k-nets embedded in a projective plane over a field

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Abstract

We investigate k-nets with $k \geq 4$ embedded in the projective plane $PG(2, \mathbb{K})$ defined over a field $\mathbb{K}$; they are line configurations in $PG(2, \mathbb{K})$ consisting of k pairwise disjoint line-sets, called components, such that any two lines from distinct families are concurrent with exactly one line from each component. The size of each component of a k-net is the same, the order of the k-net. If $\mathbb{K}$ has zero characteristic, no embedded k-net for $k \geq 5$ exists; see [10, 13]. Here we prove that this holds true in positive characteristic p as long as p is sufficiently large compared with the order of the k-net. Our approach, different from that used in [10, 13], also provides a new proof in characteristic zero.

1 Introduction

An (abstract) k-net is a point-line incidence structure whose lines are partitioned in k subsets, called components, such that any two lines from distinct components are concurrent with exactly one line from each component. The components have the same size, called the order of the k-net and denoted by

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n. A k-net has n^2 points and kn lines. A k-net (embedded) in $PG(2,\mathbb{K})$ is a subset of points and lines such that the incidence structure induced by them is a k-net.

In the complex plane, there are known plenty of examples and even infinity families of 3-nets but only one 4-net up to projectivity; see [10, 11, 12, 13]. This 4-net, called the classical 4-net, has order 3 and it exists since $PG(2,\mathbb{C})$ contains an affine subplane $AG(2,\mathbb{F}_3)$ of order 3, unique up to projectivity, and the four parallel line classes of $AG(2,\mathbb{F}_3)$ are the components of a 4-net in $PG(2,\mathbb{C})$. By a result of Stipins [10], see also [13], no k-net with $k \geq 5$ exists in $PG(2,\mathbb{C})$. Since Stipins' proof works over any algebraically closed field of characteristic zero, his result holds true in $PG(2,\mathbb{K})$ provided that $\mathbb{K}$ has zero characteristic.

Our present investigation of k-nets in $PG(2,\mathbb{K})$ includes groundfields $\mathbb{K}$ of positive characteristic p, and as a matter of fact, many more examples. This phenomena is not unexpected since $PG(2,\mathbb{K})$ with $\mathbb{K}$ of characteristic $p>0$ contains an affine subplane $AG(2,\mathbb{F}_p)$ of order p from which k-nets for $3 \leq k \leq p+1$ arise taking k parallel line classes as components. Similarly, if $PG(2,\mathbb{K})$ also contains an affine subplane $AG(2,\mathbb{F}_{p^h})$, in particular if $\mathbb{K}=\mathbb{F}_q$ with $q=p^r$ and $h|r$, then k-nets of order p^h for $3 \leq k \leq p^h+1$ exist in $PG(2,\mathbb{K})$. Actually, more families of k-nets in $PG(2,\mathbb{F}_q)$ when $q=p^r$ with $r \geq 3$ exist; see Example 5.3. On the other hand, no 5-net of order n with $p>n$ is known to exist. This suggests that for sufficiently large p compared with n, Stipins' theorem remains valid in $PG(2,\mathbb{K})$. Our Theorem 5.2 proves it for $p>3^{\varphi(n^2-n)}$ where φ is the classical Euler φ function, and in particular for $p>3^{n^2/2}$. Our approach also works in zero characteristic and provides a new proof for Stipins' result.

A key idea in our proof is to consider the cross-ratio of four concurrent lines from different components of a 4-net. Proposition 3.1 states that the cross-ratio remains constant when the four lines vary without changing component. In other words, every 4-net in $PG(2,\mathbb{K})$ has constant cross-ratio. By Theorem 4.2 in zero charactersitic, and by Theorem 4.3 in characteristic p with $p>3^{\varphi(n^2-n)}$, the constant cross-ratio is restricted to two values only, namely to the roots of the polynomial X^2-X+1. From this, the non-existence of k-nets for $k \geq 5$ easily follows both in zero characteristic and in characteristic p with $p>3^{\varphi(n^2-n)}$. It should be noted that without a suitable hypothesis on n with respect to p, the constant cross-ratio of a 4-net may assume many different values, even for finite fields, see Example 5.3.

In $PG(2,\mathbb{K})$, k-nets naturally arise from pencils of curves, the components of the k-net being the completely reducible curves in the pencil. This has given a motivation for the study of k-nets in Algebraic geometry; see [2], and [12]. We discuss this relationship in Section 2 and state an equation that will be useful in Section 3.

2 k-nets and completely irreducible curves in a pencil of curves

Let $\lambda_1, \lambda_2, \lambda_3$ be three components of a k-net of order n embedded in $PG(2, \mathbb{K})$. Let $r_i = 0$, $w_i = 0$, $t_i = 0$ $(i = 1, \ldots, n)$ be the equations of the lines in $\lambda_1, \lambda_2, \lambda_3$, respectively. The completely reducible polynomials $R = r_1 \cdots r_n$, $W = w_1 \cdots w_n$ and $T = t_1 \cdots t_n$ define three plane curves of degree n, say $\mathcal{R}$, $\mathcal{W}$ and $\mathcal{T}$. Consider the pencil Λ generated by $\mathcal{R}$ and $\mathcal{W}$. Since $\lambda_1, \lambda_2, \lambda_3$ are the components of a 3-net of order n, there exist $\alpha, \beta \in \mathbb{K}^*$ such that $\mathcal{T}$ and the curve $\mathcal{H}$ of Λ with equation $\alpha R + \beta W = 0$ have $n^2 + 1$ common points but no common components. From Bézout's theorem, $\mathcal{T} = \mathcal{H}$. Therefore,

$$\alpha r_1 \cdots r_n + \beta w_1 \cdots w_n + \gamma t_1 \cdots t_n = 0 \tag{1}$$

holds for a homogeneous triple (α, β, γ) with coordinates $\mathbb{K}^*$. Changing the projective coordinate system in $PG(2, \mathbb{K})$ the equations of the lines in the components of the 3-net change but the homogeneous triple (α, β, γ) remains invariant.

Conversely, assume that an irreducible pencil Λ of plane curves of degree n contains k curves each splitting into n distinct lines, that is, k completely reducible curves. Let λ_i with $1 \leq i \leq k$ be the set of the n lines which are the factors of a completely reducible curve. Then $\lambda_1, \lambda_2, \ldots, \lambda_k$ are the components of a k-net embedded in $PG(2, \mathbb{K})$.

3 The invariance of the cross-ratio of a 4-net

Consider a 4-net of order n embedded in $PG(2, \mathbb{K})$ and label their components with λ_i for $i = 1, 2, 3, 4$. We say that the 4-net $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ has *constant cross-ratio* if for every point P of λ the cross-ratio $(\ell_1, \ell_2, \ell_3, \ell_4)$ of the four lines $\ell_i \in \lambda_i$ through P is constant.

Proposition 3.1. *Every 4-net in $PG(2, \mathbb{K})$ has constant cross-ratio.*

Proof. In a projective reference system, let $r_i = 0$, $w_i = 0$, $t_i = 0$, $s_i = 0$ with $1 \leq i \leq n$ be the lines of a 4-net $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ respectively. Then there exist $\alpha, \beta, \gamma \in \mathbb{K}^*$ such that (1) holds and $\alpha', \beta', \gamma' \in \mathbb{K}$ such that

$$\alpha' r_1 r_2 \cdots r_n + \beta' w_1 w_2 \cdots w_n + \gamma' s_1 s_2 \cdots s_n = 0. \tag{2}$$

As observed in Section 2, the coefficients $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$ remain invariant when the reference system is changed. Take a point P of λ and relabel the lines of λ such that $r_1 = 0$, $w_1 = 0$, $t_1 = 0$ and $s_1 = 0$ are the four lines of λ passing through P. We temporarily introduce the notation (x_1, x_2, x_3) for the homogeneous coordinates of a point, and we arrange the reference system in such a way that P coincides with the point $(0, 0, 1)$, the line $x_3 = 0$ contains no point from λ_1 or λ_2 while $r_1 = x_1$ and $w_1 = x_2$. Also, non-homogeneous coordinates $x = x_1/x_3$ and $y = x_2/x_3$ can be used so that $r_1 = x$ and $w_1 = y$. Note that we have arranged the coordinates so that r_i, w_i, t_i, s_i have a zero constant term if and only if $i = 1$. Let

$$\rho = \prod_{i=2}^{n} r_i(0,0), \quad \omega = \prod_{i=2}^{n} w_i(0,0), \quad \tau = \prod_{i=2}^{n} t_i(0,0), \quad \sigma = \prod_{i=2}^{n} s_i(0,0).$$

Observe that

$$0 = \alpha r_1 \cdots r_n + \beta w_1 \cdots w_n + \gamma t_1 \cdots t_n = \alpha \rho x + \beta \omega y + \gamma \tau t_1 + [\cdots],$$

where $[\ldots]$ stands for the sum of terms of degree at least 2. From (1),

$$\frac{\alpha \rho}{\gamma \tau} x + \frac{\beta \omega}{\gamma \tau} y + t_1 = 0.$$

Similarly,

$$\frac{\alpha' \rho}{\gamma' \sigma} x + \frac{\beta' \omega}{\gamma' \sigma} y + s_1 = 0.$$

Therefore, the cross-ratio of the lines of λ passing through P is equal to

$$\kappa = \frac{\alpha \beta'}{\alpha' \beta} \tag{3}$$

and hence it is independent of the choice of the point P. ■

As an illustration of Proposition 3.1 we compute the constant cross-ratio of the known 4-net embedded in the complex plane.

Example 3.2. Let $n = 3$, and take a primitive third root of unity ξ. In homogeneous coordinates (x, y, z) of $PG(2, \mathbb{K})$, let

$$\begin{array}{lll} r_1 := x, & r_2 := y, & r_3 := z, \\ w_1 := x + y + z, & w_2 := x + \xi y + \xi^2 z, & w_3 := x + \xi^2 y + \xi z, \\ t_1 := \xi x + y + z, & t_2 := x + \xi y + z, & t_3 := x + y + \xi z, \\ s_1 := \xi^2 x + y + z, & s_2 := x + \xi^2 y + z, & s_3 := x + y + \xi^2 z. \end{array}$$

Then these lines form a 4-net λ order 3. Moreover,

$$t_1 t_2 t_3 = 3(2\xi + 1) r_1 r_2 r_3 + \xi w_1 w_2 w_3,$$
$$s_1 s_2 s_3 = -3(2\xi + 1) r_1 r_2 r_3 + \xi^2 w_1 w_2 w_3.$$

Hence, the constant cross-ratio of λ is $\kappa = -1/\xi$.

4 Some constraints on the constant cross-ratio of a 4-net

It is well known that the cross-ratio of four distinct concurrent lines can take six possible different values depending on the order in which the lines are given. If κ is one of them then $\kappa \neq 0, 1$ and these six cross-ratios are

$$\kappa, \quad \frac{1}{\kappa}, \quad 1 - \kappa, \quad \frac{1}{1 - \kappa}, \quad \frac{\kappa}{\kappa - 1}, \quad 1 - \frac{1}{\kappa}.$$

It may happen, however, that some of these values coincide, and this is the case if and only if either $\kappa \in \{-1, 1/2, 2\}$, or

$$\kappa^2 - \kappa + 1 = 0. \tag{4}$$

Proposition 3.1 says that the cross-ratio of four concurrent lines of a 4-net takes the above six values for a given $\kappa \neq 0, 1$, and each of these values can be considered as the *constant cross-ratio* of the 4-net. Now, the problem consists in computing κ. We are able to do it in zero characteristic showing that κ satisfies Equation (4). In positive characteristic there are more possibilities. This will be discussed after proving the following result.

Proposition 4.1. *Let λ be a 4-net of order n embedded in $PG(2, \mathbb{K})$. Then the cross-ratio κ of λ is an N–th root of unity of $\mathbb{K}$ such that $N = n(n - 1)$ and*

$$(\kappa - 1)^N = 1. \tag{5}$$

Proof. We prove first that $\kappa^N = 1$. Let P_{ij} be the common point of the lines r_i and w_j with $1 \le i, j \le n$. Then the unique line from λ_3 through P_{ij} has equation $\sigma_{ij} r_i + \tau_{ij} w_j$ with $\sigma_{ij}, \tau_{ij} \in \mathbb{K}^*$. Moreover, for any $k = 1, \dots, n$ there is a unique index j such that $t_k = \sigma_{ij} r_i + \tau_{ij} w_j$. For every $i = 1, \dots, n$,

$$\alpha r_1 \cdots r_n + \beta w_1 \cdots w_n + \gamma[(\sigma_{i1} r_i + \tau_{i1} w_1) \cdots (\sigma_{in} r_i + \tau_{in} w_n)] = 0. \tag{6}$$

Take a point Q on the line $r_i = 0$ such that $w_j(Q) \neq 0$ for every $1 \le j \le n$. Then

$$w_1(Q) \cdots w_n(Q)(\beta + \gamma \prod_{j=1}^{n} \tau_{ij}) = 0$$

yields

$$-\frac{\beta}{\gamma} = \prod_{j=1}^{n} \tau_{ij} \tag{7}$$

for any fixed index i. The above argument applies to any line w_j and gives

$$-\frac{\alpha}{\gamma} = \prod_{i=1}^{n} \sigma_{ij} \tag{8}$$

for any fixed index j. Therefore,

$$\left(\frac{\beta}{\alpha}\right)^n = \prod_{i=1}^{n} \prod_{j=1}^{n} \frac{\tau_{ij}}{\sigma_{ij}}. \tag{9}$$

A similar argument can be carried out for λ_4. The unique line from λ_4 through P_{ij} has equation $\delta_{ij} r_i + \omega_{ij} w_j$ with $\delta_{ij}, \omega_{ij} \in \mathbb{K}^*$. Then

$$\left(\frac{\beta'}{\alpha'}\right)^n = \prod_{i=1}^{n} \prod_{j=1}^{n} \frac{\omega_{ij}}{\delta_{ij}}. \tag{10}$$

From Lemma 3.1,

$$\frac{\tau_{ij}}{\sigma_{ij}} \cdot \frac{\delta_{ij}}{\omega_{ij}} = \kappa$$

for every $1 \le i, j \le n$. Then Equations (9) and (10) yield $\kappa^n = \kappa^{n^2}$ whence

$$\kappa^N = 1. \tag{11}$$

From the discussion at the beginning of this section, Equation (11) holds true when κ is replaced with any of the other five cross-ratio values. Therefore, (5) also holds. ■

In the complex plane, the cross-ratio equation has only two solutions, namely the roots of (4). In fact, let $\kappa = x + yi$ with $x, y \in \mathbb{R}$. Then with respect to the complex norm, (11) and (5) imply $|x + iy| = x^2 + y^2 = 1$ and $|x - 1 + iy| = (x-1)^2 + y^2 = 1$. It hence follows that $\kappa = \frac{1}{2}(1 \pm \sqrt{3}i)$, or equivalently (4).

To extend this result to any field of characteristic zero, and discuss the positive characteristic case, look at

$$f(X) = \frac{X^N - 1}{X - 1} \text{ and } g(X) = \frac{(X-1)^N - 1}{X}$$

as polynomials in $\mathbb{Z}[X]$. From the preceding discussion on the complex case, their maximum common divisor is either $X^2 - X + 1$, or 1 according as 6 divides N or does not. In the former case, divide both by $X^2 - X + 1$ and then replace $f(X)$ and $g(X)$ by them accordingly. Now, $f(X)$ and $g(X)$ are coprime, and hence their resultant is a non-zero integer R. Using a basic formula on resultants, see [4, Lemma 2.3], R may be computed in terms of a primitive N-th root of unity ξ, namely

$$R = \prod_{1 \le i,j \le N-1} (1 + \xi^i - \xi^j), \text{ when } 6 \nmid N,$$

and

$$R = \prod_{\substack{1 \le i,j \le N-1 \\ i,j \ne N/6,\, 5N/6}} (1 + \xi^i - \xi^j), \text{ when } 6 \mid N,$$

hold in the N-th cyclotomic field $\mathbb{Q}(\xi)$. Therefore, $R \neq 0$ provided that $\mathbb{K}$ has zero characteristic.

Theorem 4.2. *Let $\mathbb{K}$ be a field of characteristic 0. If a 4-net λ is embedded in $PG(2, \mathbb{K})$ then -3 is a square in $\mathbb{K}$ and the constant cross-ratio κ of λ satisfies (4).*

To investigate the positive characteristic case, we will use the well known result that $\mathbb{Q}(\xi)$ is a cyclic Galois extension of $\mathbb{Q}$ of degree $\varphi(N)$ where φ is the classical Euler function. Let α be a generator of the Galois group. Then $\alpha(\xi) = \xi^m$ for a positive integer m prime to N. Therefore, α permutes the factors in the right hand side. Given such a factor $1 + \xi^i - \xi^j$, its cyclotomic norm

$$\| 1 + \xi^i - \xi^j \| = (1 + \xi^i - \xi^j) \cdot (1 + \xi^i - \xi^j)^{\alpha} \cdot \ldots \cdot (1 + \xi^i - \xi^j)^{\alpha^{N-1}}$$

is in $\mathbb{Q}$. Actually, it is an integer since the factors are algebraic integers. Hence, the prime divisors of R come from the prime divisors of the norms $\| 1+\xi^i-\xi^j \|$. Therefore, to find an upper bound on the largest prime divisor of R it is enough to find an upper bound on these norms. Obviously,

$$| \| 1+\xi^i-\xi^j \| | \leq |1+\xi^i-\xi^j|\cdot|1+\xi^{im}-\xi^{jm}|\cdot\ldots\cdot|1+\xi^{i(\varphi(N)-1))}-\xi^{j(\varphi(N)-1))}|.$$

Since $|1-\xi^i+\xi^j| \leq 3$, this shows that $| \| 1+\xi^i-\xi^j \| | \leq 3^{\varphi(N)}$. Hence the largest prime divisor of R is at most $3^{\varphi(N)}$. Therefore, the following result is proven.

Theorem 4.3. *Let $\mathbb{K}$ be a field of characteristic $p > 0$. If $p > 3^{\varphi(n^2-n)}$ then Theorem 4.2 holds.*

For planes over finite fields, Equations (11) and (5) may provide further non-existence results on embedded 4-nets.

Theorem 4.4. *Let $\mathbb{K} = \mathbb{F}_q$ be a finite field of order $q = p^h$ with p prime. If $p \neq 3$, then there exists no 4-net of order n embedded in $PG(2, \mathbb{F}_q)$ for*

$$\gcd(n(n-1), q-1) \leq 2.$$

Proof. From Equation (11), either $\kappa = 1$ and $p = 2$ or $\kappa^2 = 1$ and $p > 2$. On the other hand, $\kappa \neq 1$. Hence $\kappa = -1$ and $p > 2$. Now, Equation (5) yields $p = 3$, a contradiction. ■

The following example shows that the hypothesis $p \neq 3$ in Theorem 4.4 is essential.

Example 4.5. Let $q = 3^r$, and regard $PG(2, \mathbb{F}_q)$ as the projective closure of the affine plane $AG(2, \mathbb{F}_q)$. The four line sets $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ form a 4-net of order q embedded in $AG(2, \mathbb{F}_q)$ where λ_1 and λ_2 consist of all horizontal and vertical lines respectively, while λ_3 and λ_4 consist of all lines with slope 1 or -1, respectively. The constant cross-ratio of this 4-net equals -1.

5 Nets with more than four components

We prove the non-existence of 5-nets embedded in $PG(2, \mathbb{K})$ over a field $\mathbb{K}$ of characteristic 0. This result was previously proved by Stipins [10]; see also

[13]. Those authors used results and techniques from Algebraic geometry. Here, we present a simple combinatorial proof depending on Theorem 4.2. Our proof also works in positive characteristic p whenever p is big enough compared to the order n of 4-net; for example, when $p > 3^{\varphi(n^2-n)}$ so that Theorem 4.3 holds. However, the non-existence result fails in general. This will be illustrated by means of some examples.

We begin with a technical lemma.

Lemma 5.1. *Let A, B, C, D, D' be collinear points in $PG(2, \mathbb{K})$ with cross-ratios $\kappa = (ABCD)$ and $1 - \kappa = (ABCD')$. If (4) holds then $(ABDD') = -\kappa$.*

Proof. Without loss of generality, $A = (1, 0, 0)$, $B = (0, 1, 0)$, $C = (1, 1, 0)$. Then $D = (\kappa, 1, 0)$, $D' = (1 - \kappa, 1, 0)$, and the result follows by a direct computation. ■

Theorem 5.2. *If the characteristic of the field $\mathbb{K}$ is either 0 or greater than $3^{\varphi(n^2-n)}$, then there exists no 5-net of order n embedded in $PG(2, \mathbb{K})$.*

Proof. Let $\Lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$ be a 5-net of order n embedded in $PG(2, \mathbb{K})$. Then $\Lambda_5 = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$, $\Lambda_4 = (\lambda_1, \lambda_2, \lambda_3, \lambda_5)$, and $\Lambda_{45} = (\lambda_1, \lambda_2, \lambda_4, \lambda_5)$ are three different 4-nets and so we can compare their cross-ratios, say

$$\kappa_5 = (l_1, l_2, l_3, l_4),\ \kappa_4 = (l_1, l_2, l_3, l_5),\ \kappa_{45} = (l_1, l_2, l_4, l_5),$$

for five lines from different components and concurrent at a point of Λ. From Proposition 4.1 each of them is a root of the polynomial $X^2 - X + 1$. Since Λ_5 and Λ_4 only differ in the last component, $\kappa_5 \neq \kappa_4$. Therefore, $\kappa_4 = 1 - \kappa_5$. From Lemma 5.1, $\kappa_{45} = -\kappa_5$. This shows that κ_{45} is not a root of $X^2 - X + 1$ contradicting Proposition 4.1. ■

Example 4.5 can be generalized for finite fields $\mathbb{F}_q$ with $q = p^r$ showing that k-nets arise from affine subplanes of $PG(2, \mathbb{K})$. Such k-nets have order p^h with $h|r$. Here, we give further k-nets of p-power order. The construction relies on an idea of G. Lunardon [7]. For the sake of simplicity, we describe the construction in terms of a dual k-net, that is, the components are sets of points such that a line connecting two points of different components hits any third component in precisely one point.

Example 5.3. Let $\mathbb{K} = \mathbb{F}_q$ such that $q = r^s$ with $s \geq 3$. Take elements $u, v \in \mathbb{F}_q$ such that $1, u, v$ are linearly independent over the subfield $\mathbb{F}_r$. Take a basis $\mathbf{b}_1, \mathbf{b}_2$ of $\mathbb{F}_q^2$ and put $\mathbf{b}_0 = u\mathbf{b}_1 + v\mathbf{b}_2$. For any $\alpha \in \mathbb{F}_r$, we define the points sets

$$A_\alpha = \{\alpha\mathbf{b}_0 + \lambda\mathbf{b}_1 + \mu\mathbf{b}_2 \mid \lambda, \mu \in \mathbb{F}_r\}$$

in $AG(2, q)$. Then the A_α's ($\alpha \in \mathbb{F}_r$) are components of a dual r-net of order r^2. In order to see this, take the points

$$P_i = \alpha_i\mathbf{b}_0 + \lambda_i\mathbf{b}_1 + \mu_i\mathbf{b}_2, \qquad i = 1, 2, 3.$$

P_1, P_2, P_3 are collinear in $AG(2, q)$ if and only if the vectors

$$(\alpha_1-\alpha_2)\mathbf{b}_0+(\lambda_1-\lambda_2)\mathbf{b}_1+(\mu_1-\mu_2)\mathbf{b}_2 \text{ and } (\alpha_1-\alpha_3)\mathbf{b}_0+(\lambda_1-\lambda_3)\mathbf{b}_1+(\mu_1-\mu_3)\mathbf{b}_2 \tag{12}$$

are linearly dependent over $\mathbb{F}_q$. By the definition of $\mathbf{b}_0$ and the independence of $\mathbf{b}_1, \mathbf{b}_2$, (12) is equivalent with

$$\begin{aligned} &((\alpha_1 - \alpha_2)u + \lambda_1 - \lambda_2)((\alpha_1 - \alpha_3)v + \mu_1 - \mu_3) - \\ &\qquad ((\alpha_1 - \alpha_3)u + \lambda_1 - \lambda_3)((\alpha_1 - \alpha_2)v + \mu_1 - \mu_2) = 0. \end{aligned} \tag{13}$$

Sorting by u and v, we obtain

$$\begin{aligned} 0 \;=\; & u[(\alpha_1 - \alpha_2)(\mu_1 - \mu_3) - (\alpha_1 - \alpha_3)(\mu_1 - \mu_2)] \\ & +v[(\alpha_1 - \alpha_3)(\lambda_1 - \lambda_2) - (\alpha_1 - \alpha_2)(\lambda_1 - \lambda_3)] \\ & +(\lambda_1 - \lambda_2)(\mu_1 - \mu_3) - (\mu_1 - \mu_2)(\lambda_1 - \lambda_3). \end{aligned}$$

The independence of $1, u, v$ over $\mathbb{F}_r$ implies the system of equations

$$0 = (\alpha_1 - \alpha_2)(\mu_1 - \mu_3) - (\alpha_1 - \alpha_3)(\mu_1 - \mu_2), \tag{14}$$
$$0 = (\alpha_1 - \alpha_3)(\lambda_1 - \lambda_2) - (\alpha_1 - \alpha_2)(\lambda_1 - \lambda_3), \tag{15}$$
$$0 = (\lambda_1 - \lambda_2)(\mu_1 - \mu_3) - (\mu_1 - \mu_2)(\lambda_1 - \lambda_3). \tag{16}$$

With given points P_1, P_2, $\alpha_1 \neq \alpha_2$, (14) and (15) has the unique solution

$$\lambda_3 = \frac{\lambda_1(\alpha_3 - \alpha_2) + \lambda_2(\alpha_1 - \alpha_3)}{\alpha_1 - \alpha_2},$$
$$\mu_3 = \frac{\mu_1(\alpha_3 - \alpha_2) + \mu_2(\alpha_1 - \alpha_3)}{\alpha_1 - \alpha_2},$$

which is a solution for (16), as well. This means that the line P_1P_2 hits A_{α_3} in the unique point

$$P_3 = \frac{\alpha_3 - \alpha_2}{\alpha_1 - \alpha_2} P_1 + \frac{\alpha_1 - \alpha_3}{\alpha_1 - \alpha_2} P_2.$$

This formula further shows that the constant cross-ratio can take any value in $\mathbb{F}_r \setminus \{0, 1\}$.

We are able to describe the geometric structure of k-nets ($k \geq 4$) where one component is contained in a line pencil.

Theorem 5.4. *Let $\lambda = (\lambda_1, \ldots, \lambda_k)$, $k \geq 4$, be a k-net of order n embedded in $PG(2, \mathbb{K})$. Assume that the component λ_1 is contained in a line pencil. Then the following hold.*

1. *The order of λ is $n = p^e$ where $p > 0$ is the characteristic of $\mathbb{K}$.*
2. *For each component λ_i, $i > 1$, there is an elementary Abelian p-group of collineations acting regularly on the lines of λ_i.*
3. *The components $\lambda_2, \ldots, \lambda_k$ are projectively equivalent.*
4. *If any other component is contained in a line pencil then all components are, and the base points of the pencils are collinear.*

Proof. It suffices to prove the theorem for $k = 4$. We give the proof for the dual k-net by assuming that the component λ_1 is contained in the line ℓ. Let κ be the constant cross-ratio of $(\lambda_2, \lambda_3, \lambda_4, \lambda_1)$ and for any point $S \notin \ell$ denote by u_S the (S, ℓ)-perspectivity such that for any point P and its image $P' = u_S(P)$, the cross-ratio of S, P, P' and $PP' \cap \ell$ is κ. Then, for any $S \in \lambda_2$, u_S induces a bijection between λ_3 and λ_4. In particular, λ_3 and λ_4 are projectively equivalent. Let $S, T \in \lambda_2$, $S \neq T$, and assume that $u_S^{-1} u_T$ has a fixed point $R \notin \ell$, that is, $u_S(R) = u_T(R) = R'$. Then, $S, T \in RR'$ and with $R'' = RR' \cap \ell$ the cross-ratios (S, R, R', R''), (T, R, R', R'') are equal to κ. This implies $S = T$, a contradiction. This means that for all $S, T \in \lambda_2$, $S \neq T$, the collineation $u_S^{-1} u_T$ is an elation with axis ℓ, and $\{u_S^{-1} u_T \mid S, T \in \lambda_2\}$ generate an elementary Abelian p-group U of collineations, leaving λ_3 invariant. Moreover, U acts transitively, hence regularly on λ_3. This finishes the proof. ∎

Example 5.5. In Example 5.3, we constructed a dual r-net of order r^2 in $AG(2, r^s)$, $s \geq 3$. For $P_1 \in A_{\alpha_1}$, $P_2 \in A_{\alpha_2}$, the line P_1P_2 has direction vectors

$$(u + \lambda)\mathbf{b}_1 + (v + \mu)\mathbf{b}_2.$$

These are linearly independent for different choices of $\lambda, \mu \in \mathbb{F}_r$, hence they determine r^2 points at infinity. Let λ_0 be the set of corresponding infinite points. Then, $(\lambda_0, \lambda_1, \dots, \lambda_r)$ is a dual $(r+1)$-net with component λ_0 contained in a line.

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