lambda_{j}, \mu_{j} \leq 1$ for $1 \leq j \leq r$ and $\max _{1 \leq j \leq r}\left|\lambda_{j}-\mu_{j}\right| \leq \delta$ then

$$
\left|f\left(\lambda_{1} u_{1}, \ldots, \lambda_{r} u_{r}, v_{1}, \ldots, v_{s}\right)-f\left(\mu_{1} u_{1}, \ldots, \mu_{r} u_{r}, v_{1}, \ldots, v_{s}\right)\right|<\epsilon
$$

whenever $\left(u_{1}, \ldots, u_{r}, v_{1}, \ldots, v_{s}\right) \in \Sigma_{<\infty}\left(A_{\infty}\right)$.
The following Lemma is easy and its proof is omitted:
Lemma 3.2. (i) Suppose $f: \Sigma_{<\infty}\left(A_{\infty}\right) \rightarrow[0, \infty)$ is bounded and uniformly $\mathcal{T}_{b x}$ continuous; then $f$ is admissible.
(ii) Suppose $f: \Sigma_{<\infty}\left(S_{\infty}\right) \rightarrow[0, \infty)$ is uniformly $\mathcal{T}_{b x}$-continuous; then $g: \Sigma_{<\infty}\left(A_{\infty}\right) \rightarrow$ $[0, \infty)$ is admissible where $g\left(u_{1}, \ldots, u_{n}\right)=f\left(u_{1} /\left\|u_{1}\right\|_{\infty}, \ldots, u_{n} /\left\|u_{n}\right\|_{\infty}\right)$.

Lemma 3.3. If $f: \Sigma_{<\infty}\left(A_{\infty}\right) \rightarrow[0, \infty)$ is admissible then for each $m \in \mathbb{N}$ the map $F_{m}:(0,1]^{m} \times \Sigma_{m}\left(S_{\infty}\right) \rightarrow[0, \infty)$ defined by

$$
F\left(\lambda_{1}, \ldots, \lambda_{m}, u_{1}, \ldots, u_{m}\right)=f\left(\lambda_{1} u_{1}, \ldots, \lambda_{m} u_{m}\right)
$$

is continuous when $\Sigma_{m}\left(S_{\infty}\right) \subset(E, \mathcal{T})^{m}$ is given the subset topology.
Proof. Suppose $\epsilon>0$. We pick $\mathcal{T}$-neighborhoods of zero in $E, U_{1}, \ldots, U_{m}$ so that $u_{j}-v_{j} \in U_{j}$ for $1 \leq j \leq m$ implies that

$$
\left|f\left(\lambda_{1} v_{1}, \ldots, \lambda_{m} v_{m}\right)-f\left(\lambda_{1} u_{1}, \ldots, \lambda_{m} u_{m}\right)\right|<\epsilon / 2
$$

for every $\left(\lambda_{1}, \ldots, \lambda_{m}\right) \in(0,1]^{m}$. If $\left(v_{1}, \ldots, v_{m}\right) \in \Sigma_{m}\left(E_{n} \cap S_{\infty}\right)$ we then pick $\delta=$ $\delta\left(v_{1}, \ldots, v_{r}\right)>0$ so that if $\left|\lambda_{j}-\mu_{j}\right|<\delta$ for $1 \leq j \leq m$ we have

$$
\left|f\left(\lambda_{1} v_{1}, \ldots, \lambda_{m} v_{m}\right)-f\left(\mu_{1} v_{1}, \ldots, \mu_{m} v_{m}\right)\right|<\epsilon / 2 .
$$

Combining gives

$$
\left|f\left(\lambda_{1} u_{1}, \ldots, \lambda_{m} u_{m}\right)-f\left(\mu_{1} v_{1}, \ldots, \mu_{m} v_{m}\right)\right|<\epsilon
$$

whenever $\max _{1 \leq j \leq m}\left|\lambda_{j}-\mu_{j}\right|<\delta$ and $u_{j}-v_{j} \in U_{j}$ for $1 \leq j \leq m$. Thus $F$ is continuous at each point $\left(\mu_{1}, \ldots, \mu_{r}, v_{1}, \ldots, v_{m}\right)$.

Suppose $f: \Sigma_{<\infty}\left(A_{\infty}\right) \rightarrow[0, \infty)$ is any admissible function. Let us adopt the convention that the function $f$ takes the value $+\infty$ at any point of $E^{\mathbb{N}} \backslash \Sigma_{<\infty}\left(A_{\infty}\right)$. For any $V \in \mathcal{B}(E)$ let

$$
\begin{aligned}
& f_{V}^{\prime}\left(u_{1}, \ldots, u_{n}\right)= \\
& \lim _{m \rightarrow \infty} \inf \left\{f\left(u_{1}, \ldots, u_{n}, v_{1}, \ldots, v_{s}\right): v_{1}, \ldots, v_{s} \in V \cap E^{(m)}, s \geq 1\right\} .
\end{aligned}
$$

Note that $V \subset_{a} W$ implies that $f_{V}^{\prime} \geq f_{W}^{\prime}$.
The following is more or less immediate:
Lemma 3.4. If $f: \Sigma_{<\infty}\left(A_{\infty}\right) \rightarrow[0, \infty)$ is admissible, then the each of the functions $f_{V}^{\prime}: \Sigma_{<\infty}\left(A_{\infty}\right) \rightarrow[0, \infty)$ is admissible.

Lemma 3.5. If $\mathcal{F}$ is a countable family of admissible functions, then there exists $V \in \mathcal{B}(E)$ so that for every $W \in \mathcal{B}(V)$ and every $f \in \mathcal{F}$ we have $f_{W}^{\prime}=f_{V}^{\prime}$.
Proof. For $W \in \mathcal{B}(E)$ and $m<n$ define $g_{m, n, W}:(0,1]^{m} \times \Sigma_{m}\left(S_{\infty} \cap E^{n}\right) \rightarrow \mathbb{R}$ by

$$
g_{m, n, W}\left(\lambda_{1}, \ldots, \lambda_{m}, u_{1}, \ldots, u_{m}\right)=f_{W}^{\prime}\left(\lambda_{1} u_{1}, \ldots, \lambda_{m} u_{m}\right) .
$$

Thus $g_{m, n, W}$ is continuous by Lemma [3.3. Since $(0,1]^{m} \times \Sigma_{m}\left(S_{\infty} \cap E^{n}\right)$ is separable for each $m, n$, we can apply the Stabilization Lemma 3.1,

We can thus assume, under the hypotheses of the Lemma, (by passing to a block subspace) that $f$ has the property that $f_{V}^{\prime}=f_{E}^{\prime}$ for all block subspaces $V$. If this happens we shall say that $f$ is stable and we write $f^{\prime}$ for $f_{E}^{\prime}$. Note that $f^{\prime}$ is admissible.

Proposition 3.6. Let $f$ be a stable admissible function. Suppose $\left(u_{1}, \ldots, u_{r}\right) \in$ $\Sigma_{<\infty}\left(A_{\infty}\right)$ and $V$ is a block subspace. Then for any $\epsilon>0$ there exists $\xi \in V \backslash\{0\}$ so that either:
(a) $f\left(u_{1}, \ldots, u_{r}, \xi\right)<f^{\prime}\left(u_{1}, \ldots, u_{r}\right)+\epsilon$ or
(b) $f^{\prime}\left(u_{1}, \ldots, u_{r}, \xi\right)<f^{\prime}\left(u_{1}, \ldots, u_{r}\right)+\epsilon$.

Proof. Let us assume that $\left(u_{1}, \ldots, u_{r}\right) \in \Sigma_{r}(E)$. Let us further assume that $V$ is a block subspace so that for any $\xi \in V$ we have

$$
\begin{equation*}
f\left(u_{1}, \ldots, u_{r}, \xi\right) \geq f^{\prime}\left(u_{1}, \ldots, u_{r}\right)+\epsilon, \quad f^{\prime}\left(u_{1}, \ldots, u_{r}, \xi\right) \geq f^{\prime}\left(u_{1}, \ldots, u_{r}\right)+\epsilon \tag{3.1}
\end{equation*}
$$

Let $\left(v_{j}\right)_{j=1}^{\infty}$ be a block basis which is a basis of $V$. We choose an increasing sequence of integers $\left(q_{k}\right)_{k=1}^{\infty}$ as follows. Let $q_{1}=1$. Assume $q_{1}, \ldots, q_{k}$ have been chosen. Let $m_{0}$ be the smallest integer so that $v_{q_{k}} \in E_{m_{0}}$. Then for every $\xi \in S_{\infty} \cap\left[v_{q_{1}}, \ldots, v_{q_{k}}\right]$ (3.1) holds. We may pick a neighborhood $U$ of 0 in $(E, \mathcal{T})$ so that

$$
\left|f\left(u_{1}, \ldots, u_{r}, \lambda \xi, w_{1} \ldots, w_{s}\right)-f\left(u_{1}, \ldots, u_{r}, \lambda \eta, w_{1}, \ldots, w_{s}\right)\right|<\epsilon / 8
$$

when $\xi-\eta \in U,\left(w_{1}, \ldots, w_{s}\right) \in \Sigma_{<\infty}\left(A_{\infty} \cap E^{\left(m_{0}\right)}\right)$ and $0<\lambda \leq 1$. By compactness there is a finite subset $\left(\xi_{1}, \ldots, \xi_{t}\right)$ of $S_{\infty} \cap\left[v_{q_{1}}, \ldots, v_{q_{k}}\right]$ such that $\eta \in$ $S_{\infty} \cap\left[v_{q_{1}}, \ldots, v_{q_{k}}\right]$ implies $\eta-\xi_{j} \in U$ for some $j$. Now pick an integer $N$ large enough so that $|\lambda-\mu|<1 / N$ implies

$$
\left|f\left(u_{1}, \ldots, u_{r}, \lambda \xi_{j}, w_{1}, \ldots, w_{s}\right)-f\left(u_{1}, \ldots, u_{r}, \mu \xi_{j}, w_{1}, \ldots, w_{s}\right)\right|<\epsilon / 8
$$

whenever $\left(w_{1}, \ldots, w_{s}\right) \in \Sigma_{<\infty}\left(E^{\left(m_{0}\right)}\right)$. Now by our assumptions we can pick $m \geq m_{0}$ so that

$$
f\left(u_{1}, \ldots, u_{r}, \frac{k}{N} \xi_{j}, w_{1}, \ldots, w_{s}\right)>f^{\prime}\left(u_{1}, \ldots, u_{r}\right)+\frac{3}{4} \epsilon
$$

whenever $1 \leq j \leq t, 1 \leq k \leq N$ and $\left(w_{1}, \ldots, w_{s}\right) \in \Sigma_{<\infty}\left(E^{(m)}\right)$. Hence

$$
f\left(u_{1}, \ldots, u_{r}, \lambda \xi_{j}, w_{1}, \ldots, w_{s}\right)>f^{\prime}\left(u_{1}, \ldots, u_{r}\right)+\frac{5}{8} \epsilon
$$

whenever $1 \leq j \leq t, 0<\lambda \leq 1$ and $\left(w_{1}, \ldots, w_{s}\right) \in \Sigma_{<\infty}\left(E^{(m)}\right)$, and thus

$$
\begin{equation*}
f\left(u_{1}, \ldots, u_{r}, \lambda \xi, w_{1}, \ldots, w_{s}\right)>f^{\prime}\left(u_{1}, \ldots, u_{r}\right)+\frac{1}{2} \epsilon \tag{3.2}
\end{equation*}
$$

whenever $0<\lambda \leq 1$ and $\xi \in S_{\infty} \cap\left[v_{q_{1}}, \ldots, v_{q_{k}}\right]$.
Then we pick $q_{k+1}>q_{k}$ so that $v_{q_{k+1}} \in E^{(m)}$. This completes the inductive construction. Now let $W=\left[v_{q_{1}}, v_{q_{2}}, \ldots\right]$. There exists $\left(w_{1}, \ldots, w_{s}\right) \in \Sigma_{s}(W)$ (where $s>0)$ so that $f\left(u_{1}, \ldots, u_{r}, w_{1}, \ldots, w_{s}\right)<f^{\prime}\left(u_{1}, \ldots, u_{r}\right)+\epsilon / 2$.

If $s=1$ then $\xi=w_{1}$ contradicts (3.1). If $s>1$ let $\lambda \xi=w_{1}=\sum_{j=1}^{l} a_{j} v_{q_{j}}$ where $a_{l} \neq 0$. Then by the selection of $q_{l+1}$ and (3.2) we see that $f\left(u_{1}, \ldots, u_{r}, w_{1}, \ldots, w_{s}\right)>$ $f^{\prime}\left(u_{1}, \ldots, u_{r}\right)+\epsilon / 2$, a contradiction.
Lemma 3.7. If $f: \Sigma_{<\infty}\left(A_{\infty}\right) \rightarrow[0, \infty)$ is admissible then the function $g: \Sigma_{<\infty}\left(A_{\infty}\right) \rightarrow$ $[0, \infty)$ given by $g\left(u_{1}, \ldots, u_{r}\right)=1$ if $r=0$ or $r=1$ and
$g\left(u_{1}, \ldots, u_{r}\right)=\inf \left\{f\left(v_{1}, \ldots, v_{s}\right):\left(v_{1}, \ldots, v_{s}\right) \prec\left(u_{1}, \ldots, u_{r}\right), 1 \leq s<r,\right\} \quad(r>1)$
is admissible.
Proof. Note that $g$ satisfies $g\left(\lambda_{1} u_{1}, \ldots, \lambda_{m} u_{m}\right)=g\left(u_{1}, \ldots, u_{m}\right)$ if $\left(u_{1}, \ldots, u_{m}\right) \in$ $\Sigma_{<\infty}\left(S_{\infty}\right)$ and $\left(\lambda_{1}, \ldots, \lambda_{m}\right) \in(0,1]^{m}$. Hence it suffices to show that $g$ is uniformly $\mathcal{T}_{b x}$-continuous on $\Sigma_{<\infty}\left(S_{\infty}\right)$. Note that, if for $\operatorname{all}\left(u_{1}, \ldots, u_{m}\right) \in \Sigma_{<\infty}\left(S_{\infty}\right)$ we define

$$
h\left(u_{1}, \ldots, u_{m}\right)=\inf \left\{f\left(\lambda_{1} u_{1}, \ldots, \lambda_{m} u_{m}\right):\left(\lambda_{1}, \ldots, \lambda_{m}\right) \in(0,1]^{m}\right\},
$$

then $h$ is uniformly $\mathcal{T}_{b x}$-continuous and
$g\left(u_{1}, \ldots, u_{r}\right)=\inf \left\{h\left(v_{1}, \ldots, v_{s}\right):\left(v_{1}, \ldots, v_{s}\right) \prec\left(u_{1}, \ldots, u_{r}\right), 1 \leq s<r,\right\} \quad(r>1)$
Suppose $\epsilon>0$. Then there is a sequence $\left(U_{n}\right)_{n=1}^{\infty}$ of $\mathcal{T}$-neighborhoods of zero so that if $\left(u_{1}, \ldots, u_{n}\right),\left(v_{1}, \ldots, v_{n}\right) \in \Sigma_{<\infty}\left(S_{\infty}\right)$ and $u_{j}-v_{j} \in U_{j}$ for $1 \leq j \leq n$ then

$$
\left|h\left(u_{1}, \ldots, u_{n}\right)-h\left(v_{1}, \ldots, v_{n}\right)\right|<\epsilon .
$$

Pick a sequence of circled neighborhoods of zero, $\left(U_{n}^{\prime}\right)_{n=1}^{\infty}$, so that $U_{n}^{\prime}+U_{n+1}^{\prime}+\cdots+$ $U_{n+k}^{\prime} \subset U_{n}$ whenever $k \geq n$. Then suppose $\left(u_{1}, \ldots, u_{n}\right),\left(v_{1}, \ldots, v_{n}\right) \in \Sigma_{<\infty}\left(S_{\infty}\right)$ with $u_{j}-v_{j} \in U_{j}^{\prime}$ for $1 \leq j \leq n$. Assume $\eta>0$ and pick $\left(x_{1}, \ldots, x_{r}\right) \prec\left(u_{1}, \ldots, u_{n}\right)$ with $r<n$ so that $h\left(x_{1}, \ldots, x_{r}\right)<g\left(u_{1}, \ldots, u_{n}\right)+\eta$. If $x_{j}=\sum_{i=k+1}^{l} a_{i} u_{i}$ let $y_{j}=\sum_{i=k+1}^{l} a_{i} v_{i}$. Then $\left|a_{i}\right| \leq 1$ and so $x_{j}-y_{j} \in U_{k+1}^{\prime}+\cdots+U_{l}^{\prime} \subset U_{j}$ since $j \leq k+1$. Hence

$$
h\left(y_{1}, \ldots, y_{r}\right)<g\left(u_{1}, \ldots, u_{n}\right)+\epsilon+\eta
$$

and so

$$
g\left(v_{1}, \ldots, v_{n}\right) \leq g\left(u_{1}, \ldots, u_{n}\right)+\epsilon
$$

By symmetry

$$
\left|g\left(v_{1}, \ldots, v_{n}\right)-g\left(u_{1}, \ldots, u_{n}\right)\right| \leq \epsilon
$$

and hence $g$ is uniformly $\mathcal{T}_{b x}$-continuous.
We shall say that a strategy $\Phi$ is $(\epsilon, V)$-effective for $f$ where $\epsilon>0$ and $V \in \mathcal{B}(E)$ if for every sequence of block subspaces $V_{j} \subset V$ there exists $n \in \mathbb{N}$ so that

$$
f\left(\Phi\left(\emptyset, V_{1}, \ldots, V_{n}\right)\right)<\sup _{W \in \mathcal{B}(E)} f_{W}^{\prime}(\emptyset)+\epsilon .
$$

If $\left(u_{1}, \ldots, u_{r}\right) \in \Sigma_{<\infty}(E)$ we shall say that a strategy $\Phi$ is $\left(\epsilon, u_{1}, \ldots, u_{r}, V\right)$-effective for $f$ where $\epsilon>0$ and $V \in \mathcal{B}(E)$ if for every sequence of block subspaces $V_{j} \subset V$ there exists $n \in \mathbb{N}$ so that

$$
f\left(\Phi\left(u_{1}, \ldots, u_{r}, V_{1}, \ldots, V_{n}\right)\right)<\sup _{W \in \mathcal{B}(E)} f_{W}^{\prime}\left(u_{1}, \ldots, u_{r}\right)+\epsilon
$$

Theorem 3.8. Suppose $f: \Sigma_{<\infty}\left(A_{\infty}\right) \rightarrow[0, \infty)$ is admissible. Then, given $\epsilon>0$ there is a block subspace $V$ and a strategy $\Phi$ which is $(\epsilon, V)$-effective for $f$.

Proof. We assume (after stabilization) that $f_{V}^{\prime}(\emptyset)=0$ for all $V \in \mathcal{B}(E)$; indeed if $a=\sup _{V \in \mathcal{B}(E)} f_{V}^{\prime}(\emptyset)$ then replace $f$ by max $(f-a, 0)$. We consider the admissible function

$$
h\left(u_{1}, \ldots, u_{r}\right)=f\left(u_{1}, \ldots, u_{r}\right)+2\left(\epsilon-2 g\left(u_{1}, \ldots, u_{r}\right)\right)_{+} .
$$

By Lemma 3.4 we can pass to a block subspace where $h$ is stable; so let us assume $h$ is stable on $E$.

We first claim that if $\left(u_{1}, \ldots, u_{n}, \ldots\right) \in \Sigma_{\infty}\left(A_{\infty}\right)$ then there exists $n$ so that $h^{\prime}\left(u_{1}, \ldots, u_{n}\right)>\epsilon$. Indeed, let $W=\left[u_{1}, \ldots, u_{n}, \ldots\right]$. Then, since $f_{W}^{\prime}(\emptyset)=0$, there exists $\left(v_{1}, \ldots, v_{s}\right) \in \Sigma_{<\infty}(W)$ with $f\left(v_{1}, \ldots, v_{s}\right)<\epsilon / 4$. Then we may find $r>s$ so that $\left(v_{1}, \ldots v_{s}\right) \prec\left(u_{1}, \ldots, u_{r}\right)$. Hence for any $\left(x_{1}, \ldots, x_{t}\right)$ such that $\left(u_{1}, \ldots, u_{r}, x_{1}, \ldots, x_{t}\right) \in \Sigma_{<\infty}\left(A_{\infty}\right)$ we have

$$
h\left(u_{1}, \ldots, u_{r}, x_{1}, \ldots, x_{t}\right) \geq 2\left(\epsilon-2 f\left(v_{1}, \ldots, v_{s}\right)\right)
$$

so that

$$
h^{\prime}\left(u_{1}, \ldots, u_{r}\right) \geq 2\left(\epsilon-2 f\left(v_{1}, \ldots, v_{s}\right)\right)>\epsilon,
$$

which proves the claim.
On the other hand, given any block subspace $V$, there exists a minimal $s \geq 1$ so that we can find $\left(v_{1}, \ldots, v_{s}\right) \in \Sigma_{<\infty}(V)$ with $f\left(v_{1}, \ldots, v_{s}\right)<\epsilon / 2$. Thus $g\left(v_{1}, \ldots, v_{s}\right) \geq$ $\epsilon / 2$ which implies $h\left(v_{1}, \ldots, v_{s}\right)<\epsilon / 2$. Hence $h^{\prime}(\emptyset)<\epsilon / 2$.

We now use a strategy for $h$ indicated by Proposition [3.6. Suppose $\epsilon_{j}>0$ for each $j \geq 0$ and $\sum \epsilon_{r}<\epsilon / 2$. Given $\left(u_{1}, \ldots, u_{r}\right) \in \Sigma_{<\infty}(E)$ and $V \in \mathcal{B}(E)$ we define $\Phi\left(u_{1}, \ldots, u_{r}, V\right)$ to be $\left(u_{1}, \ldots, u_{r}, \xi\right)$ where $\xi \in V \backslash\{0\}$ is chosen so that $h^{\prime}\left(u_{1}, \ldots, u_{r}, \xi\right)<h^{\prime}\left(u_{1}, \ldots, u_{r}\right)+\epsilon_{r}$ or $h\left(u_{1}, \ldots, u_{r}, \xi\right)<h^{\prime}\left(u_{1}, \ldots, u_{r}\right)+\epsilon_{r}$. Let $\left(V_{j}\right)_{j=1}^{\infty}$ be a sequence in $\mathcal{B}(E)$ and let $\left(u_{1}, \ldots, u_{r}, \ldots\right)=\Phi\left(\emptyset ; V_{1}, \ldots,\right)$. Then, by our previous claim there exists a first $n \geq 1$ so that $h^{\prime}\left(u_{1}, \ldots, u_{n}\right)>h^{\prime}\left(u_{1}, \ldots, u_{n-1}\right)+$ $\epsilon_{n-1}$. Hence

$$
h\left(u_{1}, \ldots, u_{n}\right)<h^{\prime}\left(u_{1}, \ldots, u_{n-1}\right)+\epsilon_{n-1}<h^{\prime}(\emptyset)+\sum_{j=0}^{n-1} \epsilon_{j}<\epsilon .
$$

We need an obvious extension of this result.
Theorem 3.9. Suppose $f: \Sigma_{<\infty}\left(A_{\infty}\right) \rightarrow[0, \infty)$ is admissible. Then there is a block subspace $V$ such that for each $\left(u_{1}, \ldots, u_{r}\right) \in \Sigma_{<\infty}\left(A_{\infty}\right)$ there is a strategy $\Phi_{u_{1}, \ldots, u_{r}}$ which is $\left(\epsilon, u_{1}, \ldots, u_{r}, V\right)$-effective for $f$.

Proof. It is easy to obtain, for each $\left(u_{1}, \ldots, u_{r}\right)$ the existence of a block subspace $V_{u_{1}, \ldots, u_{r}}$ and a strategy $\Psi_{u_{1}, \ldots, u_{r}}$ which is $\left(\epsilon / 2, u_{1}, \ldots, u_{r}, V_{u_{1}, \ldots, u_{r}}\right)$-effective for $f$. Indeed, suppose $u_{r} \in E_{m}$; define

$$
f_{1}\left(v_{1}, \ldots, v_{s}\right)=f\left(u_{1}, \ldots, u_{r}, v_{1}, \ldots, v_{s}\right) \quad\left(v_{1}, \ldots, v_{s}\right) \in \Sigma_{<\infty}\left(A_{\infty} \cap E^{(m)}\right)
$$

and apply the preceding theorem to $f_{1}$. $\left(\Psi_{u_{1}, \ldots, u_{r}}\right.$ has to be defined in some arbitrary fashion for $\left(w_{1}, \ldots, w_{s}\right)$ which do not have $\left(u_{1}, \ldots, u_{r}\right)$ as an initial segment.) Furthermore it can be seen that for each block subspace $W$ we can choose $V_{u_{1}, \ldots, u_{r}} \subset W$.

To obtain a single block subspace $V$ we first construct a dense countable subset $D_{m, r}$ in each $\Sigma_{r}\left(E_{m} \cap \mathcal{A}_{\infty}\right)$. We arrange the elements of $D=\cup_{m, r} D_{m, r}$ as a sequence and hence find a descending sequence of subspaces $\left(V_{n}\right)$ so that the strategy $\Phi_{u_{1}, \ldots, u_{r}}$ is $\left(\epsilon / 2, u_{1}, \ldots, u_{r}, V_{n}\right)$-effective for $f$ when $\left(u_{1}, \ldots, u_{r}\right)$ is the $n$ th. member of $D$. If we select $V$ to be block subspace so that $V \subset V_{n}+F_{n}$ for some finite-dimensional $F_{n}$ for each $n$, then (via a simple modification) each $\Phi_{u_{1}, \ldots, u_{r}}$ is $\left(\epsilon / 2, u_{1}, \ldots, u_{r}, V\right)$-effective for $f$. Finally we observe that if $\left(v_{1}, \ldots, v_{r}\right) \in \Sigma_{r}\left(E_{m}\right)$ is arbitrary and we then choose $\left(u_{1}, \ldots, u_{r}\right) \in D$ close enough, we can define a strategy by

$$
\Phi_{v_{1}, \ldots, v_{r}}\left(v_{1}, \ldots, v_{r}, w_{1}, \ldots, w_{s}\right)=\Psi_{u_{1}, \ldots, u_{r}}\left(u_{1}, \ldots, u_{r}, w_{1}, \ldots, w_{s}\right)
$$

(and arbitrarily otherwise) then we will have that $\Phi_{v_{1}, \ldots, v_{n}}$ is $\left(\epsilon, v_{1}, \ldots, v_{r}, V\right)$-effective for $f$.

## 4. The infinite case

We now turn to the infinite case. Suppose $f: \Sigma_{\infty}\left(A_{\infty}\right) \rightarrow[0, \infty)$ is a bounded uniformly $\mathcal{T}_{b x}$-continuous function. We may define $f_{V}^{\prime}: \Sigma_{<\infty}\left(A_{\infty}\right) \rightarrow[0, \infty)$ in a precisely analogous way. As before we adopt the convention that $f=+\infty$ on $E^{\mathbb{N}} \backslash \Sigma_{\infty}\left(A_{\infty}\right)$. Let

$$
\begin{aligned}
& f_{V}^{\prime}\left(u_{1}, \ldots, u_{r}\right)= \\
& \lim _{m \rightarrow \infty} \inf \left\{f\left(u_{1}, \ldots, u_{r}, v_{1}, \ldots, v_{s}, \ldots\right): v_{j} \in V \cap E^{(m)}, j=1,2, \ldots\right\} .
\end{aligned}
$$

It is clear that the functions $\left\{f_{V}^{\prime}: V \in \mathcal{B}(E)\right\}$ are equi-uniformly $\mathcal{T}_{b x}$-continuous.
Proceeding in the same manner as before we can show:
Proposition 4.1. If $f_{n}: \Sigma_{\infty}\left(A_{\infty}\right) \rightarrow \mathbb{R}$ is any countable family of bounded $\mathcal{T}_{b x}$ uniformly continuous functions, there is a block subspace $V$ of $E$ so that $f_{W}^{\prime}=f_{V}^{\prime}$ whenever $W \in \mathcal{B}(V)$.

We shall say that $f$ is stable if $f_{E}^{\prime}=f_{V}^{\prime}$ for every $V \in \mathcal{B}(E)$.

Lemma 4.2. Let $f_{n}: \Sigma_{\infty}\left(A_{\infty}\right) \rightarrow[0, \infty)$ be a sequence of bounded uniformly $\mathcal{T}_{b x}{ }^{-}$ continuous functions and suppose $f=\inf f_{n}$ is also $\mathcal{T}_{b x}$-uniformly continuous. Assume that each $f_{n}$ and $f$ are stable. Let $h: \Sigma_{<\infty}\left(A_{\infty}\right)=\inf _{n} f_{n}^{\prime}$. Then for every $V \in \mathcal{B}(E)$ we have $h_{V}^{\prime} \leq f^{\prime}$.

Proof. Let $V \in \mathcal{B}(E)$. Let us assume $h_{V}^{\prime}\left(u_{1}, \ldots, u_{r}\right)>\lambda>f^{\prime}\left(u_{1}, \ldots, u_{r}\right)$ for some $\left(u_{1}, \ldots, u_{r}\right) \in \Sigma_{<\infty}\left(A_{\infty} \cap V\right)$. Then there exists $m$ so that if $\left(v_{1}, \ldots, v_{s}\right) \in$ $\Sigma_{<\infty}\left(E^{(m)} \cap V\right)$ with $s \geq 1$ we have

$$
h\left(u_{1}, \ldots, u_{r}, v_{1}, \ldots, v_{s}\right)>\lambda .
$$

Thus

$$
f_{n}^{\prime}\left(u_{1}, \ldots, u_{r}, v_{1}, \ldots, v_{s}\right)>\lambda, \quad n=1,2 \ldots
$$

Let us pick $w_{1} \in A_{\infty} \cap E^{(m)} \cap V$. We will construct a sequence $\left(w_{n}\right)_{n=1}^{\infty}$ in $V$ by induction. Suppose $\left(w_{1}, \ldots, w_{n}\right)$ have been selected and let $W_{n}=\left[w_{1}, \ldots, w_{n}\right]$. Then by compactness we can find $p$ so that $W_{n} \subset E_{p}$ and if $1 \leq k \leq n$ and $\left(v_{1}, \ldots, v_{k}\right) \in \Sigma_{k}\left(W_{n} \cap A_{\infty}\right)$ then

$$
f_{k}\left(u_{1}, \ldots, u_{r}, v_{1}, \ldots, v_{k}, x_{1}, \ldots\right)>\lambda
$$

for all choices of $\left(x_{j}\right)_{j=1}^{\infty}$ in $E^{(p)}$. Pick $w_{n+1} \in E^{(p)} \cap V$.
Now let $W=\left[w_{j}\right]_{j=1}^{\infty}$. Our construction guarantees that for every $n$ it is true that

$$
f_{n}\left(u_{1}, \ldots, u_{r}, x_{1}, \ldots\right)>\lambda \quad x_{j} \in W, j=1,2, \ldots
$$

Indeed we have $\left(x_{1}, \ldots, x_{n}\right) \in \Sigma_{n}\left(\left[w_{1}, \ldots, w_{m}\right]\right)$ where $m>n$ and so this follows from our inductive construction. Thus

$$
f^{\prime}\left(u_{1}, \ldots, u_{r}\right) \geq \lambda,
$$

contradicting our initial hypothesis.
We now use the space $\mathbb{N}^{\mathbb{N}}$ with the usual product topology; this can be regarded as the space of all infinite words of the natural numbers. We will write $\mathbb{N}^{<\infty}=\cup_{n=0}^{\infty} \mathbb{N}^{k}$ which is the space of all finite words of the natural numbers, including the empty word. We will use ( $n_{1}, \ldots, n_{k}$ ) or ( $n_{1}, n_{2}, \ldots$ ) to denote a typical member of $\mathbb{N}^{<\infty}$ or $\mathbb{N}^{\mathbb{N}}$.

Theorem 4.3. Suppose $F: \mathbb{N}^{\mathbb{N}} \times \Sigma_{\infty}\left(A_{\infty}\right) \rightarrow[0, \infty)$ is a bounded map. Define $f_{n_{1}, n_{2}, \ldots}: \Sigma_{\infty}\left(A_{\infty}\right) \rightarrow[0, \infty)$ by

$$
f_{n_{1}, n_{2}, \ldots}\left(u_{1}, u_{2}, \ldots\right)=F\left(n_{1}, n_{2}, \ldots ; u_{1}, u_{2}, \ldots\right) .
$$

Suppose
(i) The maps $\left\{f_{n_{1}, n_{2}, \ldots}:\left(n_{1}, n_{2}, \ldots\right) \in \mathbb{N}^{\mathbb{N}}\right\}$ are equi-uniformly $\mathcal{T}_{b x}$-continuous.
(ii) The map $F: \mathbb{N}^{\mathbb{N}} \times \Sigma_{\infty}\left(A_{\infty}\right) \rightarrow[0, \infty)$ is lower semi-continuous for the product topology on $\mathbb{N}^{\mathbb{N}} \times\left(\Sigma_{\infty}\left(A_{\infty}\right), \mathcal{T}_{p}\right)$.

Let

$$
f\left(u_{1}, u_{2}, \ldots\right)=\inf _{\left(n_{1}, n_{2}, \ldots\right) \in \mathbb{N}^{\mathbb{N}}} F\left(n_{1}, n_{2}, \ldots ; u_{1}, u_{2}, \ldots\right) .
$$

If $f_{V}^{\prime}(\emptyset)=0$ for every block subspace $V$, then given $\epsilon>0$ there is a block subspace $V$ so that $\{f<\epsilon\}$ is $V$-strategically large.

Proof. For each $\left(n_{1}, \ldots, n_{k}\right) \in \mathbb{N}^{<\infty}$ we define

$$
f_{n_{1}, n_{2}, \ldots, n_{k}}\left(u_{1}, u_{2}, \ldots\right)=\inf _{m_{1}, m_{2} \ldots} F\left(n_{1}, \ldots, n_{k}, m_{1}, m_{2}, \ldots ; u_{1}, u_{2}, \ldots\right) .
$$

The family $f_{n_{1}, n_{2}, \ldots, n_{k}}$ is $\mathcal{T}_{b x}$-equi-uniformly continuous. By passing to a block subspace we can assume that each $f_{n_{1}, n_{2}, \ldots, n_{k}}$ is stable. Of course the family $f_{n_{1}, \ldots, n_{k}}^{\prime}$ is also $\mathcal{T}_{b x}$-equi-uniformly continuous. Let

$$
h_{n_{1}, \ldots, n_{k}}\left(u_{1}, \ldots, u_{r}\right)=\inf _{m \in \mathbb{N}} f_{n_{1}, \ldots, n_{k}, m}^{\prime}\left(u_{1}, \ldots, u_{r}\right) \quad\left(u_{1}, \ldots, u_{r}\right) \in \Sigma_{<\infty}\left(A_{\infty}\right) .
$$

This family is also equi-uniformly $\mathcal{T}_{b x}$-continuous. Passing to a further block subspace we can suppose that this family is also stable.

By Lemma 4.2 we have that $h_{n_{1}, \ldots, n_{k}}^{\prime} \leq f_{n_{1}, \ldots, n_{k}}^{\prime}$.
Let us choose a sequence $\left(\epsilon_{r}\right)_{r=0}^{\infty}$ with $\sum \epsilon_{r}=\epsilon^{\prime}<\epsilon$. Again exploiting the countability of the family $h_{n_{1}, \ldots, n_{k}}$ we can pass to a further block subspace and, by relabelling as $E$, suppose that for each $\left(n_{1}, \ldots, n_{k}\right) \in \mathbb{N}^{<\infty}$ and $\left(u_{1}, \ldots, u_{r}\right) \in \Sigma_{<\infty}\left(A_{\infty}\right)$ there is a strategy $\Phi_{n_{1}, \ldots, n_{k}, u_{1}, \ldots, u_{r}}$ with the property that if $\left(V_{j}\right)_{j=1}^{\infty}$ is any sequence of subspaces then for some $p \geq 1$,

$$
h_{n_{1}, \ldots, n_{k}} \Phi_{n_{1}, \ldots, n_{k}, u_{1}, \ldots, u_{r}}\left(u_{1}, \ldots, u_{r}, V_{1}, \ldots, V_{p}\right)<h_{n_{1}, \ldots, n_{k}}^{\prime}\left(u_{1}, \ldots, u_{r}\right)+\epsilon_{k} .
$$

We will now define a strategy $\Psi$. To do this we first define maps $\theta: \Sigma_{<\infty}\left(A_{\infty}\right) \rightarrow$ $\mathbb{N}^{<\infty}$ and $\varphi: \Sigma_{<\infty}\left(A_{\infty}\right) \rightarrow \mathbb{N}$ such that $\varphi\left(u_{1}, \ldots, u_{r}\right) \leq r$. This is done inductively on the length of $\left(u_{1}, \ldots, u_{r}\right)$. We define $\theta(\emptyset)=\emptyset$ and $\varphi(\emptyset)=0$. Suppose that $\theta$ and $\varphi$ have been defined for all ranks up to $r$ and consider ( $u_{1}, \ldots, u_{r+1}$ ).

Let $\theta\left(u_{1}, \ldots, u_{r}\right)=\left(n_{1}, \ldots, n_{k}\right)$ and $\varphi\left(u_{1}, \ldots, u_{r}\right)=s$. If

$$
h_{n_{1}, \ldots, n_{k}}\left(u_{1}, \ldots, u_{r+1}\right)<h_{n_{1}, \ldots, n_{k}}^{\prime}\left(u_{1}, \ldots, u_{s}\right)+\epsilon_{k}
$$

then we can choose $m \in \mathbb{N}$ so that

$$
f_{n_{1}, \ldots, n_{k}, m}^{\prime}\left(u_{1}, \ldots, u_{r+1}\right)<h_{n_{1}, \ldots, n_{k}}^{\prime}\left(u_{1}, \ldots, u_{s}\right)+\epsilon_{k}
$$

Let $\theta\left(u_{1}, \ldots, u_{r+1}\right)=\left(n_{1}, n_{2}, \ldots, n_{k}, m\right)$ and $\varphi\left(u_{1}, \ldots, u_{r+1}\right)=r+1$.
Otherwise we simply put $\theta\left(u_{1}, \ldots, u_{r+1}\right)=\left(n_{1}, \ldots, n_{k}\right)$ and $\varphi\left(u_{1}, \ldots, u_{r+1}\right)=s$.
To define $\Psi$ we set

$$
\Psi\left(u_{1}, \ldots, u_{r}, V\right)=\Phi_{n_{1}, \ldots, n_{k}, u_{1}, \ldots, u_{s}}\left(u_{1}, \ldots, u_{r}, V\right)
$$

where $\left(n_{1}, \ldots, n_{k}\right)=\theta\left(u_{1}, \ldots, u_{r}\right)$ and $s=\varphi\left(u_{1}, \ldots, u_{r}\right)$.
Finally, we must show that if $\left(V_{j}\right)_{j=1}^{\infty}$ is any sequence of subspaces the sequence $\left(u_{1}, u_{2}, \ldots\right)=\Psi\left(\emptyset, V_{1}, V_{2}, \ldots\right)$ is in $\{f<\epsilon\}$.

Let $k(r)$ be the length of $\theta\left(u_{1}, \ldots, u_{r}\right)$. Then $k(r) \leq r$ for all $r$. Suppose $k(r)$ remains bounded. Then there exists $s$ so that $\varphi\left(u_{1}, \ldots, u_{r}\right)=s$ for all $r \geq s$ and $\theta\left(u_{1}, \ldots, u_{k}\right)=\left(n_{1}, \ldots, n_{t}\right)$ for some fixed $\left(n_{1}, n_{2}, \ldots, n_{t}\right) \in N^{<\infty}$ when $k \geq s$. Thus

$$
\left(u_{1}, \ldots, u_{r}\right)=\Phi_{n_{1}, \ldots, n_{t}, u_{1}, \ldots, u_{s}}\left(u_{1}, \ldots, u_{s}, V_{s+1}, \ldots, V_{r}\right) \quad r \geq s
$$

It follows that for some $r>s$ we have

$$
h_{n_{1}, \ldots, n_{t}}\left(u_{1}, \ldots, u_{r}\right)<h_{n_{1}, \ldots, n_{t}}^{\prime}\left(u_{1}, \ldots, u_{s}\right)+\epsilon_{t}
$$

which implies that $\varphi(r)=r$ which is a contradiction. Hence $k(r) \uparrow \infty$. Let $r_{j}$ be the first natural number at which $k(r)=j$. Then there exists $\left(n_{1}, n_{2}, \ldots\right) \in \mathbb{N}^{\mathbb{N}}$ so that

$$
\theta\left(u_{1}, \ldots, u_{r_{j}}\right)=\left(n_{1}, \ldots, n_{j}\right) .
$$

By construction

$$
f_{n_{1}}^{\prime}\left(u_{1}, \ldots, u_{r_{1}}\right)<h^{\prime}(\emptyset)+\epsilon_{0}
$$

and then for $j \geq 1$,

$$
f_{n_{1}, \ldots, n_{j+1}}^{\prime}\left(u_{1}, \ldots, u_{r_{j+1}}\right)<h_{n_{1}, \ldots, n_{j}}^{\prime}\left(u_{1}, \ldots, u_{r_{j}}\right)+\epsilon_{j} .
$$

Since $h_{n_{1}, \ldots, n_{j}}^{\prime} \leq f_{n_{1}, \ldots, n_{j}}^{\prime}$ we conclude that

$$
f_{n_{1}, \ldots, n_{j}}^{\prime}\left(u_{1}, \ldots, u_{k_{j}}\right)<\epsilon^{\prime}
$$

for all $j$. But this implies the existence of $\left(n_{j, 1}, n_{j, 2} \ldots\right) \in \mathbb{N}^{\mathbb{N}}$ and $\left(u_{j, 1}, u_{j, 2}, \ldots\right) \in$ $\Sigma_{\infty}\left(A_{\infty}\right)$ so that $n_{j, i}=n_{i}$ for $i \leq j$ and $u_{j, i}=u_{i}$ for $i \leq k_{j}$ and

$$
F\left(n_{j, 1}, n_{j, 2}, \ldots ; u_{j, 1}, u_{j, 2}, \ldots\right)<\epsilon^{\prime} \quad j=1,2, \ldots
$$

Finally we invoke lower semi-continuity:

$$
F\left(n_{1}, n_{2}, \ldots, ; u_{1}, u_{2}, \ldots\right)<\epsilon
$$

and so $f\left(u_{1}, u_{2}, \ldots,\right)<\epsilon$.
We now recall (Lemma 2.1) that $\Sigma_{\infty}(E)$ is a Polish space for the topology $\mathcal{T}_{p}$. Thus every Borel set is analytic (i.e. a continuous image of $\mathbb{N}^{\mathbb{N}}$ ).

Theorem 4.4. Let $\sigma$ be a large subset of $\Sigma_{\infty}(E)$. Suppose:
(a) There is a sequence of absolutely convex sets $C_{n}$ such that $C_{n} \cap F$ is compact for all finite-dimensional subspaces $F$ and $\sigma \subset \prod_{n=1}^{\infty} C_{n}$.
(b) $\sigma$ is analytic as a subset of $\left(\Sigma_{\infty}(E), \mathcal{T}_{p}\right)$.

Let $\rho_{n}$ be any sequence of $F$-norms on $E$ and define for $u=\left(u_{1}, u_{2}, \ldots\right), v=$ $\left(v_{1}, v_{2}, \ldots\right) \in \Sigma_{\infty}(E)$

$$
d(u, v)=\sum_{j=1}^{\infty} \rho_{j}\left(u_{j}-v_{j}\right) .
$$

Let $\sigma_{\epsilon}=\left\{u=\left(u_{j}\right)_{j=0}^{\infty}: d(u, \sigma)=\inf _{v \in \sigma} d(u, v)<\epsilon\right\}$. Then for every $\epsilon>0$ there is a block subspace $V$ so that $\sigma_{\epsilon}$ is strategically large for $V$.

Proof. We start by reducing this to the case when $C_{n}=\left\{x:\|x\|_{\infty} \leq 1\right\}$. To do this first observe that each $C_{n}$ is $\mathcal{T}$-closed. Since $\sigma$ is large the linear space generated by $C_{n}$ is of finite codimension; if $E_{n}$ is a complementary space we can replace $C_{n}$ by the bigger set $C_{n}+K_{n}$ where $K_{n}$ is a compact absolutely convex neighborhood of the origin in $E_{n}$. So we can suppose $C_{n}$ is absorbent and hence generates a norm $\|\cdot\|_{n}$ on $E$. By induction, we can find a sequence of positive numbers $\delta_{n}$ so that $\|x\|=\sum_{n=1}^{\infty} \delta_{n}\|x\|_{n}<\infty$ for all $x \in E$. Thus we can assume that each $C_{n}=\left\{x:\|x\| \leq M_{n}\right\}$ for a single norm $\|\cdot\|$.

We can now pass to block basis which is a normalized basic sequence in the completion of $(E,\|\cdot\|)$. Intersecting $\sigma$ with $\Sigma_{\infty}(V)$ for a block subspace gives again an analytic set since $\Sigma_{\infty}(V)$ is closed; thus we can relabel so that the block subspace
is already $E$. It now follows that each $C_{n}$ is included in a set $\left\{x:\|x\|_{\infty} \leq M_{n}^{\prime}\right\}$ where $M_{n}^{\prime}$ is some sequence of positive numbers. Finally we put $\sigma^{\prime}=\left\{\left(u_{1}, u_{2}, \ldots\right)\right.$ : $\left.\left(M_{1}^{\prime} u_{1}, M_{2}^{\prime} u_{2}, \ldots\right) \in \sigma\right\}$ and note that $\sigma^{\prime} \subset \Sigma_{\infty}\left(A_{\infty}\right)$. Clearly it is enough to prove the result for $\sigma^{\prime}$ with $\rho_{j}$ replaced by $\rho_{j}^{\prime}(x)=\rho_{j}\left(M_{j}^{\prime} x\right)$.

We therefore assume that $\sigma \subset \Sigma_{\infty}\left(A_{\infty}\right)$.
Now there is a continuous surjective map $g: \mathbb{N}^{\mathbb{N}} \rightarrow \sigma$ for the $\mathcal{T}_{p}$-topology. We will define

$$
F: \mathbb{N}^{\mathbb{N}} \times \Sigma_{\infty}\left(A_{\infty}\right) \rightarrow[0, \infty)
$$

by

$$
F\left(n_{1}, n_{2}, \ldots ; u_{1}, u_{2}, \ldots\right)=\min \left(1, d\left(\left(u_{1}, u_{2}, \ldots\right), g\left(n_{1}, n_{2}, \ldots\right)\right)\right) .
$$

It is clear that the family $f_{n_{1}, n_{2}, \ldots,}$, given by

$$
f_{n_{1}, n_{2}, \ldots .}\left(u_{1}, u_{2}, \ldots\right)=F\left(n_{1}, n_{2}, \ldots ; u_{1}, u_{2}, \ldots\right)
$$

is equi-uniformly $\mathcal{T}_{b x}$-continuous. It is also clear that $F$ is lower semi-continuous for the $\mathcal{T}_{p}$-topology in the second factor.

The result now follows directly from Theorem 4.3,

## 5. Applications to F-spaces

We now apply these results to obtain the Gowers dichotomy for F-spaces. Before doing this we make some remarks on basic sequences in F-spaces. There is an Fspace (indeed a quasi-Banach space) which contains no basic sequence [10]. It turns out that there is a dichotomy result for the existence of basic sequences with a very similar flavor to that of the Gowers dichotomy, which has been known for some time.

We will need some background (see [11). Let $X$ be an F-space and let $\rho$ be an F-norm inducing the topology. A basic sequence $\left(x_{n}\right)_{n=1}^{\infty}$ is called regular if $\inf _{n} \rho\left(x_{n}\right)>0$. We denote by $\omega$ the space of all sequences (i.e. the countable product of lines). The canonical basis of $\omega$ is not regular, and $\omega$ contains no regular basic sequence. The following Lemma is elementary.

Proposition 5.1. Suppose $X$ contains no subspace isomorphic to $\omega$. Then given a basic sequence $\left(x_{n}\right)_{n=1}^{\infty}$ we may choose $a_{n}>0$ so that $\left(a_{n} x_{n}\right)_{n=1}^{\infty}$ is regular.
Proof. Indeed if not we have $\inf _{n \in \mathbb{N}} \sup _{t \in \mathbb{R}} \rho\left(t e_{n}\right)=0$. Then some subsequence of $\left(e_{n}\right)_{n=1}^{\infty}$ is equivalent to the canonical basis of $\omega$.

Two subspaces $Y, Z$ of an F-space $X$ are called separated if $Y \cap Z=\{0\}$ and the canonical projection $Y+Z \rightarrow Y$ is continuous. An F-space is called HI if no two infinite dimensional subspaces are separated.
Proposition 5.2. Suppose $X$ has a regular basis $\left(e_{n}\right)_{n=1}^{\infty}$. If there exist two separated infinite-dimensional subspaces $Y, Z$ of $X$ then there exist two separated block subspaces of $X$.
Proof. Since $X$ is regular the seminorm $\|x\|_{\infty}=\sup \left|e_{n}^{*}(x)\right|$ defines a continuous norm on $X$. Now, fixing $0<\epsilon<1 / 8$ we may inductively define a sequence $\left(x_{n}\right)_{n=1}^{\infty} \in$ $X$ and a block basic sequence $\left(u_{n}\right)_{n=1}^{\infty}$ such that:
(i) $\left\|x_{n}\right\|_{\infty}=1$ for all $n$,
(ii) $\rho\left(x_{n}-u_{n}\right)+\left\|x_{n}-u_{n}\right\|_{\infty}<\epsilon / 2^{n}$ for all $n$ and
(iii) $x_{n} \in Y$ for $n$ odd, $x_{n} \in Z$ for $n$ even.

Note that $\left\|u_{n}\right\|_{\infty} \geq 1-\epsilon>1 / 2$.
Let $P_{Y}$ be the canonical projections of $Y+Z$ onto $Y$ with kernel $Z$. For $v=$ $\sum_{j=1}^{n} a_{j} u_{j}$ in the linear span of the sequence $\left(u_{n}\right)_{n=1}^{\infty}$ we define $K v=\sum_{j=1}^{n} a_{j}\left(x_{j}-\right.$ $\left.u_{j}\right)$. Then

$$
\rho(K v) \leq \epsilon \max _{1 \leq j \leq n}\left|a_{j}\right| \leq 2 \epsilon\|v\|_{\infty}
$$

so that $K$ is continuous. Furthermore

$$
\|K v\|_{\infty} \leq 2 \epsilon \max _{1 \leq j \leq n}\left|a_{j}\right| \sup \left\|u_{n}\right\|_{\infty} \leq 4 \epsilon\|v\|_{\infty}
$$

Thus $T=I+K$ extends to a continuous operator $T:\left[u_{n}\right]_{n=1}^{\infty} \rightarrow Y+Z$. Now

$$
\|T v\|_{\infty} \geq(1-4 \epsilon)\|v\|_{\infty} \geq 1 / 2\|v\|_{\infty}, \quad v \in\left[u_{n}\right]_{n=1}^{\infty}
$$

If $\left(v_{n}\right)_{n=1}^{\infty}$ is a sequence such that $\lim _{n \rightarrow \infty} \rho\left(T v_{n}\right)=0$ then $\lim _{n \rightarrow \infty}\left\|T v_{n}\right\|_{\infty}=0$ and so $\lim _{n \rightarrow \infty}\left\|v_{n}\right\|_{\infty}=0$. Thus $\lim _{n \rightarrow \infty} \rho\left(K v_{n}\right)=0$ and hence $\lim _{n \rightarrow \infty} \rho\left(v_{n}\right)=0$. Hence $T$ is an isomorphism of $\left[u_{n}\right]_{n=1}^{\infty}$ into $Y+Z$. Consider the operator $S=$ $T^{-1} P_{Y} T$ : then $S$ is a projection of $\left[u_{n}\right]_{n=1}^{\infty}$ onto $\left[u_{2 n-1}\right]_{n=1}^{\infty}$ and the block subspaces $V=\left[u_{2 n-1}\right]_{n=1}^{\infty}$ and $W=\left[u_{2 n}\right]_{n=1}^{\infty}$ are separated.

Theorem 5.3. Let $X$ be an $F$-space with a regular basis containing no unconditional basic sequence. Then $X$ has an HI subspace $Y$.

Proof. We assume that $X$ has a regular basis $\left(e_{n}\right)_{n=1}^{\infty}$.
We now consider the countable dimensional $E$ with Hamel basis $\left(e_{n}\right)_{n=1}^{\infty}$. Note that the norm $\|\cdot\|_{\infty}$ on $E$ is continuous with respect to the F -space topology since $\left(e_{n}\right)_{n=1}^{\infty}$ is regular. For any block basic sequence $\left(u_{n}\right)_{n=1}^{\infty}$ we say that $\left(u_{n}\right)_{n=1}^{\infty}$ is somewhat unconditional if the map

$$
\sum_{j=1}^{\infty} a_{j} u_{j} \rightarrow \sum_{j=1}^{\infty}(-1)^{j} a_{j} u_{j}
$$

(defined for $\left(a_{j}\right)_{j=1}^{\infty} \in c_{00}$ ) is continuous for the F -space topology restricted to $E$. Let $\sigma_{0}$ be the collection of all somewhat unconditional sequences. We claim that with respect to $\mathcal{T}_{p}$ this set is a Borel subset of $\Sigma_{\infty}(E)$. Indeed let $\left(U_{m}\right)_{m=1}^{\infty}$ be a base of open neighborhoods of zero. Let $\sigma_{0}(m, n)$ be the set of $\left(u_{j}\right)_{j=1}^{\infty}$ so that

$$
\sum_{j=1}^{\infty} a_{j} u_{j} \in U_{m} \Longrightarrow \sum_{j=1}^{\infty}(-1)^{j} a_{j} u_{j} \in \bar{U}_{n}
$$

Then $\sigma_{0}(m, n)$ is $\mathcal{T}_{p}$-closed and $\sigma_{0}=\cap_{n=1}^{\infty} \cup_{m=1}^{\infty} \sigma_{0}(m, n)$.
We define $\sigma$ to be the subset of $\Sigma_{\infty} E$ of all block basic sequences $\left(u_{n}\right)_{n=1}^{\infty}$ such that $\left\|u_{n}\right\|_{\infty}=1$ for all $n$ and $\left(u_{n}\right)_{n=1}^{\infty}$ fails to be somewhat unconditional. Then $\sigma$ is also Borel in $\mathcal{T}_{p}$. Furthermore, since $X$ contains no unconditional basic sequence we conclude that $\sigma$ is large.

Fix some $0<\epsilon<1$ and let $\sigma^{\prime}$ be the subset of $\Sigma_{\infty}(E)$ of all sequences $\left(v_{j}\right)_{j=1}^{\infty}$ such that

$$
\inf \left\{\sum_{j=1}^{\infty}\left(\left\|u_{j}-v_{j}\right\|_{\infty}+\rho\left(u_{j}-v_{j}\right)\right):\left(u_{j}\right)_{j=1}^{\infty} \in \sigma\right\}<\epsilon .
$$

Note that if $\left(v_{j}\right)_{j=1}^{\infty} \in \sigma^{\prime}$ there exists $\left(u_{j}\right)_{j=1}^{\infty} \in \sigma$ which is equivalent to $\left(v_{j}\right)_{j=1}^{\infty}$. Hence each $\left(v_{j}\right)_{j=1}^{\infty} \in \sigma^{\prime}$ fails to be somewhat unconditional.

According to Theorem4.4 we can find a block subspace $V$ so that $\sigma^{\prime}$ is strategically large for $V$. Let $Y$ be the closure of $V$. We show that $Y$ is HI . $Y$ has a regular basis $\left(u_{n}\right)_{n=1}^{\infty}$ which is a block basis of $\left(e_{n}\right)_{n=1}^{\infty}$. According to Proposition 5.2 we need only check that if $W_{1}, W_{2}$ are two block subspaces of $V$ then $W_{1}$ and $W_{2}$ cannot be separated.

Let $\Phi$ be the strategy guaranteed by the fact that $\sigma^{\prime}$ is strategically large. Then $\Phi\left(W_{1}, W_{2}, W_{1}, W_{2}, \ldots\right)=\left(v_{1}, v_{2}, \ldots\right)$ and the sequence $\left(v_{j}\right)_{j=1}^{\infty}$ fails to be somewhat unconditional so that $W_{1}, W_{2}$ are not separated.

Let us now recall the criterion of the existence of basic sequences given in [8] (see also [12]). An F-space $X$ is called minimal if there is no strictly weaker Hausdorff topology on $X$.

Proposition 5.4. If $X$ is a non-minimal $F$-space then $X$ contains a regular basic sequence.

Let us call an infinite-dimensional F-space $X$ strongly $H I$ (SHI) if it contains a non-zero vector $e$ so that $e \in L$ for every infinite-dimensional closed subspace $L$ of $X$. We remark that it is possible to consider spaces $X$ which satisfy the slightly stronger condition that any two infinite-dimensional closed subspaces have nontrivial intersection; this condition implies $X$ contains no basic sequence, but it is not clear if it implies that $X$ is SHI. The problem is that we do not know if, under this condition, the intersection of any three infinite-dimensional closed subspaces is non-trivial. This is related to the fact, discussed later, that the sum of two strictly singular operators need not be strictly singular (see the discussion after Theorem 6.1).

Theorem 5.5. Let $X$ be an F-space containing no basic sequence. Then $X$ has an SHI-subspace Y.

Proof. We may assume that $X$ is separable. Let us say that a collection $\mathcal{L}$ of closed subspaces of $X$ is a subspace-filter in $X$ if each $L \in \mathcal{L}$ is infinite-dimensional and $L_{1} \cap L_{2} \in \mathcal{L}$ whenever $L_{1}, L_{2} \in \mathcal{L}$; we say that a subspace-filter $\mathcal{L}$ is a subspaceultrafilter if it is not contained properly in any other subspace-filter. We then pick $\mathcal{L}$ to be a subspace-filter such that $H=\cap\{L: L \in \mathcal{L}\}$ has minimal dimension $(1 \leq \operatorname{dim} H \leq \infty)$.

We will argue that $\operatorname{dim} H>0$. Indeed if $H=\{0\}$ then we define a topology $\tau$ on $X$ by taking as a base of neighborhoods sets of the form $U+L$ where $U$ is a neighborhood of zero in the F-space topology and $L \in \mathcal{L}$. If $H=\{0\}$ then $\tau$ is Hausdorff. By Proposition 5.4 we have that $\tau$ coincides with the original topology. Then we may find a strictly decreasing sequence $L_{n} \in \mathcal{L}$ so that $L_{n} \subset\{x: \rho(x)<$
$\left.2^{-n}\right\}$. If we pick $x_{n} \in L_{n} \backslash L_{n+1}$, it is easy to verify that $\left(x_{n}\right)_{n=1}^{\infty}$ is a basic sequence equivalent to the canonical basis of $\omega$.

If $\operatorname{dim} H=\infty$ then it follows from maximality that $H$ has no proper closed infinite-dimensional subspace and so we may take $Y=H$ and $y$ any non-zero element of $Y$. If $\operatorname{dim} H<\infty$ we first argue by Lindelof's theorem that since $X$ is separable we can find a descending sequence of subspaces $L_{n} \in \mathcal{L}$ so that $\cap L_{n}=H$. We may suppose this sequence is strictly descending and take $x_{n} \in L_{n} \backslash L_{n+1}$ for $n \geq 1$. Let $V_{n}=\left[x_{k}\right]_{k \geq n}$ so that $V_{n} \subset L_{n}$. Suppose $W$ is any closed infinite-dimensional subspace of $V_{1}$; then $\operatorname{dim} V_{n} \cap W=\infty$ for each $n$. Let $\mathcal{L}^{\prime}$ be any subspace-ultrafilter containing each $V_{n}$ and $W$. Then $\cap\left\{L: L \in \mathcal{L}^{\prime}\right\} \subset H$ but the inclusion cannot be strict because the original minimality assumption on $\operatorname{dim} H$. Hence $H \subset W$. Thus we can take $Y=V_{1}$ and $y \in H \backslash\{0\}$.

An examination of the proof shows that we have actually proved a slightly stronger result:
Corollary 5.6. Let $X$ be an F-space containing no basic sequence. Then $X$ has an SHI-subspace $Y$ with the property that if $E$ is the intersection of all infinitedimensional subspaces of $Y$ then there is a descending sequence of infinite-dimensional subspaces $\left(L_{n}\right)_{n=1}^{\infty}$ of $Y$ with $\cap_{n=1}^{\infty} L_{n}=E$.

We are now ready to establish the full force of the Gowers dichotomy for F-spaces.
Theorem 5.7. Let $X$ be an $F$-space. If $X$ contains no unconditional basic sequence, then $X$ contains an HI subspace.

Proof. If $X$ contains no basic sequence then $X$ contains a SHI subspace (Theorem 5.5). So we may assume $X$ has a basis. Clearly $X$ cannot contain a copy of $\omega$ so we can assume the basis is regular (Proposition 5.1). Now apply Theorem 5.3.

We conclude this section with:
Theorem 5.8. Let $X$ be an HI F-space. Suppose $X$ has a closed infinite-dimensional subspace containing no basic sequence. Then $X$ contains no basic sequence.
Proof. We will show that if $\left(V_{n}\right)_{n=1}^{\infty}$ is any descending sequence of closed infinitedimensional subspaces of $X$ then $\cap_{n=1}^{\infty} V_{n} \neq\{0\}$. We use Corollary 5.6 to deduce the existence of a descending sequence of infinite-dimensional closed subspaces $\left(L_{n}\right)_{n=1}^{\infty}$ such that, if $E=\cap_{n=1}^{\infty} L_{n}$, then $E \neq\{0\}$ and $E$ is contained in any infinite-dimensional subspace of $L_{1}$. Consider the sequence $\left(L_{n} \cap V_{n}\right)_{n=1}^{\infty}$. Then if $\operatorname{dim} L_{n} \cap V_{n}=\infty$ for all $n$ we have $E \subset \cap_{n=1}^{\infty} L_{n} \cap V_{n} \subset \cap_{n=1}^{\infty} V_{n}$.

If not then there exists $n_{0}$ such that $\operatorname{dim}\left(L_{n} \cap V_{n}\right)$ is finite and constant for $n \geq n_{0}$. Hence $L_{n} \cap V_{n}=F$ some fixed finite dimensional subspace for $n \geq n_{0}$. We show $\operatorname{dim} F>0$. If for some $n \geq n_{0}$ we have $L_{n} \cap V_{n}=\{0\}$ then $L_{n}+V_{n}$ cannot be closed since $X$ is HI. Thus there are sequences $\left(x_{k}\right)_{k=1}^{\infty}$ in $L_{n}$ and $\left(v_{k}\right)_{k=1}^{\infty} \in V_{n}$ so that $\lim \rho\left(x_{k}+v_{k}\right)=0$ but $\rho\left(x_{k}\right) \geq \delta>0$ for all $k$. Consider the metric topology on $L_{n}$ defined by the F-norm $x \rightarrow d\left(x, V_{n}\right):=\inf \{\rho(x+v): v \in V\}$. This is topology is Hausdorff on $L_{n}$ and strictly weaker than the $\rho$-topology. Hence $L_{n}$ contains a basic sequence by Proposition 5.4, and this is a contradiction. Hence $\operatorname{dim} F>0$ and $F \subset \cap_{n=1}^{\infty} V_{n}$.

## 6. Strictly singular maps

In [7] the following Theorem is shown:
Theorem 6.1. Let $X$ be a complex Banach space. If $X$ is HI then every bounded linear operator $T: X \rightarrow X$ is of the form $T=\lambda I+S$ where $S$ is strictly singular.

We do not know whether such a theorem can hold for a complex F-space but we show that it holds equally for complex quasi-Banach spaces. There are some small wrinkles in the proof as the reader will see.
¿From now on we will deal with quasi-Banach space $X$ (or $Y$, etc.) with a given quasi-norm which is assumed to be $p$-subadditive (for a suitable $0<p \leq 1$ ) i.e.

$$
\|x+y\|^{p} \leq\|x\|^{p}+\|y\|^{p}, \quad x, y \in X
$$

A linear operator $T: X \rightarrow Y$ is an isomorphic embedding if there exists $c>0$ so that $\|T x\| \geq c\|x\|$ for $x \in X$. $T$ is called strictly singular if $\left.T\right|_{V}$ fails to be an isomorphic embedding for every infinite-dimensional subspace $V$ of $X . T$ is called semi-Fredholm if ker $T$ is finite-dimensional and $T(X)$ is closed. $T$ is called Fredholm if $T$ is semi-Fredholm and $\operatorname{dim} Y / T(X)<\infty$.
$T$ is semi-Fredholm if and only if for every bounded sequence $\left(x_{n}\right)_{n=1}^{\infty}$ such that $\lim _{n \rightarrow \infty}\left\|T x_{n}\right\|=0$ we can extract a convergent subsequence. Thus it is clear the restriction of a semi-Fredholm operator to an infinite-dimensional closed subspace remains semi-Fredholm.

Let us make some remarks. Suppose $X$ is a SHI space and let $E_{X}$ be the intersection of all closed infinite-dimensional subspaces of $X$. If $\operatorname{dim} E_{X}=\infty$ then $E_{X}$ is an atomic space i.e. it has no proper closed infinite-dimensional subspace. The existence of atomic spaces is still open (the only known results in this direction are in [15]). However it is known that there exist quasi-Banach spaces $X$ for which $E_{X}$ is finite-dimensional and non-trivial, even with $\operatorname{dim} E_{X}>1$ (9], Theorem 5.5). The quotient $\operatorname{map} Q: X \rightarrow X / E_{X}$ is then both semi-Fredholm and strictly singular (this cannot happen for operators on Banach spaces). Furthermore if $\operatorname{dim} E_{X}>1$ then let $L_{1}, L_{2}$ be two distinct one-dimensional subspaces of $E_{X}$. Then the quotient maps $Q_{1}: X \rightarrow X / L_{1}$ and $Q_{2}: X \rightarrow X / L_{2}$ are both strictly singular and semi-Fredholm. However the map $x \rightarrow\left(Q_{1} x, Q_{2} x\right)$ from $X$ into $X / L_{1} \oplus X / L_{2}$ is an isomorphism. Thus the sum of two strictly singular operators need not be strictly singular!

The key fact we will need is the following:
Theorem 6.2. Let $X$ be an infinite-dimensional complex quasi-Banach space and suppose $T: X \rightarrow X$ is a bounded operator. Then there exists $\lambda \in \mathbb{C}$ so that $T-\lambda I$ is not semi-Fredholm.

This Theorem is proved for Banach spaces by Gowers and Maurey [7]. The proof for quasi-Banach spaces requires some additional tricks. These tricks are necessitated by the fact that finite-dimensional subspaces are not always complemented.

We list the relevant facts we need:
Proposition 6.3. If $X$ is a complex quasi-Banach space and $T: X \rightarrow X$ is a bounded linear operator then $S p(T)=\{\lambda \in \mathbb{C}: T-\lambda I$ is not invertible $\}$ is a non-empty compact set and $\max _{\lambda \in S p(T)}|\lambda|=\lim _{n \rightarrow \infty}\left\|T^{n}\right\|^{1 / n}$.

This is due to Zelazko [16]. We point out that the key ideas in the proof involve a reduction to the Banach algebra case. One starts with the fact ([16) that on a commutative quasi-Banach algebra the formula $r(x)=\lim _{n \rightarrow \infty}\left\|T^{n}\right\|^{1 / n}$ defines a seminorm. Using this one can prove the Gelfand-Mazur theorem (see e.g. [11]) in this context and develop the basic theory of commutative quasi-Banach algebras. The Proposition is obtained by looking at the double commutant of $T$.

Proposition 6.4. Let $X$ be a complex quasi-Banach space and let $\mathcal{G}_{1}$ denote the subset of $\mathcal{L}(X)$ consisting of all isomorphic embeddings and $\mathcal{G}_{2}$ be the collection of all surjections. Then $\mathcal{G}_{1}$ and $\mathcal{G}_{2}$ are both open sets and $\mathcal{G}_{1} \cap \mathcal{G}_{2}$ is a clopen subset relative to $\mathcal{G}_{1}$ and relative to $\mathcal{G}_{2}$.

See 11] pp. 132-134.
Proposition 6.5. Let $X$ be an infinite-dimensional complex Banach space and suppose $T: X \rightarrow X$ is quasi-nilpotent, i.e. $S p(T)=\{0\}$. Then $T$ cannot be semiFredholm.

See [7] Lemma 19. We will now need to prove this Proposition for a general complex quasi-Banach space. We do this is in several very simple steps. Assume throughout that $X$ is an infinite-dimensional complex quasi-Banach space.

Lemma 6.6. Suppose $T: X \rightarrow X$ is any bounded operator and $\lambda \in \partial S p(T)$. Then $T-\lambda I$ can be neither an isomorphic embedding nor a surjection.

Proof. This follows from Proposition 6.4.
Lemma 6.7. Suppose $X$ has trivial dual. If $T: X \rightarrow X$ is quasi-nilpotent then $T$ cannot be Fredholm.

Proof. If $T(X)$ has finite codimension in $X$ then $T$ is onto in this case. We then use Lemma 6.6

Lemma 6.8. If $X$ is any infinite-dimensional complex quasi-Banach space and $T: X \rightarrow X$ is quasi-nilpotent then $T$ cannot be Fredholm.

Proof. Denote by $X^{*}$ the dual of $X$; this is a Banach space but it can be quite small (even $\{0\}$ ). We assume $X^{*} \neq\{0\}$ as this case is covered in Lemma 6.7. Assume $T: X \rightarrow X$ is quasi-nilpotent and Fredholm. Then $T^{*}: X^{*} \rightarrow X^{*}$ is Fredholm. In fact $T^{*}\left(X^{*}\right)=\operatorname{ker}(T)^{\perp}$; this depends on the fact that every continuous linear functional $y^{*}$ on $T(X)$ can be extended to $x^{*} \in X^{*}$ since $\operatorname{dim} X / T(X)<\infty$. Since $\left\|\left(T^{*}\right)^{n}\right\| \leq\left\|T^{n}\right\|$ the spectral radius formula shows that $T^{*}$ is quasi-nilpotent. By Proposition 6.5 we must have $\operatorname{dim} X^{*}<\infty$. Let $X_{0}=\left\{x \in X: x^{*}(x)=\right.$ $\left.0 \forall x^{*} \in X^{*}\right\}$. Then $X_{0}$ is invariant for $T$ and of finite-codimension in $X$. Clearly $X_{0}^{*}=\{0\}$ and $\left.T\right|_{X_{0} \rightarrow X_{0}}$ remains Fredholm so we can apply Lemma 6.7 to get a contradiction.

Lemma 6.9. If $X$ is any infinite-dimensional complex quasi-Banach space and $T: X \rightarrow X$ is quasi-nilpotent then $T$ cannot be semi-Fredholm.

Proof. Assume $T$ is semi-Fredholm. Then by a Baire Category argument there exists $x \in X$ so that $T^{n} x \neq 0$ for every $n \in \mathbb{N}$. Let $Y=\left[T^{n} x\right]_{n=1}^{\infty}$. Then $T: Y \rightarrow Y$ is Fredholm and remains quasi-nilpotent (using Proposition 6.3). Clearly $\left.T\right|_{Y \rightarrow Y}$ is not nilpotent so $\operatorname{dim} Y=\infty$. This is a contradiction by Lemma 6.8,

Proof of Theorem 6.2. The remaining steps in the proof of Theorem 6.2 are very similar to those in [7] for the Banach space case. Assume $T-\lambda I$ is semi-Fredholm for all $\lambda \in \mathbb{C}$. We suppose $\lambda \in \partial \operatorname{Sp}(T)$ is an accumulation point of $\partial \operatorname{Sp}(T)$. Let $\lambda_{n} \rightarrow \lambda$ with $\lambda_{n} \neq \lambda$ and $\lambda_{n} \in \partial \operatorname{Sp}(T)$. Each $\lambda_{n}$ is an eigenvalue of $T$ (by Lemma 6.6, since $T-\lambda_{n} I$ is semi-Fredholm), say with eigenvector $x_{n}$. Let $Y=\left[x_{n}\right]_{n=1}^{\infty}$. Then $Y$ is invariant for $T$ and $\lambda \in \partial \operatorname{Sp}\left(\left.T\right|_{Y \rightarrow Y}\right)$. However $\left.(T-\lambda I)\right|_{Y \rightarrow Y}$ has dense range and is semi-Fredholm. Hence $\left.(T-\lambda I)\right|_{Y \rightarrow Y}$ is surjective and we have a contradiction by Lemma 6.6.

It follows that $\partial \mathrm{Sp}(T)$ has no accumulation points and hence is a finite set. Thus $\operatorname{Sp}(T)$ is also finite say $\operatorname{Sp}(\mathrm{T})=\left\{\lambda_{1}, \ldots, \lambda_{n}\right\}$. Then $S=\prod_{k=1}^{n}\left(T-\lambda_{k} I\right)$ is semiFredholm and $\operatorname{Sp}(S)=\{0\}$. This contradicts Lemma 6.9,

Theorem 6.10. Let $X$ be an infinite-dimensional complex quasi-Banach space. If $T: X \rightarrow X$ is strictly singular then $T$ cannot be semi-Fredholm.

Remark. Note that this is false for operators $T: X \rightarrow Y$ by the remarks above.
Proof. In fact $T-\lambda I$ is always semi-Fredholm if $\lambda \neq 0$ (Theorem 7.10 of [11]). The result follows from Theorem 6.2,

Theorem 6.11. Let $X$ be an infinite-dimensional complex quasi-Banach space. If $T: X \rightarrow X$ is a bounded linear operator then exactly one of the following two conditions holds:
(i) For every $\epsilon>0$ there is an infinite-dimensional closed subspace $V$ of $X$ such that $\left\|\left.T\right|_{V}\right\|<\epsilon$.
(ii) $T$ is semi-Fredholm.

If $X$ is HI then (i) is equivalent to
(i) ${ }^{\prime} T$ is strictly singular.

Proof. Assume (ii). Then there is a constant $c>0$ so that $\|T x\| \geq c d(x, F)$ for $x \in X$, where $F=\operatorname{ker} T$. If $V$ is an infinite-dimensional closed subspace we can find a sequence $\left(v_{n}\right)_{n=1}^{\infty}$ in the unit ball so $\left\|v_{m}-v_{n}\right\| \geq 1 / 2$ for $m \neq n$. Assuming that the norm is $p$-convex, by a simple compactness argument we can then show the existence of a pair $m \neq n$ so that $\left(d\left(v_{m}, F\right)^{p}+d\left(v_{n}, F\right)^{p}\right)^{1 / p} \geq 1 / 4$. Hence $\left\|T v_{m}\right\|^{p}+\left\|T v_{n}\right\|^{p} \geq(1 / 4)^{p} . c^{p}$ This implies a lower bound on $\left\|\left.T\right|_{V}\right\|$.

Now assume (ii) fails and that $F=\operatorname{ker}(T)$ is finite dimensional. Then $T$ factors in the form $T=T_{0} Q$ where $Q: X \rightarrow X / F$ is the quotient map and $T_{0}: X / F \rightarrow X$ is one-one but not an isomorphic embedding. Then there is a normalized sequence $\xi_{n} \in X / F$ so that $\left\|T_{0} \xi_{n}\right\|<2^{-n}$. Now using Theorem 4.6 of [11] we can assume by passing to a subsequence that $\left(\xi_{n}\right)_{n=1}^{\infty}$ satisfies an estimate

$$
\max _{1 \leq k \leq n}\left|a_{k}\right| \leq C\left\|\sum_{k=1}^{n} a_{k} \xi_{k}\right\|, \quad a_{1}, \ldots, a_{n} \in \mathbb{C}
$$

In particular if $V_{k}=Q^{-1}\left[\xi_{j}\right]_{j \geq k}$ then each $V_{k}$ is infinite-dimensional and $\left\|\left.T\right|_{V_{k}}\right\| \rightarrow 0$. Thus (i) holds.

Now assume $X$ is HI. Suppose $T$ satisfies (i) and is not strictly singular. Then there is an infinite-dimensional subspace $W$ so that $\|T w\| \geq \delta\|w\|$ for $w \in W$ where $\delta>0$. Pick $\epsilon=\delta / 2$ and then choose $V$ as in (i) for this $\epsilon$. Clearly $V \cap W=\{0\}$.

Now assume $v \in V, w \in W$ with $\|v+w\|=1$. Then

$$
\begin{aligned}
\|v\|^{p} & \leq 1+\|w\|^{p} \\
& \leq 1+2^{-p}\|v\|^{p}+\|w\|^{p}-2^{-p}\|v\|^{p} \\
& \leq 1+2^{-p}\|v\|^{p}+\delta^{-p}\|T w\|^{p}-2^{-p} \epsilon^{-p}\|T v\|^{p} \\
& =1+2^{-p}\|v\|^{p}+\delta^{-p}\left(\|T w\|^{p}-\|T v\|^{p}\right) \\
& \leq 1+2^{-p}\|v\|^{p}+\delta^{-p}\|T(v+w)\|^{p} \\
& \leq 1+2^{-p}\|v\|^{p}+\delta^{-p}\|T\|^{p} .
\end{aligned}
$$

Thus

$$
\|v\| \leq\left(\frac{1+\delta^{-p}\|T\|^{p}}{1-2^{-p}}\right)^{1 / p}
$$

This contradicts the fact that $X$ is HI.
Conversely if $T$ is strictly singular it cannot be semi-Fredholm by Theorem 6.10 and so (i) must hold.

Theorem 6.12. Let $X$ be an infinite-dimensional complex quasi-Banach space. If $X$ is HI then every bounded linear operator $T: X \rightarrow X$ is of the form $T=\lambda I+S$ where $S$ is strictly singular.

Proof. There exists $\lambda$ so that $T-\lambda I$ is not semi-Fredholm by Theorem 6.2, By Theorem 6.11 this means $T-\lambda I$ is strictly singular.

In the case when $X$ is SHI this result is much simpler. Indeed we have:
Theorem 6.13. Let $X$ be an SHI space and suppose $E$ is the intersection of all infinite-dimensional subspaces of $X$. Let $Q: X \rightarrow X / E$ be the quotient map (which is strictly singular). Then if $T: X \rightarrow X$ is a bounded operator, there exists $\lambda \in \mathbb{C}$ and a bounded operator $S: X / E \rightarrow X$ so that $T=\lambda I+S Q$.

Proof. Let us first give a simpler proof of Theorem 6.12, It is clearly that if $R$ : $X \rightarrow X$ is an invertible operator then $R(E) \subset E$ and this implies that $E$ is invariant for all operators on $X$. If $E$ is atomic then $E$ is rigid ([11] Theorem 7.22, p. 155). Otherwise $E$ is finite-dimensional. In either case $\left.T\right|_{E}$ has an eigenvalue $\lambda$ and so $T-\lambda I$ factors through a quotient map $Q^{\prime}: X \rightarrow X / F^{\prime}$ where $F^{\prime}$ is a non-trivial subspace of $E$. Hence $T-\lambda I$ is strictly singular.

Now using Theorem 6.11 it is clear any strictly singular operator on $X$ vanishes on $E$ and so we get the desired factorization.

## References

[1] J. Bagaria and J. López-Abad, Weakly Ramsey sets in Banach spaces, Adv. Math. 160 (2001), 133-174.
[2] , Determinacy and weakly Ramsey sets in Banach spaces, Trans. Amer. Math. Soc. 354 (2002), 1327-1349 (electronic).
[3] T. Figiel, R. Frankiewicz, R. Komorowski, and C. Ryll-Nardzewski, On hereditarily indecomposable Banach spaces, Ann. Pure Appl. Logic 126 (2004), 293-299.
[4] W. T. Gowers, A new dichotomy for Banach spaces, Geom. Funct. Anal. 6 (1996), 1083-1093.
[5] , An infinite Ramsey theorem and some Banach-space dichotomies, Ann. of Math. (2) 156 (2002), 797-833.
[6] , Ramsey methods in Banach spaces, Handbook of the geometry of Banach spaces, Vol. 2, North-Holland, Amsterdam, 2003, pp. 1071-1097.
[7] W. T. Gowers and B. Maurey, The unconditional basic sequence problem, J. Amer. Math. Soc. 6 (1993), 851-874.
[8] N. J. Kalton, Basic sequences in F-spaces and their applications, Proc. Edinburgh Math. Soc. (2) 19 (1974/75), 151-167.
[9] , An elementary example of a Banach space not isomorphic to its complex conjugate, Canad. Math. Bull. 38 (1995), 218-222.
[10] , The basic sequence problem, Studia Math. 116 (1995), 167-187.
[11] N. J. Kalton, N. T. Peck, and J. W. Roberts, An F-space sampler, London Mathematical Society Lecture Note Series, vol. 89, Cambridge University Press, Cambridge, 1984.
[12] N. J. Kalton and J. H. Shapiro, Bases and basic sequences in F-spaces, Studia Math. 56 (1976), 47-61.
[13] B. Maurey, A note on Gowers' dichotomy theorem, Convex geometric analysis (Berkeley, CA, 1996), Math. Sci. Res. Inst. Publ., vol. 34, Cambridge Univ. Press, Cambridge, 1999, pp. 149157.
[14] A. M. Pelczar, Some version of Gowers' dichotomy for Banach spaces, Univ. Iagel. Acta Math. (2003), no. 41, 235-243.
[15] M. L. Reese, Almost-atomic spaces, Illinois J. Math. 36 (1992), 316-324.
[16] W. Żelazko, On the radicals of p-normed algebras, Studia Math. 21 (1961/1962), 203-206.
Department of Mathematics, University of South Carolina, Columbia, SC 29208
E-mail address: giorgis@math.sc.edu
Department of Mathematics, University of South Carolina, Columbia, SC 29208
E-mail address: dilworth@math.sc.edu
Department of Mathematics, University of Missouri-Columbia, Columbia, MO 65211
E-mail address: nigel@math.missouri.edu


[^0]:    1991 Mathematics Subject Classification. 46A16, 91A05, 91A80.
    The second author was supported by NSF grant DMS-0701552. The third author was supported by NSF grant DMS-0555670.

