# On the limit of large girth graph sequences* 

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#### Abstract

Let $d \geq 2$ be given and let $\mu$ be an involution-invariant probability measure on the space of trees $T \in \mathcal{T}_{d}$ with maximum degrees at most $d$. Then $\mu$ arises as the local limit of some sequence $\left\{G_{n}\right\}_{n=1}^{\infty}$ of graphs with all degrees at most $d$. This answers Question 6.8 of Bollobás and Riordan [4].


## 1 Introduction

Let $\mathbf{G r a p h}_{d}$ denote the set of all finite simple graphs $G$ (up to isomorphism) for which $\operatorname{deg}(x) \leq d$ for every $x \in V(G)$. For a graph $G$ and $x, y \in V(G)$ let $d_{G}(x, y)$ denote the distance of $x$ and $y$, that is the length of the shortest path from $x$ to $y$. A rooted $(r, d)$-ball is a graph $G \in \mathbf{G r a p h}_{d}$ with a marked vertex $x \in V(G)$ called the root such that $d_{G}(x, y) \leq r$ for every $y \in V(G)$. By $U^{r, d}$ we shall denote the set of rooted $(r, d)$-balls.

If $G \in \mathbf{G r a p h}_{d}$ is a graph and $x \in V(G)$ then $B_{r}(x) \in U^{r, d}$ shall denote the rooted $(r, d)$-ball around $x$ in $G$. For any $\alpha \in U^{r, d}$ and $G \in \mathbf{G r a p h}_{d}$ we define the set $T(G, \alpha) \stackrel{\text { def }}{=}\left\{x \in V(G): B_{r}(x) \cong \alpha\right\}$ and let $p_{G}(\alpha) \stackrel{\text { def }}{=}$ $\frac{|T(G, \alpha)|}{|V(G)|}$. A graph sequence $\mathbf{G}=\left\{G_{n}\right\}_{n=1}^{\infty} \subset \mathbf{G r a p h}_{d}$ is weakly convergent if $\lim _{n \rightarrow \infty}\left|V\left(G_{n}\right)\right|=\infty$ and for every $r$ and every $\alpha \in U^{r, d}$ the limit $\lim _{n \rightarrow \infty} p_{G_{n}}(\alpha)$ exists (see [3]).
Let $\mathbf{G r}_{d}$ denote the set of all countable, connected rooted graphs $G$ for which $\operatorname{deg}(x) \leq d$ for every $x \in V(G)$. If $G, H \in \mathbf{G r}_{d}$ let $d_{g}(G, H)=2^{-r}$, where $r$ is the maximal number such that the $r$-balls around the roots of $G$ resp. $H$ are rooted isomorphic. The distance $d_{g}$ makes $\mathbf{G r}_{d}$ a compact metric space. Given an $\alpha \in U^{r, d}$ let $T\left(\mathbf{G r}_{d}, \alpha\right)=\left\{(G, x) \in \mathbf{G r}_{d}: B_{r}(x) \cong \alpha\right\}$. The sets $T\left(\mathbf{G r}_{d}, \alpha\right)$ are closed-open sets. A convergent graphs sequence $\left\{G_{n}\right\}_{n=1}^{\infty}$ define a local limit measure $\mu_{\mathbf{G}}$ on $\mathbf{G r}_{d}$, where $\mu_{\mathbf{G}}\left(T\left(\mathbf{G r}_{d}, \alpha\right)\right)=\lim _{n \rightarrow \infty} p_{G_{n}}(\alpha)$. However, not all the probability measures on $\mathbf{G r}_{d}$ arise as local limits. A necessary condition for a measure $\mu$ being a local limit is its involution invariance (see Section (2). The goal of this paper is to answer a question of Bollobás and Riordan (Question 6.8 [4]):

[^0]Theorem 1 Any involution-invariant measure $\mu$ on $\boldsymbol{G} \boldsymbol{r}_{d}$ concentrated on trees arises as a local limit of some convergent graph sequence.

As it was pointed out in [4] such graph sequences are asymptotically treelike, thus $\mu$ must arise as the local limit of a convergent large girth sequence.

## 2 Involution invariance

Let $\mathbf{G r}_{d}$ be the compact space of all connected countable rooted graphs $\vec{G}$ (up to isomorphism) of vertex degree bound $d$ with a distinguished directed edge pointing out from the root. Note that $\vec{G}$ and $\vec{H}$ are considered isomorphic if there exists a rooted isomorphism between them mapping distinguished edges into each other. Let $\vec{U}^{r, d}$ be the isomorphism classes of all rooted $(r, d)$-graphs $\vec{\alpha}$ with a distinguished edge $e(\vec{\alpha})$ pointing out from the root. Again, $T\left(\mathbf{G r}_{d}, \vec{\alpha}\right)$ is well-defined for any $\vec{\alpha} \in \vec{U}^{r, d}$ and defines a closed-open set in $\overrightarrow{\mathbf{G r}}_{d}$. Clearly, the forgetting map $\mathcal{F}: \overrightarrow{\mathbf{G r}}_{d} \rightarrow \mathbf{G r}_{d}$ is continuous. Let $\mu$ be a probability measure on $\mathbf{G r}_{d}$. Then we define a measure $\vec{\mu}$ on $\overrightarrow{\mathbf{G r}}_{d}$ the following way.
Let $\vec{\alpha} \in \vec{U}^{r, d}$ and let $\mathcal{F}(\vec{\alpha})=\alpha \in U^{r, d}$ be the underlying rooted ball. Clearly, $\mathcal{F}\left(T\left(\mathbf{G r}_{d}, \vec{\alpha}\right)\right)=T\left(\mathbf{G r}_{d}, \alpha\right)$. Let

$$
\vec{\mu}\left(T\left(\overrightarrow{\mathbf{G}}_{d}, \vec{\alpha}\right)\right):=l
$$

where $l$ is the number of edges $e$ pointing out from the root such that there exists a rooted automorphism of $\alpha$ mapping $e(\vec{\alpha})$ to $e$. Observe that

$$
\vec{\mu}\left(\mathcal{F}^{-1}\left(T\left(\mathbf{G r}_{d}, \alpha\right)\right)=\operatorname{deg}(\alpha) \mu\left(T\left(\mathbf{G r}_{d}, \alpha\right)\right)\right.
$$

We define the map $T: \overrightarrow{\mathbf{G r}}_{d} \rightarrow \mathbf{G r}_{d}$ as follows. Let $T(\vec{G})=\vec{H}$, where :

- the underlying graphs of $\vec{G}$ and $\vec{H}$ are the same,
- the root of $\vec{H}$ is the endpoint of $e(\vec{G})$,
- the distinguished edge of $\vec{H}$ is pointing to the root of $\vec{G}$.

Note that $T$ is a continuous involution. Following Aldous and Steele [2], we call $\mu$ involution-invariant if $T_{*}(\vec{\mu})=\vec{\mu}$. It is important to note [2, [1] that the limit measure of convergent graphs sequences are always involution-invariant. We need to introduce the notion of edge-balls. Let $\vec{G} \in \mathbf{G r}_{d}$. The edge-ball $B_{r}^{e}(\vec{G})$ of radius $r$ around the root of $\vec{G}$ is the following spanned rooted subgraph of $\vec{G}$ :

- The root of $B_{r}^{e}(\vec{G})$ is the same as the root of $\vec{G}$.
- $y$ is a vertex of $B_{r}^{e}(\vec{G})$ if $d(x, y) \leq r$ or $d\left(x^{\prime}, y\right) \leq r$, where $x$ is the root of $\vec{G}$ and $x^{\prime}$ is the endpoint of the directed edge $e(\vec{G})$.
- The distinguished edge of $B_{r}^{e}(\vec{G})$ is $\left(x, \overrightarrow{x^{\prime}}\right)$.

Let $\vec{E}^{r, d}$ be the set of all edge-balls of radius $r$ up to isomorphism. Then if $\vec{\phi} \in \vec{E}^{r, d}$, let $s(\vec{\phi}) \in \vec{U}^{r, d}$ be the rooted ball around the root of $\vec{\phi}$. Also, let $t(\vec{\phi}) \in \vec{U}^{r, d}$ be the $r$-ball around $x^{\prime}$ with distinguished edge $\left(x^{\prime \prime}, x\right)$.
The involution $T^{r, d}: \vec{E}^{r, d} \rightarrow \vec{E}^{r, d}$ is defined the obvious way and $t\left(T^{r, d}(\vec{\phi})\right)=$ $s(\vec{\phi}), s\left(T^{r, d}(\vec{\phi})\right)=t(\vec{\phi})$. Since $\vec{\mu}$ is a measure we have

$$
\begin{equation*}
\vec{\mu}\left(T\left(\overrightarrow{\mathbf{G}}_{d}, \vec{\alpha}\right)\right)=\sum_{\vec{\phi}, s(\vec{\phi})=\vec{\alpha}} \vec{\mu}\left(T\left(\overrightarrow{\mathbf{G}}_{d}, \vec{\phi}\right)\right) \tag{1}
\end{equation*}
$$

Also, by the involution-invariance

$$
\begin{equation*}
\vec{\mu}\left(T\left(\mathbf{G r}_{d}, \vec{\phi}\right)\right)=\vec{\mu}\left(T\left(\mathbf{G r}_{d}, T^{r, d}(\vec{\phi})\right)\right. \tag{2}
\end{equation*}
$$

since $T\left(T\left(\mathbf{G r}_{d}, \vec{\phi}\right)\right)=T\left(\mathbf{G r}_{d}, T^{r, d}(\vec{\phi})\right.$. Therefore by (1),

$$
\begin{equation*}
\vec{\mu}\left(T\left(\overrightarrow{\mathbf{G r}}_{d}, \vec{\alpha}\right)\right)=\sum_{\vec{\phi}, t(\vec{\phi})=\vec{\alpha}} \vec{\mu}\left(T\left(\overrightarrow{\mathbf{G}}_{d}, \vec{\phi}\right)\right) \tag{3}
\end{equation*}
$$

## 3 Labeled graphs

Let $\mathbf{G r}_{d}^{n}$ be the isomorphism classes of

- connected countable rooted graphs with vertex degree bound $d$
- with a distinguished edge pointing out from the root
- with vertex labels from the set $\{1,2, \ldots, n\}$.

Note that if $\vec{G}_{*}$ and $\vec{H}_{*}$ are such graphs then they called isomorphic if there exists a map $\rho: V\left(\vec{G}_{*}\right) \rightarrow V\left(\vec{H}_{*}\right)$ preserving both the underlying $\overrightarrow{\mathbf{G r}}_{d}$-structure and the the vertex labels. The labeled $r$-balls $\vec{U}_{n}^{r, d}$ and the labeled $r$-edgeballs $\vec{E}_{n}^{r, d}$ are defined accordingly. Again, $\mathbf{G r}_{d}^{n}$ is a compact metric space and $T\left(\mathbf{G r}_{d}^{n}, \vec{\alpha}_{*}\right), T\left(\mathbf{G}_{d}^{n}, \vec{\phi}_{*}\right)$ are closed-open sets, where $\vec{\alpha}_{*} \in \vec{U}^{r, d}, \vec{\phi}_{*} \in \vec{E}_{n}^{r, d}$. Now let $\mu$ be an involution-invariant probability measure on $\mathbf{G r} \mathbf{r}_{d}$ with induced measure $\vec{\mu}$. The associated measure $\vec{\mu}_{n}$ on $\mathbf{G r}_{d}^{n}$ is defined the following way.
Let $\vec{\alpha} \in \overrightarrow{U^{r}, d}$ and $\kappa_{1}, \kappa_{2}$ be vertex labelings of $\vec{\alpha}$ by $\{1,2, \ldots, n\}$. We say that $\kappa_{1}$ and $\kappa_{2}$ are equivalent if there exists a rooted automorphism of $\vec{\alpha}$ preserving the distinguished edge and mapping $\kappa_{1}$ to $\kappa_{2}$. Let $C(\kappa)$ be the equivalence class of the vertex labeling $\kappa$ of $\vec{\alpha}$. Then we define

$$
\vec{\mu}_{n}\left(T\left(\overrightarrow{\mathbf{G}}_{d}^{n},[\kappa]\right)\right):=\frac{|C(\kappa)|}{n^{|V(\vec{\alpha})|}} \vec{\mu}\left(T\left(\overrightarrow{\mathbf{G}}_{d}, \vec{\alpha}\right)\right)
$$

Lemma 3.1 a) $\vec{\mu}_{n}$ extends to a Borel-measure.
b) $\vec{\mu}\left(T\left(\overrightarrow{\boldsymbol{G r}_{d}}, \vec{\alpha}\right)\right)=\sum_{\vec{\alpha}_{*}, \mathcal{F}\left(\vec{\alpha}_{*}\right)=\vec{\alpha}} \vec{\mu}_{n}\left(T\left(\overrightarrow{\boldsymbol{G r}}_{d}^{n}, \vec{\alpha}_{*}\right)\right)$.

Proof. The second equation follows directly from th definition. In order to prove that $\vec{\mu}_{n}$ extends to a Borel-measure it is enough to prove that

$$
\vec{\mu}_{n}\left(T\left(\mathbf{G r}_{d}^{n}, \vec{\alpha}_{*}\right)\right)=\sum_{\vec{\beta}_{*} \in N_{r+1}\left(\vec{\alpha}_{*}\right)} \vec{\mu}_{n}\left(T\left(\mathbf{G r}_{d}^{n}, \vec{\beta}_{*}\right)\right)
$$

where $\vec{\alpha}_{*} \in \vec{U}_{n}^{r, d}$ and $N_{r+1}\left(\vec{\alpha}_{*}\right)$ is the set of elements $\vec{\beta}_{*}$ in $\vec{U}_{n}^{r+1, d}$ such that the $r$-ball around the root of $\vec{\beta}_{*}$ is isomorphic to $\vec{\alpha}_{*}$. Let $\vec{\alpha}=\mathcal{F}\left(\vec{\alpha}_{*}\right) \in \vec{U}^{r, d}$ and let $N_{r+1}(\vec{\alpha}) \subset \vec{U}^{r, d}$ be the set of elements $\vec{\beta}$ such that the $r$-ball around the root of $\vec{\beta}$ is isomorphic to $\vec{\alpha}$. Clearly

$$
\begin{equation*}
\vec{\mu}\left(T\left(\mathbf{G r}_{d}, \vec{\alpha}\right)\right)=\sum_{\vec{\beta} \in N_{r+1}(\vec{\alpha})} \vec{\mu}\left(T\left(\mathbf{G r}_{d}, \vec{\beta}\right)\right) \tag{4}
\end{equation*}
$$

Let $\kappa$ be a labeling of $\vec{\alpha}$ by $\{1,2, \ldots, n\}$ representing $\vec{\alpha}_{*}$. For $\vec{\beta} \in N_{r+1}(\vec{\alpha})$ let $L(\vec{\beta})$ be the set of labelings of $\vec{\beta}$ that extends some labeling of $\vec{\alpha}$ that is equivalent to $\kappa$.
Note that

$$
\vec{\mu}_{n}\left(T\left(\mathbf{G r}_{d}^{n}, \vec{\alpha}_{*}\right)\right)=\vec{\mu}\left(T\left(\mathbf{G r}_{d}, \vec{\alpha}\right)\right) \frac{|C(\kappa)|}{n^{|V(\vec{\alpha})|}}
$$

Also,

$$
\sum_{\vec{\beta}_{*} \in N_{r+1}\left(\vec{\alpha}_{*}\right)} \vec{\mu}_{n}\left(T\left(\overrightarrow{\mathbf{G}}_{d}^{n}, \vec{\beta}_{*}\right)\right)=\sum_{\vec{\beta} \in N_{r+1}(\vec{\alpha})} \vec{\mu}\left(T\left(\mathbf{G}_{d}, \vec{\beta}\right)\right) \frac{|L(\vec{\beta})|}{n^{|V(\vec{\beta})|}}
$$

Observe that $|L(\vec{\beta})|=|C(\kappa)| n^{|V(\vec{\beta})|-V(\vec{\alpha}) \mid}$. Hence

$$
\sum_{\vec{\beta}_{*} \in N_{r+1}\left(\vec{\alpha}_{*}\right)} \vec{\mu}_{n}\left(T\left(\mathbf{G}_{d}^{n}, \vec{\beta}_{*}\right)\right)=\sum_{\vec{\beta} \in N_{r+1}(\vec{\alpha})} \vec{\mu}\left(T\left(\mathbf{G r}_{d}, \vec{\beta}\right)\right) \frac{|C(\kappa)|}{n^{|V(\vec{\alpha})|}}
$$

Therfore using equation (4) our lemma follows.

The following proposition shall be crucial in our construction.
Proposition 3.1 For any $\vec{\alpha}_{*} \in \vec{U}_{n}^{r, d}$ and $\vec{\psi}_{*} \in \vec{E}_{n}^{r, d}$

- $\vec{\mu}_{n}\left(T\left(\overrightarrow{\boldsymbol{G r}_{d}^{n}}, \vec{\alpha}_{*}\right)\right)=\sum_{\vec{\phi}_{*} \in \vec{E}_{n}^{r, d}, s\left(\vec{\phi}_{*}\right)=\vec{\alpha}_{*}} \vec{\mu}_{n}\left(T\left(\overrightarrow{\left.\left.\boldsymbol{G r}_{d}^{n}, \vec{\phi}_{*}\right)\right)}\right.\right.$
- $\vec{\mu}_{n}\left(T\left(\overrightarrow{\boldsymbol{G r}}{ }_{d}^{n}, \vec{\alpha}_{*}\right)\right)=\sum_{\vec{\phi}_{*} \in \vec{E}_{n}^{r, d}, t\left(\vec{\phi}_{*}\right)=\vec{\alpha}_{*}} \vec{\mu}_{n}\left(T\left(\overrightarrow{\boldsymbol{G r}}{ }_{d}^{n}, \vec{\phi}_{*}\right)\right)$
- $\vec{\mu}_{n}\left(T\left(\overrightarrow{\boldsymbol{G r}}_{d}^{n}, \vec{\psi}_{*}\right)\right)=\vec{\mu}_{n}\left(T\left(\overrightarrow{\boldsymbol{G r}}_{d}^{n}, T_{n}^{r, d}\left(\vec{\psi}_{*}\right)\right)\right.$.

Proof. The first equation follows from the fact that $\vec{\mu}_{n}$ is a Borel-measure. Thus the second equation will be an immediate corollary of the third one. So, let us
turn to the third equation. Let $\mathcal{F}\left(\vec{\psi}_{*}\right)=\vec{\psi} \in \vec{E}^{r, d}$ and let $\kappa$ be a vertex-labeling of $\vec{\psi}$ representing $\vec{\psi}_{*}$. It is enough to prove that

$$
\vec{\mu}_{n}\left(T\left(\overrightarrow{\mathbf{G}}_{d}^{n}, \vec{\psi}_{*}\right)\right)=\frac{|C(\kappa)|}{n^{|V(\vec{\psi})|}} \vec{\mu}\left(T\left(\overrightarrow{\mathbf{G}}_{d}, \vec{\psi}\right)\right)
$$

where $C(\kappa)$ is the set of labelings of $\vec{\psi}$ equivalent to $\kappa$. Let $N_{r+1}(\vec{\psi}) \in \vec{U}^{r, d}$ be the set of elements $\vec{\beta}$ such that the edge-ball of radius $r$ around the root of $\vec{\beta}$ is isomorphic to $\vec{\psi}$. Then

$$
\begin{equation*}
\vec{\mu}\left(T\left(\overrightarrow{\mathbf{G}}_{d}, \vec{\psi}\right)\right)=\sum_{\vec{\beta} \in N_{r+1}(\vec{\psi})} \vec{\mu}\left(T\left(\overrightarrow{\mathbf{G}}_{d}, \vec{\beta}\right)\right) \tag{5}
\end{equation*}
$$

Observe that

$$
\vec{\mu}_{n}\left(T\left(\overrightarrow{\mathbf{G}}_{d}^{n}, \vec{\psi}_{*}\right)\right)=\sum_{\vec{\beta} \in N_{r+1}(\vec{\psi})} \vec{\mu}\left(T\left(\overrightarrow{\mathbf{G}}_{d}, \vec{\beta}\right)\right) \frac{k\left(\vec{\beta}, \vec{\psi}_{*}\right)}{n^{|V(\vec{\beta})|}}
$$

where $k\left(\vec{\beta}, \vec{\psi}_{*}\right)$ is the number of labelings of $\vec{\beta}$ extending an element that is equivalent to $\kappa$. Notice that $k\left(\vec{\beta}, \vec{\psi}_{*}\right)=|C(\kappa)| n^{|V(\vec{\beta})|-|V(\vec{\psi})|}$. Hence by (5) $\vec{\mu}_{n}\left(T\left(\overrightarrow{\mathbf{G}}_{d}^{n}, \vec{\psi}_{*}\right)\right)=\frac{|C(\kappa)|}{n^{|V(\overrightarrow{)})|}} \vec{\mu}\left(T\left(\overrightarrow{\mathbf{G r}}_{d}, \vec{\psi}\right)\right)$, thus our proposition follows.

## 4 Label-separated balls

Let $\mathbf{G r}_{d}^{n}$ be the isomorphism classes of

- connected countable rooted graphs with vertex degree bound $d$
- with vertex labels from the set $\{1,2, \ldots, n\}$.

Again, we define the space of labeled $r$-balls $U_{n}^{r, d}$. Then $\mathbf{G r}_{d}^{n}$ is a compact space with closed-open sets $T\left(\mathbf{G r}_{d}^{n}, M\right), M \in U_{n}^{r, d}$. Similarly to the previous section we define an associated probability measure $\mu_{n}$, where $\mu$ in an involutioninvariant probability measure on $\mathbf{G r}_{d}$.
Let $M \in U_{n}^{r, d}$ and let $R(M)$ be the set of elements of $\vec{U}_{n}^{r, d}$ with underlying graph $M$. If $A \in R(M)$, then the multiplicity of $A, l_{A}$ is the number of edges $e$ pointing out from the root of $A$ such that there is a label-preserving rooted automorphism of $A$ moving the distinguished edge to $e$. Now let

$$
\mu_{n}(M):=\frac{1}{\operatorname{deg}(M)} \sum_{A \in R(M)} l_{A} \vec{\mu}_{n}(A)
$$

The following lemma is the immediate consequence of Lemma 3.1
Lemma $4.1 \mu_{n}$ is a Borel-measure on $\boldsymbol{G r} r_{d}^{n}$ and $\sum_{M \in M(\alpha)} \mu_{n}(M)=\mu(A)$ if $\alpha \in U^{r, d}$ and $M(\alpha)$ is the set of labelings of $\alpha$ by $\{1,2, \ldots, n\}$.

Definition 4.1 $M \in U_{n}^{r, d}$ is called label-separated if all the labels of $M$ are different.

Lemma 4.2 For any $\alpha \in U^{r, d}$ and $\delta>0$ there exists an $n>0$ such that

$$
\sum_{M \in M(\alpha), M} \sum_{i s \text { label-separated }} \mu_{n}\left(T\left(\boldsymbol{G} \boldsymbol{r}_{d}, M\right)\right)-\mu\left(T\left(\boldsymbol{G} \boldsymbol{r}_{d}, \alpha\right)\right) \mid<\delta
$$

Proof. Observe that

$$
\sum_{M \in M(\alpha), M} \mu_{n}\left(T\left(\mathbf{G r}_{d}, M\right)\right)=\frac{T(n, \alpha)}{n^{|V(\alpha)|}} \mu\left(T\left(\mathbf{G r}_{d}, \alpha\right)\right)
$$

where $T(n, \alpha)$ is the number of $\{1,2, \ldots, n\}$-labelings of $\alpha$ with different labels. Clearly, $\frac{T(n, \alpha)}{n^{|V(\alpha)|}} \rightarrow 1$ as $n \rightarrow \infty$.

## 5 The proof of Theorem 1

Let $\mu$ be an involution-invariant probability measure on $\mathbf{G r}_{d}$ supported on trees. It is enough to prove that for any $r \geq 1$ and $\epsilon>0$ there exists a finite graph $G$ such that for any $\alpha \in U^{r, d}$

$$
\left|p_{G}(\alpha)-\mu\left(T\left(\mathbf{G r}_{d}, \alpha\right)\right)\right|<\epsilon .
$$

The idea we follow is close to the one used by Bowen in 5. First, let $n>0$ be a natural number such that

$$
\begin{equation*}
\left|\sum_{M \in M(\alpha), M} \mu_{n}\left(T\left(\mathbf{G r}_{d}, M\right)\right)-\mu\left(T\left(\mathbf{G r}_{d}, \alpha\right)\right)\right|<\frac{\epsilon}{10} \tag{6}
\end{equation*}
$$

Then we define a directed labeled finite graph $H$ to encode some information on $\vec{\mu}_{n}$. If $A \in \vec{U}_{n}^{r+1, d}$ then let $L_{A}$ be the unique element of $\vec{E}_{n}^{r, d}$ contained in $A$. The set of vertices of $H ; V(H):=\vec{U}_{n}^{r+1, d}$. If $A, B \in \vec{U}_{n}^{r+1, d}$ and $L_{A}=L_{B}^{-1}$ (we use the inverse notation instead of writing out the involution operator) then there is a directed edge $\left(A, L_{A}, B\right)$ from $A$ to $B$ labeled by $L_{A}$ and a directed edge $\left(B, L_{B}, A\right)$ from $B$ to $A$ labeled by $L_{B}=L_{A}^{-1}$. Note that we might have loops. We define the weight function $w$ on $H$ by

- $w(A)=\vec{\mu}_{n}\left(T\left(\mathbf{G r}_{d}^{n}, A\right)\right)$.
- $w\left(A, L_{A}, B\right)=\mu\left(T\left(\mathbf{G r}_{d}^{n}, L_{A, B}\right)\right)$, where $L_{A, B} \in \vec{E}_{n}^{r+1, d}$ the unique element such that $s\left(L_{A, B}\right)=A, t\left(L_{A, B}\right)=B$.

By Proposition 3.1 we have the following equation for all $A, B$ that are connected in $H$ :

$$
\begin{equation*}
w\left(A, L_{A}, B\right)=w\left(B, L_{A}^{-1}, A\right) \tag{7}
\end{equation*}
$$

Also,

$$
\begin{align*}
w(A) & =\sum_{w\left(A, L_{A}, B\right) \in E(H)} w\left(A, L_{A}, B\right)  \tag{8}\\
w(A) & =\sum_{w\left(B, L_{A}^{-1}, A\right) \in E(H)} w\left(B, L_{A}^{-1}, A\right) \tag{9}
\end{align*}
$$

Also if $M \in U_{n}^{r+1, d}$ then

$$
\begin{equation*}
\mu_{n}(M)=\frac{1}{\operatorname{deg}(M)} \sum_{A \in R(M)} l_{A} w(A) \tag{10}
\end{equation*}
$$

where $l_{A}$ is the multiplicity of $w(A)$.

Since the equations (7), (8), (9) have rational coefficients we also have weight functions $w_{\delta}$ on $H$

- taking only rational values
- satisfying equations (7), (8), (9)
- such that $\left|w_{\delta}(A)-w(A)\right|<\delta$ for any $A \in V(H)$, where the exact value of $\delta$ will be given later.

Now let $N$ be a natural number such that

- $\frac{N w_{\delta}(A)}{l_{A}} \in \mathbb{N}$ if $A \in V(H)$.
- $N w_{\delta}\left(A, L_{A}, B\right) \in \mathbb{N}$ if $\left(A, L_{A}, B\right) \in E(H)$.

Step 1. We construct an edge-less graph $Q$ such that:

- $V(Q)=\cup_{A \in V(H)} Q(A) \quad$ (disjoint union)
- $|Q(A)|=N w_{\delta}(A)$
- each $Q(A)$ is partitioned into $\cup_{\left(A, L_{A}, B\right) \in E(H)} Q\left(A, L_{A}, B\right)$ such that $\left|Q\left(A, L_{A}, B\right)\right|=N w_{\delta}\left(A, L_{A}, B\right)$.

Since $w_{\delta}$ satisfy our equations such $Q$ can be constructed.

Step 2. We add edges to $Q$ in order to obtain the graph $R$. For each pair $A, B$ that are connected in the graph $H$ form a bijection $Z_{A, B}: Q\left(A, L_{A}, B\right) \rightarrow$ $Q\left(B, L_{B}, A\right)$. If there is a loop in $H$ consider a bijection $Z_{A, A}$. Then draw an edge between $x \in Q\left(A, L_{A}, B\right)$ and $y \in Q\left(B, L_{B}, A\right)$ if $Z_{A, B}(x)=y$.

Step 3. Now we construct our graph $G$. If $M \in U_{n}^{r+1, d}$ is a rooted labeled tree such that $\mu_{n}(M) \neq 0$ let $Q(M)=\cup_{A \in R(M)} Q(A)$. We partition $Q(M)$ into
$\cup_{i=1}^{S_{M}} Q_{i}(M)$ such a way that each $Q_{i}(M)$ contains exactly $l_{A}$ elements from the set $Q(A)$. By the definition of $N$, we can make such partition.
The elements of $V(G)$ will be the sets $\left\{Q_{i}(M)\right\}_{M \in U_{n}^{r+1, d}, 1 \leq i \leq s_{M}}$. We draw one edge between $Q_{i}(M)$ and $Q_{j}\left(M^{\prime}\right)$ if there exists $x \in Q_{i}(\bar{M}), y \in Q_{j}\left(M^{\prime}\right)$ such that $x$ and $y$ are connected in $R$. We label the vertex $Q_{i}(M)$ by the label of the root of $M$. Let $Q_{i}(M)$ be a vertex of $G$ such that $M$ is a label-separated tree. Note that if $M$ is not a rooted tree then $\mu_{n}(M)=0$. It is easy to see that the $r+1$-ball around $Q_{i}(M)$ in the graph $G$ is isomorphic to $M$ as rooted labeled balls. Also if $M$ is not label-separated then the $r+1$-ball around $Q_{i}(M)$ can not be a label-separated tree. Therefore

$$
\begin{align*}
& \sum \quad p_{G}(L)=  \tag{11}\\
& L \in U_{n}^{r, d}, L \text { is not a label-separated tree } \\
& =\sum_{L \in U_{n}^{r, d}, L \text { is not a label-separated tree }} \sum_{A \in R(L)} w_{\delta}(L) \leq \frac{\epsilon}{10}+\delta d\left|U_{n}^{r, d}\right| . \tag{12}
\end{align*}
$$

Also, if $M$ is a label-separated tree then

$$
\begin{equation*}
\left|p_{G}(M)-\mu_{n}\left(T\left(\mathbf{G r}_{d}, M\right)\right)\right| \leq|R(M)| \delta \leq d \delta . \tag{13}
\end{equation*}
$$

Thus by (6), (11), (13) if $\delta$ is choosen small enough then for any $\alpha \in U^{r+1, d}$

$$
\left|p_{G}(\alpha)-\mu\left(T\left(\mathbf{G r}_{d}, \alpha\right)\right)\right|<\epsilon .
$$

Thus our Theorem follows.

## References

[1] D. Aldous and R. Lyons, Processes on Unimodular Random Networks, Electron. J. Probab. 12 (2007), no. 54, 1454-1508.
[2] D. Aldous and M. J. Steele, The objective method: probabilistic combinatorial optimization and local weak convergence. Probability on discrete structures, 1-72, Encyclopaedia Math. Sci., 110, Springer, Berlin, 2004.
[3] I. Benjamini and O. Schramm, Recurrence of distributional limits of finite planar graphs. Electron. J. Probab. 6 (2001), no. 23, 13 pp. (electronic).
[4] B. Bollobás and O. Riordan, Sparse graphs: metrics and random models (preprint) http://arxiv.org/abs/0708.1919
[5] L. Bowen, Periodicity and circle packings of the hyperbolic plane Geom. Dedicata 102 (2003) 213-236.


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