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A new approach to splitting-off

Attila Bernáth and Tamás Király

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Attila Bernáth** and Tamás Király***

Abstract

A new approach to undirected splitting-off is presented in this paper. We study the behaviour of splitting-off algorithms when applied to the problem of covering a symmetric skew-supermodular set function by a graph. This hard problem is a natural generalization of many solved connectivity augmentation problems, such as local edge-connectivity augmentation of graphs, global arc-connectivity augmentation of mixed graphs with undirected edges, or the node-to-area connectivity augmentation problem in graphs. Using a simple lemma we characterize the situation when a splitting-off algorithm can be stuck. This characterization enables us to give very simple proofs for the classical results mentioned above. Finally we apply our observations in generalizations of the above problems: we consider three connectivity augmentation problems in hypergraphs with hyperedges of minimum total size without increasing the rank. The first is local edge-connectivity augmentation of undirected hypergraphs. The second is global arc-connectivity augmentation of mixed hypergraphs with undirected hyperedges. The third is a hypergraphic generalization of the node-to-area connectivity augmentation problem. We show that a greedy approach (almost) works for these cases.

1 Introduction

Let us be given a finite ground set V. A set function $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ is called *skew-supermodular* if at least one of the following two inequalities holds for every $X, Y \subseteq V$:

$$p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y), \tag{$\cap\cup$}$$

$$p(X) + p(Y) \leq p(X - Y) + p(Y - X). \tag{$-$}$$

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**Dept. of Operations Research, Eötvös University, Pázmány P. s. 1/C, Budapest, Hungary H-1117. The author is a member of the Egerváry Research Group (EGRES). E-mail: `bernath@cs.elte.hu`.
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***MTA-ELTE Egerváry Research Group, Department of Operations Research, Eötvös University, Pázmány Péter sétány 1/C, Budapest, Hungary, H-1117. E-mail: `tkiraly@cs.elte.hu`. Supported by grants OTKA K60802 and OMFB-01608/2006.

In this paper we consider the problem of covering a symmetric skew-supermodular set function $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ by a graph, or in some cases by a hypergraph of restricted type. We distinguish two versions of this problem. In the **degree specified version** we are also given a **degree specification** $m : V \to \mathbb{Z}_+$ and the question is whether a graph (or hypergraph) G covering p exists with $d_G(v) = m(v)$ for every $v \in V$. In the **minimum version** we simply want to find a graph covering p that has a minimum number of edges (for hypergraphs we want to minimize the sum of the sizes of the hyperedges). Possibly the latter problem seems more interesting and natural, however a solution for the first always gives a solution to the second by the skew-supermodularity of p, therefore we will mainly speak about the degree specified problem.

This problem is a natural generalization of many connectivity augmentation problems. Examples include the **local edge-connectivity augmentation problem in graphs** solved by Frank [7], the **global arc-connectivity augmentation problem in mixed graphs** solved by Bang-Jensen, Frank and Jackson [1], and the **node-to-area connectivity augmentation problem** solved by Ishii and Hagiwara [10]. The general problem of covering a symmetric skew-supermodular set function $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ with a minimum number of graph edges is known to be NP-complete (see e.g. [6], where the NP-completeness of the degree specified version is also shown implicitly). However, as seen above, many special cases have been shown to be polynomially solvable. The key approach in solving this kind of questions is the technique called "splitting-off": we first find a smallest number of graph edges covering p that are incident to a new node s, and then we try to get rid of s by splitting off pairs of edges incident to it (which means that we substitute this path of length 2 with the shortcut). We have found a new approach to this second step that simplifies proofs for known results and enables us to prove new results, too. The key lemma (Lemma 2) of our results states that if there is a set $X \subseteq V$ with $p(X) \geq 2$ then there always exists an **admissible splitting** (the splitting is admissible if the resulting graph still covers p). Consider a greedy algorithm that starts with a graph containing edges incident to s and in each step performs an (arbitrary) admissible splitting as long as it is possible. Then Lemma 2 enables us to prove some interesting properties of the situation when this algorithm gets stuck.

In Section 3 we prove Lemma 2 and give some of its consequences. In Subsection 3.1 we show some observations on the stuck situation. In Subsection 3.2 we demonstrate the strength of our approach by giving simple proofs for known results. First we consider a special case of the theorem of Benczúr and Frank [2], that already includes the classical splitting lemma of Lovász. Then we give a simple proof for the classical splitting theorem of Mader [11] (used by Frank in [7]) and the undirected splitting theorem in mixed graphs used by Bang-Jensen, Frank and Jackson in [1].

We analyze the stuck situation for special symmetric skew-supermodular functions in Section 4. We consider two special symmetric skew-supermodular functions in Subsections 4.1 and 4.2. The first case is when $p(X) = \max\{q(X), q(\overline{X})\}$ with a crossing supermodular function q (which includes the global arc-connectivity augmentation of a mixed graph or hypergraph). Here we obtain a very good characterization of the stuck situation. It turns out that if we contract tight sets then $q(X) = 1$ or $q(\overline{X}) = 1$

for any nonempty $X \subsetneq V$. Introducing the notation $\mathcal{F} = \{X \subseteq V : q(X) = 1\}$ and $\text{co}(\mathcal{F}) = \{X \subseteq V : \overline{X} \in \mathcal{F}\}$ this leads to the following question: how does a crossing family $\mathcal{F} \subseteq 2^V - \{\emptyset, V\}$ look like that satisfies $\mathcal{F} \cup \text{co}(\mathcal{F}) = 2^V - \{\emptyset, V\}$? We give a complete characterization of such families in Theorem 13. This theorem enables us to show a result on global arc-connectivity augmentation of mixed hypergraphs in Section 5.2, but we find it interesting for its own sake, too.

A set function $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ is called *crossing negamodular* if $(-)$ holds whenever X and Y are crossing. The second special symmetric skew-supermodular function is defined by $p(X) = \max\{q(X), q(\overline{X})\}$ with a crossing negamodular function q (which is a generalization of the function arising in the node-to-area connectivity augmentation problem). Covering such a function with a minimum number of graph edges already includes NP-complete problems, as was observed by Miwa and Ito [12]. So we make a similar assumption to that of Ishii and Hagiwara and assume that $q = R - d_H$ where R is a crossing negamodular function that does not take 1 as value and H is an arbitrary hypergraph. We analyze the situation where a greedy algorithm would get stuck for this function and as an application we show that this algorithm can only slightly fail for the **node-to-area connectivity augmentation problem in hypergraphs with hyperedges of minimum total size without increasing the rank**. Our results imply that the greedy approach will always produce a graph that has at most one more edge than the optimum for this problem in graphs.

In Section 5 we give applications. In Subsection 5.1 we prove the following result (Theorem 17): the **local edge-connectivity augmentation problem of hypergraphs with hyperedges of minimum total size** can be solved by adding only graph edges and one hyperedge whose size is at most the rank of the original hypergraph. There is only one exceptional case when this is impossible: when the minimum total size is odd and we augment a graph. This theorem can be regarded as a common generalization of the theorem of Frank [7] on local edge-connectivity augmentation of graphs and the theorem of Szigeti [14] on local edge-connectivity augmentation of hypergraphs. After proving this result we have been informed that Ben Cosh had already proved a similar theorem in his Ph.D. thesis.

Finally, in Subsection 5.2 we consider **global arc-connectivity augmentation of a mixed hypergraph without increasing the rank** by undirected hyperedges. We show that the greedy approach can fail for this problem, but only slightly. To be more precise, we prove that a mixed hypergraph of rank at most γ can always be augmented greedily to become (k, l)-arc-connected from a specified root node r if $k, l \geq 2$ by graph edges and a hyperedge of size at most $\gamma + 1$.

2 Preliminaries

Let us be given a finite ground set V. For subsets X, Y of V let $\overline{X}$ be $V - X$ (the ground set will be clear from the context). If X has only one element x then we will call it a **singleton** and we will not distinguish between X and its only element x. Sets $X, Y \subseteq V$ are **intersecting** if $X \cap Y, X - Y$ and $Y - X$ are all nonempty. If furthermore $X \cup Y \neq V$ then we say that they are **crossing**. For a family $\mathcal{F} \subseteq 2^V$

let $\mathrm{co}(\mathcal{F}) = \{X \subseteq V : \overline{X} \in \mathcal{F}\}$. We say that $\mathcal{F}$ is a **ring family** (**crossing family**) if $X, Y \in \mathcal{F}$ implies $X \cap Y, X \cup Y \in \mathcal{F}$ for an arbitrary (crossing, resp.) pair X, Y.

Let $q : 2^V \to \mathbb{Z} \cup \{-\infty\}$ be a set function: we will require all the set functions in this paper to satisfy $q(\emptyset) \leq 0$ and $q(V) \leq 0$. Define **the complement of** q as $\overline{q}(X) = q(\overline{X})$ and **the symmetrized of** q by $q^s(X) = \max\{q(X), q(\overline{X})\}$ for any $X \subseteq V$.

A set function $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ is called *skew-supermodular* if for any $X, Y \subseteq V$ at least one of $(\cap\cup)$ or $(-)$ holds. If $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ is a skew-supermodular function and $(\cap\cup)$ holds for some sets $X, Y \subseteq V$ then we say that X *and* Y *satisfy* $(\cap\cup)$, or shortly that $X (\cap\cup) Y$: if we don't explicitly say which function is meant then we always mean p. The same notation is used for $(-)$. Observe that the symmetrized of a skew-supermodular function is skew-supermodular.

A set function $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ is called *(crossing) supermodular* if it satisfies $(\cap\cup)$ for any (crossing) pair X and Y. A set function b is called *(crossing) submodular* if $-b$ is (crossing) supermodular. A set function $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ is called *(crossing) negamodular* if it satisfies $(-)$ for any (crossing) pair X and Y. Note that symmetrized of a crossing supermodular (or crossing negamodular) set function is skew-supermodular.

A set function is *symmetric* if $p(X) = p(V - X)$ for every $X \subseteq V$. Any function $m : V \to \mathbb{R}$ also induces a set function (that will also be denoted by m) with the definition $m(X) = \sum_{v \in X} m(v)$ for any $X \subseteq V$.

For a hypergraph $H = (V, \mathcal{E})$ and a set $X \subseteq V$ we say that a hyperedge $e \in \mathcal{E}$ enters X if neither $e \cap X$ nor $e \cap (V - X)$ is empty, and we define $d_H(X) = |\{e \in \mathcal{E} : e \text{ enters } X\}|$ (the **degree** of X in H). This is a symmetric submodular function. Since we will also allow loop edges if H is a graph, we need to count those in the degree specification: $d_H^+(v) = d_H(v) + 2|\{\text{loop edges incident to } v\}|$. For two set functions d, p we say that d **covers** p if $d(X) \geq p(X)$ for any $X \subseteq V$ ($d \geq p$ for short). We say that the hypergraph H **covers** p if d_H covers p. Observe that H covers p if and only if H covers p^s. The **total size of the hypergraph** is the sum of the cardinalities of the hyperedges: if our hypergraph is a graph then this is two times the number of the edges of this graph. The **rank of a hypergraph** is the size of the largest hyperedge in it. For $S, T \subseteq V$ let $\lambda_H(S, T)$ denote the maximum number of edge-disjoint paths starting in S and ending in T (we say that $\lambda_H(S, T) = \infty$ if $S \cap T \neq \emptyset$). By Menger's theorem

$$\lambda_H(S, T) = \min\{d_H(X) : T \subseteq X \subseteq V - S\}.$$

A **mixed graph** may have directed and undirected edges, too. For a mixed graph G and sets $X, Y \subseteq V$ let $d_G(X, Y)$ denote the number of (undirected or directed) edges of G with one endpoint in $X - Y$ and the other in $Y - X$.

For a set function $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ we introduce the polyhedron

$$C(p) = \{x \in \mathbb{R}^V : x(Z) \geq p(Z)\ \forall Z \subseteq V, x \geq 0\}.$$

It is known that for a skew-supermodular function p this is an (integer) contrapolymatroid (for details see [1]). In order to turn our proof into polynomial algorithms we need to assume that we can test membership in polynomial time in $C(p - d_G)$ for any graph G: though this is not necessarily true in general (as an example, let $p(X_0) = 0$ for a fixed X_0 and $-infty$ otherwise), but it will always hold in the applications given below.

In what follows let $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ be a symmetric, skew-supermodular function that satisfies $p(\emptyset) \le 0$ and $m : V \to \mathbb{Z}$ a nonnegative function satisfying $m(X) \ge p(X)$ for any $X \subseteq V$ (i.e. an integer element of $C(p)$). We would like to decide whether there is a graph (or possibly hypergraph) G covering p that satisfies $d^+_G(v) = m(v)$ for every $v \in V$. We note that, by the properties of a contrapolymatroid, a polynomial algorithm to the degree specified covering problem will give rise to a solution to the minimum version of the problem, and to more general versions such as the minimum node-cost problem. For more details we refer to [1]. Define the **greedy bound** by $gb(p) = \max\{\sum^t_{X \in \mathcal{X}} p(X) : \mathcal{X} \text{ is a subpartition of } V\} = \min\{1 \cdot x : x \in C(p)\}$: this is obviously a lower bound for the minimum total size of any hypergraph covering p. We say that $m \in C(p) \cap \mathbb{Z}^V$ is **minimal** if $m' \in C(p) \cap \mathbb{Z}^V$, $m' \le m$ implies that $m' = m$, in other words $m(V) = gb(p)$.

For a node $v \in V$ we say that v is **positive** if $m(v) > 0$, and **neutral** otherwise. The set of positive nodes will be denoted by V^+. Assume $u, v \in V^+$ are two positive nodes (possibly $u = v$, but then $m(u) \ge 2$ is assumed). The operation **splitting-off (at u and v)** is the following: let

$$m' = m - \chi_{\{u\}} - \chi_{\{v\}} \text{ and } p' = p - d_{(V,\{(uv)\})}. \tag{1}$$

One can observe that this is indeed the usual notion of splitting-off: if we introduce a graph $G = (V + s, E)$ with every edge of E incident to s and $d_G(s, v) = m(v)$ for any $v \in V$ then we are back at the well known splitting-off operation. However we found this way of presenting our results more convenient. If $m'(X) \ge p'(X)$ for any $X \subseteq V$ then we say that the splitting off is **admissible**. Clearly, splitting off at u and v is admissible if and only if there is no dangerous set X containing both u and v (a set X is **dangerous** if $m(X) - p(X) \le 1$ and it is called **tight** if $m(X) - p(X) = 0$). We will also say that such a dangerous set X **blocks the splitting at u and v**, or simply that X **blocks u and v**.

Let $M_p = \max\{p(X) : X \subseteq V\}$. A set X with $p(X) = M_p$ will be called **p-maximal**. Clearly, if $M_p \le 0$ then any splitting-off is admissible. Note that for two p-maximal sets X and Y either both of $X \cap Y$ and $X \cup Y$ or both of $X - Y$ and $Y - X$ are also p-maximal.

2.1 Contraction of tight sets

If $T \subseteq V$ then contracting T roughly means that from now on we consider it to be a singleton. Formally this means that we define $V/T = V - T + v_T$ where v_T was not in V. For any set function $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ we define $p/T : 2^{V/T} \to \mathbb{Z} \cup \{-\infty\}$ by $p/T(X) = p(X)$ if $v_T \notin X$ and $p/T(X) = p(X - v_T + T)$ if $v_T \in X$. For $m : V \to \mathbb{R}$ define $m/T : V/T \to \mathbb{R}$ with $m/T(v) = m(v)$ if $v \ne v_T$ and $m/T(v_T) = m(T)$: observe that regarding m to be a set function would give the same definition. In this contracted problem a splitting-off is admissible if it is admissible with respect to p/T.

Note that p/T will inherit the interesting properties of p investigated in this paper (e.g. symmetry, crossing supermodularity, skew-supermodularity etc.). Contraction of a hypergraph $H = (V, \mathcal{E})$ is understood in the obvious way as $H/T = (V/T, \{e \in \mathcal{E} : T \cap e = \emptyset\} \cup \{e - T + v_T : T \cap e \neq \emptyset\})$, so we avoid multiplicities of nodes in hyperedges in this paper. However, for the graph of the edges split so far we must count the multiplicity in the loop edges obtained this way in order to satisfy the degree specification: this will not cause any confusion. One can check that $d_{H/T} = d_H/T$. A useful observation is the following.

Lemma 1. *Let $u, v \in V$ with $m(u), m(v) > 0$. If we contract a tight set T then the splitting at u' and v' is admissible if and only if the splitting at u and v is admissible (where u' (v') is the contracted image of u (v, respectively)).*

Proof. By the definition of p/T if the splitting-off at u and v was admissible then it clearly stays admissible. Let us prove the other direction. Assume that u', v' becomes admissible while u, v was not admissible, i.e. there was a set $X \subseteq V$ with $p(X) \geq m(X) - 1$ with $u, v \in X$. Clearly, neither $T \subseteq X$ nor $X \cap T = \emptyset$ can hold. If $(\cap\cup)$ holds for X and T then $X \cup T$ is also dangerous, a contradiction. So $(-)$ must hold for them, meaning $X - T$ is also dangerous and $u, v \in X - T$, a contradiction again. $\square$

This lemma allows us to simplify some of the proofs by assuming that every tight set is a singleton.

3 The key lemma and its consequences

The starting point of our results is the following lemma. This lemma was also found by Nutov who sketched a proof in [13]. For the special case when p is obtained from local edge-connectivity augmentation requirements in a hypergraph, this lemma was implicitly also shown by Ben Cosh in [5]. However the proof presented here is simpler than the previous ones and its constructiveness might have further applications, too.

Lemma 2. *Let $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ be a symmetric, skew-supermodular function and $m \in C(p) \cap \mathbb{Z}^V$. If $M_p = \max\{p(X) : X \subseteq V\} > 1$ then there is an admissible splitting-off.*

Proof. Let Y be a minimal set satisfying $p(Y) = M_p$. By symmetry, $p(V - Y) = M_p$, too, so we can choose a minimal set $Z \subseteq V - Y$ satisfying $p(Z) = M_p$. Since $M_p \geq 1$ we can choose $y \in Y, z \in Z$ with $m(y), m(z) > 0$. We claim that the splitting at y and z is admissible. Assume that it is not and consider a dangerous set X containing y and z. Since $m(X - Y) \leq m(X) - m(y) \leq m(X) - 1$ and $p(Y - X) < M_p$ by the minimality of Y, X and Y cannot satisfy $(-)$, since that would mean $m(X) - 1 + M_p \leq p(X) + p(Y) \leq p(X - Y) + p(Y - X) < m(X - Y) + M_p \leq m(X) - 1 + M_p$, a contradiction. So X and Y must satisfy $(\cap\cup)$, which implies (using $m(X \cap Y) = m(X) - m(X - Y) \leq m(X) - m(z) \leq m(X) - 1$) that $p(X \cup Y) = M_p$ and $m(X - Y) = 1$, since $m(X) - 1 + M_p \leq p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y) \leq$

$m(X\cap Y)+M_p \le m(X)-1+M_p$. Now $X\cup Y$ and Z cannot satisfy $(-)$ since this would give $p(Z-(X\cup Y))=M_p$, contradicting the minimality of Z. Therefore $X\cup Y$ and Z satisfy $(\cap\cup)$ implying that $p(Z\cap(X\cup Y))=M_p$, which is only possible if $Z\subseteq X\cup Y$. But $2\le M_p=p(Z)\le m(Z)\le m(X-Y)=1$ gives a contradiction. $\square$

Let us mention an important consequence of this lemma. If there is no admissible splitting-off, then $p\le 1$ and every pair $u,v\in V^+$ is in a dangerous set X: this means that $p(X)=1$ and $m(X)=2$, hence $m\le 1$. The following corollary is immediate: it generalizes a theorem of Szigeti (see [14]) and in a special case it was also observed by Ben Cosh (see [5]).

Corollary 3. *If p is a symmetric, skew-supermodular function and $m\in C(p)\cap\mathbb{Z}^V$ then there is a hypergraph H covering p with degree function m that contains at most one hyperedge of size greater than 2.*

Consider the following greedy algorithm.

Algorithm GREEDYCOVER
begin

INPUT A **symmetric skew-supermodular** function $p:2^V\to\mathbb{Z}\cup\{-\infty\}$ (given with an oracle) and $m\in C(p)\cap\mathbb{Z}^V$.

OUTPUT A graph $G=(V,E)$ and a hyperedge e such that the hypergraph $G+e$ covers p and $d_{G+e}(v)=m(v)$ for every $v\in V$.

1.1. Initialize $G=(V,\emptyset)$.
1.2. While there exists an admissible pair u,v do
1.3. Let $m=m-\chi(u)-\chi(v)$ and $p=p-d_{(V,\{(u,v)\})}$ and $G=G+(uv)$.
1.4. Output G and e where $\chi_e=m$.

end

Clearly, if one can test membership in $C(p-d_G)$ in polynomial time for any graph G then this algorithm terminates in polynomial time. We say that the algorithm **got stuck** if the hyperedge in the output is of size greater than 0.

3.1 General observations on the stuck case

We can read out many things about the situation when the algorithm GREEDYCOVER gets stuck from Lemma 2. Assume that the procedure started with the function p_0 and $m_0\in C(p_0)\cap\mathbb{Z}^V$, performed some admissible splittings and got stuck at some point: let the graph of the edges split so far be G and let $p=p_0-d_G$ and $m(v)=m_0(v)-d^+_G(v)$ for any $v\in V$. If there is no admissible splitting, then every pair $u,v\in V^+$ is in a dangerous set X: since $p\le 1$ this means that $p(X)=1$ and $m(X)=2$, hence $m\le 1$. The interesting case for us will be the case when the splitting procedure gets stuck with $m(V)\ge 4$. In the rest of this section we assume that we are at this stuck situation with $m(V)\ge 4$.

Observe that the algorithm GREEDYCOVER can be modified in an obvious way if $G+e$ is not a feasible output (for example e is too big): we can replace e with any connected hypergraph on V^+. For example if we are only allowed to use graph edges then we can notice that with $m(V)-1$ graph edges we can finish the procedure: any spanning tree on V^+ will cover p. However, we could possibly cover p with less edges, as the example $p(X)=1$ if $|X|\in\{1,2,n-2,n-1\}$ (and $p(X)=0$ otherwise) shows. Though there is a lower bound: one needs at least $\lceil 2(m(V)-1)/3\rceil$ edges to finish the procedure. The special case of the following lemma when m is minimal was also proved in [13].

Lemma 4. *Let $p: 2^V \to \mathbb{Z}\cup\{-\infty\}$ be a symmetric, skew-supermodular function and $m\in C(p)\cap\mathbb{Z}^V$. Assume that there is no admissible splitting-off. Then any (inclusionwise) minimal graph G covering p has at least $\lceil 2(m(V)-1)/3\rceil$ edges.*

Proof. We claim that we can assume that the edges of G connect positive nodes: consider any edge $e=(xy)\in E(G)$, where at least one of x and y is not positive. Since $G-e$ does not cover p, the function $p'=p-d_{G-e}$ has positive values, however obviously $p'\le p\le 1$. We claim that the family $\mathcal{F}=\{X\subseteq V: x\in X, y\notin X, p'(X)=1\}$ is closed under intersection and union. Let $X,Y\in\mathcal{F}$: then p' cannot satisfy $(-)$ for X and Y, since then G would not cover p. So $(\cap\cup)$ holds for X and Y and function p', which implies that $X\cap Y, X\cup Y\in\mathcal{F}$, as claimed. Since $m\in C(p)\cap\mathbb{Z}^V$, there must be a positive node $x_0\in\cap\mathcal{F}$ and a positive node $y_0\in V-\cup\mathcal{F}$, so $G'=G-e+(x_0y_0)$ also covers p and iterating this we arrive at a graph that has only edges between positive nodes.

Every component of $G[V^+]$ must be of cardinality at least 3, except for at most one singleton component (however, if m is minimal, then there is no such singleton component). So if $\mathcal{C}$ denotes the set of these components then $|\mathcal{C}|\le (m(V)-1)/3+1$. Using this we have

$$|E(G)|\ge\sum_{C\in\mathcal{C}}(|V(C)|-1)\ge m(V)-(m(V)+2)/3=2(m(V)-1)/3. \qquad \square$$

Let us give a lemma that will be useful later. Assume $x_0,x_1,x_2\in V$ are three different positive nodes and X_0,X_1,X_2 are three dangerous sets blocking them with $x_i\in X_j\cap X_k$ for any $\{i,j,k\}=\{0,1,2\}$. (Since we assume that $m(V)\ge 4$ the three sets X_0,X_1,X_2 are pairwise crossing here.) We will say that X_0,X_1 and X_2 form a **blocking-triangle**. X_2 will be called **slim** if $X_0\cap X_1\cap X_2=\emptyset$ and $X_2-(X_0\cup X_1)=\emptyset$.

Lemma 5 (Slimming Lemma)**.** *Assume that X_0 and X_1 satisfy $(\cap\cup)$. Then $(X_2-(X_0\cap X_1))\cap(X_0\cup X_1)$ is also dangerous and blocks x_0,x_1.*

Proof. Since X_0 and X_1 satisfy $(\cap\cup)$, $p(X_0\cap X_1)=p(X_0\cup X_1)=1$. Now $X_0\cap X_1$ and X_2 cannot satisfy $(\cap\cup)$, since that would imply that $p(X_0\cap X_1\cap X_2)=1$, but $m(X_0\cap X_1\cap X_2)=0$. This implies that $p(X_2')=1$ where $X_2'=X_2-(X_0\cap X_1)$. Now X_2' and $X_0\cup X_1$ cannot satisfy $(-)$, since that would give $p(X_2'-(X_0\cup X_1))=1$ contradicting $m(X_2'-(X_0\cup X_1))=0$. So we obtain from $(\cap\cup)$ that $p(X_2'\cap(X_0\cup X_1))=1$ and clearly $x_0,x_1\in X_2'\cap(X_0\cup X_1)$. $\square$

We note that the family of sets blocking a fixed pair of nodes $u,v\in V^+$ is closed under union and intersection. Let us denote the unique minimal member of this family by X_{uv}. Observe, that for 4 different nodes $u,v,x,y\in V^+$ we have $X_{uv}\cap X_{xy}=\emptyset$: they cannot satisfy $(\cap\cup)$ since $m(X_{uv}\cap X_{xy})=0$, so they satisfy $(-)$, and then by minimality they must be disjoint.

3.2 Simple proofs

In this subsection we give simple proofs of classical results in order to demonstrate the simplicity of our approach. First we give a simple proof of a special case of a theorem of Benczúr and Frank. They proved in [2] that the problem of covering a symmetric, crossing supermodular set function by a minimum number of graph edges can be solved in polynomial time. In a special case this problem can be solved greedily. Many proofs below consider the situation when the Algorithm GREEDYCOVER gets stuck. In most of the cases we can assume that this is already the case in the beginning, since after some steps we are again at an instance of our starting problem: an example of this is Theorem 6. Note that a symmetric crossing supermodular function is also skew-supermodular (which is not necessarily the case without the symmetry). Furthermore, a symmetric crossing supermodular function satisfies both $(\cap\cup)$ and $(-)$ if X and Y are crossing.

Theorem 6. *Let $p' : 2^V \to \mathbb{Z}\cup\{-\infty\}$ be a symmetric, crossing supermodular function that does not take 1 as value, and $G = (V, E)$ be an arbitrary graph. Then the Algorithm GREEDYCOVER does not get stuck with input $p = p' - d_G$ and arbitrary $m \in C(p) \cap \mathbb{Z}^V$ if $m(V)$ is even.*

Proof. Assume that the algorithm GREEDYCOVER gets stuck (at start). Then $m(V) \geq 4$ must hold. Consider a blocking triangle X, Y, Z. By Lemma 2 and the observations above any pair of this three sets must satisfy $(\cap\cup)$ and $(-)$ for p with equality. Using the Slimming Lemma we can assume that X, Y and Z are all slim. However, $p(X \cap Y) = p'(X \cap Y) - d_G(X \cap Y) = 1$ and $p'(X \cap Y) \neq 1$ implies that there must be an edge in G leaving $X \cap Y$. But in presence of such an edge we are able to find two sets out of X, Y, Z that cannot satisfy $(-)$ or $(\cap\cup)$ with equality. □

Observe that Benczúr and Frank prove their theorem for *symmetric positively crossing supermodular* functions. A function $p : 2^V \to \mathbb{Z}_+$ is **positively crossing supermodular** if it satisfies $(\cap\cup)$ for any crossing pair $X, Y \subseteq V$ with $p(X), p(Y) > 0$. However our proof of Theorem 6 clearly works for this more general class, too. The above theorem includes the classical splitting theorem of Lovász that can be used for global edge-connectivity augmentation of graphs.

Lemma 7 (Lovász' lemma)**.** *Let $G = (V + s, E)$ be k-edge-connected in V, where $k \geq 2$. Assume $d_G(s)$ is even. Then there exists a splitting-off at s that preserves k-edge-connectivity in V.*

Proof. Let $G' = G[V]$ and $p : 2^V \to \mathbb{Z}$ defined by $p(X) = k - d_{G'}(X)$ for any $\emptyset \neq X \neq V$ and $p(\emptyset) = p(V) = 0$. Let $m(v) = d_G(s, v)$ for any $v \in V$. With these notations the lemma follows from Theorem 6. □

Next we give a simple proof of Mader's classical splitting lemma.

Lemma 8 (Mader's lemma)**.** *Let $G = (V + s, E)$ be such that there is no cut edge incident to s and $d_G(s) > 3$. Then there exists a splitting-off at s that preserves the local edge-connectivities in V.*

Proof. If there is no cut edge incident to s then $\lambda_G(u,v) \geq 2$ for any pair of s-neighbours u,v. Let us define $R(X) = \max\{\lambda_G(x,y) : x \in X, y \in V - X\}$ for any X with $\emptyset \neq X \neq V$ and $R(\emptyset) = R(V) = 0$ and $p(X) = R(X) - d_{G[V]}(X)$ for any $X \subseteq V$. Let $m(v) = d_G(s,v)$ for any $v \in V$. It is well known and easy to check that (R and) p is a symmetric and skew-supermodular function. By assumption, m covers p. Assume that there is no splitting-off and take a blocking triangle X,Y,Z consisting of **maximal** dangerous sets. Consider the following two cases.
Case I.: Assume that two of these three sets (wlog. X and Y) satisfy $(\cap\cup)$. Then, using the Slimming Lemma, substitute Z by $Z' = (Z - (X \cap Y)) \cap (X \cup Y)$. Let $R(Z') = \lambda_G(z,v)$ with $z \in Z'$ and $v \in V - Z'$ and assume wlog. that $z \in X \cap Z'$ implying $R(Z') \leq R(X \cap Z')$. Since there is no cut edge incident to s, $d_G(Y \cap Z') \geq R(Y \cap Z') \geq 2$. Then $d_G(Z') - 1 \leq R(Z') \leq R(X \cap Z') \leq d_G(X \cap Z') = d_G(Z') - d_G(Y \cap Z') + d_G(X \cap Z', Y \cap Z') \leq d_G(Z') - 2 + d_G(X \cap Z', Y \cap Z')$ implies that $d_G(X \cap Z', Y \cap Z') > 0$, but then X and Y cannot satisfy $(\cap\cup)$ with equality.
Case II.: Assume that X,Y and Z pairwise satisfy $(-)$. This implies that $p(X-Y) = 1$, consequently Z and $X-Y$ cannot satisfy $(-)$, since $m((X-Y)-Z) = 1$. Thus they satisfy $(\cap\cup)$ which implies by the maximality of Z that $X - (Y \cup Z) = \emptyset$. Similarly we can prove that $Y - (Z \cup X) = Z - (X \cup Y) = \emptyset$. Using that there is a neighbour of s not in $X \cup Y \cup Z$ we can deduce that $R(X \cup Y \cup Z) \geq 2$. However, since X, Y and Z pairwise satisfy $(-)$ with equality, there must not be an edge of $G[V]$ leaving $X \cup Y \cup Z$. But this would imply that $p(X \cup Y \cup Z) \geq 2$, contradicting Lemma 2. $\square$

Finally we will give a simple proof of a theorem of Bang-Jensen, Frank and Jackson [1] on undirected splitting-off in mixed graphs: the $k = l$ case is a special case of Theorem 3.2 of [1], so we also manage to extend slightly this special case.

Theorem 9 (Bang-Jensen, Frank, Jackson)**.** *Let $M = (V + s, E)$ be a mixed graph and assume that s is only incident with undirected edges. Let $r \in V$ and $k,l \geq 2$ integers and assume that $\lambda_M(r,v) \geq k$ and $\lambda_M(v,r) \geq l$ for any $v \in V$. Then there exists a splitting-off at s preserving this property, provided that $d_M(s) > 3$.*

Proof. We can assume that $M - s$ is a digraph (by substituting undirected edges by oppositely directed pairs of arcs): let us denote this digraph by $D = (V, A)$ and let $m(v) = d_M(s,v)$ for any $v \in V$. Let the function q be defined by $q(\emptyset) = q(V) = 0$, $q(X) = k - \varrho_D(X)$ for any nonempty $X \subseteq V - r$ and $q(X) = l - \varrho_D(X)$ for any $X \subsetneq V$ with $r \in X$. Then one can check that q is crossing supermodular and thus $p = q^s$ is skew-supermodular. Since M is (k,l)-arc-connected from r (apart from s), $m(X) \geq p(X)$ for any $X \subseteq V$. Assume that there is no splitting-off. Consider a blocking triangle X,Y,Z. We can assume without loss of generality that either $q(X) = q(Y) = 1$ or $\overline{q}(X) = \overline{q}(Y) = 1$ so X and Y must satisfy $(\cap\cup)$ with equality, implying that $d_D(X,Y) = 0$. By Lemma 5 we can assume that Z is slim. If $r \notin Z$ then either $\varrho_D(Z) = k - 1$ or $\delta_D(Z) = l - 1$: assume the former, the other case being analogous. But $\varrho_D(Z \cap X) \geq k - 1$ and $\varrho_D(Z \cap Y) \geq k - 1$ together with $k \geq 2$ implies that $d_D(X,Y) > 0$, a contradiction. If $r \in Z$ (wlog. $r \in Z \cap X$) then either $\varrho_D(Z) = l - 1$ or $\delta_D(Z) = k - 1$: assume the former, and observe that $\varrho_D(Z \cap X) \geq l - 1$ and $\varrho_D(Z \cap Y) \geq k - 1 > 0$ again implie that $d_D(X,Y) > 0$, thus yield a contradiction. $\square$

4 Stuck situation for special skew-supermodular functions

In this section we want to characterize the stuck situation if the symmetric skew-supermodular function p is of form q^s with some special function q. In this section we will assume that tight sets are singletons. Recall that for a pair $u, v \in V^+$ the unique minimal set blocking them is denoted by X_{uv}. Observe that for four nodes $x, y, u, v \in V^+$

$$X_{xy}(-)\, X_{yu} \text{ and } X_{yu}(-)\, X_{uv} \Rightarrow |X_{xy}| = |X_{yu}| = |X_{uv}| = 2. \tag{2}$$

For the subsequent two subsections let us introduce some notations. If p is the symmetrized of a function q then for any set X either $p(X) = q(X)$ or $p(X) = q(\overline{X})$ (possibly both). In the former case we say that X is of **q-type**, in the latter we say that X is of **$\overline{q}$-type** (so X can be of both types). We introduce two (undirected, simple) graphs on the set of positive nodes: the edge set of the **q-graph** (**$\overline{q}$-graph**) consists of the pairs u, v of positive nodes having $q(X_{uv}) = 1$ ($\overline{q}(X_{uv}) = 1$, respectively). Since there is no admissible splitting, the union of these two graphs is the complete graph (on the set of positive nodes), and an edge may belong to both graphs. We will call this 2-edge-coloured complete graph **the $q\overline{q}$-graph**.

4.1 Crossing supermodular functions

In this subsection we characterize the stuck situation if p is the symmetrized of a crossing supermodular function q. Recall that a set function $q : 2^V \to \mathbb{Z} \cup \{-\infty\}$ is called *crossing supermodular* if it satisfies $(\cap\cup)$ whenever X and Y are crossing. One can check that the complement of a crossing supermodular function is also crossing supermodular, and the symmetrized of a crossing supermodular function is skew-supermodular.

If two crossing sets X and Y are of the same type then they will satisfy $(\cap\cup)$. If furthermore $p(X) = p(Y) = 1$ then their intersection and union is also of the same type as X and Y (here we use that $p \leq 1$). On the other hand if X and Y are of different types then $p(X - Y) = p(Y - X) = 1$. Also note that from any three sets there are two of the same type.

If p is symmetric and crossing supermodular, then it is easy to check that every node is positive (one can find examples showing that this does not hold in general, if only the skew-supermodularity of p is assumed). However we will prove this in a more general case, namely when p is the symmetrized of a crossing supermodular function q. First it is useful to prove the following lemma.

Lemma 10. *If p is the symmetrized of a crossing supermodular function q then $|X_{uv}| = 2$ for any $u, v \in V^+$.*

Proof. Assume that there are nodes $x, z \in V^+$ such that $|X_{xz}| > 2$. By possibly complementing q we can assume that X_{xz} is of q-type. Let $y \in V - X_{xz}$ be another positive node. We claim that X_{xy} must be of q-type, too. If not, then $X_{xz} - X_{xy} = z$,

$X_{xy} - X_{xz} = y$, since they are tight. But then X_{yz} cannot be of q-type (since this would imply $X_{yz} \cap X_{xz} = z$ and $X_{xy} - X_{yz} = y$, a contradiction), neither of $\overline{q}$-type (for a similar reason). So we have proved that for any $y \in V^+ - \{x, z\}$ the set X_{xy} is of q-type. So the union of these sets $Y = \cup_{y \in V^+ - \{x,z\}} X_{xy}$ is also of q-type, and has $p(Y) = q(Y) = 1$. However this implies that $1 = p(V - Y) = m(z) = m(V - Y)$, so it is tight, which contradicts $|X_{xz}| > 2$ (note that $Y \cap X_{xz} = x$). □

The lemma implies that the edge set of the q-**graph** ($\overline{q}$-**graph**) consists of the pairs u, v of positive nodes having $q(\{u, v\}) = 1$ ($\overline{q}(\{u, v\}) = 1$, respectively). Observe that a non-singleton connected component $X \neq V$ of the q-graph is also of q-type and has $q(X) = 1$ (and similarly for the $\overline{q}$-graph). This immediately implies the result promised before.

Lemma 11. *If p is the symmetrized of a crossing supermodular function q then every node is positive.*

Proof. Suppose not, then the set of positive nodes $V^+ \neq V$ must be connected in at least one of the two graphs (since the union of two disconnected graphs cannot be the complete graph), so $p(V^+) = 1$. But then $p(V - V^+) = 1$ by the symmetry, contradicting $m(V - V^+) = 0$. □

What is more, this implies the following surprising observation.

Lemma 12. *If p is the symmetrized of a supermodular function q, then $p(X) = 1$ for any X with $\emptyset \neq X \neq V$ (i.e. $q(X) = 1$ or $q(V - X) = 1$ for every such set).*

Proof. By the preceding argument, any non-singleton $X \subsetneq V$ must be connected in at least one of the two graphs, so has $p(X) = 1$ (it is also easy to see for singletons, using $m(V) \geq 4$). □

Consequently we have a crossing family $\mathcal{F}$ containing all sets with q value 1, and the family of the complements of this family $\mathrm{co}(\mathcal{F})$ (these are the sets with $\overline{q}$ value 1), and the union of these two families is $2^V - \{\emptyset, V\}$. In the following theorem we will characterize such families (for sake of brevity we will also add $\emptyset$ and V in the family: these sets can always be added to or removed from a crossing family).

Let $x \in V$ and let $X_1, \ldots, X_t$ be $t \geq 1$ pairwise disjoint subsets of $V - x$ (possibly $t = 1$ and $X_1 = \emptyset$). We introduce the following family:

$$\mathcal{F}_{x, X_1, \ldots, X_t} = \{X \subseteq V : x \in X \text{ or } X \subseteq X_i \text{ for some } i \in 1, \ldots, t\}.$$

Theorem 13. *Let $\mathcal{F} \subseteq 2^V$ be a crossing family with $\emptyset, V \in \mathcal{F}$ that satisfies $\mathcal{F} \cup \mathrm{co}(\mathcal{F}) = 2^V$. Then either V has exactly four elements and $\mathcal{F} = 2^V \setminus \{\{y, z\}\}$ for some $y \neq z$, $y, z \in V$ or there exists a node $x \in V$ and $X_1, \ldots, X_t$ pairwise disjoint subsets of $V - x$ for some $t \geq 1$ such that either $\mathcal{F}$ or $\mathrm{co}(\mathcal{F})$ is equal to $\mathcal{F}_{x, X_1, \ldots X_t}$ or $\mathcal{F}_{x, X_1, \ldots X_t} \cup \{V - x\}$.*

Proof. We can clearly assume that V has at least 3 elements. We introduce 2 (simple undirected) graphs on V: for sake of simplicity we call them **blue** and **red**. The blue graph is $B = (V, \{(u,v) : \{u,v\} \in \mathcal{F}\})$, and the red is $R = (V, \{(u,v) : \{u,v\} \in \text{co}(\mathcal{F})\})$ (so some edges might belong to both graphs). It might seem that these graphs don't have every information on $\mathcal{F}$, but it turns out that they almost do. Again, we have that the union of these two graphs is the complete graph, and a non-singleton connected component $X \neq V$ of B is in $\mathcal{F}$ (so $V - X$ is in $\text{co}(\mathcal{F})$). This implies that if $B[V - \{u,v\}]$ is connected for nodes $u \neq v$, then $(u,v) \in R$, and vice versa. If $(u,v) \in B$ then we will say that this edge is **blue**, if $(u,v) \notin R$ then we will say that this edge is **pure blue**.

Claim 1. *There is a node $x \in V$ such that either B or R contains every edge (x,v) for any $v \in V - x$.*

Proof. Assume indirectly that every node $v \in V$ is entered by a pure red edge and by a pure blue edge, too. Consider an edge (u,v) that is pure blue: this means that $B[V - \{u,v\}]$ is disconnected, so there is a bipartition X, Y of $V - \{u,v\}$ such that every edge is pure red between X and Y. Assume wlog. that the pure red neighbour x of v is in X and consider two cases.
CASE I. $|X| \geq 2$. Since $R[V - \{v,x\}]$ must be disconnected, every edge of the form (u,y) must be pure blue for any $y \in V - \{u,v,x\}$. So the pure red edge entered by u must be the edge (u,x). Now consider any $x' \in X - x$: since $B[V - \{x,x'\}]$ is connected, this edge is red, but then x is not entered by a pure blue edge, a contradiction.
CASE II. $X = \{x\}$. Then there is a bipartition Y_1, Y_2 of $Y + u$ such that every edge between Y_1 and Y_2 is pure blue. Assume that $u \in Y_1$ and consider any $y \in Y_1 - u$: since $R[V - \{u,y\}]$ is connected, this edge is blue. Then the only possibility for a pure red edge incident to u is necessarily the edge (u,x) which again means that there is no pure blue edge leaving x, finishing the proof of the claim.

So consider the vertex x given by this claim and assume wlog. that (x,v) is blue for any $v \in V - x$. We distinguish again two cases.
CASE I. There are two intersecting sets $Y, Z \in \mathcal{F}$ such that $Y \cup Z = V - x$. We claim that they can be chosen such that their symmetric difference is of cardinality two. Indeed, for any $y \in Y - Z$ the set $V - \{x,y\}$ also belongs to $\mathcal{F}$ since Z and $Y - y = (Y - y + x) \cap Y$ both belong to $\mathcal{F}$, they are crossing and this is their union. So Z can be substituted by $Z' = V - \{x,y\}$ and similarly Y can be substituted by $Y' = V - \{x,z\}$ for some $z \in Z - Y$. Now if $|V| > 4$ then this implies that $\mathcal{F} = 2^V$, as one can check, and if $|V| = 4$ then this $\mathcal{F}$ can also be $2^V \setminus \{\{y,z\}\}$.
CASE II. There aren't two intersecting sets $X, Y \in \mathcal{F}$ such that $X \cup Y = V - x$. Let the maximal sets of $\mathcal{F}$ properly contained in $V - x$ be $X_1, X_2, \ldots, X_t$: these are pairwise disjoint and since $\emptyset \in \mathcal{F}$, we have $t \geq 1$. One can simply check that $\mathcal{F}$ is either $\mathcal{F}_{x,X_1,\ldots X_t}$ or $\mathcal{F}_{x,X_1,\ldots X_t} \cup \{V - x\}$, as claimed above. □

A simple corollary that is worth mentioning is the following.

Theorem 14. *Let $\mathcal{F} \subseteq 2^V$ be a ring family with $\emptyset, V \in \mathcal{F}$ that satisfies $\mathcal{F} \cup \mathrm{co}(\mathcal{F}) = 2^V$. Then there exists a node $x \in V$ and a (possibly empty) set $X_1 \subseteq V - x$ such that either $\mathcal{F}$ or $\mathrm{co}(\mathcal{F})$ is equal to $\mathcal{F}_{x,X_1}$.*

4.2 Crossing negamodular functions

Recall that a set function $q : 2^V \to \mathbb{Z} \cup \{-\infty\}$ is called *crossing negamodular* if it satisfies $(-)$ whenever X and Y are crossing. Note that the symmetrized of a crossing negamodular function is skew-supermodular, but the complement of a crossing negamodular function is not necessarily crossing negamodular. An important special case is a **monotone decreasing** function: by that we mean a function q that satisfies $q(\emptyset) \le 0$ but $q(X) \ge q(Y)$ for any $\emptyset \subsetneq X \subseteq Y \subseteq V$.

In this section we want to characterize the stuck situation if $p = q^s$ with a crossing negamodular function q. An important observation is the following: if $q : 2^V \to \mathbb{Z} \cup \{-\infty\}$ is crossing negamodular and $X, Y \subseteq V$ are crossing sets with $q(X) = \overline{q}(Y) = M_q = 1$ (i.e. X and Y are of different type), then $q(X \cap Y) = 1$ and $\overline{q}(X \cup Y) = 1$. Recall that for a pair $u, v \in V^+$ the unique minimal set blocking them is denoted by X_{uv}. If X_{xy}, X_{yu} and X_{uv} are of the same type for four different positive nodes x, y, u, v then they all must be of cardinality two by (2).

As an example consider the node-to-area connectivity augmentation problem (NA-augmentation problem for short) in graphs solved by Ishii and Hagiwara [10]. The problem is the following. Given a graph $G = (V, E)$, a collection of subsets $\mathcal{W}$ of V (called *areas*) and a requirement function $r : \mathcal{W} \to \mathbb{Z}_+$, find a minimum number of new edges F such that $\lambda_{G+F}(x, W) \ge r(W)$ for any $W \in \mathcal{W}$ and $x \in V$. This problem is in general NP-complete (even if G is the empty graph and $r(W) = 1$ for every $W \in \mathcal{W}$), so the authors of [10] assume that $r \ge 2$ and surprisingly the problem becomes tractable: they give a polynomial time algorithm that solves it. Let us show why this problem is a special case of the problem investigated in this section. Define

$$R_{N2A}(X) = \max\{r(W) : W \in \mathcal{W},\ W \cap X = \emptyset\} \text{ for any } \emptyset \ne X \subseteq V \text{ and } R_{N2A}(\emptyset) = 0. \tag{3}$$

This is a monotone decreasing function, so it is crossing negamodular, and it does not take 1 as value and an edge set F is a feasible solution to our problem if and only if d_F covers $R_{N2A} - d_G$. We mention that R^s_{N2A} is the function that was called a **symmetric semi-monotone function** in [9].

This example shows that the problem of covering a crossing negamodular function with a minimum number of graph edges is in general NP-complete (even for monotone decreasing functions). So, similarly to [10], assume that $R : 2^V \to \mathbb{Z} \cup \{-\infty\}$ is crossing negamodular, R does not take 1 as value, and let $q = R - d_H$ with a hypergraph $H = (V, \mathcal{E})$. The following lemma characterizes the stuck situation of the Algorithm GREEDYCOVER with the input $p = q^s$ and a minimal $m \in C(p) \cap \mathbb{Z}^V$. Note however that it is not known how to implement the algorithm to run in polynomial time, since it is not yet known how to maximize a crossing negamodular function in polynomial time. A hyperedge of H is called a *large hyperedge* if it contains at least two nodes of V^+.

Lemma 15. *If there is no admissible splitting-off and $m(V) \geq 5$ then there exists a large hyperedge. Furthermore, the number of positive nodes that are avoided by a large hyperedge is at most one.*

Proof. Assume that there is no large hyperedge. By the minimality of m, an arbitrary $x \in V^+$ is contained in a non-singleton hyperedge e. We claim that neither the q-graph nor the $\overline{q}$-graph can contain a path consisting of 3 edges. Assume indirectly that for some four nodes $x, y, u, v \in V^+$ the sets X_{xy}, X_{yu}, X_{uv} are all of the same type: then (2) gives that they all are of cardinality 2. But then X_{xy} and X_{yu} cannot satisfy $(-)$ with equality by the nonsingleton hyperedge containing y, proving our claim. One can check that the edge set of a complete graph on at least 5 nodes cannot be decomposed into 2 sets such that neither of them contains a path of 3 edges, so there must be a large hyperedge.

Assume that there is a large hyperedge e that avoids $x \in V^+$. Since e is large, there exist $u, v \in V^+ \cap e$. X_{xu} and X_{xv} must be of the same type by the crossing negamodularity. If e avoids another positive node y then X_{xu} and X_{yu} cannot be of the same type for similar reasons. This implies that e cannot avoid a third positive node, so it contains at least 3 positive nodes, since $m(V) \geq 5$. Then the type of X_{uv} and X_{ux} must be different, since they cannot satisfy $(-)$ with equality because of the edge e that is not contained in X_{uv}. But then the type of X_{uv} and X_{uy} would be the same, which cannot hold for the same reason, so e cannot avoid the second positive node y. Furthermore, these observations on the $q\overline{q}$-graph show that x can be the only positive node that is avoided by a large hyperedge. □

We mention that if $m(V) = 4$ then we don't necessarily have large hyperedges: an example can be found in [10]. One can also check that even if there are large hyperedges, they might contain 2 positive nodes if $m(V)$ is only 4.

As an application of this lemma consider the following generalization of the node-to-area connectivity augmentation problem. Given a hypergraph $H = (V, \mathcal{E})$ of rank at most γ, a collection of subsets $\mathcal{W}$ of V and a function $r : \mathcal{W} \to \mathbb{Z}_+$ satisfying $r \geq 2$, find a hypergraph H' of minimum total size such that $\lambda_{H+H'}(x, W) \geq r(W)$ for any $W \in \mathcal{W}$ and $x \in V$ and the rank of $H + H'$ is at most γ. We will call this problem the **node-to-area connectivity augmentation problem in hypergraphs without increasing the rank**. If we define R_{N2A} with (3) and set $q = R_{N2A} - d_H$ then it is clear that $H + H'$ satisfies the area requirements if and only H' covers q. Since R_{N2A} does not take 1 as value, we can apply Lemma 15 and obtain that the Algorithm GREEDYCOVER fails only slightly for this problem. Note that the Algorithm GREEDYCOVER can be implemented to run in polynomial time for this special function R_{N2A}.

Theorem 16. *Let an instance of the minimum total size node-to-area connectivity augmentation problem in hypergraphs be given by the hypergraph $H = (V, \mathcal{E})$ of rank at most $\gamma \geq 2$, $\mathcal{W} \subseteq 2^V$ and $r : \mathcal{W} \to \mathbb{Z}_+$ with $r \geq 2$. Then the Algorithm GREEDY-COVER gives a solution that contains only graph edges and one hyperedge of size at most $\gamma + 1$, if $\gamma > 2$ and $\gamma + 2$ if $\gamma = 2$.*

We mention that, though our proof does not rely on this, after contraction of a set T the function R_{N2A}/T can be defined with a node-to-area requirement function as follows: if R_{N2A} was defined with $\mathcal{W}$ and r then let $\mathcal{W}' = \{W \in \mathcal{W} : T \cap W = \emptyset\} \cup \{W - T + v_T : T \cap W \neq \emptyset\}$ and let $r'(W) = r(W)$ if $v_T \notin W$ and $r'(W') = r(W)$ if $W' = W - T + v_T$. One can check that $\mathcal{W}'$ and r' define R_{N2A}/T.

Note that for any γ there are examples where the Algorithm GREEDYCOVER would output a hyperedge of size greater than γ. For $\gamma = 2$ an example can be found in [10], for bigger values consider the following example. Let V contain $\gamma + 2$ nodes $x_0, x_1, \ldots, x_\gamma, y$ and the hypergraph H contain two hyperedges $\{x_0, y\}$ and $\{x_1, \ldots, x_\gamma\}$. The areas are of the form $\mathcal{W} = \{\{x_0, y, x_i\} : i = 1, 2, \ldots, \gamma\} + \{V - x_0\}$ and $r(W) = 2$ for any $W \in \mathcal{W}$. One can check that the (only) minimal degree-specification is $m = \chi_{V-y}$ and there is no admissible splitting-off. Also note that the greedy bound cannot be achieved in this example without increasing the rank.

Figure 4.2 is an illustration with $\gamma = 4$. The (hyper)edges are with a solid line, some of the areas are illustrated with dashed lines. The empty ball is the neutral node y.

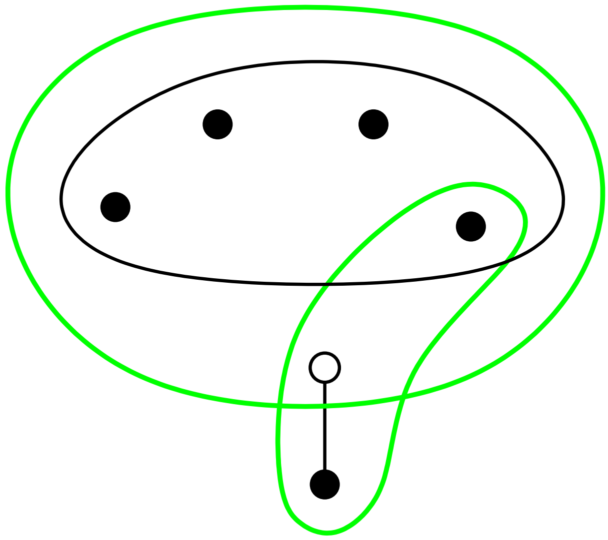

Figure 1: A NA-augmentation problem in hypergraphs where the greedy bound cannot be achieved

If we specialize our results for graphs ($\gamma = 2$) we obtain that a greedy algorithm (the obvious modification of GREEDYCOVER) uses at most one more edge (i.e. at most two more total size) than necessary. Our results do not characterize the cases when the greedy bound in the node-to-area augmentation problem in graphs can be achieved, this can be found in [10] and [9], but they imply that a greedy algorithm can only fail by at most one (edge) for this problem.

A more careful analysis of the stuck situation shows that a slight modification of the Algorithm GREEDYCOVER will solve the *node-to-area connectivity augmentation problem in hypergraphs without increasing the rank* optimally for $\gamma \geq 4$. Details will be given in [3].

5 Further applications

5.1 Local edge-connectivity augmentation of hypergraphs

In this section we consider the **local edge-connectivity augmentation of hypergraphs without increasing the rank**. Let $H = (V, \mathcal{E})$ be a hypergraph of rank at most γ, and let $r : V \times V \to \mathbb{Z}_+ \setminus \{1\}$ be a symmetric edge-connectivity requirement that does not take 1 as value. Let us define the set function R as $R(\emptyset) = R(V) = 0$ and

$$R(X) = \max_{u \in X, v \notin X} r(u, v) \qquad (\emptyset \neq X \subsetneq V). \tag{4}$$

Our aim is to find a hypergraph H' of minimum total size such that $H + H'$ covers R, that is, $\lambda_{H+H'}(u, v) \geq r(u, v)$ for every pair of nodes u, v. Since R is a skew supermodular function, the Algorithm GREEDYCOVER gives a solution that contains graph edges and at most one hyperedge. The question we want to answer is whether the size of this hyperedge is at most γ. One case when this is obviously not true is when $\gamma = 2$ and the greedy bound is odd: then the size of the hyperedge will be 3. The following theorem shows that this is the only exceptional case. Note that this theorem generalizes the theorem of Frank [7] on local edge-connectivity augmentation of graphs. After having proved this theorem we have been informed that Ben Cosh had also proved it in his PhD thesis [5]. We have noticed, that combining our ideas with those in [5] the following simple proof can be given. Our original proof can be found in [4].

Theorem 17. *Let an instance of the minimum total size local edge-connectivity augmentation problem be given by a hypergraph $H = (V, \mathcal{E})$ of rank at most $\gamma > 2$, and the symmetric edge-connectivity requirement $r : V \times V \to \mathbb{Z}_+ \setminus \{1\}$. Then the Algorithm GREEDYCOVER gives a solution to this problem that contains only graph edges and one hyperedge of size at most γ.*

Proof. We will prove more, namely that the hyperedge in the output of the algorithm GREEDYCOVER is of size at most γ for any **minimal** input $m \in C(p) \cap \mathbb{Z}^V$: observe that this contains more general augmentation problems, e.g. the minimum node-cost version, too. We can assume that the Algorithm GREEDYCOVER is stuck already at the beginning and assume indirectly that $m(V) > \gamma$. This proof is easier told if we think of the classical description of splitting-off: instead of the degree specification m, add a new node s to the hypergraph and introduce $m(v)$ parallel edges between s and any $v \in V$ (of course, from the results above $m(v) \leq 1$ for any v, so there are no parallel edges: the positive nodes become the neighbours of s) and denote the hypergraph obtained this way with H' (that has node set $V + s$ and only graph edges incident to s). By the assumptions, H' is r-edge-connected in V. We can further assume that $r(u, v) = \lambda_{H'}(u, v)$ for any $u, v \in V$, since if we increase $r(u, v)$ then we do not create new admissible splittings (note, that possibly new sets become tight). So we assume that R is defined by (4) with $\lambda_{H'}$ substituted in the place of r and $p = R - d_H$ as before. We further assume that tight sets are singletons, which implies that $\lambda_{H'}(u, v) = \min(d_{H'}(u), d_{H'}(v))$ for any $u, v \in V$. Let t be a neighbour of s such that $d_{H'}(t) = \min\{d_{H'}(v) :\ v$ is a neighbour of $s\}$. Let u and v be neighbours of s (distinct from each other and from t). The following claim was already used in [8] (Claim 4.1): for completeness, we include a proof.

Claim 2. *X_{tu} and X_{tv} satisfy $(-)$ (with function R and thus with p, too).*

Proof. Assume that they do not satisfy $(-)$ with R: then they must satisfy $(\cap\cup)$. Thus their intersection is tight, so it is t. On the other hand, since $d_{H'}(t) \le d_{H'}(u)$, $R(X_{tu}) \le R(X_{tu} - t)$, and similarly $R(X_{tv}) \le R(X_{tv} - t)$, implying that $R(X_{tu}) + R(X_{tv}) \le R(X_{tu} - t) + R(X_{tv} - t) = R(X_{tu} - X_{tv}) + R(X_{tv} - X_{tu})$, so they satisfy $(-)$ after all, a contradiction.

So there exists a set $X \subseteq V$ containing t, such that $X_{tu} = X + u$ for any neighbour u of s (using that tight sets are singletons). Since $p(X) \le 1$ (by Lemma 2) and $R(X) \ge 2$ (using that $t \in X$), there must be a hyperedge e in H entering X. We claim that this hyperedge must contain every neighbour of s (except possibly t), contradicting the hypothesis that the rank of H is at most γ. Assume that it excludes a neighbour u of s and fix an arbitrary other neighbour v of s (distinct from u and t). Since X_{tu} and X_{tv} must satisfy $(-)$ with equality for p, this implies that $e \subseteq X_{tv}$. But then X_{tu} and X_{tw} will not satisfy $(-)$ with equality for a fourth neighbour w of s. $\square$

We mention that the minimality of m is crucial in the proof above: if m is not minimal then a simple example shows that the greedy algorithm can fail and produce a hyperedge of size $\gamma + 1$.

5.2 Global arc-connectivity augmentation of mixed hypergraphs

A **mixed hypergraph** $M = (V, \mathcal{A})$ is a pair of a finite set V and a family $\mathcal{A}$ of subsets of V (repetitions are allowed). For an $a \in \mathcal{A}$ every $v \in a$ can be either a **head node**, a **tail node** or even both (**head-tail node**), such that every hyperarc contains at least one head and one tail. More formally we could say that $\mathcal{A}$ contains nonempty ordered set-pairs (T, H) (T being the set of tails, H being the set of heads, possibly $H \cap T \neq \emptyset$). An undirected hypergraph can be considered (for our purposes) as a special mixed hypergraph where every node in a hyperarc is a head-tail node of this hyperarc. The set V is called the **node set** of the mixed hypergraph, the family $\mathcal{A}$ is called the **hyperarc set** (or sometimes shortly the **arc set**) of the mixed hypergraph. **Reversing a hyperarc in $\mathcal{A}$** means switching the roles of the nodes in it, i.e. head nodes become tail nodes and vice versa (so head-tail nodes remain like that). When we say that v is a tail node of a hyperarc a then we also allow that it is a head-tail node (and similarly for head nodes).

In a mixed hypergraph M, a **path** between nodes s and t is an alternating sequence of distinct nodes and hyperarcs $s = v_0, a_1, v_1, a_2, \ldots, a_k, v_k = t$, such that v_{i-1} is a tail node of a_i and v_i is a head node of a_i for all i between 1 and k. A hyperarc a **enters a set** X if there is a head node of a in X and there is a tail node of a in $V - X$. A hyperarc **leaves a set** if it enters the complement of this set. For a set X we define $\varrho_M(X) = |\{a \in \mathcal{A} : a \text{ enters } X\}|$ (the **in-degree** of X) and $\delta_M(X) = \varrho_M(V - X)$ (the **out-degree** of X). It is easy to check that the functions ϱ and δ are submodular functions. Given a mixed hypergraph $M = (V, \mathcal{A})$ and sets $S, T \subseteq V$, let $\lambda_M(S, T)$ denote the maximum number of arc-disjoint paths starting in S and ending in T (we say that $\lambda_M(S, T) = \infty$ if $S \cap T \neq \emptyset$). By Menger's theorem:

$$\lambda_M(S, T) = \min\{\varrho_M(X) : T \subseteq X \subseteq V - S\}.$$

If $M = (V, \mathcal{A})$ is a mixed hypergraph, $r \in V$ is a designated root node and k, l are nonnegative integers, then we say that M **is (k, l)-arc-connected from** r if $\lambda_M(r, v) \geq k$ and $\lambda_M(v, r) \geq l$ for any $v \in V$. Let us define the set function $q = q_{M,r,k,l}$ by $q(\emptyset) = q(V) = 0$, $q(X) = k - \varrho_M(X)$ for any nonempty $X \subseteq V - r$ and $q(X) = l - \varrho_M(X)$ for any $X \subsetneq V$ with $r \in X$: this function is the straightforward extension of the function introduced in the proof of the theorem on mixed graphs (Theorem 9). Then one can check that q is crossing supermodular. For a hypergraph H one can prove that $M + H$ is (k, l)-arc-connected from r if and only if d_H covers q (or equivalently q^s).

If $M = (V, \mathcal{A})$ is a mixed hypergraph and $X \subseteq V$ then contracting X yields the mixed hypergraph $M/X = (V/X, \mathcal{A}/X)$ the following way: for every $a = (T_a, H_a) \in \mathcal{A}$ let $T'_a = T_a$ if $T_a \cap X = \emptyset$ and let $T'_a = T_a - X + v_X$ otherwise, similarly let $H'_a = H_a$ if $H_a \cap X = \emptyset$ and let $H'_a = H_a - X + v_X$ otherwise. Then $\mathcal{A}/X = \{a' = (T'_a, H'_a) : a \in \mathcal{A}\}$. Observe that $\varrho_M/X = \varrho_{M/X}$. If the root node r is in X then the contracted node v_X will become the new root node. This shows that contracting a set defines a contracted problem the natural way.

Let $M = (V, \mathcal{A})$ be a mixed hypergraph and let $k, l \geq 2$ be integers. We assume that M is of rank at most γ. We want to make M (k, l)-arc-connected by adding an undirected, degree specified hypergraph that also has rank at most γ. Is it true that the Algorithm GREEDYCOVER will output such a hypergraph? The answer is "almost yes": an example shows that sometimes this can only be done by adding a hyperedge of cardinality $\gamma + 1$ (even for $k = l = 2$). Consider the following mixed hypergraph $M = (V, \mathcal{A})$: let $|V| \geq 3$ and $x, y \in V$ be two nodes. There are 3 hyperarcs in $\mathcal{A}$: one is a digraph arc (x, y), the second is $(y, V - x - y)$ and the third is $(V - x - y, x)$. Finally let $k = l = 2$ and $\gamma = |V| - 1$. It is easy to see that the greedy bound is $|V|$ and the only way to achieve it is to add the hyperedge V.

Figure 5.2 is an illustration: the digraph arc and one of the other two hyperarcs are drawn with solid lines, the third hyperarc is drawn with dashed lines. We use the convention that the tails of a hyperarc are denoted by an "o" and heads by an "x" (except for the digraph arc, which is denoted by an arrow).

However, we can prove the following result.

Theorem 18. *If M is of rank at most $\gamma \geq 2$ and $k, l \geq 2$ are integers, then we can make M (k, l)-arc-connected greedily by the addition of graph edges and one hyperedge of size at most $\gamma + 1$.*

Proof. Let $q = q_{M,r,k,l}$ and $p = q^s$ and let $m \in C(p) \cap \mathbb{Z}^V$. We can assume that the Algorithm GREEDYCOVER is stuck already at start. We have to prove that $m(V)$ is at most $\gamma + 1$. We can also assume that tight sets are singletons (and delete singleton hyperedges, since they are irrelevant for connectivity), so by the observations

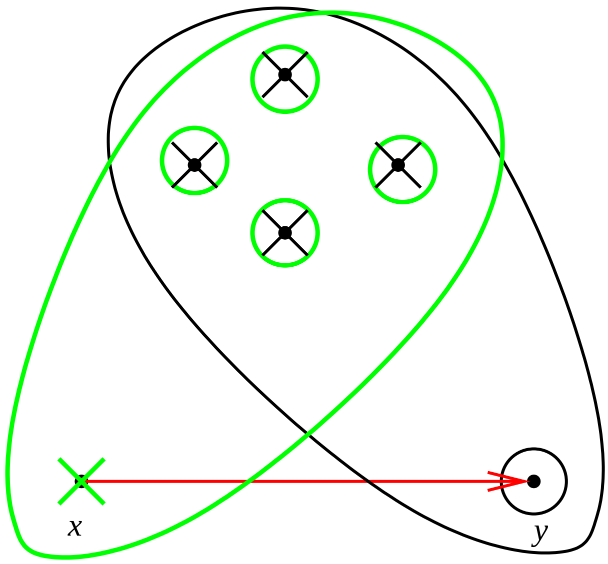

Figure 2: A mixed hypergraph that cannot be made 2-arc-connected with a hypergraph meeting the greedy bound without increasing the rank

in Section 4.1 every node is positive. By Theorem 13, there is an $x \in V$ such that (by possibly reversing every hyperarc of M and switching the role of k and l) every set $X \neq V$ with $x \in X$ has $q(X) = 1$ (observe that this consequence is also true for the sporadic example on 4 nodes). First we claim that $V - x$ cannot contain hyperarcs. Assume that it does contain a hyperarc a, let v be an arbitrary head node of a, and let $X = a - v + x$ and $Y = \{v, x\}$. These sets are crossing (since $|a| < |V - x|$ by the assumption) and of q-type, but $(\cap\cup)$ cannot hold with equality for them, a contradiction. So every hyperarc of M contains x. We claim that if $v \neq x$ is a tail of a hyperarc $a = (T_a, H_a)$ satisfying $|a| \geq 3$, then $x \in H_a$ and $T_a - v - x = \emptyset$. To see this consider the crossing sets $X = a - v$ and $Y = \{v, x\}$. Then $q(X) = q(Y) = 1$ but one can check that $(\cap\cup)$ cannot hold with equality for X and Y, a contradiction. So the hyperarcs leaving any $v \in V - x$ all enter x and such a hyperarc cannot leave two such nodes. This implies that $\varrho(x) = \sum_{v \in V-x} \delta(v)$. If $x = r$ then $l - 1 = \varrho(x) = \sum_{v \in V-x} \delta(v) = |V - x|(l - 1)$ contradicting that $|V| > 2$ and $l > 1$. On the other hand, if $x \neq r$ then $k - 1 = \varrho(x) = \sum_{v \in V-x} \delta(v) = (|V| - 2)(l - 1) + (k - 1)$, again contradicting that $|V| > 2$ and $l > 1$. □

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