Almost all triangle-free triple systems are tripartite

József Balogh* and Dhruv Mubayi†

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Abstract

A triangle in a triple system is a collection of three edges isomorphic to $\{123, 124, 345\}$. A triple system is triangle-free if it contains no three edges forming a triangle. It is tripartite if it has a vertex partition into three parts such that every edge has exactly one point in each part. It is easy to see that every tripartite triple system is triangle-free. We prove that almost all triangle-free triple systems with vertex set $[n]$ are tripartite.

Our proof uses the hypergraph regularity lemma of Frankl and Rödl [13], and a stability theorem for triangle-free triple systems due to Keevash and the second author [15].

1 Introduction

Let $[V]^k$ denote the collection of all k-element subsets of a set V (if $V = [n] = \{1, 2 \ldots, n\}$, then we write $[n]^k$ instead of $[[n]]^k$). Say that $\mathcal{H}$ is a k-uniform hypergraph (k-graph for short) with vertex set $V = V(\mathcal{H})$ if $\mathcal{H} \subset [V]^k$. If $k = 2$, then $\mathcal{H}$ is a graph. Let F be a k-graph. A k-graph is F-*free* if it contains no copy of F as a (not necessarily induced) subhypergraph.

Beginning with a result of Erdős-Kleitman-Rothschild [11], there has been much work concerning the number and structure of F-free graphs with vertex set $[n]$ (see, e.g. [10, 18, 22, 1, 2, 3, 6]). The strongest of these results essentially states that for a large class of graphs F, most of the F-free graphs with vertex set $[n]$ have a similar structure to the F-free graph with the maximum number of edges. Many of these results use the Szemerédi regularity lemma. With the development of the hypergraph regularity lemma, some of these problems can be attacked for hypergraphs.

Definition. *For a* 3*-graph* F *let* $\mathrm{Forb}(n, F)$ *denote the collection of* F*-free* 3*-graphs with vertex set* $[n]$.

*University of Illinois, 1409 W. Green Street, Urbana, IL 61801, USA; and Department of Mathematics, U.C. California at San Diego, 9500 Gilmann Drive, La Jolla, Department of Mathematics; e-mail: jobal@math.uiuc.edu; research supported in part by NSF CAREER Grant DMS-0745185 and DMS-0600303, UIUC Campus Research Board Grants 09072 and 08086, and OTKA Grant K76099.

†Department of Mathematics, Statistics, and Computer Science, University of Illinois, Chicago, IL 60607; email: mubayi@math.uic.edu; research supported in part by NSF grants DMS 0653946 and 0969092.

Given a k-graph F, let $\text{ex}(n, F)$ be the maximum number of edges in an F-free k-graph on $[n]$. As all subhypergraphs of an F-free hypergraph are also F-free, we get

$$2^{\text{ex}(n,F)} \leq |\text{Forb}(n, F)|.$$

Since the seminal work of Erdős, Kleitman and Rothschild [11], substantial effort has been applied to prove matching upper bounds for $|\text{Forb}(n, F)|$. The first result in this direction for hypergraphs was due to Nagle and Rödl [19] who proved that for a fixed 3-graph F,

$$|\text{Forb}(n, F)| \leq 2^{\text{ex}(n,F)+o(n^3)}. \tag{1}$$

Subsequent to [19], with the development of the hypergraph regularity method, there was some progress on these problems for hypergraphs, e.g. Nagle, Rödl and Schacht [20] extended (1) to k-graphs, and Dotson and Nagle [9] to counting induced F-free k-graphs.

Since there is no extremal result for hypergraphs similar to Turán's theorem for graphs, one cannot expect a general result that characterizes the structure of almost all F-free 3-graphs on $[n]$ for large classes of F. Nevertheless, much is known about the extremal numbers for a few specific 3-graphs F and one could hope to obtain characterizations for these F. Recently, Person and Schacht [21] proved the first result of this kind, by showing that almost all 3-graphs on $[n]$ not containing a Fano configuration are 2-colorable.

A 3-graph is *tripartite* or 3-*partite* if it has a vertex partition into three parts such that every edge has exactly one point in each part. Let

$$s(n) := \left\lfloor \frac{n}{3} \right\rfloor \left\lfloor \frac{n+1}{3} \right\rfloor \left\lfloor \frac{n+2}{3} \right\rfloor = \frac{n^3}{27} + O(n^2)$$

be the maximum number of edges in a 3-partite 3-graph with n vertices. A 3-graph is *cancellative* if $A \cup B = A \cup C$ implies that $B = C$ for edges A, B, C. Every tripartite 3-graph is cancellative. It is easy to see that a cancellative 3-graph is one that contains no copy of

$$F_5 = \{123, 124, 345\} \quad \text{and} \quad K_4^- = \{123, 124, 234\}.$$

Katona conjectured, and Bollobás [7] proved that the maximum number of edges in a cancellative 3-graph with n vertices is $s(n)$. Let us call F_5 a 3-graph triangle. Later Frankl and Füredi [12] sharpened Bollobás' theorem by proving that $\text{ex}(n, F_5) = s(n)$ for $n > 3000$ (this was improved to $n > 33$ in [15]). Moreover, both results proved that the unique triangle-free 3-graph on $[n]$ with $s(n)$ edges is 3-partite with parts sizes differing by at most one. This leads us to a natural structure for most triangle-free 3-graphs. Denote by $T(n)$ the number of 3-partite 3-graphs on $[n]$. Our main result is the upper bound in the following theorem.

Theorem 1. *Almost all F_5-free 3-graphs on $[n]$ are 3-partite. More precisely there is a constant $C > 0$ such that*

$$T(n) \leq |\text{Forb}(n, F_5)| < \left(1 + 2^{Cn - \frac{2n^2}{45}}\right) T(n). \tag{2}$$

Using the fact that

$$\frac{3^n}{4n^2}2^{s(n)} < T(n) < 3^n 2^{s(n)}$$

(see Lemma 8), for n large, we get the following improvement over (1).

Corollary 1. *As $n \to \infty$,*

$$\log_2 |\mathrm{Forb}(n, F_5)| = s(n) + n \log_2 3 + \Theta(\log n).$$

In a forthcoming paper [5], we shall characterize the structure of almost all F-free 3-graphs, where $F = \{123, 124, 125, 345\}$. Note that such a precise statement as Theorem 1 is rare. As mentioned earlier, the only similar hypergraph result is due to Person and Schacht. Even for graphs, there are rather few results of this type: Prömel and Steger [22] characterized the structure of almost all F-free graphs when F has a color-critical edge; the analogous statement for induced containment was recently proved by Balogh and Butterfield [4]. Recently, Balogh, Bollobás and Simonovits [3] studied several other F's, including $F = K(2, 2, 2)$.

2 Outline of the Proof

The proof of Theorem 1 consists of two parts.

Part 1. In this part our goal is to reduce the problem to 3-graphs that are almost 3-partite. This is formalized in Theorem 2 below. For a 3-graph $\mathcal{H}$ with a 3-partition $P = X \cup Y \cup Z$ of its vertices, say that an edge is *crossing* if it has exactly one point in each part, otherwise say that it is *non-crossing*. Let D_P be the set of non-crossing edges. An *optimal partition* $X \cup Y \cup Z$ of a 3-graph $\mathcal{H}$ is a 3-partition of the vertices of $\mathcal{H}$ which minimizes the number of non-crossing edges. Let $D = D_{\mathcal{H}}$ be the number of non-crossing edges in an optimal partition of $\mathcal{H}$. Define

$$\mathrm{Forb}(n, F_5, \eta) := \{\mathcal{H} \subset [n]^3 : F_5 \not\subset \mathcal{H} \text{ and } D_{\mathcal{H}} \le \eta n^3\}.$$

In words, this is the number of triangle-free 3-graphs $\mathcal{H}$ on $[n]$ that are almost 3-partite, where almost 3-partite means that $D_{\mathcal{H}} \le \eta n^3$. The goal of Part 1 is the following theorem.

Theorem 2. *For every $\eta > 0$, there exists $\nu > 0$ and n_0 such that if $n > n_0$, then*

$$|\mathrm{Forb}(n, F_5) - \mathrm{Forb}(n, F_5, \eta)| < 2^{(1-\nu)\frac{n^3}{27}}.$$

We will prove Theorem 2 in Section 5. The proof uses the strong hypergraph regularity lemma. Before going further, let us briefly discuss the method of Person and Schacht [21] (the next section contains more details about this): the key property that they used was the linearity of the Fano plane, namely the fact that every two edges of the Fano plane share at most one vertex. This enabled them to apply the (weak) 3-graph regularity lemma, which is almost identical to Szemerédi's regularity lemma. They then applied an embedding lemma for linear hypergraphs proved recently by Kohayakawa-Nagle-Rödl-Schacht [17].

It is well-known that such an embedding lemma fails to hold for non-linear 3-graphs unless one uses the (strong) 3-graph regularity lemma, and operating in this environment is more complicated. Our first contribution (via Theorem 2) is to address the situation for the particular non-linear F_5 using the strong 3-graph regularity lemma.

Part 2. In this part we fix a 3-partition P and consider hypergraphs $\mathcal{H} \in \mathrm{Forb}(n, F_5, \eta)$ for which P is an optimal partition. We then prove various properties about the edge-distribution of most of these $\mathcal{H}$. Many of these properties are enjoyed by random 3-graphs with positive density. For example, in Section 6.2 (Lemma 9) we show that most $\mathcal{H} \in \mathrm{Forb}(n, F_5, \eta)$ have some lower-regularity properties. In Section 6.3 (Lemma 10) we prove that in most $\mathcal{H} \in \mathrm{Forb}(n, F_5, \eta)$, no vertex lies in many non-crossing edges. Finally, in Section 6.4 we prove by induction on n, that in most $\mathcal{H} \in \mathrm{Forb}(n, F_5, \eta)$ there is no non-crossing edge; the idea here is that a non-crossing edge forces too many restrictions on the edges incident to it.

As an introduction to these ideas and the general structure of the argument, we will devote the next section to sketching the ideas of the Person-Schacht [21] proof.

3 Warmup: Counting Fano-free 3-graphs

Say that a 3-graph is 2-*colorable* if it has a vertex partition into two parts so that every edge intersects both parts. In this section we will illustrate our general proof structure by sketching the Person-Schacht proof that almost all 3-graphs with vertex set $[n]$ that contain no copy of the Fano plane are 2-colorable.

Let us first introduce the weak Szemerédi regularity lemma for 3-graphs. In a 3-graph $\mathcal{H}$, let X, Y, Z be disjoint sets of vertices. Write $e(X, Y, Z)$ for the number of edges with one point in each of X, Y, Z. The *density* of (X, Y, Z) is

$$d(X, Y, Z) = \frac{e(X, Y, Z)}{|X||Y||Z|}.$$

Given $\epsilon > 0$, the triple (X, Y, Z) is ϵ-*regular* if for every choice of $X' \subset X$, $Y' \subset Y$, $Z' \subset Z$ with

$$|X'| \geq \epsilon|X|, \qquad |Y'| \geq \epsilon|Y|, \qquad |Z'| \geq \epsilon|Z|$$

we have

$$d(X', Y', Z') = d(X, Y, Z) \pm \epsilon.$$

Consider a partition $P = V_0 \cup V_1 \cup \cdots \cup V_k$ of $V(\mathcal{H})$. We say that P is an ϵ-*regular partition* if

1) $|V_0| < \epsilon|V|$,

2) $|V_1| = \cdots = |V_k|$,

3) the triple (V_i, V_j, V_l) is ϵ-regular for all but ϵk^3 choices of i, j, l.

With this notation we can state the weak version of the celebrated Szemerédi regularity lemma for 3-graphs.

Lemma 1. (regularity lemma – weak version) *For every $\epsilon > 0$ and every integer $k_0 \geq 1$ there exist integers K, N_0 such that every 3-graph of order at least N_0 admits an ϵ-regular partition $V_0, \ldots, V_k$ with $k_0 \leq k \leq K$.*

Associated to a regularity partition and any $\alpha_0 \in (0,1]$, there is a cluster 3-graph $\mathcal{J} = \mathcal{J}(\mathcal{H})$ with parameters ϵ, α_0, N_0: $V(\mathcal{J}) = [k]$ and $ijl \in \mathcal{J}$ if (V_i, V_j, V_l) is ϵ-regular with density at least α_0. In addition to the regularity lemma we need the following embedding lemma recently proved by Kohayakawa-Nagle-Rödl-Schacht [17]. Recall that a hypergraph is linear if every two edges have at most one point in common.

Lemma 2. (embedding lemma) *For all $\alpha_0 \in (0,1]$ and linear 3-graph $\mathcal{F}$, there exists ϵ_0 and n_0 such that if $\mathcal{J}$ is the cluster 3-graph of $\mathcal{H}$ with parameters $\epsilon \leq \epsilon_0, \alpha_0$ and $N_0 \geq n_0$, and $\mathcal{F} \subset \mathcal{J}$, then $\mathcal{F} \subset \mathcal{H}$.*

Now let us recall some definitions needed for the proof of the Person-Schacht theorem. For a 3-graph $\mathcal{H}$ with a 2-partition P of its vertices, say that an edge is *crossing* if it intersects both parts, otherwise say that it is *non-crossing*. Let D_P be the set of non-crossing edges. An *optimal partition* $X \cup Y$ of a 3-graph $\mathcal{H}$ is a 2-partition of the vertices of $\mathcal{H}$ which minimizes the number of non-crossing edges. Let $D = D_{\mathcal{H}}$ be the number of non-crossing edges in an optimal partition. Let us denote the Fano plane by $\mathbf{F}$. Define

$$\mathrm{Forb}(n, \mathbf{F}, \eta) := \{\mathcal{H} \subset [n]^3 : \mathbf{F} \not\subset \mathcal{H} \text{ and } D_{\mathcal{H}} \leq \eta n^3\}.$$

In words, this is the number of $\mathbf{F}$-free 3-graphs $\mathcal{H}$ on $[n]$ that are almost 2-colorable, where almost 2-colorable means that $D_{\mathcal{H}} \leq \eta n^3$.

We need two more results about the Fano plane, due to Füredi-Simonovits [14] and Keevash-Sudakov [16]. Let

$$b(n) = \binom{n}{3} - \binom{\lceil n/2 \rceil}{3} - \binom{\lfloor n/2 \rfloor}{3} \sim \frac{n^3}{8}$$

be the maximum number of edges in a 2-colorable 3-graph.

Lemma 3. (Fano extremal) *For n sufficiently large, the maximum number of edges in an $\mathbf{F}$-free 3-graph on n vertices is $b(n)$.*

Lemma 4. (Fano stability) *For all $\nu_2 > 0$ there exists $\nu_1 > 0$ such that every $\mathbf{F}$-free 3-graph with n vertices and at least $b(n) - \nu_1 n^3$ edges has a vertex 2-partition where the number of non-crossing edges is at most $\nu_2 n^3$.*

Our first goal is the following theorem.

Theorem 3. *For every $\eta > 0$, there exists $\nu_1 > 0$ and n_0 such that if $n > n_0$, then*

$$|\mathrm{Forb}(n, \mathbf{F}) - \mathrm{Forb}(n, \mathbf{F}, \eta)| < 2^{(1-\nu_1)\frac{n^3}{8}}.$$

Proof. (Sketch) Given $\eta > 0$, choose constants satisfying the following hierarchy:

$$1/n_0 \ll \epsilon \ll 1/k_0 \ll \alpha_0 \ll \nu_1 \ll \nu_2 \ll \eta.$$

For each $\mathcal{H} \in \mathrm{Forb}(n, \mathbf{F}) - \mathrm{Forb}(n, \mathbf{F}, \eta)$, take an ϵ-regular partition $P = P(\mathcal{H}) = V_0 \cup \ldots \cup V_k$ where $k_0 \leq k \leq K$. Define $\mathcal{J} = \mathcal{J}(\mathcal{H})$ to be the cluster 3-graph of $\mathcal{H}$ associated to this partition, i.e. $V(\mathcal{J}) = [k]$ and $ijl \in \mathcal{J}$ if (V_i, V_j, V_l) is ϵ-regular with density at least α_0. Now let us define an equivalence relation on $\mathrm{Forb}(n, \mathbf{F}) - \mathrm{Forb}(n, \mathbf{F}, \eta)$ as follows: $\mathcal{H}_1 \sim \mathcal{H}_2$ if $P(\mathcal{H}_1) = P(\mathcal{H}_2)$ and $\mathcal{J}(\mathcal{H}_1) = \mathcal{J}(\mathcal{H}_2)$. The number of equivalence classes is at most $n^K \times 2^{K^3} < 2^{\nu_1 n^3/8}$. Let us fix an equivalence class C and estimate $|C|$.

Pick $\mathcal{H} \in C$. Then Lemma 2 and Lemma 3 imply that $|\mathcal{J}(\mathcal{H})| \leq \mathrm{ex}(k, \mathbf{F}) = b(k) \leq k^3/8$. Let us go further and show that in fact $|\mathcal{J}(\mathcal{H})| < (1 - 8\nu_1)k^3/8$. Indeed, if this is not the case, then by Lemma 4 and Lemma 2, $\mathcal{J}(\mathcal{H})$ has a vertex partition $X \cup Y$ such that the number of edges in $[X]^3 \cup [Y]^3$ is at most $\nu_2 k^3$. Now consider the vertex partition $V_X \cup V_Y$ of $\mathcal{H}$, where $V_X = \bigcup_{i \in X} V_i$ and $V_Y = V_0 \cup \bigcup_{i \in Y} V_i$. Then short calculations shows that the number of edges in each of $\mathcal{H}[V_X]$, $\mathcal{H}[V_Y]$ is at most $3\nu_2 n^3$. Therefore $D_{\mathcal{H}} < 6\nu_2 n^3 < \eta n^3$ contradicting the fact that $\mathcal{H} \notin \mathrm{Forb}(n, \mathbf{F}, \eta)$. We have shown above that each $\mathcal{H} \in C$ satisfies $|\mathcal{J}(\mathcal{H})| < (1 - 8\nu_1)k^3/8$. Using this fact, one can show that

$$|C| \leq 2^{(1-8\nu_1)n^3/8} \times \sum_{t=1}^{2\alpha_0 n^3} \binom{n^3}{t} < 2^{(1-7\nu)n^3/8}.$$

Multiplying by the number of equivalence classes, we get that

$$|\mathrm{Forb}(n, \mathbf{F}) - \mathrm{Forb}(n, \mathbf{F}, \eta)| \leq 2^{\nu_1 n^3/8} \times 2^{(1-7\nu_1)n^3/8} < 2^{(1-\nu_1)n^3/8}.$$

This completes the proof. □

It now suffices to consider **F**-free 3-graphs that are almost 2-colorable. We will very briefly sketch the ideas in this part of the Person-Schacht proof. They prove each of the claims in the following steps.

Step 1: Most of the 3-graphs under consideration after Theorem 3 have part sizes that are very close to $n/2$. This follows from a standard counting argument.

Step 2: Most of the 3-graphs under consideration after Step 1 satisfy the following property: if our 3-graph $\mathcal{H}$ has an optimal partition $X \cup Y$, and $A \subset X, B \subset Y, C \in Z$ where $Z \in \{X, Y\}$ (A, B, C pairwise disjoint), and each of A, B, C is not too small, then $e(A, B, C) > |A||B||C|/4$. Note that this is within a factor 2 of what we would expect in random 3-graphs. Again this follows by a counting argument, which computes an upper bound for the number of 3-graphs that fail to satisfy this property.

Note that Steps 1 and 2 have not used any specific property about $\mathbf{F}$.

Step 3: Every 3-graph under consideration after Step 2 has the following property: in any optimal partition every vertex has only a few non-crossing edges. This is proved by showing that if a vertex in one part, say X, has many edges within X, then by optimality of the partition, it also has many edges that hit the other part Y. Using these edges and the edges that one is guaranteed by Step 2, we can find a copy of $\mathbf{F}$ in $\mathcal{H}$, which is a contradiction.

Once Step 3 has been proved the necessary structure on $\mathcal{H}$ has been obtained to do an induction argument that shows that most $\mathcal{H}$ have no edges within a part. Very informally, if even one edge within a part was present, then this would exclude many other edges between both parts, and the number of $\mathcal{H}$ with these excluded edges is small.

Our proof, which begins in the next section, requires more complicated versions of all the ingredients presented in this section. To begin with, we need to use the strong hypergraph regularity lemma since F_5 is not linear, and this complicates matters pertaining to cluster 3-graphs which are not present in the same way as in the weak setup; this is accomplished in Sections 4 and 5. After this, we need to prove stronger structural properties than those in Steps 2 and 3 as it is harder to find a copy of F_5 than to find a copy of $\mathbf{F}$ due to the non-linearity of F_5. Instead of Step 2, we need the more complicated notions defined in Section 6.2, and instead of Step 3 we need more detailed information about link graphs described in Section 6.3.

4 Hypergraph Regularity

In this section, we quickly define the notions required to state the hypergraph regularity lemma. These concepts will be used in Section 5 to prove Theorem 2. Further details can be found in [13] or [19].

Given a k-partite graph G with k-partition $V_1, \ldots, V_k$, we write $G = \bigcup_{i<j} G^{ij}$, where $G^{ij} = G[V_i \cup V_j]$ is the bipartite subgraph of G with parts V_i and V_j. For $B \in [k]^3$, the 3-partite graph $G(B) = \bigcup_{\{i,j\} \in [B]^2} G^{ij}$ is called a *triad*. For a bipartite graph G, the *density* of the pair V_1, V_2 with respect to G is $d_G(V_1, V_2) = \frac{|G^{12}|}{|V_1||V_2|}$.

Given an integer $l > 0$ and real $\epsilon > 0$, a k-partite graph G is called an $(\epsilon, 1/l)$-*regular* k-partite graph if for every $i < j$, G^{ij} is ϵ-regular with density $(1/l)(1 \pm \epsilon)$. For a k-partite graph G, let $\mathcal{K}_3(G)$ denote the 3-graph with vertex set $V(G)$ whose edges correspond to triangles of G. An easy consequence of these definitions is the following fact.

Lemma 5. (triangle counting lemma) *For integer $l > 0$ and real $\theta > 0$, there exists $\epsilon > 0$*

such that every $(\epsilon, 1/l)$-regular k-partite G with $|V_i| = m$ for all i satisfies

$$|\mathcal{K}_3(G)| = (1 \pm \theta)\frac{m^3}{l^3}.$$

Consider a k-partite 3-graph $\mathcal{H}$ with k-partition $V_1, \ldots, V_k$. Here k-partite means that every edge of $\mathcal{H}$ has at most one point in each V_i. Often we will say that these edges are crossing, and the edges that have at least two points in some V_i are non-crossing. Given $B \in [k]^3$, let $\mathcal{H}(B) = \mathcal{H}[\cup_{i \in B} V_i]$. Given k-partite graph G and k-partite 3-graph $\mathcal{H}$ with the same vertex partition, say that G *underlies* $\mathcal{H}$ if $\mathcal{H} \subset \mathcal{K}_3(G)$. In other words, every edge of $\mathcal{H}$ is a triangle in G. Define the density $d_{\mathcal{H}}(G(B))$ of $\mathcal{H}$ with respect to $G(B)$ as follows:

$$d_{\mathcal{H}}(G(B)) = \frac{|\mathcal{H}(B)|}{|\mathcal{K}_3(G(B))|}$$

if $|\mathcal{K}_3(G(B))| > 0$ and 0 otherwise. Informally, $d_{\mathcal{H}}(G(B))$ is the proportion of triangles in $G(B)$ that are edges of $\mathcal{H}$.

This definition leads to the more complicated definition of $\mathcal{H}$ being (δ, r)-regular with respect to the triad $G(B)$, where $r > 0$ is an integer and $\delta > 0$. We will not state this definition here and it suffices to take this definition as a "black box" that will be used later.

If $\mathcal{H}$ is (δ, r)-regular with respect to $G(B)$ and $d_{\mathcal{H}}(G(B)) = \alpha \pm \delta$, then say that $\mathcal{H}$ is (α, δ, r)-*regular* with respect to $G(B)$.

For a vertex set V, an (l, t, γ, ϵ)-*partition* $\mathcal{P}$ is a partition $V = V_0 \cup V_1 \cup \cdots \cup V_t$ together with a collection of edge-disjoint bipartite graphs P_a^{ij}, where $1 \le i < j \le t, 0 \le a \le l_{ij} \le l$ that satisfy the following properties:

(i) $|V_0| < t$ and $|V_i| = \lfloor \frac{n}{t} \rfloor := m$ for each $i > 0$,

(ii) $\bigcup_{a=0}^{l_{ij}} P_a^{ij} = K(V_i, V_j)$ for all $1 \le i < j \le t$, where $K(V_i, V_j)$ is the complete bipartite graph with parts V_i, V_j,

(iii) all but $\gamma\binom{t}{2}m^2$ pairs $\{v_i, v_j\}$, $v_i \in V_i, v_j \in V_j$, are edges of ϵ-regular bipartite graphs P_a^{ij}, and

(iv) for all but $\gamma\binom{t}{2}$ pairs $\{i, j\} \in [t]^2$, we have $|P_0^{ij}| \le \gamma m^2$ and $d_{P_a^{ij}}(V_i, V_j) = (1 \pm \epsilon)\frac{1}{l}$ for all $a \in [l_{ij}]$.

Finally, suppose that $\mathcal{H} \subset [n]^3$ is a 3-graph and $\mathcal{P}$ is an (l, t, γ, ϵ)-partition. For $B = \{i, j, l\}$, say that $G(B) = P_{a_1}^{ij} \cup P_{a_2}^{jl} \cup P_{a_3}^{il}$ is a (δ, r)-*regular triad* of $\mathcal{P}$ if $\mathcal{H}$ is (δ, r)-*regular* with respect to $G(B)$. Then $\mathcal{P}$ is (δ, r)-regular if

$$\sum\{|\mathcal{K}_3(G(B))| : G(B) \text{ is not a } (\delta, r)\text{-regular triad of } \mathcal{P}\} < \delta n^3.$$

We can now state the regularity lemma due to Frankl and Rödl [13].

Theorem 4. (regularity lemma) *For every δ, γ with $0 < \gamma \le 2\delta^4$, for all integers t_0, l_0 and for all integer-valued functions $r = r(t,l)$ and all functions $\epsilon(l)$, there exist T_0, L_0, N_0 such that every 3-graph $\mathcal{H} \subset [n]^3$ with $n \ge N_0$ admits a $(\delta, r(t,l))$-regular $(l, t, \gamma, \epsilon(l))$-partition for some t, l satisfying $t_0 \le t < T_0$ and $l_0 \le l < L_0$.*

To apply the regularity lemma above, we need to define a cluster hypergraph and state an accompanying embedding lemma, sometimes called the key lemma. Given a 3-graph $\mathcal{F}$, let $\partial\mathcal{F}$ be the set of pairs that lie in an edge of $\mathcal{F}$.

Cluster 3-graph. For given constants $k, \delta, l, r, \epsilon$ and sets $\{\alpha_B : B \in [k]^3\}$ of non-negative reals, let $\mathcal{H}$ be a k-partite 3-graph with parts $V_1, \ldots, V_k$, each of size m. Let G be a graph, and $\mathcal{F} \subset [k]^3$ be a 3-graph such that the following conditions are satisfied.

(i) $G = \bigcup_{\{i,j\} \in \partial\mathcal{F}} G^{ij}$ underlies $\mathcal{H}$ and for all $\{i,j\} \in \partial\mathcal{F}$, G^{ij} is $(\epsilon, 1/l)$-regular.

(ii) For each $B \in \mathcal{F}$, $\mathcal{H}(B)$ is (α_B, δ, r)-regular with respect to the triad $G(B)$.

Then we say that $\mathcal{F}$ is a *cluster 3-graph* of $\mathcal{H}$.

Lemma 6. (embedding lemma) *Let $k \ge 4$ be fixed. For all $\alpha > 0$, there exists $\delta > 0$ such that for all integers $l > \frac{1}{\delta}$, there exists an integer $r > 0$ and $\epsilon > 0$ such that the following holds: Suppose that $\mathcal{F}$ is a cluster 3-graph of $\mathcal{H}$ with underlying graph G and parameters $k, \delta, l, r, \epsilon, \{\alpha_B : B \in [k]^3\}$ where $\alpha_B \ge \alpha$ for all $B \in \mathcal{F}$. Then $\mathcal{F} \subset \mathcal{H}$.*

The embedding lemma is an easy consequence of the counting lemma, which finds not just one but many copies of $\mathcal{F}$ in $\mathcal{H}$. Though for our purposes we need only the weaker statement of the embedding lemma (for a proof of the embedding lemma, see [19]).

5 Most F_5-free 3-graphs are almost tripartite

In this section we will prove Theorem 2. We will need the following stability result proved in [15].

Theorem 5. (Keevash-Mubayi [15]) *For every $\nu'' > 0$, there exist $\nu' > 0$ and an integer t_2 such that every F_5-free 3-graph on $t > t_2$ vertices and at least $(1 - 2\nu')\frac{t^3}{27}$ edges has a 3-partition for which the number of non-crossing edges is at most $\nu'' t^3$.*

Given $\eta > 0$, we will define a set of constants that will obey the following hierarchy:

$$\eta \gg \nu'' \gg \nu' \gg \nu \gg \sigma, \theta \gg \alpha_0, \frac{1}{t_0} \gg \delta \gg \gamma > \frac{1}{l_0} \gg \frac{1}{r}, \epsilon \gg \frac{1}{n_0}.$$

Before proceeding with further details regarding our constants, we define the *binary entropy function* $H(x) := -x \log_2 x - (1-x)\log_2(1-x)$. We use the fact that for $0 < x < 0.5$ we have

$$\binom{n}{xn} < 2^{H(x)n}.$$

Additionally, if x is sufficiently small then

$$\sum_{i=0}^{xn} \binom{n}{i} < 2^{H(x)n}. \tag{3}$$

Detailed definition of constants.

Set

$$\nu'' = \frac{\eta}{1000} \tag{4}$$

and suppose that ν_1' and t_2 are the outputs of Theorem 5 with input ν''. Put

$$\nu' = \min\{\nu_1', \nu''\} \quad \text{and} \quad \nu = (\nu')^4. \tag{5}$$

We choose

$$\theta = \frac{\nu}{4(1-\nu)}. \tag{6}$$

Choose σ_1 small enough so that

$$\left(1 - \frac{\nu}{2}\right)\frac{n^3}{27} + o(n^3) + H(\sigma_1)n^3 \leq \left(1 - \frac{\nu}{3}\right)\frac{n^3}{27} \tag{7}$$

holds for sufficiently large n. In fact the function denoted by $o(n^3)$ will actually be seen to be of order $O(n^2)$ so (7) will hold for sufficiently large n. Choose σ_2 small enough so that (3) holds for σ_2. Let

$$\sigma = \min\{\sigma_1, \sigma_2\}.$$

Next we consider the triangle counting lemma (Lemma 5) which provides an ϵ for each θ and l. Since θ is fixed, we may let $\epsilon_1 = \epsilon_1(l)$ be the output of Lemma 5 for each integer l.

For σ defined above, set

$$\delta_1 = \alpha_0 = \frac{\sigma}{100} \quad \text{and} \quad t_1 = \left\lceil \frac{1}{\delta_1} \right\rceil. \tag{8}$$

Let

$$t_0 = \max\{t_1, t_2, 33\}.$$

Now consider the embedding lemma (Lemma 6) with inputs $k = 5$ and α_0 defined above. The embedding lemma gives $\delta_2 = \delta_2(\alpha_0)$, and we set

$$\delta = \min\{\delta_1, \delta_2\}, \qquad \gamma = \delta^4, \qquad l_0 = \frac{2}{\delta}. \tag{9}$$

For each integer $l > \frac{1}{\delta}$, let $r = r(l)$ and $\epsilon_2 = \epsilon_2(l)$ be the outputs of Lemma 6. Set

$$\epsilon = \epsilon(l) = \min\{\epsilon_1(l), \epsilon_2(l)\}. \tag{10}$$

With these constants, the regularity lemma (Theorem 4) outputs N_0. We choose n_0 such that $n_0 > N_0$ and every $n > n_0$ satisfies (7).

Proof of the Theorem 2.

We will prove that

$$|\mathrm{Forb}(n, F_5) - \mathrm{Forb}(n, F_5, \eta)| < 2^{(1-\frac{\nu}{3})\frac{n^3}{27}}.$$

This is of course equivalent to Theorem 2. The initial part of the proof that follows is similar to the proof of [19], though there is a slight difference in how we define equivalence classes. Starting from Lemma 7 most of the ideas are new.

For each $\mathcal{G} \in \mathrm{Forb}(n, F_5) - \mathrm{Forb}(n, F_5, \eta)$, we use the hypergraph regularity lemma, Theorem 4, to obtain a (δ, r)-regular (l, t, γ, ϵ)-partition $\mathcal{P} = \mathcal{P}(\mathcal{G})$. The input constants for Theorem 4 are as defined above and then Theorem 4 guarantees constants T_0, L_0, N_0 so that every 3-graph $\mathcal{G}$ on $n > N_0$ vertices admits a (δ, r)-regular (l, t, γ, ϵ)-partition $\mathcal{P}$ where $t_0 \leq t \leq T_0$ and $l_0 \leq l \leq L_0$. We may assume that $\mathcal{P}$ has vertex partition $[n] = V_0 \cup V_1 \cup \cdots \cup V_t$, $|V_i| = m = \lfloor \frac{n}{t} \rfloor$ for all $i \geq 1$, and system of bipartite graphs P_a^{ij}, where $1 \leq i < j \leq t, 0 \leq a \leq l_{ij} \leq l$.

Let $\mathcal{E}_0 \subset \mathcal{G}$ be the set of triples that either

(i) intersect V_0, or

(ii) have at least two points in some $V_i, i \geq 1$, or

(iii) contain a pair in P_0^{ij} for some i, j, or

(iv) contain a pair in some P_a^{ij} that is not ϵ-regular with density $\frac{1}{l}$.

By the properties of an (l, t, γ, ϵ)-partition

$$|\mathcal{E}_0| \leq tn^2 + t\left(\frac{n}{t}\right)^2 n + \gamma \binom{t}{2} m^2 n + 2\gamma \binom{t}{2} \left(\frac{n}{t}\right)^2 n.$$

Let $\mathcal{E}_1 \subset \mathcal{G} - \mathcal{E}_0$ be the set of triples $\{v_i, v_j, v_k\}$ such that either

• the three bipartite graphs of $\mathcal{P}$ associated with the pairs within the triple form a triad $G(B)$ that is not (δ, r)-regular with respect to $\mathcal{G}(\{i, j, k\})$, or

• the density $d_{\mathcal{G}}(G(B)) < \alpha_0$.

Then

$$|\mathcal{E}_1| \leq \delta n^3 + \alpha_0 n^3.$$

Let $\mathcal{E}_{\mathcal{G}} = \mathcal{E}_0 \cup \mathcal{E}_1$. Now (8) and (9) imply that

$$|\mathcal{E}_{\mathcal{G}}| \leq \sigma n^3.$$

Set $\mathcal{G}' = \mathcal{G} - \mathcal{E}_{\mathcal{G}}$. Next define $\mathcal{J} = \mathcal{J}(\mathcal{G}) \subset [t]^3 \times [l] \times [l] \times [l]$ as follows: For $1 \leq i < j < k \leq t$, $1 \leq a, b, c \leq l$, we have $(\{i, j, k\}, a, b, c) \in \mathcal{J}$ if and only if

• $G = P_a^{ij} \cup P_b^{jk} \cup P_c^{ik}$ is $(\epsilon, 1/l)$-regular, and

• $\mathcal{G}'(\{i, j, k\})$ is $(\overline{\alpha}, \delta, r)$-regular with respect to G, where $\overline{\alpha} \geq \alpha_0$.

From now on we shall replace the cumbersome notation $(\{i,j,k\},a,b,c)$ by (ijk,abc).

For each $\mathcal{G} \in \mathrm{Forb}(n,F_5,\eta) - \mathrm{Forb}(n,F_5,\eta)$, choose one (δ,r)-regular (l,t,γ,ϵ)-partition $\mathcal{P} = \mathcal{P}(\mathcal{G})$ guaranteed by Theorem 4, and let $\mathcal{P} = \{\mathcal{P}_1,\ldots,\mathcal{P}_p\}$ be the set of all such partitions over the family $\mathrm{Forb}(n,F_5,\eta) - \mathrm{Forb}(n,F_5,\eta)$. Note also that once we have defined $\mathcal{P}(\mathcal{G})$ we have also defined $\mathcal{J} = \mathcal{J}(\mathcal{G})$. Define an equivalence relation on $\mathrm{Forb}(n,F_5,\eta) - \mathrm{Forb}(n,F_5,\eta)$ by letting $\mathcal{G}_1 \sim \mathcal{G}_2$ iff

- $\mathcal{P}(\mathcal{G}_1) = \mathcal{P}(\mathcal{G}_2)$ and
- $\mathcal{J}(\mathcal{G}_1) = \mathcal{J}(\mathcal{G}_2)$.

The number of equivalence classes q is the number of partitions times the number of choices of $\mathcal{J} \subset [t]^3 \times [l] \times [l] \times [l]$. The number of partitions satisfies

$$p \le \left(\sum_{t=t_0}^{T_0} t^n\right)\left(\binom{T_0+1}{2}\sum_{l=l_0}^{L_0}(l+1)\right)^{\binom{n}{2}}.$$

Consequently,

$$q \le T_0^{n+1}\left(\binom{T_0+1}{2}(L_0+1)^2\right)^{\binom{n}{2}} 2^{\binom{T_0+1}{3}(L_0+1)^3} < 2^{O(n^2)}.$$

We will show that each equivalence class $C(\mathcal{P}_i,\mathcal{J})$ satisfies

$$|C(\mathcal{P}_i,\mathcal{J})| = 2^{(1-\frac{\nu}{2})\frac{n^3}{27}+H(\sigma)n^3}. \tag{11}$$

Combined with the upper bound for q above and (7), we obtain

$$|\mathrm{Forb}(n,F_5,\eta)| \le 2^{O(n^2)}2^{(1-\frac{\nu}{2})\frac{n^3}{27}+H(\sigma)n^3} \le 2^{(1-\frac{\nu}{3})\frac{n^3}{27}}.$$

For the rest of the proof, we fix an equivalence class $C = C(\mathcal{P}_i,\mathcal{J})$ and we will show the upper bound in (11).

We view $\mathcal{J}$ as a multiset of triples on $[t]$. For each $\phi : [t]^2 \to [l]$, let $\mathcal{J}_\phi \subset \mathcal{J}$ be the 3-graph on $[t]$ with edge set

$$\{\{i,j,k\} : (ijk, \phi(\{i,j\})\phi(\{j,k\})\phi(\{i,k\})) \in \mathcal{J}\}.$$

In other words, $\{i,j,k\} \in \mathcal{J}_\phi$ iff the triples of $\mathcal{G}$ that lie on top of the triangles of $P_a^{ij} \cup P_b^{jk} \cup P_c^{ik}$, $a = \phi(ij)$, $b = \phi(jk)$, $c = \phi(ik)$, are $(\overline{\alpha},\delta,r)$-regular and the underlying bipartite graphs $P_a^{ij}, P_b^{jk}, P_c^{ik}$ are all $(\epsilon,1/l)$-regular.

By our choice of the constants in (9) and (10), and by the construction of $\mathcal{J}$, for a fixed ϕ, any five vertex 3-graph $\mathcal{F} \subset \mathcal{J}_\phi$ is a cluster 3-graph for $\mathcal{G}$, and hence by the embedding lemma

$\mathcal{F} \subset \mathcal{G}$. Since $F_5 \not\subset \mathcal{G}$, we conclude that $F_5 \not\subset \mathcal{J}_\phi$. It was shown in [15] that for $t \geq 33$, we have $\mathrm{ex}(t, F_5) \leq \frac{t^3}{27}$. Since we know that $t \geq 33$, we conclude that

$$|\mathcal{J}_\phi| \leq \mathrm{ex}(t, F_5) \leq \frac{t^3}{27}$$

for each $\phi : [t]^2 \to [l]$. Recall from (5) that $\nu' = \nu^{1/4}$.

Lemma 7. *Suppose that* $|\mathcal{J}| > (1-\nu)\frac{l^3 t^3}{27}$. *Then for at least* $(1-\nu')l^{\binom{t}{2}}$ *of the functions* $\phi : [t]^2 \to [l]$ *we have*

$$|\mathcal{J}_\phi| \geq (1-\nu')\frac{|\mathcal{J}|}{l^3}.$$

Proof. Form the following bipartite graph: the vertex partition is $\Phi \cup \mathcal{J}$, where

$$\Phi = \left\{\phi : [t]^2 \to [l]\right\}$$

and the edges are of the form $\{\phi, (ijk, abc)\}$ if and only if $\phi \in \Phi$, $(ijk, abc) \in \mathcal{J}$ where $\phi(\{i,j\}) = a,\ \phi(\{j,k\}) = b,\ \phi(\{i,k\}) = c$. Let E denote the number of edges in this bipartite graph. Since each $(ijk, abc) \in \mathcal{J}$ has degree precisely $l^{\binom{t}{2}-3}$, we have

$$E = |\mathcal{J}| l^{\binom{t}{2}-3}.$$

Note that the degree of ϕ is $|\mathcal{J}_\phi|$. Suppose for contradiction that the number of ϕ for which $|\mathcal{J}_\phi| \geq (1-\nu')\frac{|\mathcal{J}|}{l^3}$ is less than $(1-\nu')l^{\binom{t}{2}}$. Then since $|\mathcal{J}_\xi| \leq \frac{t^3}{27}$ for each $\xi \in \Phi$, we obtain the upper bound

$$E \leq (1-\nu')l^{\binom{t}{2}}\frac{t^3}{27} + \nu' l^{\binom{t}{2}}(1-\nu')\frac{|\mathcal{J}|}{l^3}.$$

Dividing by $l^{\binom{t}{2}-3}$ then yields

$$|\mathcal{J}| \leq (1-\nu')l^3\frac{t^3}{27} + \nu'(1-\nu')|\mathcal{J}|.$$

Simplifying, we obtain

$$(1-\nu'(1-\nu'))|\mathcal{J}| \leq (1-\nu')l^3\frac{t^3}{27}.$$

The lower bound $|\mathcal{J}| > (1-\nu)\frac{l^3 t^3}{27}$ then gives

$$(1-\nu'(1-\nu'))(1-\nu) < 1-\nu'.$$

Since $\nu' = \nu^{1/4}$, the left hand side expands to

$$1 - \nu' + \nu^{1/2} - \nu + \nu^{5/4} - \nu^{3/2} > 1 - \nu'.$$

This contradiction completes the proof. $\square$

Using Lemma 7 we will prove the following claim.

Claim 1.

$$|\mathcal{J}| \leq (1-\nu)\frac{l^3t^3}{27}.$$

Once we have proved Claim 1, the proof of Theorem 2 is completed by the following argument which is very similar to that in [19]. Define

$$S^C = \bigcup_{(ijk,abc)\in\mathcal{J}} \mathcal{K}_3(P_a^{ij} \cup P_b^{jk} \cup P_c^{ik}).$$

The triangle counting lemma implies that for each $(ijk, abc) \in \mathcal{J}$, $|\mathcal{K}_3(P_a^{ij} \cup P_b^{jk} \cup P_c^{ik})| < \frac{m^3}{l^3}(1+\theta)$. Now Claim 1 and (6) give

$$|S^C| \leq \frac{m}{l^3}(1+\theta)|\mathcal{J}| \leq m^3(1+\theta)(1-\nu)\frac{t^3}{27} < m^3\frac{t^3}{27}\left(1-\frac{\nu}{2}\right) \leq \frac{n^3}{27}\left(1-\frac{\nu}{2}\right).$$

Since $\mathcal{G}' \subset S^C$ for every $\mathcal{G} \in C$,

$$|\{\mathcal{G}' : \mathcal{G} \in C\}| \leq 2^{(1-\frac{\nu}{2})\frac{n^3}{27}}.$$

Each $\mathcal{G} \in C$ can be written as $\mathcal{G} = \mathcal{G}' \cup \mathcal{E}_{\mathcal{G}}$. In view of (3) and $|\mathcal{E}_{\mathcal{G}}| \leq \sigma n^3$, the number of $\mathcal{E}_{\mathcal{G}}$ with $\mathcal{G} \in C$ is at most $\sum_{i\leq\sigma n^3}\binom{n^3}{i} \leq 2^{H(\sigma)n^3}$. Consequently,

$$|C| \leq 2^{(1-\frac{\nu}{2})\frac{n^3}{27}+H(\sigma)n^3}$$

and we are done proving (11).

Proof of Claim 1. Suppose to the contrary that $|\mathcal{J}| > (1-\nu)\frac{l^3t^3}{27}$. We apply Lemma 7 and conclude that for most functions ϕ the corresponding triple system $\mathcal{J}_\phi$ satisfies

$$|\mathcal{J}_\phi| \geq (1-\nu')\frac{|\mathcal{J}|}{l^3} > (1-\nu')(1-\nu)\frac{t^3}{27} > (1-2\nu')\frac{t^3}{27}.$$

By Theorem 5, we conclude that for all of these ϕ, the 3-graph $\mathcal{J}_\phi$ has a 3-partition where the number of non-crossing edges is at most $\nu''t^3$. We also conclude that the number of crossing triples that are not edges of $\mathcal{J}_\phi$ is at most

$$\left(\frac{2\nu'}{27}+\nu''\right)t^3 < \frac{5}{3}\nu''t^3. \tag{12}$$

Fix one such ϕ and let the optimal partition of $\mathcal{J}_\phi$ be $Q_\phi = X \cup Y \cup Z$. Let $Q = V_X \cup V_Y \cup V_Z$ be the corresponding vertex partition of $[n] - V_0$. In other words, V_X consists of the union of all those parts V_i for which $i \in X$ etc. Let Q' be the partition obtained from Q by arbitrarily distributing the vertices of V_0 into the three parts of Q. We will show that Q' is a partition of $[n]$ where the number of non-crossing edges $|D_{Q'}|$ is fewer than ηn^3. This contradicts the fact that $\mathcal{G} \in \mathrm{Forb}(n, F_5) - \mathrm{Forb}(n, F_5, \eta)$ and completes the proof of Theorem 2.

We have argued earlier that $|\mathcal{E}_{\mathcal{G}}| \leq \sigma n^3 \leq \frac{\eta}{2} n^3$. The number of edges of $\mathcal{G}$ that intersect V_0 is at most $|V_0| n^2 \leq t n^2$, so

$$|D_{Q'} - \mathcal{E}_{\mathcal{G}}| \leq |D_Q - \mathcal{E}_{\mathcal{G}}| + t n^2.$$

Consequently, it suffices to prove that

$$|D_Q - \mathcal{E}_{\mathcal{G}}| + t n^2 \leq \frac{\eta}{2} n^3. \tag{13}$$

For each $\xi : [t]^2 \to [l]$, define

$$\mathcal{G}_\xi = \mathcal{G}' \cap \bigcup \left\{ \mathcal{K}_3 \left(P^{ij}_{\xi(\{i,j\})} \cup P^{jk}_{\xi(\{j,k\})} \cup P^{ik}_{\xi(\{i,k\})} \right) : \{i,j,k\} \in \mathcal{J}_\xi \right\}.$$

In other words, $\mathcal{G}_\xi$ is the union, over all $\{i,j,k\} \in \mathcal{J}_\xi$, of the edges of $\mathcal{G}$ that lie on top of the triangles in $P^{ij}_{\xi(\{i,j\})} \cup P^{jk}_{\xi(\{j,k\})} \cup P^{ik}_{\xi(\{i,k\})}$.

Let D_ξ be the set of edges in $\mathcal{G}_\xi$ that are non-crossing with respect to $Q = V_X \cup V_Y \cup V_Z$. We will estimate $|D_Q - \mathcal{E}_{\mathcal{G}}|$ by summing $|D_\xi|$ over all ξ. Please note that each $e \in D_Q - \mathcal{E}_{\mathcal{G}}$ lies in exactly $l^{\binom{t}{2}-3}$ different D_ξ due to the definition of $\mathcal{J}$. Call a $\xi : [t]^2 \to [l]$ *good* if it satisfies the conclusion of Lemma 7, otherwise call it *bad*. In other words, ξ is good iff

$$|\mathcal{J}_\xi| \geq (1 - \nu') \frac{|\mathcal{J}|}{l^3}.$$

Summing over all ξ gives

$$l^{\binom{t}{2}-3} |D_Q - \mathcal{E}_{\mathcal{G}}| = \sum_{\xi : [t]^2 \to [l]} |D_\xi| = \sum_{\xi \ good} |D_\xi| + \sum_{\xi \ bad} |D_\xi|.$$

Note that for a given $\{i,j,k\} \in \mathcal{J}_\xi$ the number of edges in $\mathcal{G}_\xi$ corresponding to $\{i,j,k\}$ is the number of edges in $V_i \cup V_j \cup V_k$ on top of triangles formed by the three bipartite graphs, each of which is ϵ-regular of density $(1/l)(1 \pm \epsilon)$. By the triangle counting lemma, the total number of such triangles is at most

$$2 |V_i| |V_j| |V_k| \left(\frac{1}{l} \right)^3 < 2 \left(\frac{n}{t} \right)^3 \left(\frac{1}{l} \right)^3 := R.$$

By Lemma 7, the number of bad ξ is at most $\nu' l^{\binom{t}{2}}$. So we have

$$\sum_{\xi \ bad} |D_\xi| \leq \nu' l^{\binom{t}{2}} \binom{t}{3} R < \nu' l^{\binom{t}{2}-3} n^3.$$

It remains to estimate $\sum_{\xi \ good} |D_\xi|$.

Fix a good ξ and let the optimal partition of $\mathcal{J}_\xi$ be $Q_\xi = A \cup B \cup C$ (recall that we know the number of non-crossing edges with respect to this partition is less than $\nu'' t^3$).

Claim 2. The number of crossing edges of Q_ξ that are non-crossing edges of Q_ϕ is at most $100 \nu'' t^3$.

Suppose that Claim 2 is true. Then we obtain

$$\sum_{\xi \text{ good}} |D_\xi| \le l^{\binom{t}{2}} \left[100\nu'' t^3 R + \nu'' t^3 R\right] \le l^{\binom{t}{2}-3} \left(202\nu'' n^3\right).$$

Explanation: We consider the contribution from the non-crossing edges of Q_ϕ that are (i) crossing edges of Q_ξ and (ii) non-crossing edges of Q_ξ. We do not need to consider the contribution from the crossing edges of Q_ϕ since by definition, these do not give rise to edges of D_Q.

Altogether, using (4) we obtain

$$|D_Q - \mathcal{E}_\mathcal{G}| + tn^2 \le (202\nu'' + \nu')n^3 + tn^2 < \frac{\eta}{2}n^3$$

which proves (13). We now prove Claim 2.

Proof of Claim 2. Suppose for contradiction that the number of crossing edges of Q_ξ that are non-crossing edges of Q_ϕ is more than $100\nu'' t^3$. Each of these edges intersects at most $3\binom{t}{2}$ other edges of $\mathcal{J}_\xi$, so by the greedy algorithm we can find a collection of at least $50\nu'' t$ of these edges that form a matching M. Pick one such edge $e = \{k, k', k''\} \in M$ and assume that k and k' lie in same part U of Q_ϕ. Let d be the number of ways to choose a set of two 3-element sets $\{f, f'\}$ with the following properties:

- $f = \{i, j, k\}, f' = \{i, j, k'\}$, $i, j \notin U \cup \{k''\}$ and
- i and j lie in distinct parts of Q_ϕ.

In particular, both f, f' lie in the complete 3-partite 3-graph with parts X, Y, Z.

Since $|\mathcal{J}_\phi| > (1 - 2\nu')\frac{t^3}{27}$, $|D_{Q_\phi}| \le \nu'' t^3$ and ν', ν'' are sufficiently small, each part of Q_ϕ has size very close to $t/3$. In particular,

$$d \ge (\min\{|X|, |Y|, |Z|\} - 1)^2 \ge \frac{t^2}{10}.$$

As $\{e, f, f'\} \cong F_5$ there are at least d potential copies of F_5 that we can form using e and two crossing triples f, f' of Q_ϕ. Suppose that $f = \{i, j, k\}, f' = \{i, j, k'\}$ are both in $\mathcal{J}_\phi$ for one such choice of $\{f, f'\}$. Consider the following eight bipartite graphs:

$$G^{ij} = Q^{ij}_{\phi(\{i,j\})}, \quad G^{jk} = Q^{jk}_{\phi(\{j,k\})} \quad G^{ik} = Q^{ik}_{\phi(\{i,k\})} \quad G^{jk'} = Q^{jk'}_{\phi(\{j,k'\})} \quad G^{ik'} = Q^{ik'}_{\phi(\{i,k'\})}$$

$$G^{kk'} = Q^{kk'}_{\xi(\{k,k'\})} \quad G^{k'k''} = Q^{k'k''}_{\xi(\{k',k''\})} \quad G^{kk''} = Q^{kk''}_{\xi(\{k,k''\})}.$$

Set $G = \bigcup G^{uv}$ where the union is over the eight bipartite graphs defined above. Since $\{e, f, f'\} \subset \mathcal{J}_\phi \cup \mathcal{J}_\xi$, the 3-graph $J = \{e, f, f'\}$ associated with G and $\mathcal{G}$ is a cluster 3-graph. By (9) and (10), we may apply the embedding lemma and obtain the contradiction $F_5 \subset \mathcal{G}$. We conclude that $f'' \notin \mathcal{J}_\phi$ for some $f'' \in \{f, f'\}$.

To each $e \in M$ we have associated at least d triples $f'' \notin \mathcal{J}_\phi$. Since M is a matching and $|e \cap f''| = 1$, each such f'' is counted at most three times. Summing over all $e \in M$, we obtain at least $\frac{|M|d}{3} \geq \frac{5}{3}\nu''t^3$ triples f'' that are crossing with respect to P_ϕ but are not edges of $\mathcal{J}_\phi$. This contradicts (12) and completes the proof. □

6 Proof of Theorem 1

In this section we complete the proof of Theorem 1. We begin with some preliminary statements in Section 6.1, and then we follow the outline described in Section 2.

6.1 Inequalities

We shall use Chernoff's inequality as follows:

Theorem 6. *Let $X_1, \ldots, X_m$ be independent $\{0,1\}$ random variables with $P(X_i = 1) = p$ for each i. Let $X = \sum_i X_i$, i.e., X has binomial distribution $B(m,p)$. Then the following inequality holds for $a > 0$:*

$$P(X < \mathbb{E}X - a) < \exp(-a^2/(2pm)).$$

Recall that $T(n)$ is the number of 3-partite 3-graphs with vertex set $[n]$ and $s(n) = \lfloor \frac{n+2}{3} \rfloor \cdot \lfloor \frac{n+1}{3} \rfloor \cdot \lfloor \frac{n}{3} \rfloor$. For a 3-partition (A, B, C) of a 3-graph, and $u \in A, v \in B$, write $L_C(u,v)$ or simply $L(u,v)$ for the set of $w \in C$ such that uvw is an edge. As usual, the multinomial coefficient $\binom{n}{a,b,c} = \frac{n!}{a!b!c!}$.

Lemma 8. *For n sufficiently large, we have*

$$\frac{3^n 2^{s(n)}}{4n^2} < \left(\frac{1}{6} - 0.001\right) \binom{n}{\lfloor \frac{n+2}{3} \rfloor, \lfloor \frac{n+1}{3} \rfloor, \lfloor \frac{n}{3} \rfloor} 2^{s(n)} \; < T(n) \; < \; 3^n 2^{s(n)}. \tag{14}$$

In addition,

$$T(n-2) < \left(n^2 2^{-\frac{2n^2}{9} + n}\right) T(n). \tag{15}$$

Proof. For the upper bound in (14), observe that 3^n counts the number of 3-partitions of the vertices, and the exponent of 2 is the maximum number of crossing edges that a 3-partite 3-graph can have, and each crossing triples can either be an edge or not.

For the lower bound we count the number of (unordered) 3-partitions where this equality can be achieved. Each such 3-partition gives rise to $2^{s(n)}$ 3-partite 3-graphs. The number of such 3-partitions of $[n]$ is at least

$$\frac{1}{6}\binom{n}{\lfloor \frac{n+2}{3} \rfloor, \lfloor \frac{n+1}{3} \rfloor, \lfloor \frac{n}{3} \rfloor}.$$

We argue next that most of the 3-partite 3-graphs obtained in this way are different. More precisely, we show below that for any given 3-partition P as above, most 3-partite 3-graphs with 3-partition P have a unique 3-partition (which must be P). Define the link graph $L_X(u,v)$ to be the set of edges of the form uvx, with $x \in X$. Given a 3-partition (U_1, U_2, U_3) of $[n]$, if the crossing edges are added randomly, then Chernoff's inequality gives that almost all 3-graphs generated satisfy the following two conditions:

(i) for all $u \in U_i, v \in U_j$, where $\{i,j,k\} = \{1,2,3\}$ we have $|L_{U_k}(u,v)| > n/10$,

(ii) for $\{i,j,k\} = \{1,2,3\}$ and for every $A_i \subset U_i, A_j \subset U_j$ with $|A_i|, |A_j| > n/10$ and $v \in U_k$, the number of crossing edges intersecting each of A_i, A_j and containing v is at least $|A_1||A_2|/10$.

If $\mathcal{H}$ has 3-partition (U_1, U_2, U_3) of $[n]$, and it satisfies conditions (i) and (ii), then the 3-partition is unique. Indeed, take u, v lying in an edge, then u, v and $L(u,v)$ are in different parts, where $|L(u,v)| > n/10$, so for $w \in L(u,v)$, $L(u,w)$ is in the same part as v and $L(v,w)$ is in the same part as u. Now by (ii) the rest of the vertices must lie in a unique part. This completes the proof of the middle inequality in (14).

To prove the first inequality in (14) and (15) first note that if $a+b+c=n$, then $\binom{n}{a,b,c}$ is maximized for $a = \lfloor (n+2)/3 \rfloor, b = \lfloor (n+1)/3 \rfloor, c = \lfloor n/3 \rfloor$. This implies that

$$3^n = \sum_{a+b+c=n} \binom{n}{a,b,c} \leq \binom{n+2}{2} \binom{n}{\lfloor \frac{n+2}{3} \rfloor, \lfloor \frac{n+1}{3} \rfloor, \lfloor \frac{n}{3} \rfloor} < (0.6)n^2 \binom{n}{\lfloor \frac{n+2}{3} \rfloor, \lfloor \frac{n+1}{3} \rfloor, \lfloor \frac{n}{3} \rfloor}.$$

This immediately proves the first inequality in (14). Using (14) we now obtain

$$\frac{T(n-2)}{T(n)} < \frac{3^{n-2} 2^{s(n-2)}}{(\frac{1}{6} - o(1)) \binom{n}{\lfloor \frac{n+2}{3} \rfloor, \lfloor \frac{n+1}{3} \rfloor, \lfloor \frac{n}{3} \rfloor} 2^{s(n)}} < n^2 2^{s(n-2)-s(n)}.$$

It is easy to see that $s(n) - s(n-2) \geq 2n^2/9 - n$, and the result follows. □

6.2 Lower Density

In this subsection, we first define a set of four structural conditions tailored to our proof (Definition 7 below) and then prove that most 3-graphs that are almost 3-partite satisfy all of these conditions. Part of the motivation for defining these conditions comes from the fact that in a typical 3-partite 3-graph the density of the crossing edges is close to that in a random 3-graph (so each crossing edge would be present with probability 1/2), but we can afford to relax this condition in order to make the computations somewhat simpler. Our condition (iv) just requires the class sizes to be about the same, (i) is a weak-regularity condition, which actually follows from (ii). Conditions (ii) and (iii) are some versions of the strong (lower) regularity. Recall that $uv \in G$ means that uv is an edge of the graph G, and we write $|G|$ for the number of edges of G.

Definition 7. *A vertex partition (U_1, U_2, U_3) of a 3-graph $\mathcal{F}$ is μ-lower dense if each of the following conditions are satisfied:*

(i) For every i if $A_i \subset U_i$ with $|A_i| \geq \mu n$ then

$$|\{E \in \mathcal{F} : \; |E \cap A_i| = 1, \; \text{for } 1 \leq i \leq 3\} \; | > \; |A_1| \cdot |A_2| \cdot |A_3| \cdot 2^{-2}.$$

(ii) Let $\{i, j, \ell\} = \{1, 2, 3\}$, $A_i \subset U_i$ with $|A_i| \geq \mu n$, $G \subset U_j \times U_\ell$ with $|G| \geq \mu^2 n^2$. Then

$$|\{E \in \mathcal{F} : \; |E \cap A_i| = 1, \; E - A_i \in G\}| \; > \; |A_i| \cdot |G| \cdot 2^{-2}.$$

(iii) Let $\{i, j, \ell\} = \{1, 2, 3\}$, $A_i \subset U_i$ and $A_j \subset U_j$ with $|A_i|, |A_j| \geq \mu n$, and G be a matching on U_ℓ with $|G| \geq \mu n$. Set

$$\mathcal{F}_{A_i,A_j,G} = \{\{C, D\} \in [\mathcal{F}]^2 : C - U_\ell = D - U_\ell, \; |C \cap A_i| = |C \cap A_j| = 1, \{(C \cap U_\ell)(D \cap U_\ell)\} \in G\}.$$

In words, $\mathcal{F}_{A_i,A_j,G}$ is the number of pairs of edges $\{C, D\}$, $C, D \in \mathcal{F}$ where $C = a_i a_j c_\ell$, $D = a_i a_j d_\ell$, $a_i \in A_i, a_j \in A_j$, $\{c_\ell, d_\ell\} \subset U_\ell$, $c_\ell d_\ell \in G$. Then

$$|\mathcal{F}_{A_i,A_j,G}| \geq |A_i| \cdot |A_j| \cdot |G| \cdot 2^{-3}.$$

(iv) For every i we have $||U_i| - n/3| < \mu n$.

For $\mu > 0$ let $\mathrm{Forb}(n, F_5, \eta, \mu) \subset \mathrm{Forb}(n, F_5, \eta)$ be the family of μ-lower dense hypergraphs.

Lemma 9. *For every $\eta > 0$ with $\mu^3 \geq 10^3 H(6\eta)$ if n is large enough then*

$$|\mathrm{Forb}(n, F_5, \eta) - \mathrm{Forb}(n, F_5, \eta, \mu)| \; < \; 2^{n^3(1/27 - \mu^3/40)}.$$

Proof. We wish to count the number of $\mathcal{H} \in \mathrm{Forb}(n, F_5, \eta) - \mathrm{Forb}(n, F_5, \eta, \mu)$. The number of ways to choose a 3-partition of $\mathcal{H}$ is at most 3^n.

Given a particular 3-partition $P = (U_1, U_2, U_3)$, the number of ways the at most ηn^3 non-crossing edges could be placed is at most

$$\sum_{i \leq \eta n^3} \binom{\binom{n}{3}}{i} < 2^{H(6\eta)\binom{n}{3}}.$$

Let

$$f(n, \eta) = 3^n \cdot 2^{H(6\eta)\binom{n}{3}},$$

which upper bounds the number of ways choosing a 3-partition of the vertex set, and for a given partition the number of ways of placing the at most ηn^3 non-crossing edges.

If $|U_i - n/3| > \mu n$ for some i, then the number of possible crossing edges is at most

$$n^3(1/27 - \mu^2/4 + \mu^3/4) < n^3(1/27 - \mu^2/5).$$

We conclude that the number of $\mathcal{H} \in \mathrm{Forb}(n, F_5, \eta) - \mathrm{Forb}(n, F_5, \eta, \mu)$ for which there exists a partition that fails property (iv) is at most

$$f(n, \eta) 2^{n^3(1/27 - \mu^2/5)}.$$

Each $\mathcal{H} \notin \mathrm{Forb}(n, F_5, \eta, \mu)$ fails to satisfy one of the four conditions in Definition 7. For a fixed partition P and choice of non-crossing edges, we may view $\mathcal{H}$ as a probability space where we choose each crossing edge with respect to P independently with probability $1/2$. The total number of ways to choose the crossing edges is at most $2^{n^3/27}$ (an upper bound on the size of the probability space) so we obtain that $|\mathrm{Forb}(n, F_5, \eta) - \mathrm{Forb}(n, F_5, \eta, \mu)|$ is upper bounded by

$$f(n,\eta) \cdot 2^{n^3/27} \cdot \mathit{Prob}(\mathcal{H} \text{ fails (i) or (ii) or (iii)}) + f(n,\eta) 2^{n^3(1/27-\mu^2/5)}.$$

We will consider each of these probabilities separately and then use the union bound. First however, note that the number of choices for $A_i \subset U_i$ as in Definition 7 is at most 2^n and the number of ways G could be chosen is at most 2^{n^2}.

(i) This follows from (ii), for given A_2, A_3 let G be the complete bipartite graph on (A_2, A_3).
(ii) Since $|A_i||G| \geq \mu^3 n^3$, Chernoff's inequality gives

$$\mathit{Prob}(\mathcal{H} \text{ fails (ii)}) \leq 2^n \cdot 2^{n^2} \cdot \exp(-\mu^3 n^3/16).$$

(iii) Since $|A_i||A_j||G| \geq \mu^3 n^3$ and both edges C and D must be present, we apply Chernoff's inequality with $m = |A_i||A_j||G|/2$ and $p = 1/4$. The number of matchings G is at most $(n^2)^{n/2} = 2^{n \log_2 n}$, so

$$\mathit{Prob}(\mathcal{H} \text{ fails (iii)}) \leq 2^{2n} \cdot 2^{n \log_2 n} \cdot \exp(-\mu^3 n^3/32).$$

The lemma now follows since $10^3 H(6\eta) \leq \mu^3$, and n is sufficiently large. □

6.3 There is no vertex with too many non-crossing edges

Let $\mathcal{H} \in \mathrm{Forb}(n, F_5, \eta, \mu)$, where n is large enough, and let (U_1, U_2, U_3) be an optimal partition of $\mathcal{H}$, with $x \in U_1$. For a vertex y let $L_{i,j}(y)$ denote the set of edges of $\mathcal{H}$ containing y, and additionally intersecting U_i and U_j. In particular, $L_{i,i}(y)$ is the set of edges of $\mathcal{H}$ which contain y, and their other vertices are in U_i.

The aim of this subsection is to prove the following lemma, which shows that the number of non-crossing edges containing a vertex is small.

Lemma 10. *Each of the following is satisfied for $x \in U_1$.*
(i) $|L_{1,1}(x)| < 2\mu n^2$.
(ii) $|L_{1,2}(x)| < 2\mu n^2$.
(iii) $|L_{2,2}(x)| < 2\mu n^2$.
(iv) $|L_{1,3}(x)| < 2\mu n^2$.
(v) $|L_{3,3}(x)| < 2\mu n^2$.

Before proving the lemma, let us state the following easy fact which follows from building up a matching using the greedy algorithm.

Lemma 11. *Every graph G with n vertices contains a matching of size at least $\frac{|G|}{2n}$.*

Proof. (of Lemma 10) (i) If $|L_{1,1}(x)| > 2\mu n^2$ then by Lemma 11 $\{E-x :\ E \in L_{1,1}(x)\}$ contains a matching G with size at least μn. Then using Definition 7 (iii) (with $G, A_i = U_2, A_j = U_3$) for $\mathcal{H}$, we find $y, z \in U_1, a \in U_2, b \in U_3$ such that $xyz, yab, zab \in \mathcal{H}$, yielding an $F_5 \subset \mathcal{H}$, a contradiction.

(ii) Suppose for contradiction that $|L_{1,2}(x)| \geq 2\mu n^2$. By the optimality of the partition $|L_{1,2}(x)| \leq |L_{2,3}(x)|$, otherwise x could be moved to U_3 to decrease the number of non-crossing edges. We shall use property (ii) in Definition 7. We use it with $G = \{E - x :\ E \in L_{1,2}(x)\}$ and

$$A_3 = \{z \in U_3 :\ \exists \text{ crossing edges } E_1, E_2 \in \mathcal{H} \text{ with } \{x, z\} \subset E_1 \cap E_2\}.$$

Note that $|A_3| \geq \mu n$ as $|L_{2,3}(x)| \geq 2\mu n^2$. Since $\mathcal{H}$ is μ-lower dense, we find $abz \in \mathcal{H}$ with $xab \in L_{1,2}(x)$ and $z \in A_3$. By definition of A_3, there exists $b' \in U_2 - \{b\}$ such that $xb'z \in L_{2,3}(x)$. This gives us $abx, abz, xb'z \in \mathcal{H}$, forming an F_5.

(iii) Suppose for contradiction that $|L_{2,2}(x)| \geq 2\mu n^2$. By Lemma 11 $\{E - x :\ E \in L_{2,2}(x)\}$ contains a matching G with size at least μn. Then using Definition 7 (iii) (with $G, A_i = U_1 - x, A_j = U_3$) we find $b, b' \in U_2,\ a \in U_1,\ c \in U_3$ such that $abc, ab'c, xbb' \in \mathcal{H}$, forming an $F_5 \subset \mathcal{H}$, a contradiction.

The proof of (iv) is identical to (ii) and of (v) is to (iii). □

6.4 Getting rid of the non-crossing edges

In this section we complete the proof of Theorem 1. Our arguments are similar to those of the previous section, however, as we get rid of only a few edges, the computation needed here is more delicate. The general idea is that we remove some vertices of a non-crossing edge, and count the number of ways it could have been joined to the rest of the hypergraph.

We shall prove (2) via induction on n. Fix an n_0 such that $1/n_0$ is much smaller than any of our constants, and all of our prior lemmas and theorems are valid for every $n \geq n_0$. Let $C > 10$ be sufficiently large that (2) is true for every $n \leq n_0$.

Let $\mathrm{Forb}'(n, F_5, \eta, \mu)$ be the set of hypergraphs $\mathcal{H} \in \mathrm{Forb}(n, F_5, \eta, \mu)$ having an optimal partition with a non-crossing edge. Our final step is to give an upper bound $|\mathrm{Forb}'(n, F_5, \eta, \mu)|$. There are two types of non-crossing edge, one which is completely inside of a class, and the one which intersects two classes.

Let the non-crossing edge be xyz, and the optimal partition be U_1, U_2, U_3. Without loss of generality assume that $x, y \in U_1$.

In an $\mathcal{H} \in \mathrm{Forb}'(n, F_5, \eta, \mu)$, x, y, z could be chosen at most n^3 ways, the optimal partition of $\mathcal{H}$ in at most 3^n ways and the hypergraph $\mathcal{H} - \{x, y\}$ in at most $|\mathrm{Forb}(n-2, F_5)|$ ways.

By Lemma 10 each of $|L_{1,1}(x)|$, $|L_{1,1}(y)|$, $|L_{1,2}(x)|$, $|L_{1,2}(y)|$, $|L_{1,3}(x)|$, $|L_{1,3}(y)|$, $|L_{2,2}(x)|$, $|L_{2,2}(y)|$, $|L_{3,3}(x)|$, $|L_{3,3}(y)|$ is at most $2\mu n^2$, therefore the number of ways the non-crossing edges could be joined to x, y is at most

$$\left(\sum_{i\leq 2\mu n^2}\binom{n^2/2}{i}\right)^{10} \leq 2^{10H(4\mu)n^2}.$$

The key point is that for any $(u,v)\in(U_2-z)\times(U_3-z)$, we cannot have both $xuv, yuv\in\mathcal{H}$ otherwise they form with xyz a copy of F_5. Together with Definition 7 part (iv), we conclude that the number of ways to choose the crossing edges containing x or y is at most

$$3^{|U_2||U_3|}2^{2n}\leq 3^{\frac{n^2}{9}+\mu n^2}.$$

Note that the 2^{2n} estimates the number of ways having edges containing $\{u,z\}$ or $\{v,z\}$, as for these pairs we do not have any restriction.

Putting this together,

$$|\mathrm{Forb}'(n,F_5,\eta,\mu)| \;\leq\; n^3 3^n|\mathrm{Forb}(n-2,F_5)|\cdot 2^{10H(4\mu)n^2}3^{\frac{n^2}{9}+\mu n^2}. \tag{16}$$

By the induction hypothesis, this is at most

$$n^3 3^n(1+2^{C(n-2)-\frac{2(n-2)^2}{45}})T(n-2)2^{10H(4\mu)n^2}3^{\frac{n^2}{9}+\mu n^2}.$$

Using (15) this is upper bounded by

$$\left(1+2^{C(n-2)-\frac{2(n-2)^2}{45}}\right)2^{\left(\frac{5\log n}{n^2}+\frac{\log_2 3}{n^2}+90H(4\mu)+\log_2 3+9\mu-2+\frac{9}{n}\right)\frac{n^2}{9}}\cdot T(n).$$

As mentioned before, the crucial point in the expression above is that $\log_2 3-2<0$. More precisely, since $n>n_0$, $\log_2 3<1.59$ and $\frac{5\log n}{n^2}+\frac{\log_2 3}{n^2}+90H(4\mu)+9\mu<0.001$, we have

$$\left(\frac{5\log n}{n^2}+\frac{\log_2 3}{n^2}+90H(4\mu)+\log_2 3+9\mu-2+\frac{9}{n}\right)\frac{n^2}{9}<-\frac{2n^2}{45}.$$

Consequently,

$$|\mathrm{Forb}'(n,F_5,\eta,\mu)| \;\leq\; \left(1+2^{C(n-2)-\frac{2(n-2)^2}{45}}\right)\cdot 2^{-\frac{2n^2}{45}}T(n)<\frac{1}{10}2^{Cn-\frac{2n^2}{45}}T(n).$$

Now we can complete the proof of (2) by upper bounding $|\mathrm{Forb}(n,F_5)|$ as follows:

$$\begin{aligned}
&|\mathrm{Forb}(n,F_5)-\mathrm{Forb}(n,F_5,\eta)|+|\mathrm{Forb}(n,F_5,\eta)-\mathrm{Forb}(n,F_5,\eta,\mu)|+|\mathrm{Forb}'(n,F_5,\eta,\mu)|+T(n)\\
&\quad<\;2^{(1-\nu)\frac{n^3}{27}}+2^{n^3(\frac{1}{27}-\frac{\mu^3}{40})}+\frac{1}{10}2^{Cn-\frac{2n^2}{45}}T(n)+T(n)\\
&\quad<(1+2^{Cn-\frac{2n^2}{45}})T(n),
\end{aligned}$$

where the last inequality holds due to $T(n)>2^{s(n)}>2^{\frac{n^3}{27}-O(n^2)}$. This completes the proof of the theorem. $\square$

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