

ANALOGUES OF THE CENTRAL POINT THEOREM FOR FAMILIES WITH d -INTERSECTION PROPERTY IN \mathbb{R}^d

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ABSTRACT. In this paper we consider families of compact convex sets in \mathbb{R}^d such that any subfamily of size at most d has a nonempty intersection. We prove some analogues of the central point theorem and Tverberg's theorem for such families.

1. INTRODUCTION

Let us start with a definition:

Definition 1.1. A family of sets \mathcal{F} has *property* Π_k if for any nonempty $\mathcal{G} \subseteq \mathcal{F}$ such that $|\mathcal{G}| \leq k$ the intersection $\bigcap \mathcal{G}$ is not empty.

Helly's theorem [7] states that a finite family of convex sets (or any family of convex compact sets) with Π_{d+1} property in \mathbb{R}^d has a common point. In the review [4] Helly's theorem and many its generalizations are considered in detail.

In this paper we concentrate on the families with Π_d property in \mathbb{R}^d , the "almost" Helly property. The typical example of a family with Π_d property is any family of affine hyperplanes in general position. It can be easily seen already in the case of affine hyperplanes that such a family need not have a common point, and even need not have a bounded *piercing number*, which is the smallest size of a finite set intersecting any set of the family. The reader may also consult [10], where some bounds on the piercing number following from the Π_d property are given for particular families of sets, for example, balls of equal radii, balls of arbitrary radii, or translates of a single convex compact set in the plane.

An important consequence of Helly's theorem is the central point theorem [5, 18, 19] for measures: For every absolute continuous probability measure μ on \mathbb{R}^d one can find a *central point*, that is a point x such that any halfspace $H \ni x$ has $\mu(H) \geq \frac{1}{d+1}$. Here we discuss the discrete central point theorem for finite point sets instead of measures: For a finite set $X \subset \mathbb{R}^d$ there exists a *central point* $x \in \mathbb{R}^d$ such that any half-space $H \ni x$ contains at least

$$r = \left\lceil \frac{|X|}{d+1} \right\rceil$$

points of X . Here $|X|$ denotes the cardinality of X .

In [11] several "dual" analogues of the central point theorem were established for the families of affine hyperplanes. For example, if \mathcal{F} is a family of affine hyperplanes in \mathbb{R}^d then there exists a point $x \in \mathbb{R}^d$ such that any ray starting at x intersects at least

$$r = \left\lceil \frac{|\mathcal{F}|}{d+1} \right\rceil$$

2000 *Mathematics Subject Classification.* 52A20, 52A35, 52C35.

Key words and phrases. central point theorem, Tverberg's theorem, Helly's theorem.

Supported by the Dynasty Foundation, the President's of Russian Federation grant MD-352.2012.1, the Russian Foundation for Basic Research grants 10-01-00096 and 10-01-00139, the Federal Program "Scientific and scientific-pedagogical staff of innovative Russia" 2009–2013, and the Russian government project 11.G34.31.0053.

affine hyperplanes of \mathcal{F} . The word “dual” here does not mean that this theorem follows from the original discrete central point theorem by the projective duality or some other transformation; this “dual” theorem in fact requires a separate proof using some topology.

Here we prove an analogue of the dual central point theorem for every family of convex compact sets with Π_d property:

Theorem 1.2. *Let a finite family \mathcal{F} of convex closed sets in \mathbb{R}^d have property Π_d . Then there exists a point $x \in \mathbb{R}^d$ such that any unbounded continuous curve that passes through x intersects at least*

$$r = \left\lceil \frac{|\mathcal{F}|}{d+1} \right\rceil$$

sets in \mathcal{F} .

Similar to what is done in [11] it is natural to generalize this theorem in the spirit of Tverberg’s theorem [20]. First, we have to make a definition. For a family \mathcal{G} of $d+1$ compact convex sets in \mathbb{R}^d with Π_d property we have two alternatives: either all the sets in \mathcal{G} have a common point, or the nerve of the family \mathcal{G} (see [4] for the discussion of nerves) is a simplicial complex equal to the boundary $\partial\Delta^d$ of the standard d -simplex. By the nerve theorem the union $\bigcup \mathcal{G}$ is homotopy equivalent to $\partial\Delta^d$ (or a $(d-1)$ -dimensional sphere) and by the Alexander duality [6, Theorem 3.44] the complement $\mathbb{R}^d \setminus \bigcup \mathcal{G}$ consists of two connected components, one being bounded and the other being unbounded. Now it is natural to make a definition:

Definition 1.3. Consider a family \mathcal{G} of $d+1$ convex compact sets in \mathbb{R}^d with Π_d property. If the family \mathcal{G} has no point in common, then the complement of its union consists of two connected components: X and Y , where X is bounded and Y is unbounded. In this case for any point $x \in X$ we say that \mathcal{G} surrounds x .

Remark 1.4. A typical example is: $d+1$ facets of any d -dimensional simplex surround any point in the interior of the simplex.

Now we state the analogue of the Tverberg theorem:

Theorem 1.5. *Let a finite family \mathcal{F} of convex compact sets in \mathbb{R}^d have property Π_d . Suppose the number*

$$r = \left\lceil \frac{|\mathcal{F}|}{d+1} \right\rceil$$

is a prime power. Then there exists a point $x \in \mathbb{R}^d$ and r pairwise disjoint nonempty subfamilies $\mathcal{F}_1, \dots, \mathcal{F}_r \subseteq \mathcal{F}$ such that the following condition holds for any $i = 1, \dots, r$:

- 1) *either some member of \mathcal{F}_i contains x ;*
- 2) *or the family \mathcal{F}_i surrounds x .*

We conjecture that this result holds without the assumption that r is a prime power. In this case this would imply Theorem 1.2 directly, because any unbounded continuous curve through x must intersect some element of every \mathcal{F}_i . It turns out that in order to deduce Theorem 1.2, it is sufficient (see Section 6) to prove Theorem 1.5 only for prime numbers r .

It is also possible to give a generalization of Theorem 1.5 in the spirit of Tverberg’s transversal conjecture [21]; see also [3, 9, 11, 22, 25] for proofs of some particular cases of Tverberg’s transversal conjecture and similar results.

Definition 1.6. Consider a family \mathcal{G} of $d-m+1$ convex compact sets in \mathbb{R}^d with Π_{d-m} property and an affine m -subspace L . We say that \mathcal{G} surrounds L if $\pi(\mathcal{G})$ surrounds the point $\pi(L)$, where π is the projection along L .

Theorem 1.7. *Suppose that each of $m + 1$ families \mathcal{F}_i ($i = 0, \dots, m$) of convex compact sets in \mathbb{R}^d have property Π_{d-m} . Let the numbers*

$$r_i = \left\lceil \frac{|\mathcal{F}_i|}{d - m + 1} \right\rceil$$

be powers of the same prime p and

- a) either $p = 2$;
- b) or $d - m$ is even;
- c) or $m = 0$.

Then there exists an affine m -subspace L and, for every $i = 0, \dots, m$, some r_i pairwise disjoint nonempty subfamilies $\mathcal{F}_{i1}, \dots, \mathcal{F}_{ir_i} \subseteq \mathcal{F}_i$ such that for any $i = 0, \dots, m$ and $j = 1, \dots, r_i$ the following condition holds:

- 1) either some member of \mathcal{F}_{ij} intersects L ;
- 2) or the family \mathcal{F}_{ij} surrounds L .

The case $m = 0$ is inserted here to make a unified statement with Theorem 1.5. Actually, in this theorem the sets need not be convex, it is sufficient that all their projections to linear $(d - m)$ -subspaces are convex; this property is sometimes called $(d - m)$ -convexity.

The assumption that r_i are prime powers is essential in the proof of Theorem 1.7 since the action of a p -torus on the configuration space is required, see Section 5. Of course, it is natural to conjecture that this restriction is not necessary.

While this paper was considered and reviewed in the journal, another paper [12] with similar results was published. So the content of this paper has a large intersection with that of [12].

Acknowledgments.

The author thanks V.L. Dol'nikov for the discussions that have lead to formulation of these results and the unknown referee for numerous helpful suggestions.

2. FACTS FROM TOPOLOGY

We consider topological spaces with continuous (left) action of a finite group G and continuous maps between such spaces that commute with the action of G . We call them G -spaces and G -maps. We mostly consider groups $G = (\mathbb{Z}_p)^k$ for prime p , so-called p -tori.

For basic facts about (equivariant) topology and vector bundles the reader is referred to the books [8, 14, 17]. The cohomology is assumed with coefficients in \mathbb{F}_p (p is the same as in the definition of G), we omit the coefficients from notation. Let us start from some standard definitions. In this paper we assume Čech cohomology, it is safe to make such assumptions in results like Lemma 3.4.

Definition 2.1. Denote by EG the classifying G -space, which can be thought of as an infinite join $EG = G * \dots * G * \dots$ with diagonal left G -action. Denote $BG = EG/G$. For any G -space X denote by $X_G = (X \times EG)/G$, and put (equivariant cohomology in the sense of Borel) $H_G^*(X) = H^*(X_G)$. It is easy to verify that for a free G -space X the space X_G is homotopy equivalent to X/G .

Consider the algebra of G -equivariant cohomology of the point $A_G = H_G^*(\text{pt}) = H^*(BG)$. For a group $G = (\mathbb{Z}_p)^k$ the algebra $A_G = H_G^*(\mathbb{Z}_p)$ has the following structure (see [8]). In the case p odd it has $2k$ multiplicative generators v_i, u_i with dimensions $\dim v_i = 1$ and $\dim u_i = 2$ and relations

$$v_i^2 = 0, \quad \beta v_i = u_i,$$

where we denote by $\beta(x)$ the Bockstein homomorphism.

In the case $p = 2$ the algebra A_G is the algebra of polynomials of k variables v_1, \dots, v_k of degree one.

We are going to find the equivariant cohomology of a G -space X using the following spectral sequence (see [8, 16]):

Proposition 2.2. *The natural fiber bundle $\pi_{X_G} : X_G \rightarrow BG$ with fiber X gives the spectral sequence with the E_2 -term*

$$E_2^{x,y} = H^x(BG; \mathcal{H}^y(X)),$$

having a structure of a graded A_G -module, and converging to a graded A_G -module, associated with the filtration of $H_G^*(X)$.

The system of coefficients $\mathcal{H}^y(X)$ is obtained from the action of $G = \pi_1(BG)$ on the cohomology $H^y(X)$. The differentials of this spectral sequence are homomorphisms (of corresponding degree) of graded A_G -modules.

This proposition implies the following: If the space X is $(n - 1)$ -connected then the natural map $A_G^m \rightarrow H_G^m(X)$ is injective in dimensions $m \leq n$.

Any representation of G can be considered as a vector bundle over the point pt , and it has corresponding characteristic classes in $H_G^*(\text{pt})$. We need the following lemma, that follows from the results of [8, Chapter III, § 1] (see also [13, 23]):

Lemma 2.3. *Let $G = (\mathbb{Z}_p)^k$, and let $I[G]$ be the subspace of the group algebra $\mathbb{R}[G]$, consisting of elements*

$$\sum_{g \in G} a_g g, \quad \sum_{g \in G} a_g = 0.$$

Then the Euler class $e(I[G]) \neq 0 \in A_G$ and is not a divisor of zero in A_G .

In this lemma the assumption that $G = (\mathbb{Z}_p)^k$ is essential.

We also need the following folklore fact on the Grassmann variety (see [3, 9, 25] for its different applications). Consider the canonical bundle over the Grassmann variety $\gamma : E(\gamma) \rightarrow G_d^{d-m}$. In the case $p = 2$ we consider the variety of non-oriented linear $(d - m)$ -subspaces, and for odd p we consider the variety of oriented subspaces.

Lemma 2.4. *For the Euler class $e(\gamma)$ modulo p the following holds*

$$e(\gamma)^m \neq 0 \in H^{m(d-m)}(G_d^{d-m}; \mathbb{F}_p),$$

if either $p = 2$, or $d - m$ is even, or $m = 0$. In the latter case we put $e(\gamma)^0 = 1 \in H^0(G_d^{d-m}; \mathbb{F}_p)$ by definition.

It is hard to locate the place where this lemma was proved for the first time (for example, it follows from Schubert calculus); one particular reference for the proof is [9, Lemma 8], where this class in the oriented case is shown to be Poincaré dual to a set of two points with same signs. In the non-oriented case and mod 2 cohomology this class is Poincaré dual to a single point, which is a nontrivial 0-cycle mod 2.

3. TOPOLOGY OF TVERBERG'S THEOREM

In Tverberg's theorem and its topological generalizations (see [1, 24] for example) it is important to consider the configuration space of r -tuples of points $x_1, \dots, x_r \in \Delta^N$ with pairwise disjoint supports. Here Δ^N is a simplex of dimension N . Let us make some definitions, following the book [15].

Definition 3.1. Let K be a simplicial complex. Denote by K_Δ^r the subset of the r -fold product K^r , consisting of the r -tuples (x_1, \dots, x_r) such that every pair x_i, x_j ($i \neq j$) has disjoint supports in K . We call K_Δ^r the r -fold deleted product of K .

Definition 3.2. Let K be a simplicial complex. Denote by K_{Δ}^{*r} the subset of the r -fold join K^{*r} , consisting of convex combinations $w_1x_1 \oplus \cdots \oplus w_rx_r$ such that every pair x_i, x_j ($i \neq j$) with weights $w_i, w_j > 0$ has disjoint supports in K . We call K_{Δ}^{*r} the r -fold deleted join of K .

Note that the deleted join is a simplicial complex again, while the deleted product has no natural simplicial complex structure, although it has some cellular complex structure.

The r -fold deleted product of the simplex $\Delta^{(r-1)(d+1)}$ is the natural configuration space in Tverberg's theorem, but sometimes it is simpler to use the deleted join because of the following fact. Denote by $[r]$ the set $\{1, \dots, r\}$ with the discrete topology, the following lemma is well-known, see [15] for example.

Lemma 3.3. *The deleted join of the simplex $(\Delta^N)_{\Delta}^{*r} = [r]^{*N+1}$ is $(N-1)$ -connected.*

If r is a prime power $r = p^k$, then the group $G = (\mathbb{Z}_p)^k$ can be somehow identified with $[r]$, so a G -action on K_{Δ}^r and K_{Δ}^{*r} by permuting the r factor arises. In this case Proposition 2.2 and the above lemma imply that the natural map $A_G^l \rightarrow H_G^l((\Delta^N)_{\Delta}^{*r})$ is injective in dimensions $l \leq N$. We need a similar fact for deleted products, following from the next lemma:

Lemma 3.4. *Let $r = p^k$, $G = (\mathbb{Z}_p)^k$, and let K be a simplicial complex. If the natural map $A_G^l \rightarrow H_G^l(K_{\Delta}^{*r})$ is injective for $l \leq N$, then the similar map $A_G^l \rightarrow H_G^l(K_{\Delta}^r)$ is injective for $l \leq N - r + 1$.*

Proof. Define the map $\alpha : K^{*r} \rightarrow \mathbb{R}[G]$ as follows. Let α map a convex combination $w_1x_1 \oplus \cdots \oplus w_rx_r \in K^{*r}$ to $(w_1, \dots, w_r) \in \mathbb{R}^r$, the latter space is identified with $\mathbb{R}[G]$, if we identify the set $[r]$ with G . This map is G -equivariant.

Consider the natural orthogonal projection $\pi : \mathbb{R}[G] \rightarrow I[G]$ (the latter G -representation is defined in Lemma 2.3) and the natural inclusion $\iota : K_{\Delta}^{*r} \rightarrow K^{*r}$. The map $\beta = \pi \circ \alpha \circ \iota : K_{\Delta}^{*r} \rightarrow I[G]$ is G -equivariant, and it can be easily checked that

$$K_{\Delta}^r = \{y \in K_{\Delta}^{*r} : \beta(y) = 0\}.$$

Now assume the contrary: the image of some nonzero element $\xi \in A_G^l$ is zero in $H_G^l(K_{\Delta}^r)$ and $l \leq N - r + 1$. We denote the classes in A_G and their natural images in the equivariant cohomology of G -spaces by the same letters if it does not lead to confusion. Put $e(I[G]) = e \in A_G^{r-1}$ for brevity.

The Euler class of a vector bundle is zero outside the zero set of a section of the bundle. Indeed, if Z is the zero set of a section s of a vector bundle ν over a space X , then the restriction of ν to $X \setminus Z$ has a nonzero section s . Therefore $\nu|_{X \setminus Z}$ has zero Euler class and the needed claim follows from the naturality of the Euler class: the Euler class of the restriction is the restriction of the original Euler class.

So we know that e vanishes in $H_G^{r-1}(K_{\Delta}^{*r} \setminus K_{\Delta}^r)$. By the standard property of the cohomology product (often used to estimate the Lusternik–Schnirelman category by the cup-length) we obtain that the class $e\xi$ vanishes over $(K_{\Delta}^{*r} \setminus K_{\Delta}^r) \cup K_{\Delta}^r$, that is over the whole K_{Δ}^{*r} . By Lemma 2.3 $e\xi \neq 0 \in A_G^{l+r-1}$, and we come to contradiction with the injectivity condition in the statement of this lemma. \square

4. PROOF OF THEOREM 1.5

It would be sufficient to prove Theorem 1.7, since Theorem 1.5 is its particular case. Though we give a separate proof for Theorem 1.5 to clarify the exposition. The reasoning in this proof (and the subsequent proofs) is essentially the same as in [11, 12].

Consider the simplex $\Delta = \Delta^{n-1}$, along with some identification of its vertices with the members of \mathcal{F} . Take some large enough ball $B \subset \mathbb{R}^d$, containing all the sets of \mathcal{F}

in its interior. The configuration space that we study is $\Delta_\Delta^r \times B$, denote its elements by $(y_1, y_2, \dots, y_r, p)$. The points y_i in the simplex Δ will be considered in barycentric coordinates as functions $y_i : \mathcal{F} \rightarrow \mathbb{R}^+$ each of them having unit sum of values. The condition that an r -tuple (y_1, \dots, y_r) lies in the deleted product means that the supports of these functions are pairwise disjoint.

Put for brevity $\mathbb{R}^d = V$. Now let us map our configuration space to V^r by the following rule. Let $\pi_K(p)$ be the metric projection of p to $K \in \mathcal{F}$ sending every p to the closest to p point in K ; this map is 1-Lipschitz and therefore continuous. Put

$$f(y_1, y_2, \dots, y_r, p) = \bigoplus_{i=1}^r \sum_{K \in \mathcal{F}} y_i(K) (\pi_K(p) - p),$$

This map is evidently continuous and G -equivariant, if we identify V^r with $V[G]$ (V -valued functions on G with G -action by right multiplication by g^{-1}).

The map f can be considered as a section of a G -equivariant vector bundle $V[G] \times \Delta_\Delta^r \times B \rightarrow \Delta_\Delta^r \times B$. This bundle is trivial by definition but the action of G makes it equivariantly nontrivial. The relative Euler class of this section¹ can be decomposed according to the decomposition $V[G] = V \oplus V \otimes I[G]$, multiplicativity of the relative Euler class (see [11]), and the Künneth formula:

$$e(f) = w^d \times u \in H_G^{rd}(\Delta_\Delta^r \times B, \Delta_\Delta^r \times \partial B) = H_G^{d(r-1)}(\Delta_\Delta^r) \otimes H^d(B, \partial B).$$

Here w is the image of $e(I[G])$ in $H^{r-1}(\Delta_\Delta^r)$ and u is the generator of $H^d(B, \partial B)$. We indeed obtain u as the second factor because for any fixed (y_1, \dots, y_r) the corresponding projection f'' of the section f to the summand V corresponds to a vector pointing from p to a convex combination of vectors $\pi_K(p)$. If B contains all sets of \mathbb{F} in its interior as we have assumed, then this vector always points inside B for $p \in \partial B$. Hence the corresponding Euler class is the same as in the Brouwer fixed point theorem [2], which is the generator of $H^d(B, \partial B)$.

By Lemmas 2.3 and 3.4, $w^d \neq 0 \in H_G^{d(r-1)}(\Delta_\Delta^r)$, and the Künneth formula implies that $e(f) \neq 0$.

The map f therefore must have a zero, let it be $(y_1, y_2, \dots, y_r, p)$. For any $K \in \mathcal{F}$ there is at most one $i \in [r]$ such that $y_i(K) > 0$, since y_i 's have disjoint supports. In this case we put K to the subset \mathcal{F}_i . From the definition of f it follows that for any i the projections of p to the sets $K \in \mathcal{F}_i$ have p in their convex hull.

We use the following lemma:

Lemma 4.1. *Let a family $\mathcal{G} = \{G_1, \dots, G_{d+1}\}$ of convex compact sets in \mathbb{R}^d have property Π_d . Let a point $p \in \mathbb{R}^d$ be such that p lies in the interior of the convex hull of g_1, \dots, g_{d+1} , where g_i is the closest to p point in G_i . Then \mathcal{G} surrounds p .*

Proof of Lemma 4.1. Consider the half-spaces

$$H_i = \{x \in \mathbb{R}^d : (x, g_i - p) \geq (g_i, g_i - p)\}$$

and note that $G_i \subseteq H_i$. Clearly, $\bigcap_{i=1}^{d+1} H_i = \emptyset$.

For any $i = 1, \dots, d+1$ the nonempty intersection $\bigcap_{j \neq i} G_j$ is contained in $\bigcap_{j \neq i} H_j$, take one point $x_i \in \bigcap_{j \neq i} G_j$. The simplex $\Delta = \text{conv}_{i=1}^{d+1} \{x_i\}$ contains $\mathbb{R}^d \setminus \bigcup_{i=1}^{d+1} H_i \ni p$ (compare [10, Lemma 1]), and every its facet $\partial_i \Delta = \text{conv}_{j \neq i} \{x_j\}$ is contained in the corresponding G_i .

¹ The reader is referred to [11] for properties of the relative Euler class. It is important that the relative Euler class depends on both the vector bundle and its section.

Thus $p \notin \bigcup_{i=1}^{d+1} G_i$ and is separated from infinity by $\bigcup_{i=1}^{d+1} G_i \supseteq \partial\Delta$, so \mathcal{G} surrounds p by definition. \square

If p coincides with one of $\pi_K(p)$ (for $K \in \mathcal{F}_i$), then p is already contained in $\bigcup \mathcal{F}_i$. If p lies in the interior of the convex hull of some $d+1$ points of $\{\pi_K(p)\}_{K \in \mathcal{F}_i}$, we reduce \mathcal{F}_i so that it contains only those $d+1$ corresponding sets K and note that $\{\pi_K(p)\}_{K \in \mathcal{F}_i}$ surround p by Lemma 4.1, and therefore \mathcal{F}_i surrounds p .

If none of the above two alternatives holds, then p lies in the relative interior of the convex hull of some $n < d+1$ points $\pi_{K_1}(p), \dots, \pi_{K_n}(p)$, $K_1, \dots, K_n \in \mathcal{F}_i$. Denote the half-spaces

$$H_K = \{x \in \mathbb{R}^d : (x, \pi_K(p) - p) \geq (\pi_K(p), \pi_K(p) - p)\}.$$

Note that $K \subseteq H_K$ (since π is the projection) and the half-spaces H_{K_1}, \dots, H_{K_n} have empty intersection. So some $n < d+1$ sets of \mathcal{F}_i have an empty intersection that contradicts the Π_d property. So the case $n < d+1$ is impossible and the proof is complete.

5. PROOF OF THEOREM 1.7

For any affine m -subspace L denote the unique linear $(d-m)$ -subspace in \mathbb{R}^d , orthogonal to L , by L^\perp . It is easy to see that L is determined uniquely by L^\perp and the point $L \cap L^\perp$. So the variety of all affine m -subspaces is the total space of the canonical bundle γ_d^{d-m} over the Grassmann variety G_d^{d-m} .

For any $V \in G_d^{d-m}$ denote the orthogonal projection of \mathbb{R}^d onto V by π_V . For any $X \in \bigcup_{i=0}^m \mathcal{F}_i$, $V \in G_d^{d-m}$, and $p \in V$ denote by $\phi(V, p, X)$ the closest to p point in $\pi_V(K)$. This point depends continuously on the pair (V, p) (a standard technical argument showing this is omitted) and lies in V .

Fix some $i = 0, \dots, m$ and define a linear map $\psi_i : K_i = \Delta^{|\mathcal{F}_i|+1} \rightarrow V$ that maps the vertices of the simplex (corresponding to $X \in \mathcal{F}_i$) to the points $\phi(V, p, X) - p$ for $X \in \mathcal{F}_i$ and is piece-wise linear. Denote by $\xi_i : (K_i)_{\Delta}^{r_i} \rightarrow V^{r_i}$ the corresponding map of the deleted product. This map is the analogue of f from the previous proof, but we have to define one such map for every \mathcal{F}_i .

Let the group $G_i = (\mathbb{Z}_p)^{k_i}$, where $r_i = p^{k_i}$, act on the deleted product $L_i = (K_i)_{\Delta}^{r_i}$ and on V^{r_i} by permutations, we put $V^{r_i} = V[G_i]$ to indicate this action, the map ξ_i thus becomes G_i -equivariant.

In the sequel we put $\gamma_d^{d-m} = \gamma$ for brevity. Summing up all the maps we obtain a map

$$\xi : L_0 \times \dots \times L_m \rightarrow V[G_0] \oplus \dots \oplus V[G_m].$$

The map ξ also depends on the pair $(V, p) \in E(\gamma)$ continuously, so actually it gives a section ξ of the vector bundle ν with fiber $V[G_0] \oplus \dots \oplus V[G_m]$ over the space $E(\gamma) \times L_0 \times \dots \times L_m$. Here V as a function of the pair (V, p) can be treated as the pullback of the vector bundle $\gamma \rightarrow G_d^{d-m}$ by the map $\gamma : E(\gamma) \rightarrow G_d^{d-m}$. We denote this pullback by γ (it does not make a confusion) and therefore assume ξ to be a section for $\gamma \otimes (\mathbb{R}[G_0] \oplus \dots \oplus \mathbb{R}[G_m])$.

To prove the theorem we have to find $V \in G_d^{d-m}$, $p \in V$, $(y_0, \dots, y_m) \in L_0 \times \dots \times L_m$ such that $\xi(V, p, y_0, \dots, y_m) = 0$.

If we take the bundle of large enough balls $B(\gamma)$ in γ , the section ξ obviously has no zeros over $\partial B(\gamma) \times L_0 \times \dots \times L_m$ (this happens when all the balls $\partial B(\gamma)$ contain all the projections $\pi_V(X)$ for $X \in \bigcup_i \mathcal{F}_i$ in their interiors). To guarantee the zeros for the section ξ , we have to find the relative Euler class

$$e(\xi) \in H_{G_0 \times \dots \times G_m}^{(d-m)(r_0 + \dots + r_m)}(B(\gamma) \times L_0 \times \dots \times L_m, \partial B(\gamma) \times L_0 \times \dots \times L_m).$$

Put for brevity $G = G_0 \times \dots \times G_m$.

Let us decompose the bundle ν and its section ξ in the following way. Any $V[G_i]$ can be split $V[G_i] = V \otimes \mathbb{R}[G] = V \otimes \mathbb{R} \oplus V \otimes I[G_i] = V \oplus V \otimes I[G_i]$. So the section ξ splits into section η of the bundle $v = \gamma^{m+1}$ and ζ of the bundle $\omega = \gamma \otimes \bigoplus_{i=0}^m I[G_i]$, and $\nu = v \oplus \omega$.

The section η has no zeroes on $\partial B(\gamma) \times L_0 \times \cdots \times L_m$ and, in fact, for large enough balls in $B(\gamma)$ the homotopy $\eta_t = (1-t)\eta + t(-p, \dots, -p)$ connects it to the section $(-p, \dots, -p)$ so that η_t has no zeroes on $\partial B(\gamma) \times L_0 \times \cdots \times L_m$ for all $t \in [0, 1]$. The section η_1 does not depend on the factor $L_0 \times \cdots \times L_m$ and it can be easily seen that (see [11], the proof of Theorem 6)

$$\begin{aligned} e(\eta) &= u(\gamma)e(\gamma)^m \times 1 \in H^{(d-m)(m+1)}(B(\gamma), \partial B(\gamma)) \times H_G^0(L_0 \times \cdots \times L_m) \subset \\ &\subset H^{(d-m)(m+1)}(B(\gamma) \times L_0 \times \cdots \times L_m, \partial B(\gamma) \times L_0 \times \cdots \times L_m), \end{aligned}$$

where $u(\gamma)$ is the Thom's class of γ (in $H^*(E(\gamma))$), $e(\gamma)$ is its Euler class, and the last inclusion is the Künneth formula. Lemma 2.3 and the Thom isomorphism show that $u(\gamma)e(\gamma)^m \neq 0$ (compare [11, Proof of Theorem 6]).

Now we consider the class $e(\zeta) \in H_G^{(d-m)(r_0+\cdots+r_m-m-1)}(B(\gamma) \times L_0 \times \cdots \times L_m)$. Taking some fixed $p \in B(\gamma)$ and considering the inclusion

$$\iota_p : L_0 \times \cdots \times L_m = \{p\} \times L_0 \times \cdots \times L_m \rightarrow B(\gamma) \times L_0 \times \cdots \times L_m$$

and the induced bundle $\iota_p^*(\omega) = \bigoplus_{i=0}^m (I[G_i])^{d-m}$, we obtain

$$\begin{aligned} \iota_p^*(e(\zeta)) &= e(I[G_0])^{d-m} \times e(I[G_1])^{d-m} \times \cdots \times e(I[G_m])^{d-m} \in H_G^*(L_0 \times \cdots \times L_m) = \\ &= H_{G_0}^*(L_0) \times \cdots \times H_{G_m}^*(L_m), \end{aligned}$$

the last equality being the Künneth formula. By Lemmas 2.3 and 3.4, for any $i = 0, \dots, m$ the Euler class $e(I[G_i])^{d-m} \neq 0 \in H_{G_i}^{(d-m)(r_i-1)}(L_i)$ and, by the Künneth formula, $\iota_p^*(e(\zeta)) = a \neq 0$. From one more Künneth formula for the product $B(\gamma) \times L_0 \times \cdots \times L_m$ it follows that

$$e(\zeta) = 1 \times a + \sum_j b_j \times c_j,$$

where $b_j \in H^*(B(\gamma))$, $c_j \in H_G^*(L_0 \times \cdots \times L_m)$ are some classes such that $\dim b_j > 0$ for all j . Hence

$$e(\xi) = u(\gamma)e(\gamma)^m \times a + \sum_j u(\gamma)e(\gamma)^m b_j \times c_j,$$

and $e(\xi) \neq 0$ by the Künneth formula (its first summand cannot be eliminated by the latter sum).

Now we have a zero of ξ at (V, p, y_0, \dots, y_m) . Every point $y_i \in L_i$ is actually an r_i -tuple of points $y_{i1}, \dots, y_{ir_i} \in K_i = \Delta^{|\mathcal{F}_i|+1}$ with pairwise disjoint supports. We identify the vertices of K_i with \mathcal{F}_i and write

$$y_{ij} = \sum_{X \in \mathcal{F}_i} w(i, j, X)X.$$

Denote $\mathcal{F}_{ij} = \{X \in \mathcal{F}_i : w(i, j, X) > 0\}$, each X is assigned to no more than one of \mathcal{F}_{ij} , because y_{ij} have pairwise disjoint supports. The condition $\xi = 0$ implies that for any $i = 0, \dots, m$ and $j = 1, \dots, r_i$ the point p is a convex combination of its projections to the sets $\pi_V(X)$:

$$p = \sum_{X \in \mathcal{F}_{ij}} w(i, j, X)\phi(V, p, X).$$

Now we define L to be the affine subspace, orthogonal to V and passing through p . The rest of the proof is the same as in the previous section, because every \mathcal{F}_{ij} either intersects L

(equivalently, the family $\{\pi_V(X)\}_{X \in \mathcal{F}_{ij}}$ covers p) or surrounds L (equivalently, the family $\{\pi_V(X)\}_{X \in \mathcal{F}_{ij}}$ surrounds p).

6. PROOF OF THEOREM 1.2

In this theorem we can assume that \mathcal{F} consists of compact sets. Indeed, for a large enough ball B the family $\{X \cap B\}_{X \in \mathcal{F}}$ consists of compact sets and has property Π_d .

As it was already noted, this theorem follows from Theorem 1.5 directly when r is a prime power. Consider some other r . Obviously, it is sufficient to prove the theorem in the case $N = |\mathcal{F}| = (d+1)(r-1) + 1$.

By the Dirichlet theorem on arithmetic progressions, we can find a positive integer k such that $R = k(r-1) + 1$ is a prime. Now take the family \mathcal{F}' of size kN by simply repeating each set in \mathcal{F} exactly k times. Note that

$$kN = k(d+1)(r-1) + k = (d+1)(R-1) + k \geq (d+1)(R-1) + 1,$$

so we can apply the case of the theorem, that is already proved, to \mathcal{F}' to get some point x .

Every unbounded closed curve $C \ni x$ intersects at least $R = k(r-1) + 1$ sets of \mathcal{F}' . Each set of \mathcal{F} is counted no more than k times, then we conclude that C intersects at least r sets of \mathcal{F} .

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