On the limit of large girth graph sequences^{*}

Gábor Elek

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Abstract

Let $d \geq 2$ be given and let μ be an involution-invariant probability measure on the space of trees $T \in \mathcal{T}_d$ with maximum degrees at most d. Then μ arises as the local limit of some sequence $\{G_n\}_{n=1}^{\infty}$ of graphs with all degrees at most d. This answers Question 6.8 of Bollobás and Riordan [4].

1 Introduction

Let $\operatorname{\mathbf{Graph}}_d$ denote the set of all finite simple graphs G (up to isomorphism) for which $\operatorname{deg}(x) \leq d$ for every $x \in V(G)$. For a graph G and $x, y \in V(G)$ let $d_G(x, y)$ denote the distance of x and y, that is the length of the shortest path from x to y. A rooted (r, d)-ball is a graph $G \in \operatorname{\mathbf{Graph}}_d$ with a marked vertex $x \in V(G)$ called the root such that $d_G(x, y) \leq r$ for every $y \in V(G)$. By $U^{r,d}$ we shall denote the set of rooted (r, d)-balls.

If $G \in \mathbf{Graph}_d$ is a graph and $x \in V(G)$ then $B_r(x) \in U^{r,d}$ shall denote the rooted (r, d)-ball around x in G. For any $\alpha \in U^{r,d}$ and $G \in \mathbf{Graph}_d$ we define the set $T(G, \alpha) \stackrel{\text{def}}{=} \{x \in V(G) : B_r(x) \cong \alpha\}$ and let $p_G(\alpha) \stackrel{\text{def}}{=} \frac{|T(G,\alpha)|}{|V(G)|}$. A graph sequence $\mathbf{G} = \{G_n\}_{n=1}^{\infty} \subset \mathbf{Graph}_d$ is weakly convergent if $\lim_{n\to\infty} |V(G_n)| = \infty$ and for every r and every $\alpha \in U^{r,d}$ the limit $\lim_{n\to\infty} p_{G_n}(\alpha)$ exists (see [3]).

Let \mathbf{Gr}_d denote the set of all countable, connected rooted graphs G for which $\deg(x) \leq d$ for every $x \in V(G)$. If $G, H \in \mathbf{Gr}_d$ let $d_g(G, H) = 2^{-r}$, where r is the maximal number such that the r-balls around the roots of G resp. H are rooted isomorphic. The distance d_g makes \mathbf{Gr}_d a compact metric space. Given an $\alpha \in U^{r,d}$ let $T(\mathbf{Gr}_d, \alpha) = \{(G, x) \in \mathbf{Gr}_d : B_r(x) \cong \alpha\}$. The sets $T(\mathbf{Gr}_d, \alpha)$ are closed-open sets. A convergent graphs sequence $\{G_n\}_{n=1}^{\infty}$ define a local limit **measure** $\mu_{\mathbf{G}}$ on \mathbf{Gr}_d , where $\mu_{\mathbf{G}}(T(\mathbf{Gr}_d, \alpha)) = \lim_{n \to \infty} p_{G_n}(\alpha)$. However, not all the probability measures on \mathbf{Gr}_d arise as local limits. A necessary condition for a measure μ being a local limit is its **involution invariance** (see Section 2). The goal of this paper is to answer a question of Bollobás and Riordan (Question 6.8 [4]):

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Theorem 1 Any involution-invariant measure μ on \mathbf{Gr}_d concentrated on trees arises as a local limit of some convergent graph sequence.

As it was pointed out in [4] such graph sequences are asymptotically treelike, thus μ must arise as the local limit of a convergent large girth sequence.

2 Involution invariance

Let $\vec{\mathbf{Gr}}_d$ be the compact space of all connected countable rooted graphs \vec{G} (up to isomorphism) of vertex degree bound d with a distinguished directed edge pointing out from the root. Note that \vec{G} and \vec{H} are considered isomorphic if there exists a rooted isomorphism between them mapping distinguished edges into each other. Let $\vec{U}^{r,d}$ be the isomorphism classes of all rooted (r,d)-graphs $\vec{\alpha}$ with a distinguished edge $e(\vec{\alpha})$ pointing out from the root. Again, $T(\vec{\mathbf{Gr}}_d, \vec{\alpha})$ is well-defined for any $\vec{\alpha} \in \vec{U}^{r,d}$ and defines a closed-open set in $\vec{\mathbf{Gr}}_d$. Clearly, the forgetting map $\mathcal{F} : \vec{\mathbf{Gr}}_d \to \mathbf{Gr}_d$ is continuous. Let μ be a probability measure on \mathbf{Gr}_d . Then we define a measure $\vec{\mu}$ on $\vec{\mathbf{Gr}}_d$ the following way. Let $\vec{\alpha} \in \vec{U}^{r,d}$ and let $\mathcal{F}(\vec{\alpha}) = \alpha \in U^{r,d}$ be the underlying rooted ball. Clearly,

Let $\vec{\alpha} \in U^{r,a}$ and let $\mathcal{F}(\vec{\alpha}) = \alpha \in U^{r,a}$ be the underlying rooted ball. Clearly, $\mathcal{F}(T(\vec{\mathbf{Gr}}_d, \vec{\alpha})) = T(\mathbf{Gr}_d, \alpha).$ Let

$$\vec{\mu}(T(\vec{\mathbf{Gr}}_d, \vec{\alpha})) := l \,,$$

where l is the number of edges e pointing out from the root such that there exists a rooted automorphism of α mapping $e(\vec{\alpha})$ to e. Observe that

$$\vec{\mu}(\mathcal{F}^{-1}(T(\mathbf{Gr}_d, \alpha))) = \deg(\alpha)\mu(T(\mathbf{Gr}_d, \alpha)).$$

We define the map $T: \vec{\mathbf{Gr}}_d \to \vec{\mathbf{Gr}}_d$ as follows. Let $T(\vec{G}) = \vec{H}$, where :

- the underlying graphs of \vec{G} and \vec{H} are the same,
- the root of \vec{H} is the endpoint of $e(\vec{G})$,
- the distinguished edge of \vec{H} is pointing to the root of \vec{G} .

Note that T is a continuous involution. Following Aldous and Steele [2], we call μ **involution-invariant** if $T_*(\vec{\mu}) = \vec{\mu}$. It is important to note [2],[1] that the limit measure of convergent graphs sequences are always involution-invariant. We need to introduce the notion of **edge-balls**. Let $\vec{G} \in \mathbf{Gr}_d$. The edge-ball $B_r^e(\vec{G})$ of radius r around the root of \vec{G} is the following spanned rooted subgraph of \vec{G} :

- The root of $B_r^e(\vec{G})$ is the same as the root of \vec{G} .
- y is a vertex of $B_r^e(\vec{G})$ if $d(x, y) \leq r$ or $d(x', y) \leq r$, where x is the root of \vec{G} and x' is the endpoint of the directed edge $e(\vec{G})$.
- The distinguished edge of $B_r^e(\vec{G})$ is (\vec{x}, \vec{x}') .

Let $\vec{E}^{r,d}$ be the set of all edge-balls of radius r up to isomorphism. Then if $\vec{\phi} \in \vec{E}^{r,d}$, let $s(\vec{\phi}) \in \vec{U}^{r,d}$ be the rooted ball around the root of $\vec{\phi}$. Also, let $t(\vec{\phi}) \in \vec{U}^{r,d}$ be the *r*-ball around x' with distinguished edge (x', x).

The involution $T^{r,d}: \vec{E}^{r,d} \to \vec{E}^{r,d}$ is defined the obvious way and $t(T^{r,d}(\vec{\phi})) = s(\vec{\phi}), s(T^{r,d}(\vec{\phi})) = t(\vec{\phi})$. Since $\vec{\mu}$ is a measure we have

$$\vec{\mu}(T(\vec{\mathbf{Gr}}_d, \vec{\alpha})) = \sum_{\vec{\phi}, s(\vec{\phi}) = \vec{\alpha}} \vec{\mu}(T(\vec{\mathbf{Gr}}_d, \vec{\phi})).$$
(1)

Also, by the involution-invariance

$$\vec{u}(T(\vec{\mathbf{Gr}}_d, \vec{\phi})) = \vec{\mu}(T(\vec{\mathbf{Gr}}_d, T^{r,d}(\vec{\phi})),$$
(2)

since $T(T(\vec{\mathbf{Gr}}_d, \vec{\phi})) = T(\vec{\mathbf{Gr}}_d, T^{r,d}(\vec{\phi})$. Therefore by (1),

$$\vec{\mu}(T(\vec{\mathbf{Gr}}_d, \vec{\alpha})) = \sum_{\vec{\phi}, t(\vec{\phi}) = \vec{\alpha}} \vec{\mu}(T(\vec{\mathbf{Gr}}_d, \vec{\phi}))$$
(3)

3 Labeled graphs

Let $\vec{\mathbf{Gr}_d}^n$ be the isomorphism classes of

- connected countable rooted graphs with vertex degree bound d
- with a distinguished edge pointing out from the root
- with vertex labels from the set $\{1, 2, \ldots, n\}$.

Note that if \vec{G}_* and \vec{H}_* are such graphs then they called isomorphic if there exists a map $\rho: V(\vec{G}_*) \to V(\vec{H}_*)$ preserving both the underlying $\mathbf{G}\vec{\mathbf{r}}_d$ -structure and the the vertex labels. The labeled *r*-balls $\vec{U}_n^{r,d}$ and the labeled *r*-edge-balls $\vec{E}_n^{r,d}$ are defined accordingly. Again, $\mathbf{G}\vec{\mathbf{r}}_d^n$ is a compact metric space and $T(\mathbf{G}\vec{\mathbf{r}}_d^n, \vec{\alpha}_*), T(\mathbf{G}\vec{\mathbf{r}}_d^n, \vec{\phi}_*)$ are closed-open sets, where $\vec{\alpha}_* \in \vec{U}^{r,d}, \vec{\phi}_* \in \vec{E}_n^{r,d}$. Now let μ be an involution-invariant probability measure on $\mathbf{G}\mathbf{r}_d$ with induced measure $\vec{\mu}$. The associated measure $\vec{\mu}_n$ on $\mathbf{G}\vec{\mathbf{r}}_d^n$ is defined the following way. Let $\vec{\alpha} \in \vec{U}^{r,d}$ and κ_1, κ_2 be vertex labelings of $\vec{\alpha}$ by $\{1, 2, \ldots, n\}$. We say that

Let $\alpha \in U^{\prime,\alpha}$ and κ_1, κ_2 be vertex labelings of α by $\{1, 2, ..., n\}$. We say that κ_1 and κ_2 are equivalent if there exists a rooted automorphism of $\vec{\alpha}$ preserving the distinguished edge and mapping κ_1 to κ_2 . Let $C(\kappa)$ be the equivalence class of the vertex labeling κ of $\vec{\alpha}$. Then we define

$$\vec{\mu}_n(T(\vec{\mathbf{Gr}_d}^n, [\kappa])) := \frac{|C(\kappa)|}{n^{|V(\vec{\alpha})|}} \vec{\mu}(T(\vec{\mathbf{Gr}_d}, \vec{\alpha})).$$

Lemma 3.1 a) $\vec{\mu}_n$ extends to a Borel-measure.

b)
$$\vec{\mu}(T(\vec{Gr}_{d},\vec{\alpha})) = \sum_{\vec{\alpha}_{*}, \mathcal{F}(\vec{\alpha}_{*})=\vec{\alpha}} \vec{\mu}_{n}(T(\vec{Gr}_{d}^{n},\vec{\alpha}_{*}))$$

Proof. The second equation follows directly from th definition. In order to prove that $\vec{\mu}_n$ extends to a Borel-measure it is enough to prove that

$$\vec{\mu}_n(T(\vec{\mathbf{Gr}_d^n}, \vec{\alpha}_*)) = \sum_{\vec{\beta}_* \in N_{r+1}(\vec{\alpha}_*)} \vec{\mu}_n(T(\vec{\mathbf{Gr}_d^n}, \vec{\beta}_*)) \,,$$

where $\vec{\alpha}_* \in \vec{U}_n^{r,d}$ and $N_{r+1}(\vec{\alpha}_*)$ is the set of elements $\vec{\beta}_*$ in $\vec{U}_n^{r+1,d}$ such that the *r*-ball around the root of $\vec{\beta}_*$ is isomorphic to $\vec{\alpha}_*$. Let $\vec{\alpha} = \mathcal{F}(\vec{\alpha}_*) \in \vec{U}^{r,d}$ and let $N_{r+1}(\vec{\alpha}) \subset \vec{U}^{r,d}$ be the set of elements $\vec{\beta}$ such that the *r*-ball around the root of $\vec{\beta}$ is isomorphic to $\vec{\alpha}$. Clearly

$$\vec{\mu}(T(\vec{\mathbf{Gr}}_d, \vec{\alpha})) = \sum_{\vec{\beta} \in N_{r+1}(\vec{\alpha})} \vec{\mu}(T(\vec{\mathbf{Gr}}_d, \vec{\beta})) \,. \tag{4}$$

Let κ be a labeling of $\vec{\alpha}$ by $\{1, 2, ..., n\}$ representing $\vec{\alpha}_*$. For $\vec{\beta} \in N_{r+1}(\vec{\alpha})$ let $L(\vec{\beta})$ be the set of labelings of $\vec{\beta}$ that extends some labeling of $\vec{\alpha}$ that is equivalent to κ .

Note that

$$\vec{\mu}_n(T(\vec{\mathbf{Gr}_d}^n, \vec{\alpha}_*)) = \vec{\mu}(T(\vec{\mathbf{Gr}_d}, \vec{\alpha})) \frac{|C(\kappa)|}{n^{|V(\vec{\alpha})|}}$$

Also,

$$\sum_{\vec{\beta}_* \in N_{r+1}(\vec{\alpha}_*)} \vec{\mu}_n(T(\vec{\mathbf{Gr}}_d^n, \vec{\beta}_*)) = \sum_{\vec{\beta} \in N_{r+1}(\vec{\alpha})} \vec{\mu}(T(\vec{\mathbf{Gr}}_d, \vec{\beta})) \frac{|L(\beta)|}{n^{|V(\vec{\beta})|}}.$$

Observe that $|L(\vec{\beta})| = |C(\kappa)| n^{|V(\vec{\beta})| - V(\vec{\alpha})|}.$ Hence

$$\sum_{\vec{\beta}_* \in N_{r+1}(\vec{\alpha}_*)} \vec{\mu}_n(T(\vec{\mathbf{Gr}_d^n}, \vec{\beta}_*)) = \sum_{\vec{\beta} \in N_{r+1}(\vec{\alpha})} \vec{\mu}(T(\vec{\mathbf{Gr}_d}, \vec{\beta})) \frac{|C(\kappa)|}{n^{|V(\vec{\alpha})|}} \,.$$

Therfore using equation (4) our lemma follows.

The following proposition shall be crucial in our construction.

Proposition 3.1 For any $\vec{\alpha}_* \in \vec{U}_n^{r,d}$ and $\vec{\psi}_* \in \vec{E}_n^{r,d}$

- $\vec{\mu}_n(T(\vec{Gr}_d^n, \vec{\alpha}_*)) = \sum_{\vec{\phi}_* \in \vec{E}_n^{r,d}, s(\vec{\phi}_*) = \vec{\alpha}_*} \vec{\mu}_n(T(\vec{Gr}_d^n, \vec{\phi}_*))$
- $\vec{\mu}_n(T(\vec{Gr}_d^n, \vec{\alpha}_*)) = \sum_{\vec{\phi}_* \in \vec{E}_n^{r,d}, t(\vec{\phi}_*) = \vec{\alpha}_*} \vec{\mu}_n(T(\vec{Gr}_d^n, \vec{\phi}_*))$
- $\vec{\mu}_n(T(\vec{Gr}_d^n, \vec{\psi}_*)) = \vec{\mu}_n(T(\vec{Gr}_d^n, T_n^{r,d}(\vec{\psi}_*))).$

Proof. The first equation follows from the fact that $\vec{\mu}_n$ is a Borel-measure. Thus the second equation will be an immediate corollary of the third one. So, let us

turn to the third equation. Let $\mathcal{F}(\vec{\psi_*}) = \vec{\psi} \in \vec{E}^{r,d}$ and let κ be a vertex-labeling of $\vec{\psi}$ representing $\vec{\psi_*}$. It is enough to prove that

$$\vec{\mu}_n(T(\vec{\mathbf{Gr}_d^n}, \vec{\psi}_*)) = \frac{|C(\kappa)|}{n^{|V(\vec{\psi})|}} \vec{\mu}(T(\vec{\mathbf{Gr}_d}, \vec{\psi})),$$

where $C(\kappa)$ is the set of labelings of $\vec{\psi}$ equivalent to κ . Let $N_{r+1}(\vec{\psi}) \in \vec{U}^{r,d}$ be the set of elements $\vec{\beta}$ such that the edge-ball of radius r around the root of $\vec{\beta}$ is isomorphic to $\vec{\psi}$. Then

$$\vec{\mu}(T(\vec{\mathbf{Gr}}_d, \vec{\psi})) = \sum_{\vec{\beta} \in N_{r+1}(\vec{\psi})} \vec{\mu}(T(\vec{\mathbf{Gr}}_d, \vec{\beta})) \,.$$
(5)

Observe that

$$\vec{\mu}_n(T(\vec{\mathbf{Gr}}_d^n, \vec{\psi}_*)) = \sum_{\vec{\beta} \in N_{r+1}(\vec{\psi})} \vec{\mu}(T(\vec{\mathbf{Gr}}_d, \vec{\beta})) \frac{k(\vec{\beta}, \vec{\psi}_*)}{n^{|V(\vec{\beta})|}}$$

where $k(\vec{\beta}, \vec{\psi}_*)$ is the number of labelings of $\vec{\beta}$ extending an element that is equivalent to κ . Notice that $k(\vec{\beta}, \vec{\psi}_*) = |C(\kappa)| n^{|V(\vec{\beta})| - |V(\vec{\psi})|}$. Hence by (5) $\vec{\mu}_n(T(\vec{\mathbf{Gr}}_d^n, \vec{\psi}_*)) = \frac{|C(\kappa)|}{n^{|V(\vec{\psi})|}} \vec{\mu}(T(\vec{\mathbf{Gr}}_d, \vec{\psi}))$, thus our proposition follows.

4 Label-separated balls

Let \mathbf{Gr}_d^n be the isomorphism classes of

- connected countable rooted graphs with vertex degree bound d
- with vertex labels from the set $\{1, 2, \ldots, n\}$.

Again, we define the space of labeled *r*-balls $U_n^{r,d}$. Then \mathbf{Gr}_d^n is a compact space with closed-open sets $T(\mathbf{Gr}_d^n, M), M \in U_n^{r,d}$. Similarly to the previous section we define an associated probability measure μ_n , where μ in an involution-invariant probability measure on \mathbf{Gr}_d .

Let $M \in U_n^{r,d}$ and let R(M) be the set of elements of $\vec{U}_n^{r,d}$ with underlying graph M. If $A \in R(M)$, then the multiplicity of A, l_A is the number of edges e pointing out from the root of A such that there is a label-preserving rooted automorphism of A moving the distinguished edge to e. Now let

$$\mu_n(M) := \frac{1}{\deg(M)} \sum_{A \in R(M)} l_A \vec{\mu}_n(A) \,.$$

The following lemma is the immediate consequence of Lemma 3.1.

Lemma 4.1 μ_n is a Borel-measure on \mathbf{Gr}_d^n and $\sum_{M \in M(\alpha)} \mu_n(M) = \mu(A)$ if $\alpha \in U^{r,d}$ and $M(\alpha)$ is the set of labelings of α by $\{1, 2, \ldots, n\}$.

Definition 4.1 $M \in U_n^{r,d}$ is called label-separated if all the labels of M are different.

Lemma 4.2 For any $\alpha \in U^{r,d}$ and $\delta > 0$ there exists an n > 0 such that

$$|\sum_{M \in M(\alpha), M \text{ is label-separated}} \mu_n(T(\mathbf{Gr}_d, M)) - \mu(T(\mathbf{Gr}_d, \alpha))| < \delta.$$

Proof. Observe that

$$\sum_{\substack{M \in M(\alpha), M \text{ is label-separated}}} \mu_n(T(\mathbf{Gr}_d, M)) = \frac{T(n, \alpha)}{n^{|V(\alpha)|}} \mu(T(\mathbf{Gr}_d, \alpha)),$$

where $T(n, \alpha)$ is the number of $\{1, 2, ..., n\}$ -labelings of α with different labels. Clearly, $\frac{T(n, \alpha)}{n|V(\alpha)|} \to 1$ as $n \to \infty$.

5 The proof of Theorem 1

Let μ be an involution-invariant probability measure on \mathbf{Gr}_d supported on trees. It is enough to prove that for any $r \geq 1$ and $\epsilon > 0$ there exists a finite graph G such that for any $\alpha \in U^{r,d}$

$$|p_G(\alpha) - \mu(T(\mathbf{Gr}_d, \alpha))| < \epsilon$$

The idea we follow is close to the one used by Bowen in [5]. First, let n > 0 be a natural number such that

$$\left|\sum_{\substack{M \in M(\alpha), M \text{ is label-separated}}} \mu_n(T(\mathbf{Gr}_d, M)) - \mu(T(\mathbf{Gr}_d, \alpha))\right| < \frac{\epsilon}{10}.$$
 (6)

Then we define a directed labeled finite graph H to encode some information on $\vec{\mu}_n$. If $A \in \vec{U}_n^{r+1,d}$ then let L_A be the unique element of $\vec{E}_n^{r,d}$ contained in A. The set of vertices of H; $V(H) := \vec{U}_n^{r+1,d}$. If $A, B \in \vec{U}_n^{r+1,d}$ and $L_A = L_B^{-1}$ (we use the inverse notation instead of writing out the involution operator) then there is a directed edge (A, L_A, B) from A to B labeled by L_A and a directed edge (B, L_B, A) from B to A labeled by $L_B = L_A^{-1}$. Note that we might have loops. We define the weight function w on H by

- $w(A) = \vec{\mu}_n(T(\vec{\mathbf{Gr}_d^n}, A)).$
- $w(A, L_A, B) = \mu(T(\vec{\mathbf{Gr}_d}^n, L_{A,B}))$, where $L_{A,B} \in \vec{E}_n^{r+1,d}$ the unique element such that $s(L_{A,B}) = A, t(L_{A,B}) = B$.

By Proposition 3.1 we have the following equation for all A, B that are connected in H:

$$w(A, L_A, B) = w(B, L_A^{-1}, A).$$
(7)

Also,

$$w(A) = \sum_{w(A, L_A, B) \in E(H)} w(A, L_A, B)$$
(8)

$$w(A) = \sum_{w(B, L_A^{-1}, A) \in E(H)} w(B, L_A^{-1}, A)$$
(9)

Also if $M \in U_n^{r+1,d}$ then

$$\mu_n(M) = \frac{1}{\deg(M)} \sum_{A \in R(M)} l_A w(A), \tag{10}$$

where l_A is the multiplicity of w(A).

Since the equations (7), (8), (9) have rational coefficients we also have weight functions w_{δ} on H

- taking only rational values
- satisfying equations (7), (8), (9)
- such that $|w_{\delta}(A) w(A)| < \delta$ for any $A \in V(H)$, where the exact value of δ will be given later.

Now let N be a natural number such that

- $\frac{Nw_{\delta}(A)}{l_A} \in \mathbb{N}$ if $A \in V(H)$.
- $Nw_{\delta}(A, L_A, B) \in \mathbb{N}$ if $(A, L_A, B) \in E(H)$.

Step 1. We construct an edge-less graph Q such that:

- $V(Q) = \bigcup_{A \in V(H)} Q(A)$ (disjoint union)
- $|Q(A)| = Nw_{\delta}(A)$
- each Q(A) is partitioned into $\cup_{(A,L_A,B)\in E(H)}Q(A,L_A,B)$ such that $|Q(A,L_A,B)| = Nw_{\delta}(A,L_A,B).$

Since w_{δ} satisfy our equations such Q can be constructed.

Step 2. We add edges to Q in order to obtain the graph R. For each pair A, B that are connected in the graph H form a bijection $Z_{A,B} : Q(A, L_A, B) \to Q(B, L_B, A)$. If there is a loop in H consider a bijection $Z_{A,A}$. Then draw an edge between $x \in Q(A, L_A, B)$ and $y \in Q(B, L_B, A)$ if $Z_{A,B}(x) = y$.

Step 3. Now we construct our graph G. If $M \in U_n^{r+1,d}$ is a rooted labeled tree such that $\mu_n(M) \neq 0$ let $Q(M) = \bigcup_{A \in R(M)} Q(A)$. We partition Q(M) into

 $\bigcup_{i=1}^{s_M} Q_i(M)$ such a way that each $Q_i(M)$ contains exactly l_A elements from the set Q(A). By the definition of N, we can make such partition.

The elements of V(G) will be the sets $\{Q_i(M)\}_{M \in U_n^{r+1,d}, 1 \leq i \leq s_M}$. We draw one edge between $Q_i(M)$ and $Q_j(M')$ if there exists $x \in Q_i(M), y \in Q_j(M')$ such that x and y are connected in R. We label the vertex $Q_i(M)$ by the label of the root of M. Let $Q_i(M)$ be a vertex of G such that M is a label-separated tree. Note that if M is not a rooted tree then $\mu_n(M) = 0$. It is easy to see that the r + 1-ball around $Q_i(M)$ in the graph G is isomorphic to M as rooted labeled balls. Also if M is not label-separated tree. Therefore

$$\sum_{L \in U_n^{r,d}, L \text{ is not a label-separated tree}} p_G(L) = (11)$$

$$= \sum_{L \in U_n^{r,d}, L \text{ is not a label-separated tree}} \sum_{A \in R(L)} w_{\delta}(L) \le \frac{\epsilon}{10} + \delta d|U_n^{r,d}|.$$
(12)

Also, if M is a label-separated tree then

$$|p_G(M) - \mu_n(T(\mathbf{Gr}_d, M))| \le |R(M)|\delta \le d\delta.$$
(13)

Thus by (6),(11),(13) if δ is choosen small enough then for any $\alpha \in U^{r+1,d}$

$$|p_G(\alpha) - \mu(T(\mathbf{Gr}_d, \alpha))| < \epsilon.$$

Thus our Theorem follows.

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