# A quasi-stability result for dictatorships in $S_{n}$ 

David Ellis, Yuval Filmus, and Ehud Friedgut ${ }^{\dagger}$

December 2013


#### Abstract

We prove that Boolean functions on $S_{n}$ whose Fourier transform is highly concentrated on the first two irreducible representations of $S_{n}$, are close to being unions of cosets of point-stabilizers. We use this to give a natural proof of a stability result on intersecting families of permutations, originally conjectured by Cameron and Ku [6, and first proved in [10. We also use it to prove a 'quasistability' result for an edge-isoperimetric inequality in the transposition graph on $S_{n}$, namely that subsets of $S_{n}$ with small edge-boundary in the transposition graph are close to being unions of cosets of point-stabilizers.


## 1 Introduction

In extremal combinatorics, we are typically interested in subsets $S$ of a finite set $X$ which satisfy some property, $P$ say. Often, we wish to determine the maximum or minimum possible size of a subset $S \subset X$ which has the property $P$. The maximum or minimum-sized subsets of $X$ with the property $P$ are called the extremal sets.

In the past fifty years, discrete Fourier analysis has been used to solve a number of extremal problems where the set $X$ may be given the structure of a finite group, $G$. In this case, given a set $S$ with the property $P$, one may consider the characteristic function $\mathbf{1}_{S}$, and take the Fourier transform of $\mathbf{1}_{S}$. (Recall that the characteristic function of a subset $S \subset G$ is the real-valued function on $G$ with value 1 on $S$ and 0 elsewhere.) If we are lucky, the property $P$ gives us information about the Fourier transform of $\mathbf{1}_{S}$, which can then be used to obtain a sharp bound on $|S|$. Often, such proofs tell us that if $S$ is an extremal set, then the Fourier transform of $\mathbf{1}_{S}$ must be supported on a certain set, $T$ say; this can then be used to describe the structure of the extremal sets.

Under these conditions, it often turns out that if $S$ is 'almost-extremal', meaning that it has size close to the extremal size, then the Fourier transform of $\mathbf{1}_{S}$ is highly concentrated on $T$. If one can characterize the Boolean functions whose Fourier transform is highly concentrated on $T$, one can describe the structure of the almost-extremal

[^0]sets. Sometimes, almost-extremal sets must be close in structure to a genuine extremal set; this phenomenon is known as stability.

Characterizing the Boolean functions whose Fourier transform is highly concentrated on $T$ often turns out to be a hard problem. To date, such a characterization has been obtained in several cases where the group $G$ is Abelian, using the welldeveloped theory of Fourier analysis on Abelian groups. The simplest case is that of dictatorships: Friedgut, Kalai and Naor [19] prove that a Boolean function on $\{0,1\}^{n}$ whose Fourier transform is close to being concentrated on the first two levels, must be close to a dictatorship (a function determined by just one coordinate). This was useful for Kalai in [26] where he deduced a stability version of Arrow's theorem on social choice functions, namely that if a neutral social choice function has small probability of irrationality, then it must be close to a dictatorship. Alon, Dinur, Friedgut and Sudakov [1] proved a similar result for $\mathbb{Z}_{r}^{n}$ (a result later improved by Hatami and Ghandehari, [23]), and utilized it to describe the large independent sets in powers of a large family of graphs.

In this paper, we obtain a similar result for Boolean functions on $S_{n}$. It is easy to see that if $f: S_{n} \rightarrow \mathbb{R}$, then the Fourier transform of $f$ is supported on the first two irreducible representations of $S_{n}$ if and only if it lies in the subspace spanned by the characteristic functions of cosets of point-stabilisers. (For brevity, we refer to the cosets of point-stabilisers as 1 -cosets, and we denote the subspace spanned by their characteristic functions as $U_{1}$. Similarly, a $t$-coset is a coset of the stabilizer of an ordered $t$-tuple of distinct points.) If $f$ is Boolean, i.e. $f: S_{n} \rightarrow\{0,1\}$, and $f \in U_{1}$, then $f$ is the characteristic function of a disjoint union of 1-cosets. (This is somewhat trickier to show; a proof may be found e.g. in [15].) A disjoint union of 1-cosets is precisely a subset of $S_{n}$ whose characteristic function is determined by the image of the pre-image of just one coordinate; by analogy with the $\{0,1\}^{n}$ case, we call these subsets (or their characteristic functions) dictators.

In this paper, we consider Boolean functions on $S_{n}$ whose Fourier transform is highly concentrated on the first two irreducible representations of $S_{n}$ - equivalently, Boolean functions which are close (in Euclidean distance) to the subspace $U_{1}$. We prove the following 'quasi-stability' result.

Theorem 1. There exist absolute constants $C_{0}, \epsilon_{0}>0$ such that the following holds. Let $\mathcal{A} \subset S_{n}$ with $|\mathcal{A}|=c(n-1)$ !, where $c \leq n / 2$, and let $f=\mathbf{1}_{\mathcal{A}}: S_{n} \rightarrow\{0,1\}$ be the characteristic function of $\mathcal{A}$, so that $\mathbb{E}[f]=c / n$. Let $f_{1}$ denote the orthogonal projection of $f$ onto $U_{1}$. If $\mathbb{E}\left[\left(f-f_{1}\right)^{2}\right] \leq \epsilon c / n$, where $\epsilon \leq \epsilon_{0}$, then there exists a Boolean function $\tilde{f}: S_{n} \rightarrow\{0,1\}$ such that

$$
\begin{equation*}
\mathbb{E}\left[(f-\tilde{f})^{2}\right] \leq C_{0} c^{2}\left(\epsilon^{1 / 2}+1 / n\right) / n \tag{1}
\end{equation*}
$$

and $\tilde{f}$ is the characteristic function of a union of round $(c)$ 1-cosets of $S_{n}$. Moreover, $|c-\operatorname{round}(c)| \leq C_{0} c^{2}\left(\epsilon^{1 / 2}+1 / n\right)$. (Here, round $(c)$ denotes the nearest integer to $c$, rounding up if $c \in \mathbb{Z}+\frac{1}{2}$.)

This theorem says that a Boolean function on $S_{n}$ (of small expectation) whose Fourier transform is close to being concentrated on the first two irreducible representations of $S_{n}$, must be close in structure to the characteristic function of a union of

1-cosets. Equivalently, a (small) subset of $S_{n}$, whose characteristic function is close (in Euclidean distance) to $U_{1}$, must be close in symmetric difference to a union of 1-cosets.

This statement is not 'stability' in the strongest sense; in fact, 'genuine stability' does not occur. A 'genuine' stability result would say that a subset of $S_{n}$ whose characteristic function is close to $U_{1}$ must be close in symmetric difference to a subset of $S_{n}$ whose characteristic function lies in $U_{1}$ - i.e., close in symmetric difference to a disjoint union of 1-cosets. A union of two non-disjoint 1-cosets is not close in symmetric difference to any of these, and yet its characteristic function is close to $U_{1}$. Our result says that subsets close to unions of cosets (not necessarily disjoint) are the only possibility. We therefore call it a 'quasi-stability' result.

If we restrict our attention to subsets $\mathcal{A} \subset S_{n}$ with size close to $(n-1)$ !, Theorem 1 says that if the characteristic function $\mathbf{1}_{\mathcal{A}}$ is close to $U_{1}$, then $\mathcal{A}$ must be close in symmetric difference to a single 1-coset. This leads to our first application: a natural proof of the following conjecture of Cameron and Ku [10].

Conjecture 2. There exists $\delta>0$ such that for all $n \in \mathbb{N}$, the following holds. If $\mathcal{A} \subset S_{n}$ is an intersecting family of permutations with $|\mathcal{A}| \geq(1-\delta)(n-1)$ !, then $\mathcal{A}$ is contained within a 1-coset of $S_{n}$.
(Recall that a family $\mathcal{A} \subset S_{n}$ is said to be intersecting if any two permutations in $\mathcal{A}$ agree at some point.) Conjecture 2 is a rather strong stability statement for intersecting families of permutations. It was first proved by the first author in [10] using a different method (viz., by obtaining much weaker stability information, and then using the intersecting property to 'bootstrap' this information). As suggested by Hatami and Ghandehari [23], progress on the Cameron-Ku conjecture has indeed been linked to a greater understanding of a kind of stability phenomenon for Boolean functions on $S_{n}$.

As a second application, we obtain a structural description of subsets of $S_{n}$ of various sizes which are almost-extremal for the edge-isoperimetric inequality for $S_{n}$. If $\mathcal{A} \subset S_{n}$, we let $\partial \mathcal{A}$ denote the edge-boundary of $\mathcal{A}$ in the transposition graph, the Cayley graph on $S_{n}$ generated by the transpositions. We prove the following.

Theorem 3. For each $c \in \mathbb{N}$, there exists $n_{0}(c) \in \mathbb{N}$ and $\delta_{0}(c)>0$ such that the following holds. Let $\mathcal{A} \subset S_{n}$ with $|\mathcal{A}|=c(n-1)$ !, and with

$$
|\partial \mathcal{A}| \leq \frac{|\mathcal{A}|(n!-|\mathcal{A}|)}{(n-1)!}+\delta n|\mathcal{A}|
$$

where $n \geq n_{0}(c)$ and $\delta \leq \delta_{0}(c)$. Then there exists a family $\mathcal{B} \subset S_{n}$ such that $\mathcal{B}$ is a union of c 1-cosets of $S_{n}$, and

$$
|\mathcal{A} \backslash \mathcal{B}| \leq O(c \delta)(n-1)!+O\left(c^{2}\right)(n-2)!
$$

(We may take $\delta_{0}(c)=\Omega\left(c^{-4}\right)$ and $n_{0}(c)=O\left(c^{2}\right)$.)
Here, the almost-extremal sets include unions of 1-cosets which are not disjoint, whereas the extremal sets consist only of disjoint unions of 1-cosets. We feel that this is a good example of a problem where the class of almost-extremal sets is considerably richer than the class of extremal sets.

This paper is part one of a 'trilogy' dealing with results similar to Theorem 11 In [13, we deal with balanced Boolean functions whose Fourier transform is highly concentrated on the first two irreducible representations of $S_{n}$. We prove that if $f$ is a Boolean function on $S_{n}$ with expectation bounded away from 0 and 1 , with Fourier transform highly concentrated on the first two irreducible representations of $S_{n}$, then $f$ is close in structure to a dictatorship. Hence, in the balanced case, genuine stability occurs, as opposed to the 'quasi-stability' phenomenon for Boolean functions with expectation $O(1 / n)$, in the current paper. In both cases, however, the Boolean functions which are close to $U_{1}$ are close in structure to unions of 1-cosets. The reason for the disparity between the balanced case and the sparse case is that a union of $c$ pairwise non-disjoint 1-cosets is $\Theta\left(c^{2} / n^{2}\right)$-far from $U_{1}$. In the setting of the current paper, $c=o(n)$ and so $\Theta\left(c^{2} / n^{2}\right)=o(1)$. By contrast, in the setting of [13], $c=\Theta(n)$ and so $\Theta\left(c^{2} / n^{2}\right)=\Theta(1)$, so a union of 1-cosets can be close to $U_{1}$ only if it is essentially a disjoint union of 1-cosets. Our approaches in the two cases are completely different, and to date, we have not been able to come up with a unified approach works for the entire range $1 / n \leq \mathbb{E}(f) \leq 1-1 / n$.

The third part of our trilogy, [14, deals with Boolean functions on $S_{n}$ whose Fourier transform is highly concentrated on irreducible representations corresponding to partitions of $n$ with first row of length at least $n-t$, or equivalently, Boolean functions which are 'close' in Euclidean distance to the subspace of $\mathbb{C}\left[S_{n}\right]$ spanned by all characteristic functions of $t$-cosets. We prove that such a function must be 'close' to the characteristic function of a union of $t$-cosets, using methods similar to the ones in this paper. However, the amount of representation theory needed for this makes for a hefty treatise which deserves a separate showcase. We point out that this is analogous to the state of affairs in the theory of Boolean functions on $\{0,1\}^{n}$. There, the theorems dealing with Boolean functions whose Fourier transform is highly concentrated on sets of size at most $t$, for $t>1$ (e.g. [5], [18, [28], 31], and recently [27]), tend to be far more complicated than in the $t=1$ case. In the $\{0,1\}^{n}$ case, such theorems have proven to be quite useful, e.g. as an important component in the proof of a 'stability' version of the Simonovits-Sós conjecture [12]. We trust that the symmetric-group versions will prove useful too.

The structure of the rest of the paper is as follows. In section 2 we provide some general background and notation. In section 3 we state and prove our main theorem. In section 4 we describe our two applications. Finally, in section 5 we mention some open problems.

## 2 Notation and Background

### 2.1 General representation theory

In this section, we recall the basic notions and results we need from general representation theory. For more background, the reader may consult 33 .

Let $G$ be a finite group. A representation of $G$ over $\mathbb{C}$ is a pair $(\rho, V)$, where $V$ is a finite-dimensional complex vector space, and $\rho: G \rightarrow G L(V)$ is a group homomorphism from $G$ to the group of all invertible linear endomorphisms of $V$. The vector space $V$,
together with the linear action of $G$ defined by $g v=\rho(g)(v)$, is sometimes called a $\mathbb{C} G$ module. A homomorphism between two representations $(\rho, V)$ and $\left(\rho^{\prime}, V^{\prime}\right)$ is a linear map $\phi: V \rightarrow V^{\prime}$ such that $\phi(\rho(g)(v))=\rho^{\prime}(g)(\phi(v))$ for all $g \in G$ and $v \in V$. If $\phi$ is a linear isomorphism, the two representations are said to be equivalent, or isomorphic, and we write $(\rho, V) \cong\left(\rho^{\prime}, V^{\prime}\right)$. If $\operatorname{dim}(V)=n$, we say that $\rho$ has dimension $n$, and we write $\operatorname{dim}(\rho)=n$.

The representation $(\rho, V)$ is said to be irreducible if it has no proper subrepresentation, i.e. there is no proper subspace of $V$ which is $\rho(g)$-invariant for all $g \in G$.

It turns out that for any finite group $G$, there are only finitely many equivalence classes of irreducible complex representations of $G$, and any complex representation of $G$ is isomorphic to a direct sum of irreducible representations of $G$. Hence, we may choose a set of representatives $\mathcal{R}$ for the equivalence classes of complex irreducible representations of $G$.

If $(\rho, V)$ is a complex representation of $V$, the character $\chi_{\rho}$ of $\rho$ is the map defined by

$$
\begin{aligned}
\chi_{\rho}: G & \rightarrow \mathbb{C} \\
\chi_{\rho}(g) & =\operatorname{Tr}(\rho(g)),
\end{aligned}
$$

where $\operatorname{Tr}(\alpha)$ denotes the trace of the linear map $\alpha$ (i.e. the trace of any matrix of $\alpha$ ). Note that $\chi_{\rho}(\mathrm{Id})=\operatorname{dim}(\rho)$, and that $\chi_{\rho}$ is a class function on $G$ (meaning that it is constant on each conjugacy-class of $G$.)

The usefulness of characters lies in the following
Fact. Two complex representations are isomorphic if and only if they have the same character.

Definition. Let $\mathcal{R}$ be a complete set of non-isomorphic, irreducible representations of $G$, i.e. containing one representative from each isomorphism class of irreducible representations of $G$. Let $f: G \rightarrow \mathbb{C}$ be a complex-valued function on $G$. The Fourier transform of $f$ is defined by

$$
\begin{equation*}
\hat{f}(\rho)=\frac{1}{|G|} \sum_{\sigma \in G} f(\sigma) \rho(\sigma) \quad(\rho \in \mathcal{R}) \tag{2}
\end{equation*}
$$

it can be viewed as a map from $\mathcal{R}$ to End $(V)$, the space of all linear endomorphisms of $V$.

Let $G$ be a finite group. Let $\mathbb{C}[G]$ denote the vector space of all complex-valued functions on $G$. Let $\mathbb{P}$ denote the uniform probability measure on $G$ :

$$
\mathbb{P}(\mathcal{A})=|\mathcal{A}| /|G| \quad(\mathcal{A} \subset G)
$$

We equip $\mathbb{C}[G]$ with the inner product induced by the uniform probability measure on $G$ :

$$
\langle f, g\rangle=\frac{1}{|G|} \sum_{\sigma \in G} f(\sigma) \overline{g(\sigma)}
$$

Let

$$
\|f\|_{2}=\sqrt{\mathbb{E}\left[f^{2}\right]}=\sqrt{\frac{1}{|G|} \sum_{\sigma \in G}|f(\sigma)|^{2}}
$$

denote the induced Euclidean norm.
For each irreducible representation $\rho$ of $G$, let

$$
U_{\rho}:=\{f \in \mathbb{C}[G]: \hat{f}(\pi)=0 \text { for all irreducible representations } \pi \nsubseteq \rho\}
$$

We refer to this as 'the subspace of functions whose Fourier transform is supported on the irreducible representation $\rho^{\prime}$. Note that if $\rho^{\prime} \cong \rho$, then $U_{\rho^{\prime}}=U_{\rho}$. It turns out that the $U_{\rho}$ 's are pairwise orthogonal, and that

$$
\mathbb{C}[G]=\bigoplus_{\rho \in \mathcal{R}} U_{\rho}
$$

Moreover, we have $\operatorname{dim}\left(U_{\rho}\right)=(\operatorname{dim}(\rho))^{2}$ for all $\rho$.
For each $\rho \in \mathcal{R}$, let $f_{\rho}$ denote orthogonal projection onto the subspace $U_{\rho}$. Then we have

$$
\begin{equation*}
\|f\|_{2}^{2}=\sum_{\rho \in \mathcal{R}}\left\|f_{\rho}\right\|_{2}^{2} \tag{3}
\end{equation*}
$$

## Background on the representation theory of $S_{n}$.

Definition. $A$ partition of $n$ is a non-increasing sequence of integers summing to $n$, i.e. a sequence $\lambda=\left(\lambda_{1}, \ldots, \lambda_{k}\right)$ with $\lambda_{1} \geq \lambda_{2} \geq \ldots \geq \lambda_{k}$ and $\sum_{i=1}^{k} \lambda_{i}=n$; we write $\lambda \vdash n$. For example, $(3,2,2) \vdash 7$.

The following two orders on partitions of $n$ will be useful.
Definition. (Dominance order) Let $\lambda=\left(\lambda_{1}, \ldots, \lambda_{r}\right)$ and $\mu=\left(\mu_{1}, \ldots, \mu_{s}\right)$ be partitions of $n$. We say that $\lambda \unrhd \mu\left(\lambda\right.$ dominates $\mu$ ) if $\sum_{j=1}^{i} \lambda_{i} \geq \sum_{j=1}^{i} \mu_{i} \forall i$ (where we define $\left.\lambda_{i}=0 \forall i>r, \mu_{j}=0 \forall j>s\right)$.

It is easy to see that this is a partial order.
Definition. (Lexicographic order) Let $\lambda=\left(\lambda_{1}, \ldots, \lambda_{r}\right)$ and $\mu=\left(\mu_{1}, \ldots, \mu_{s}\right)$ be partitions of $n$. We say that $\lambda>\mu$ if $\lambda_{j}>\mu_{j}$, where $j=\min \left\{i \in[n]: \lambda_{i} \neq \mu_{i}\right\}$.

It is easy to see that this is a total order which extends the dominance order.
It is well-known that there is an explicit 1-1 correspondence between irreducible representations of $S_{n}$ (up to isomorphism) and partitions of $n$. The reader may refer to [34] for a full description of this correspondence, or to the paper [15] for a shorter description.

For each partition $\alpha$, we write $[\alpha]$ for the corresponding isomorphism class of irreducible representations of $S_{n}$, and we write $U_{\alpha}=U_{[\alpha]}$ for the vector space of complex-valued functions on $\Gamma$ whose Fourier transform is supported on $[\alpha]$. Similarly, if $f \in \mathbb{C}\left[S_{n}\right]$, we write $f_{\alpha}$ for the orthogonal projection of $f$ onto $U_{\alpha}$.

We will be particularly interested in the first two irreducible representations of $S_{n}$ (under the lexicographic order on partitions). The first, $[n]$, is the trivial representation; $U_{[n]}$ is the subspace of $\mathbb{C}\left[S_{n}\right]$ consisting of the constant functions. The second may be obtained as follows.

The permutation representation $\rho_{\mathrm{perm}}$ is the representation corresponding to the permutation action of $S_{n}$ on $\{1,2, \ldots, n\}$. It turns out that $\rho_{\text {perm }}$ decomposes into a direct sum of a copy of the trivial representation $[n]$ and a copy of $[n-1,1]$, the second irreducible representation of $S_{n}$.

As observed in 15], we have

$$
U_{(n)} \oplus U_{(n-1,1)}=\operatorname{Span}\left\{\mathbf{1}_{T_{i j}}: i, j \in[n]\right\}
$$

where

$$
T_{i j}=\left\{\sigma \in S_{n}: \sigma(i)=j\right\}
$$

The $T_{i j}$ 's are the cosets of the point-stabilizers in $S_{n}$; for brevity, we call them the 1 -cosets of $S_{n}$. We write

$$
U_{1}:=U_{(n)} \oplus U_{(n-1,1)}=\operatorname{Span}\left\{\mathbf{1}_{T_{i j}}: i, j \in[n]\right\}
$$

If $f \in \mathbb{C}\left[S_{n}\right]$, we will write $f_{1}$ for the orthogonal projection of $f$ onto the subspace $U_{1}$; note that $f_{1}=f_{(n)}+f_{(n-1,1)}$.

Similarly, if $t>1$, and if $I$ and $J$ are ordered $t$-tuples of distinct elements of $[n]$, then we write

$$
T_{I J}:=\left\{\sigma \in S_{n}: \sigma(I)=J\right\} .
$$

We call the $T_{I J}$ 's the $t$-cosets of $S_{n}$, and we define

$$
U_{t}:=\operatorname{Span}\left\{\mathbf{1}_{T_{I J}}: I, J \text { are ordered } t \text {-tuples of distinct elements of }[n]\right\}
$$

Recall the following theorem from [15], which completely characterizes the Boolean functions in $U_{1}$.

Theorem 4. If $\mathcal{A} \subset S_{n}$ has $\mathbf{1}_{\mathcal{A}} \in U_{1}$, then $\mathcal{A}$ is a disjoint union of 1-cosets of $S_{n}$.
Remark 1. If $\mathcal{A} \subset S_{n}$ is a disjoint union of 1-cosets of $S_{n}$, then we either have

$$
\mathcal{A}=\bigcup_{j \in J} T_{i j}
$$

for some $i \in[n]$ and some $J \subset[n]$, or

$$
\mathcal{A}=\bigcup_{i \in I} T_{i j}
$$

for some $j \in[n]$ and some $I \subset[n]$. Hence, $\mathbf{1}_{\mathcal{A}}$ must be determined by the image or preimage of a single element. We may therefore call $\mathbf{1}_{\mathcal{A}}$ a dictatorship, by analogy with the $\{0,1\}^{n}$ case, hence the title of this paper.

Remark 2. It is in place to remark that if $t \geq 2$, then a Boolean function in $U_{t}$ is not necessarily the characteristic function of a union of $t$-cosets. Theorem 27 in 15 ] states that a Boolean function in $U_{t}$ is the characteristic function of a disjoint union of $t$-cosets, but this is false for $t \geq 2$; a counterexample, and the error in the proof, is pointed out by the second author in [17. A counterexample when $t=2$ is as follows. Let $n \geq 8$. For any permutation $\pi \in S_{n}$, define $x=x(\pi) \in\{0,1\}^{4}$ by $x_{i}=\mathbf{1}_{\{\pi(i) \in[4]\}}$, and consider the function

$$
\left.f: S_{n} \rightarrow\{0,1\} ; \quad \pi \mapsto \mathbf{1}_{\left\{x_{1} \geq x_{2} \geq x_{3} \geq x_{4}\right.} \text { or } x_{1} \leq x_{2} \leq x_{3} \leq x_{4}\right\} .
$$

It can be checked that $f \in U_{2}$, but the value of $f$ clearly cannot be determined by fixing the images of at most two elements of $[n]$, so neither $f$ nor $1-f$ is a union of 2 -cosets. It is easy to use $f$ to construct a counterexample for each $t \geq 3$, by considering a product of $f$ with the characteristic function of the pointwise stabilizer of a $(t-2)$-set. We note that the main application of Theorem 27 in 15 was to characterize (for large $n$ ) the $t$-intersecting families in $S_{n}$ of maximum size (i.e., to characterize the cases of equality in the Deza-Frankl conjecture); fortunately, this characterization follows immediately e.g. from the Hilton-Milner type result of the first author in [11, where the proof does not depend on Theorem 27 in [15] (and indeed predates the latter).

Our main theorem describes what happens when $\mathbf{1}_{\mathcal{A}}$ is near $U_{1}$. We will show that if $f: S_{n} \rightarrow\{0,1\}$ is a Boolean function on $S_{n}$ such that

$$
\mathbb{E}\left[\left(f-f_{1}\right)^{2}\right]
$$

is small, then there exists a Boolean function $h$ such that

$$
\mathbb{E}\left[(f-h)^{2}\right]
$$

is small, and $h$ is the characteristic function of a union of 1-cosets of $S_{n}$. Note that, by (3), we have

$$
\mathbb{E}\left[\left(f-f_{1}\right)^{2}\right]=\left\|f-f_{1}\right\|_{2}^{2}=\sum_{\alpha \neq(n),(n-1,1)}\left\|f_{\alpha}\right\|^{2}=\operatorname{dist}\left(f, U_{1}\right)^{2}
$$

where dist denotes the Euclidean distance.
Our proof relies on considering the first, second and third moments of a nonnegative function $g$ which is an affine shift of the projection $f_{1}$. One may compare this with the proofs of the Abelian analogues in [1] and [19], where the fourth moment is considered in order to obtain structural information.

Throughout, if $u$ and $v$ are functions of several variables (e.g. $n, c, \epsilon_{1}$ ), the notation $u=O(v)$ will mean that there exists an absolute constant $C$ (not depending upon any of the variables) such that $|u| \leq C|v|$ pointwise. Similarly, the notation $u=\Omega(v)$ will mean that there exists a universal constant $C>0$ such that $|u| \geq C|v|$ pointwise. As usual, round $(x)$ will denote $x$ rounded to the nearest integer, rounding up if $x \in \mathbb{Z}+\frac{1}{2}$.

## 3 The quasi-stability theorem

In this section, we prove our main 'quasi-stability' theorem, Theorem 1 .
Theorem 1. There exist absolute constants $C_{0}, \epsilon_{0}>0$ such that the following holds. Let $\mathcal{A} \subset S_{n}$ with $|\mathcal{A}|=c(n-1)!$, where $c \leq n / 2$, and let $f=\mathbf{1}_{\mathcal{A}}: S_{n} \rightarrow\{0,1\}$ be the characteristic function of $\mathcal{A}$, so that $\mathbb{E}[f]=c / n$. Let $f_{1}$ denote orthogonal projection of $f$ onto $U_{1}=U_{(n)} \oplus U_{(n-1,1)}$. If $\mathbb{E}\left[\left(f-f_{1}\right)^{2}\right] \leq \epsilon c / n$, where $\epsilon \leq \epsilon_{0}$, then there exists a Boolean function $\tilde{f}: S_{n} \rightarrow\{0,1\}$ such that

$$
\begin{equation*}
\mathbb{E}\left[(f-\tilde{f})^{2}\right] \leq C_{0} c^{2}\left(\epsilon^{1 / 2}+1 / n\right) / n \tag{4}
\end{equation*}
$$

and $\tilde{f}$ is the characteristic function of a union of round $(c) 1$-cosets of $S_{n}$. Moreover, $|c-\operatorname{round}(c)| \leq C_{0} c^{2}\left(\epsilon^{1 / 2}+1 / n\right)$.
Remark 3. Observe that Theorem 1 is non-trivial only if $\epsilon=O\left(c^{-2}\right)$. Unfortunately, it does not imply Theorem 4 when we take $\epsilon=0$, due to the presence of the 'extra' term $1 / n$ in the right-hand side of (4). We conjecture in Section 5 that this term can be removed (see Conjecture 24).

While the ideas behind the proof are quite simple, the proof itself is rather long and technical. So before presenting the actual proof, we give an overview.

Proof overview. The proof concentrates on analysing the matrix $B=\left(b_{i j}\right)_{i, j \in[n]}$ defined by

$$
b_{i j}=\frac{\left|\mathcal{A} \cap T_{i j}\right|}{(n-1)!}-\frac{|\mathcal{A}|}{n!}=n\left\langle f, \mathbf{1}_{T_{i j}}\right\rangle-\langle f, \mathbf{1}\rangle,
$$

where 1 denotes the constant function with value 1 .
The $b_{i j}$ 's turn out to be quite informative. If $\mathcal{A}$ contains $T_{i j}$, and $c=o(n)$, then $b_{i j}$ is close to 1 , whereas if $\mathcal{A}=T_{k l}$, and $(k, l) \neq(i, j)$, then $b_{i j}$ is close to 0 . This is illustrated by the following example, which is a good one to keep in mind while reading the proof overview.

If $\mathcal{A}$ is a disjoint union of $c$-cosets (a dictatorship), then $B$ takes one of the following forms:
or

$$
c\left\{\left(\begin{array}{cccc}
1-\frac{c}{n} & -\frac{n-c}{n(n-1)} & \cdots & -\frac{n-c}{n(n-1)}  \tag{6}\\
\vdots & \vdots & & \vdots \\
1-\frac{c}{n} & -\frac{n-c}{n(n-1)} & \cdots & -\frac{n-c}{n(n-1)} \\
-\frac{c}{n} & \frac{c}{n(n-1)} & \cdots & \frac{c}{n(n-1)} \\
\vdots & & & \\
-\frac{c}{n} & \frac{c}{n(n-1)} & \cdots & \frac{c}{n(n-1)}
\end{array}\right) .\right.
$$

Note that in both the above matrices, the sum of the squares of the entries is approximately $c$, and also the sum of the cubes of the entries is approximately $c$. Our first step will be to show that, under the hypotheses of the theorem, the same is true for $B$. This in turn will enable us to show that $B$ contains $m$ entries which are close to 1 , where $m \approx c$, and all other entries are close to 0 .

Rather than working directly with $f_{1}$, it turns out to be easier to work with the function

$$
h=\sum_{i, j} b_{i j} \mathbf{1}_{T_{i, j}}
$$

which is an affine shift of $f_{1}$. This is because the second and third moments of $h$ are nicely related to the $b_{i j}$ 's, whereas the same is not true of $f_{1}$. Indeed, it turns out (see Lemma (5) that

$$
\mathbb{E}\left[h^{2}\right]=\frac{1}{n-1} \sum_{i, j} b_{i j}^{2}
$$

and

$$
\mathbb{E}\left[h^{3}\right]=\frac{n}{(n-1)(n-2)} \sum_{i, j} b_{i j}^{3}
$$

The bound on $\mathbb{E}\left[\left(f-f_{1}\right)^{2}\right]$ gives us a bound on $\mathbb{E}\left[h^{2}\right]$, and hence a bound on $\sum_{i, j} b_{i j}^{2}$. To obtain information about $\mathbb{E}\left[h^{3}\right]$, it is helpful to consider another affine shift of $f_{1}$, namely the function

$$
g=\sum_{i, j} \frac{\left|\mathcal{A} \cap T_{i j}\right|}{(n-1)!} \mathbf{1}_{T_{i j}}=\frac{n}{n-1} f_{1}+\frac{n-2}{n-1} c
$$

which is non-negative. We use the bound on $\mathbb{E}\left[\left(f-f_{1}\right)^{2}\right]$, together with the fact that $f$ is Boolean, to obtain a lower bound on $\mathbb{E}\left[g^{3}\right]$, which translates to a lower bound on $\mathbb{E}\left[h^{3}\right]$, finally giving us a lower bound on $\sum_{i, j} b_{i j}^{3}$.

Let $F(n, c)$ and $G(n, c)$ denote respectively the sum of the squares and the sum of the cubes of the entries of the matrix (5). We obtain

$$
\begin{equation*}
F(n, c)-c\left(1+\frac{1}{n-1}\right) \epsilon \leq \sum_{i, j} b_{i j}^{2} \leq F(n, c) \tag{7}
\end{equation*}
$$

and

$$
\begin{equation*}
\sum_{i, j} b_{i j}^{3} \geq G(n, c)-\frac{27}{4}\left(3 c^{2}+2 c\right) \epsilon_{1}^{1 / 2} \tag{8}
\end{equation*}
$$

Subtracting (8) from (7), we deduce that

$$
\sum_{i, j} b_{i j}^{2}\left(1-b_{i j}\right) \leq O\left(c^{2}\right) \sqrt{\epsilon}+O(c / n)
$$

This means that each $b_{i j}$ is either very close to 1 or very close to 0 . Since $\sum_{i, j} b_{i j}^{2}$ is close to $F(n, c) \approx c$, it follows that there are roughly $c$ entries which are very close to 1 . (This implies that $c$ must be close to an integer, $m$ say.) These entries correspond to $m$ 1-cosets of $S_{n}$, whose union is almost contained within $\mathcal{A}$. These 1-cosets need not
be disjoint, but provided $c=o(n)$, their union has size roughly $c(n-1)$ ! (the error is of order $c^{2}(n-2)$ !), so it gives a good approximation to $\mathcal{A}$. This will complete the proof.

Proof of Theorem 11: First, note that for any absolute constant $n_{0}$, we may choose $C_{0}$ sufficiently large that the conclusion of the theorem holds for all $n \leq n_{0}$. Hence, we may assume throughout that $n>n_{0}$, for any fixed $n_{0} \in \mathbb{N}$.

Let

$$
a_{i j}=\frac{\left|\mathcal{A} \cap T_{i j}\right|}{(n-1)!}
$$

and let

$$
b_{i j}=a_{i j}-c / n
$$

Let $B$ denote the matrix $\left(b_{i j}\right)_{i, j \in[n]}$. Note that

$$
a_{i j}=n\left\langle f, \mathbf{1}_{T_{i j}}\right\rangle
$$

so

$$
b_{i j}=n\left\langle f, \mathbf{1}_{T_{i j}}\right\rangle-c / n
$$

Moreover,

$$
\begin{equation*}
\sum_{j=1}^{n} b_{i j}=0 \text { for all } i \in[n], \quad \text { and } \quad \sum_{i=1}^{n} b_{i j}=0 \text { for all } j \in[n] \tag{9}
\end{equation*}
$$

i.e. the matrix $B$ has all its row and column sums equal to 0 .

Instead of working directly with the function $f_{1}$, it will be convenient to work with the functions

$$
\begin{equation*}
g=\sum_{i, j} a_{i j} \mathbf{1}_{T_{i j}} \tag{10}
\end{equation*}
$$

and

$$
\begin{equation*}
h=\sum_{i, j} b_{i j} \mathbf{1}_{T_{i j}} . \tag{11}
\end{equation*}
$$

These are both affine shifts of $f_{1}$; indeed, we have

$$
\begin{equation*}
g=n \frac{f_{1}+(n-2) \mathbb{E}\left[f_{1}\right]}{n-1}=\left(1+\frac{1}{n-1}\right) f_{1}+\left(1-\frac{1}{n-1}\right) c \tag{12}
\end{equation*}
$$

since the functions on both sides lie in $U_{1}$, and they both have the same inner product with $\mathbf{1}_{T_{i j}}$, for every $i, j \in[n]$. (Recall that $U_{1}=\operatorname{Span}\left\{\mathbf{1}_{T_{i j}}: i, j \in[n]\right\}$.) Observe that

$$
\begin{equation*}
h=g-c=n \frac{f_{1}+(n-2) \mathbb{E}\left[f_{1}\right]}{n-1}-c=\left(1+\frac{1}{n-1}\right) f_{1}-\frac{c}{n-1} . \tag{13}
\end{equation*}
$$

We now proceed to translate the information we know about $f_{1}$ to information about $h$; this in turn will give us information about the matrix $B=\left(b_{i j}\right)$. We have $\mathbb{E}\left[f_{1}\right]=\mathbb{E}[f]=c / n$, and therefore

$$
\mathbb{E}[g]=c,
$$

$$
\mathbb{E}[h]=0
$$

Write $\mathbb{E}\left[\left(f-f_{1}\right)^{2}\right]=\epsilon_{1} c / n$; by assumption, $\epsilon_{1} \leq \epsilon$. Since $f_{1}$ is an orthogonal projection of $f$, we have

$$
\begin{equation*}
\mathbb{E}\left[f_{1}^{2}\right]=\mathbb{E}\left[f^{2}\right]-\mathbb{E}\left[\left(f-f_{1}\right)^{2}\right]=\mathbb{E}[f]-\mathbb{E}\left[\left(f-f_{1}\right)^{2}\right]=\left(1-\epsilon_{1}\right) c / n \tag{14}
\end{equation*}
$$

From (12), we have

$$
g^{2}=\left(\frac{n}{n-1}\right)^{2}\left(f_{1}^{2}+2(n-2) f_{1} \mathbb{E}\left[f_{1}\right]+(n-2)^{2}\left(\mathbb{E}\left[f_{1}\right]\right)^{2}\right)
$$

Taking expectations, we obtain

$$
\mathbb{E}\left[g^{2}\right]=\left(\frac{n}{n-1}\right)^{2}\left(\mathbb{E}\left[f_{1}^{2}\right]+n(n-2)\left(\mathbb{E}\left[f_{1}\right]\right)^{2}\right)
$$

Substituting $\mathbb{E}\left[f_{1}\right]=c / n$ and (14) to this expression yields:

$$
\begin{equation*}
\mathbb{E}\left[g^{2}\right]=c^{2}\left(1-\frac{1}{(n-1)^{2}}\right)+\left(1+\frac{1}{n-1}\right)^{2} \frac{c}{n}\left(1-\epsilon_{1}\right) \tag{15}
\end{equation*}
$$

Since $h=g-\mathbb{E}[g]$, we obtain

$$
\begin{equation*}
\mathbb{E}\left[h^{2}\right]=\mathbb{E}\left[g^{2}\right]-(\mathbb{E}[g])^{2}=\left(1+\frac{1}{n-1}\right)^{2} \frac{c}{n}\left(1-\epsilon_{1}\right)-\frac{c^{2}}{(n-1)^{2}} \tag{16}
\end{equation*}
$$

We now proceed to express $\mathbb{E}\left[h^{2}\right]$ and $\mathbb{E}\left[h^{3}\right]$ in terms of the coefficients $b_{i j}$.
Lemma 5. Let $\left(b_{i j}\right)_{i, j=1}^{n}$ be a real matrix satisfying

$$
\begin{equation*}
\sum_{j=1}^{n} b_{i j}=0 \text { for all } i \in[n], \quad \text { and } \quad \sum_{i=1}^{n} b_{i j}=0 \text { for all } j \in[n], \tag{17}
\end{equation*}
$$

and let

$$
h=\sum_{i, j} b_{i j} \mathbf{1}_{T_{i j}}
$$

Then

$$
\mathbb{E}\left[h^{2}\right]=\frac{1}{n-1} \sum_{i, j} b_{i j}^{2},
$$

and

$$
\begin{equation*}
\mathbb{E}\left[h^{3}\right]=\frac{n}{(n-1)(n-2)} \sum_{i, j} b_{i j}^{3} . \tag{18}
\end{equation*}
$$

Proof. We first consider $\mathbb{E}\left[h^{2}\right]$. Squaring (11), we obtain

$$
h^{2}=\sum_{i, j, k, l} b_{i j} b_{k l} \mathbf{1}_{T_{i j}} \mathbf{1}_{T_{k l}}=\sum_{i, j} b_{i j}^{2} \mathbf{1}_{T_{i j}}+\sum_{\substack{i, j, k, l: \\ i \neq k, j \neq l}} b_{i j} b_{k l} \mathbf{1}_{T_{i j} \cap T_{k l}}
$$

Taking expectations, we obtain:

$$
\begin{align*}
\mathbb{E}\left[h^{2}\right] & =\frac{1}{n} \sum_{i, j} b_{i j}^{2}+\frac{1}{n(n-1)} \sum_{\substack{i, j, k, l: \\
i \neq k, j \neq l}} b_{i j} b_{k l} \\
& =\frac{1}{n} \sum_{i, j} b_{i j}^{2}+\frac{1}{n(n-1)}\left(\sum_{i, j, k, l} b_{i j} b_{k l}-\sum_{i, j, l} b_{i j} b_{i l}-\sum_{i, j, k} b_{i j} b_{k j}+\sum_{i, j} b_{i j}^{2}\right) \\
& =\frac{1}{n-1} \sum_{i, j} b_{i j}^{2} \tag{19}
\end{align*}
$$

where the last equality follows from (17).
We now consider $\mathbb{E}\left[h^{3}\right]$. Cubing (11), we obtain

$$
\begin{aligned}
h^{3} & =\sum_{i, j, k, l, p, q} b_{i j} b_{k l} b_{p q} \mathbf{1}_{T_{i j}} \mathbf{1}_{T_{k l}} \mathbf{1}_{T_{p q}} \\
& =\sum_{i, j} b_{i j}^{3} \mathbf{1}_{T_{i j}}+3 \sum_{\substack{i, j, k, l: \\
i \neq k, j \neq l}} b_{i j}^{2} b_{k l} \mathbf{1}_{T_{i j} \cap T_{k l}} \\
& +\sum_{\substack{i, k, p \text { distinct, } \\
j, l, q \text { distinct }}} b_{i j} b_{k l} b_{p q} \mathbf{1}_{T_{i j} \cap T_{k l} \cap T_{p q}}
\end{aligned}
$$

Taking the expectation of the above gives:

$$
\begin{align*}
\mathbb{E}\left[g^{3}\right] & =\frac{1}{n} \sum_{i, j} b_{i j}^{3}+\frac{3}{n(n-1)} \sum_{\substack{i, j, k, l: \\
i \neq k, j \neq l}} b_{i j}^{2} b_{k l} \\
& +\frac{1}{n(n-1)(n-2)} \sum_{\substack{i, k, p \text { distinct, } \\
j, l, q \text { distinct }}} b_{i j} b_{k l} b_{p q} \tag{20}
\end{align*}
$$

Observe that

$$
\begin{align*}
\sum_{\substack{i, j, k, l: \\
i \neq k, j \neq l}} b_{i j}^{2} b_{k l} & =\sum_{i, j, k, l} b_{i j}^{2} b_{k l}-\sum_{i, j, l} b_{i j}^{2} b_{i l}-\sum_{i, j, k} b_{i j}^{2} b_{k j}+\sum_{i, j} b_{i j}^{3} \\
& =\sum_{i, j} b_{i j}^{3} \tag{21}
\end{align*}
$$

using (17).

Similarly, we have

$$
\begin{align*}
\sum_{\substack{i, k, p \\
j, l, q \\
j \text { distinctinct }}} b_{i j} b_{k l} b_{p q} & =\sum_{\substack{i, k, p, p \\
j, l, q}} b_{i j} b_{k l} b_{p q}\left(1-1_{i=k}\right)\left(1-1_{k=p}\right)\left(1-1_{p=i}\right)\left(1-1_{j=l}\right)\left(1-1_{l=q}\right)\left(1-1_{q=j}\right) \\
& =\sum_{\substack{i, k, p, p \\
j, l, q}} b_{i j} b_{k l} b_{p q}-3 \sum_{\substack{i, p \\
j, l, q}} b_{i j} b_{i l} b_{p q}-3 \sum_{\substack{i, k, p \\
j, q}} b_{i j} b_{k j} b_{p q} \\
& +2 \sum_{\substack{i \\
j, l, q}} b_{i j} b_{i l} b_{i q}+2 \sum_{i, k, p} b_{i j} b_{k j} b_{p j} \\
& +6 \sum_{\substack{i, p \\
j, l}} b_{i j} b_{i l} b_{p l}+3 \sum_{\substack{i, k \\
j, l}} b_{i j} b_{k l}^{2} \\
& -6 \sum_{\substack{i \\
j, l}} b_{i j} b_{i l}^{2}-6 \sum_{\substack{i, p}} b_{i j}^{2} b_{p j}+4 \sum_{i, j} b_{i j}^{3} \\
& =4 \sum_{i, j} b_{i j}^{3}, \tag{22}
\end{align*}
$$

again using (17). Substituting (21) and (22) into (20) gives:

$$
\begin{equation*}
\mathbb{E}\left[h^{3}\right]=\frac{n}{(n-1)(n-2)} \sum_{i, j} b_{i j}^{3}, \tag{23}
\end{equation*}
$$

completing the proof of Lemma 5 .
Combining (19) and (16) yields

$$
\begin{equation*}
\sum_{i, j} b_{i j}^{2}=\left(1+\frac{1}{n-1}\right)\left(1-\epsilon_{1}\right) c-\frac{c^{2}}{n-1} . \tag{24}
\end{equation*}
$$

We now require a lower bound on $\mathbb{E}\left[h^{3}\right]$. In fact, it is more convenient to deal with the non-negative function $g$; since $g$ is an affine shift of $h$, and $\mathbb{E}[h]$ and $\mathbb{E}\left[h^{2}\right]$ are both known, a lower bound on $\mathbb{E}\left[g^{3}\right]$ will immediately yield a lower bound on $\mathbb{E}\left[h^{3}\right]$.
Lemma 6. Under the hypotheses of Theorem 亿, if $g=\left(1+\frac{1}{n-1}\right) f_{1}+\left(1-\frac{1}{n-1}\right) c$, then

$$
\mathbb{E}\left[g^{3}\right] \geq c^{3} \frac{(n-2)^{2}(n+1)}{(n-1)^{3}}+3 c^{2} \frac{n(n-2)}{(n-1)^{3}}+c \frac{n^{2}}{(n-1)^{3}}-\frac{27}{4}\left(3 c^{2}+2 c\right) \epsilon_{1}^{1 / 2} / n .
$$

Proof. Recall that

$$
\mathbb{E}\left[\left(f-f_{1}\right)^{2}\right]=\epsilon_{1} c / n,
$$

and

$$
g=\left(1+\frac{1}{n-1}\right) f_{1}+\left(1-\frac{1}{n-1}\right) c
$$

Let

$$
F=\left(1+\frac{1}{n-1}\right) f+\left(1-\frac{1}{n-1}\right) c .
$$

Observe that $F$ takes just two values, $L:=\left(1-\frac{1}{n-1}\right) c$ and $H:=\left(1-\frac{1}{n-1}\right) c+1+\frac{1}{n-1}$. Moreover, $\mathbb{E}[F]=c$, and

$$
\mathbb{E}\left[(g-F)^{2}\right]=\left(\frac{n}{n-1}\right)^{2} \epsilon_{1} c / n
$$

These conditions suffice to obtain a lower bound on $\mathbb{E}\left[g^{3}\right]$. Indeed, we will now solve the following optimization problem.

Problem $P$. Let $\theta \in(0,1)$ and let $H, L, \eta \in \mathbb{R}_{\geq 0}$ be such that $H>L$. Define a function $F:[0,1] \rightarrow\{H, L\}$ by

$$
F(x)= \begin{cases}H & \text { if } 0 \leq x<\theta \\ L & \text { if } \theta \leq x \leq 1\end{cases}
$$

Among all (measurable) functions $g:[0,1] \rightarrow \mathbb{R}_{\geq 0}$ such that $\mathbb{E}[g]=\mathbb{E}[F]$ and $\mathbb{E}[(g-$ $\left.F)^{2}\right] \leq \eta$, find the minimum value of $\mathbb{E}\left[g^{3}\right]$.

Observe that if $g:[0,1] \rightarrow \mathbb{R}_{\geq 0}$ is feasible for $P$, then the function

$$
\tilde{g}(x)= \begin{cases}\frac{1}{\theta} \int_{0}^{\theta} g(x) \mathrm{d} x & \text { if } 0 \leq x<\theta \\ \frac{1}{1-\theta} \int_{\theta}^{1} g(x) \mathrm{d} x & \text { if } \theta \leq x \leq 1\end{cases}
$$

obtained by averaging $g$ first over $[0, \theta]$ and then over $[\theta, 1]$, is also feasible. Indeed, we clearly have $\mathbb{E}[\tilde{g}]=\mathbb{E}[g]$, and

$$
\begin{aligned}
\mathbb{E}\left[(g-F)^{2}\right] & =\int_{0}^{\theta}(g(x)-H)^{2} \mathrm{~d} x+\int_{\theta}^{1}(g(x)-L)^{2} \mathrm{~d} x \\
& \geq \frac{1}{\theta}\left(\int_{0}^{\theta}(g(x)-H) \mathrm{d} x\right)^{2}+\frac{1}{1-\theta}\left(\int_{\theta}^{1}(g(x)-L) \mathrm{d} x\right)^{2} \\
& =\theta\left(\frac{1}{\theta} \int_{0}^{\theta} g(x) \mathrm{d} x-H\right)^{2}+(1-\theta)\left(\frac{1}{1-\theta} \int_{\theta}^{1} g(x) \mathrm{d} x-L\right)^{2} \\
& =\mathbb{E}\left[(\tilde{g}-F)^{2}\right]
\end{aligned}
$$

by the Cauchy-Schwarz inequality. Moreover, we have

$$
\begin{aligned}
\mathbb{E}\left[g^{3}\right] & =\int_{0}^{\theta} g(x)^{3} \mathrm{~d} x+\int_{\theta}^{1} g(x)^{3} \mathrm{~d} x \\
& =\theta \cdot \frac{1}{\theta} \int_{0}^{\theta} g(x)^{3} \mathrm{~d} x+(1-\theta) \cdot \frac{1}{1-\theta} \int_{\theta}^{1} g(x)^{3} \mathrm{~d} x \\
& \geq \theta\left(\frac{1}{\theta} \int_{0}^{\theta} g(x) \mathrm{d} x\right)^{3}+(1-\theta)\left(\frac{1}{1-\theta} \int_{\theta}^{1} g(x) \mathrm{d} x\right)^{3} \\
& =\mathbb{E}\left[\tilde{g}^{3}\right]
\end{aligned}
$$

by the convexity of $y \mapsto y^{3}$. Hence, replacing $g$ with $\tilde{g}$ if necessary, we may assume that $g$ is constant on $[0, \theta)$ and on $[\theta, 1]$. In other words, we may assume that $g$ has the following form:

$$
g(x)= \begin{cases}r & \text { if } 0 \leq x<\theta \\ s & \text { if } \theta \leq x \leq 1\end{cases}
$$

Therefore, $P$ is equivalent to the following problem:

## Problem Q:

$$
\begin{aligned}
\text { Minimize } & \theta r^{3}+(1-\theta) s^{3} \\
\text { subject to } & \theta r+(1-\theta) s=\theta H+(1-\theta) L \\
& \theta(r-H)^{2}+(1-\theta)(s-L)^{2} \leq \eta \\
& r, s \geq 0
\end{aligned}
$$

Or, writing $r=H-(1-\theta) \delta$ (so that $s=L+\theta \delta$ ), we obtain the following reformulation:

## Problem $Q^{\prime}$ :

$$
\begin{aligned}
\text { Minimize } & \theta(H-(1-\theta) \delta)^{3}+(1-\theta)(L+\theta \delta)^{3} \\
\text { subject to } & \theta(1-\theta) \delta^{2} \leq \eta \\
& -L / \theta \leq \delta \leq H /(1-\theta)
\end{aligned}
$$

When $\delta=H-L$, the function $g$ is constant. By the strict convexity of the function $y \mapsto y^{3}$ (on $\mathbb{R}_{\geq 0}$ ), the objective function is strictly decreasing on $[0, H-L]$ as a function of $\delta 1$ Hence, provided $\sqrt{\eta /(\theta(1-\theta))} \leq H-L$, the minimum is attained at $\delta=\sqrt{\eta /(\theta(1-\theta))}$, at which point the value of the objective function is

$$
\begin{aligned}
& \theta H^{3}+(1-\theta) L^{3}-3\left(H^{2}-L^{2}\right) \sqrt{\theta(1-\theta)} \eta^{1 / 2}+3((1-\theta) H+\theta L) \eta-\frac{1-2 \theta}{\sqrt{\theta(1-\theta)}} \eta^{3 / 2} \\
& =\mathbb{E}\left[F^{3}\right]-3\left(H^{2}-L^{2}\right) \sqrt{\theta(1-\theta)} \eta^{1 / 2}+3((1-\theta) H+\theta L) \eta-\frac{1-2 \theta}{\sqrt{\theta(1-\theta)}} \eta^{3 / 2}
\end{aligned}
$$

using the fact that $\theta H^{3}+(1-\theta) L^{3}=\mathbb{E}\left[F^{3}\right]$. Substituting in our values, namely $\eta=\left(\frac{n}{n-1}\right)^{2} \epsilon_{1} c / n, H=\left(1-\frac{1}{n-1}\right) c+\left(1+\frac{1}{n-1}\right), L=\left(1-\frac{1}{n-1}\right) c$, and $\theta=c / n$, we see that provided $\epsilon_{1} \leq 1-c / n$ (which holds provided $\epsilon_{0} \leq 1 / 2$ ), the optimum value of $Q^{\prime}$ is

$$
\begin{aligned}
& \mathbb{E}\left[F^{3}\right]-3\left(1+\frac{1}{n-1}\right)\left(\left(1+\frac{1}{n-1}\right)^{2}+2 c\left(1-\frac{1}{(n-1)^{2}}\right)\right)(1-c / n)^{1 / 2} \epsilon_{1}^{1 / 2} c / n \\
& +3\left(\left(1-\frac{1}{n-1}\right) c+(1-c / n)\left(1+\frac{1}{n-1}\right)\right) \frac{n c}{(n-1)^{2}} \epsilon_{1}-\frac{1-2 c / n}{\sqrt{1-c / n}}\left(1+\frac{1}{n-1}\right)^{3} \epsilon_{1}^{3 / 2} c / n
\end{aligned}
$$

[^1]Hence, we have

$$
\begin{aligned}
\mathbb{E}\left[g^{3}\right] & \geq \mathbb{E}\left[F^{3}\right]-3\left(1+\frac{1}{n-1}\right)\left(\left(1+\frac{1}{n-1}\right)^{2}+2 c\left(1-\frac{1}{(n-1)^{2}}\right)\right)(1-c / n)^{1 / 2} \epsilon_{1}^{1 / 2} c / n \\
& +3\left(\left(1-\frac{1}{n-1}\right) c+(1-c / n)\left(1+\frac{1}{n-1}\right)\right) \frac{n c}{(n-1)^{2}} \epsilon_{1}-\frac{1-2 c / n}{\sqrt{1-c / n}}\left(1+\frac{1}{n-1}\right)^{3} \epsilon_{1}^{3 / 2} c / n \\
& \geq \mathbb{E}\left[F^{3}\right]-3\left(1+\frac{1}{n-1}\right)^{3}\left(2 c^{2}+c\right) \epsilon_{1}^{1 / 2} / n-\left(1+\frac{1}{n-1}\right)^{3} \epsilon_{1}^{3 / 2} c / n \\
& \geq \mathbb{E}\left[F^{3}\right]-\left(1+\frac{1}{n-1}\right)^{3}\left(6 c^{2}+4 c\right) \epsilon_{1}^{1 / 2} / n \\
& \geq \mathbb{E}\left[F^{3}\right]-\frac{27}{4}\left(3 c^{2}+2 c\right) \epsilon_{1}^{1 / 2} / n,
\end{aligned}
$$

using the facts that $\epsilon_{1} \leq 1, c / n \leq 1 / 2$ and $n \geq 3$. Observe that

$$
\mathbb{E}\left[F^{3}\right]=c^{3} \frac{(n-2)^{2}(n+1)}{(n-1)^{3}}+3 c^{2} \frac{n(n-2)}{(n-1)^{3}}+c \frac{n^{2}}{(n-1)^{3}},
$$

so we have

$$
\mathbb{E}\left[g^{3}\right] \geq c^{3} \frac{(n-2)^{2}(n+1)}{(n-1)^{3}}+3 c^{2} \frac{n(n-2)}{(n-1)^{3}}+c \frac{n^{2}}{(n-1)^{3}}-\frac{27}{4}\left(3 c^{2}+2 c\right) \epsilon_{1}^{1 / 2} / n,
$$

as required.
Since $g=h+c$, we have

$$
\mathbb{E}\left[g^{3}\right]=\mathbb{E}\left[(h+c)^{3}\right]=\mathbb{E}\left[h^{3}\right]+3 c \mathbb{E}\left[h^{2}\right]+3 c^{2} \mathbb{E}[h]+c^{3}
$$

Combining this with Lemma 6 yields:

$$
\mathbb{E}\left[h^{3}\right] \geq c^{3} \frac{(n-2)^{2}(n+1)}{(n-1)^{3}}+3 c^{2} \frac{n(n-2)}{(n-1)^{3}}+c \frac{n^{2}}{(n-1)^{3}}-\frac{27}{4}\left(3 c^{2}+2 c\right) \epsilon_{1}^{1 / 2} / n-3 c \mathbb{E}\left[h^{2}\right]-3 c^{2} \mathbb{E}[h]-c^{3}
$$

Using $\mathbb{E}[h]=0$ and (16), we obtain:

$$
\begin{aligned}
\mathbb{E}\left[h^{3}\right] & \geq \frac{c n^{2}}{(n-1)^{3}}-\frac{3 c^{2} n}{(n-1)^{3}}+\frac{2 c^{3}}{(n-1)^{3}}+\epsilon_{1} \frac{3 c^{2} n}{(n-1)^{2}}-\frac{27}{4} \epsilon_{1}^{1 / 2}\left(3 c^{2}+2 c\right) / n \\
& \geq \frac{c n^{2}}{(n-1)^{3}}-\frac{3 c^{2} n}{(n-1)^{3}}+\frac{2 c^{3}}{(n-1)^{3}}-\frac{27}{4} \epsilon_{1}^{1 / 2}\left(3 c^{2}+2 c\right) / n
\end{aligned}
$$

Combining this with (18) yields:

$$
\begin{aligned}
\sum_{i, j} b_{i j}^{3} & \geq c\left(1-\frac{1}{(n-1)^{2}}\right)-c^{2} \frac{3(n-2)}{(n-1)^{2}}+c^{3} \frac{2(n-2)}{n(n-1)^{2}}-\epsilon_{1}^{1 / 2}\left(3 c^{2}+2 c\right) \frac{27(n-1)(n-2)}{4 n^{2}} \\
& \geq c\left(1-\frac{1}{(n-1)^{2}}\right)-c^{2} \frac{3(n-2)}{(n-1)^{2}}+c^{3} \frac{2(n-2)}{n(n-1)^{2}}-\frac{27}{4}\left(3 c^{2}+2 c\right) \epsilon_{1}^{1 / 2}
\end{aligned}
$$

To summarize, we now know that the $b_{i j}$ 's satisfy:

$$
\begin{equation*}
\sum_{i, j} b_{i j}^{2}=\left(1+\frac{1}{n-1}\right)\left(1-\epsilon_{1}\right) c-\frac{c^{2}}{n-1} \tag{25}
\end{equation*}
$$

$$
\begin{equation*}
\sum_{i, j} b_{i j}^{3} \geq c\left(1-\frac{1}{(n-1)^{2}}\right)-c^{2} \frac{3(n-2)}{(n-1)^{2}}+c^{3} \frac{2(n-2)}{n(n-1)^{2}}-\frac{27}{4}\left(3 c^{2}+2 c\right) \epsilon_{1}^{1 / 2} \tag{26}
\end{equation*}
$$

To illuminate the above, we now calculate the values of $\sum_{i, j} b_{i j}^{2}$ and $\sum_{i, j} b_{i j}^{3}$ when $\mathcal{A}$ is a disjoint union of $c 1$-cosets of $S_{n}$, for some $c \in[n]$. Such a set $\mathcal{A}$ must be of the form

$$
T_{i_{1} j} \cup T_{i_{2} j} \cup \ldots \cup T_{i_{c} j}
$$

for some $j \in[n]$ and some distinct $i_{1}, i_{2}, \ldots, i_{c} \in[n]$, or of the form

$$
T_{i j_{1}} \cup T_{i j_{2}} \cup \ldots \cup T_{i j_{c}}
$$

for some $i \in[n]$ and some distinct $i_{1}, i_{2}, \ldots, i_{c} \in[n]$. Clearly, all these families have the same $\sum_{i, j} b_{i j}^{2}$ and the same $\sum_{i, j} b_{i j}^{3}$; we may therefore assume that $\mathcal{A}=$ $T_{11} \cup T_{12} \cup \ldots \cup T_{1 c}$. For this family, the matrix $\left(b_{i j}\right)$ is as follows:

$$
\left(\begin{array}{cccccc}
1-\frac{c}{n} & \cdots & 1-\frac{c}{n} & -\frac{c}{n} & \cdots & -\frac{c}{n} \\
-\frac{n-c}{n(n-1)} & \cdots & -\frac{n-c}{n(n-1)} & \frac{c}{n(n-1)} & \cdots & \frac{c}{n(n-1)} \\
\vdots & & \vdots & \vdots & & \vdots \\
-\frac{n-c}{n(n-1)} & \cdots & -\frac{n-c}{n(n-1)} & \frac{c}{n(n-1)} & \cdots & \frac{c}{n(n-1)}
\end{array}\right)
$$

we have

$$
\begin{aligned}
\sum_{i, j} b_{i j}^{2} & =c\left(1-\frac{c}{n}\right)^{2}+c(n-1)\left(\frac{n-c}{n(n-1)}\right)^{2}+(n-c)(n-1)\left(\frac{c}{n(n-1)}\right)^{2} \\
& =\frac{c(n-c)}{n-1} \\
& =c\left(1+\frac{1}{n-1}\right)-\frac{c^{2}}{n-1} \\
& :=F(n, c)
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_{i, j} b_{i j}^{3} & =c\left(1-\frac{c}{n}\right)^{3}-(n-c)\left(\frac{c}{n}\right)^{3}-c(n-1)\left(\frac{c-1}{n(n-1)}\right)^{3}+(n-c)(n-1)\left(\frac{c}{n(n-1)}\right)^{3} \\
& =c\left(1-\frac{1}{(n-1)^{2}}\right)-c^{2} \frac{3(n-2)}{(n-1)^{2}}+c^{3} \frac{2(n-2)}{n(n-1)^{2}} \\
& :=G(n, c)
\end{aligned}
$$

Hence, if $c$ is a fixed integer, and $\mathcal{A}$ is a family of size $c(n-1)$ ! whose characteristic function has Fourier transform which is highly concentrated on the first two levels, then (25) says that $\sum_{i, j} b_{i j}^{2}$ is close to $F(n, c)$, the value it takes when $\mathcal{A}$ is a disjoint union of $c 1$-cosets of $S_{n}$. Similarly, (26) says that $\sum_{i, j} b_{i j}^{3}$ is not too far below $G(n, c)$, the value it takes when $\mathcal{A}$ is a disjoint union of $c 1$-cosets of $S_{n}$. Formally, we have

$$
\begin{equation*}
\sum_{i, j} b_{i j}^{2}=F(n, c)-c\left(1+\frac{1}{n-1}\right) \epsilon_{1} \tag{27}
\end{equation*}
$$

and

$$
\begin{equation*}
\sum_{i, j} b_{i j}^{3} \geq G(n, c)-\frac{27}{4}\left(3 c^{2}+2 c\right) \epsilon_{1}^{1 / 2} \tag{28}
\end{equation*}
$$

These two facts will suffice to show that $\mathcal{A}$ is close to being a union of 1-cosets of $S_{n}$. We now observe that cannot be too small.

## Claim 1.

$$
c \geq 1-O(1 / n)-\frac{135}{2} \epsilon_{0}^{1 / 2}
$$

Proof of claim: Suppose that $c \leq 1$. By the convexity of $y \mapsto y^{3 / 2}$, we have

$$
\begin{aligned}
\sum_{i, j} b_{i j}^{3} & \leq\left(\sum_{i, j} b_{i j}^{2}\right)^{3 / 2} \\
& \leq(c+2 c / n)^{3 / 2} \\
& \leq c^{3 / 2}(1+2 / n)^{3 / 2} \\
& \leq c^{3 / 2}(1+O(1 / n))
\end{aligned}
$$

Combining this with (26) yields:

$$
\begin{aligned}
c^{3 / 2}(1+O(1 / n)) & \geq c-O\left(c / n^{2}\right)-O\left(c^{2} / n\right)-\frac{27}{4}\left(3 c^{2}+2 c\right) \epsilon_{1}^{1 / 2} \\
& \geq c\left(1-O(1 / n)-\frac{135}{4} \epsilon_{1}^{1 / 2}\right) \\
& \geq c\left(1-O(1 / n)-\frac{135}{4} \epsilon_{0}^{1 / 2}\right)
\end{aligned}
$$

Rearranging, we obtain:

$$
c^{1 / 2} \geq 1-O(1 / n)-\frac{135}{4} \epsilon_{0}^{1 / 2}
$$

Squaring yields

$$
c \geq 1-O(1 / n)-\frac{135}{2} \epsilon_{0}^{1 / 2}
$$

as required.
Provided $n_{0}$ is sufficiently large, and $\epsilon_{0}$ is sufficiently small, Claim 1 implies that $c \geq 1 / 2$, so $c \leq 2 c^{2}$. Hence, (25) and (26) imply the following.

$$
\begin{align*}
& \sum_{i, j} b_{i j}^{2} \leq c+2 c / n=c+O(c / n)  \tag{29}\\
& \sum_{i, j} b_{i j}^{3} \geq c-O\left(c^{2}\right) \epsilon_{1}^{1 / 2}-O\left(c^{2} / n\right) \tag{30}
\end{align*}
$$

Let $x_{1}, \ldots, x_{N}$ denote the entries $\left(b_{i j}\right)_{i, j \in[n]}$ in non-increasing order. We have

$$
\begin{equation*}
\sum_{k=1}^{N} x_{k}^{2} \leq c+O(c / n) \tag{31}
\end{equation*}
$$

and

$$
\begin{equation*}
\sum_{k=1}^{N} x_{k}^{3} \geq c-O\left(c^{2}\right) \epsilon_{1}^{1 / 2}-O\left(c^{2} / n\right) \tag{32}
\end{equation*}
$$

Subtracting (32) from (31) yields:

$$
\begin{equation*}
\sum_{k=1}^{N} x_{k}^{2}\left(1-x_{k}\right) \leq O\left(c^{2}\right) \epsilon_{1}^{1 / 2}+O\left(c^{2} / n\right) \tag{33}
\end{equation*}
$$

Let $m$ be the largest index $k$ such that $x_{k} \geq 1 / 2$ (recall that the $x_{k}$ are arranged in non-increasing order). Then

$$
\sum_{k=m+1}^{N} x_{k}^{2} \leq 2 \sum_{k=m+1}^{N} x_{k}^{2}\left(1-x_{k}\right) \leq O\left(c^{2}\right) \epsilon_{1}^{1 / 2}+O\left(c^{2} / n\right)
$$

Therefore

$$
\begin{equation*}
m \geq \sum_{k=1}^{m} x_{k}^{2} \geq c-O\left(c^{2}\right) \epsilon_{1}^{1 / 2}-O\left(c^{2} / n\right) \tag{34}
\end{equation*}
$$

On the other hand, we have

$$
\sum_{k=1}^{m}\left(1-x_{k}\right) \leq 4 \sum_{k=1}^{m} x_{k}^{2}\left(1-x_{k}\right) \leq O\left(c^{2}\right) \epsilon_{1}^{1 / 2}+O\left(c^{2} / n\right)
$$

Rearranging,

$$
\begin{equation*}
\sum_{k=1}^{m} x_{k} \geq m-O\left(c^{2}\right) \epsilon_{1}^{1 / 2}-O\left(c^{2} / n\right) \tag{35}
\end{equation*}
$$

Since $2 x_{k}-1 \leq x_{k}^{2}$, we have

$$
\begin{equation*}
c+O(c / n) \geq \sum_{k=1}^{m} x_{k}^{2} \geq 2 \sum_{k=1}^{m} x_{k}-m \geq m-O\left(c^{2}\right) \epsilon_{1}^{1 / 2}-O\left(c^{2} / n\right) \tag{36}
\end{equation*}
$$

Combining (34) and (36) yields:

$$
\begin{equation*}
|c-m| \leq O\left(c^{2}\right) \epsilon_{1}^{1 / 2}+O\left(c^{2} / n\right) \tag{37}
\end{equation*}
$$

Our aim is now to replace $m$ by an integer $m^{\prime}$ which satisfies the analogues of (35) and (37), and which in addition has $\left|c-m^{\prime}\right|<1$. If $m \geq c$, then let $m^{\prime}=\lceil c\rceil$. Certainly, the analogue of (37) is satisfied, and we have

$$
\sum_{k=1}^{m^{\prime}} x_{k} \geq \frac{m^{\prime}}{m}\left(m-O\left(c^{2}\right) \epsilon_{1}^{1 / 2}-O\left(c^{2} / n\right)\right) \geq m^{\prime}-O\left(c^{2}\right) \epsilon_{1}^{1 / 2}-O\left(c^{2} / n\right)
$$

If $m<c$, then let $m^{\prime}=\lfloor c\rfloor$. Again, the analogue of (37) is satisfied, and using $x_{k} \geq-c / n$, we have

$$
\sum_{k=1}^{m^{\prime}} x_{k} \geq \sum_{k=1}^{m} x_{k}-c^{2} / n \geq m-O\left(c^{2}\right) \epsilon_{1}^{1 / 2}-O\left(c^{2} / n\right) \geq m^{\prime}-O\left(c^{2}\right) \epsilon_{1}^{1 / 2}-O\left(c^{2} / n\right)
$$

Summarising, we have

$$
\begin{gather*}
\left|c-m^{\prime}\right|<\min \left(1, O\left(c^{2}\right) \epsilon_{1}^{1 / 2}+O\left(c^{2} / n\right)\right)  \tag{38}\\
\sum_{k=1}^{m^{\prime}} x_{k} \geq m^{\prime}-O\left(c^{2}\right) \epsilon_{1}^{1 / 2}-O\left(c^{2} / n\right) \tag{39}
\end{gather*}
$$

Now observe that (38) and (39) hold with $m^{\prime}$ replaced by round (c). Indeed, we have $|c-\operatorname{round}(c)| \leq\left|c-m^{\prime}\right|$ always. Moreover, if $m^{\prime} \neq \operatorname{round}(c)$, then $\left(\right.$ since $\left.\left|m^{\prime}-c\right|<1\right)$ we have

$$
1=\left|m^{\prime}-\operatorname{round}(c)\right| \leq 2\left|m^{\prime}-c\right| \leq O\left(c^{2}\right) \epsilon_{1}^{1 / 2}+O\left(c^{2} / n\right)
$$

It follows that

$$
\sum_{k=1}^{\operatorname{round}(c)} x_{k} \geq \operatorname{round}(c)-1-c / n-O\left(c^{2}\right) \epsilon_{1}^{1 / 2}-O\left(c^{2} / n\right) \geq \operatorname{round}(c)-O\left(c^{2}\right) \epsilon_{1}^{1 / 2}-O\left(c^{2} / n\right)
$$

since $-c / n \leq x_{k} \leq 1$ for all $k$. We may therefore redefine $m^{\prime}=\operatorname{round}(c)$.
Let $b_{i_{l} j_{1}}, b_{i_{2} j_{2}}, \ldots, b_{i_{m^{\prime}} j_{m^{\prime}}}$ be the entries of the matrix $B$ corresponding to $x_{1}, \ldots, x_{m^{\prime}}$, and let

$$
\mathcal{C}=\bigcup_{l=1}^{m^{\prime}} T_{i_{l} j_{l}}
$$

denote the corresponding union of $m^{\prime} 1$-cosets of $S_{n}$. We have

$$
\begin{aligned}
\sum_{l=1}^{m^{\prime}}\left|\mathcal{A} \cap T_{i_{l} j_{l}}\right| & \geq(n-1)!\sum_{l=1}^{m^{\prime}} b_{i_{l} j_{l}} \\
& =(n-1)!\sum_{l=1}^{m^{\prime}} x_{l} \\
& \geq\left(m^{\prime}-O\left(c^{2}\right) \epsilon_{1}^{1 / 2}-O\left(c^{2} / n\right)\right)(n-1)! \\
& \geq\left(c-O\left(c^{2}\right) \epsilon_{1}^{1 / 2}-O\left(c^{2} / n\right)\right)(n-1)!
\end{aligned}
$$

Since $\left|T_{i_{l} j_{l}} \cap T_{i_{k} j_{k}}\right| \leq(n-2)$ ! whenever $k \neq l$, we have

$$
|\mathcal{A} \cap \mathcal{C}| \geq \sum_{l=1}^{m^{\prime}}\left|\mathcal{A} \cap T_{i_{l} j_{l}}\right|-\binom{m^{\prime}}{2}(n-2)!\geq\left(c-O\left(c^{2}\right) \epsilon_{1}^{1 / 2}-O\left(c^{2} / n\right)\right)(n-1)!
$$

i.e. $\mathcal{A}$ contains almost all of $\mathcal{C}$. Since $|\mathcal{A}|=c(n-1)$ !, we must have

$$
|\mathcal{A} \triangle \mathcal{C}|=|\mathcal{A}|+|\mathcal{C}|-2|\mathcal{A} \cap \mathcal{C}| \leq\left(O\left(c^{2}\right) \epsilon_{1}^{1 / 2}+O\left(c^{2} / n\right)\right)(n-1)!
$$

Let $\tilde{f}=1_{\mathcal{C}}$ denote the characteristic function of $\mathcal{C}$; then we have

$$
\mathbb{E}\left[(f-\tilde{f})^{2}\right]=|\mathcal{A} \triangle \mathcal{C}| / n!\leq\left(O\left(c^{2}\right) \epsilon_{1}^{1 / 2}+O\left(c^{2} / n\right)\right) / n
$$

This completes the proof of Theorem 1

## 4 Applications

We now give two applications of Theorem 1 The first application is a short proof of a conjecture of Cameron and Ku (Conjecture 2) on the structure of large intersecting families of permutations in $S_{n}$. The second application (Theorem 3) describes the structure of families of permutations which have small edge-boundary in the transposition graph. Both of these applications involve normal Cayley graphs on $S_{n}$, so we will first give some background on normal Cayley graphs on finite groups.

Definition. Let $G$ be a finite group, and let $S \subset G \backslash\{I d\}$ be symmetric (meaning that $S^{-1}=S$ ). The Cayley graph on $G$ with generating set $S$ is the undirected graph with vertex-set $G$, where we join $g$ to gs for every $g \in G$ and $s \in S$; we denote it by $\operatorname{Cay}(G, S)$. Formally,

$$
V(\operatorname{Cay}(G, S))=G, \quad E(\operatorname{Cay}(G, S))=\{\{g, g s\}: g \in G, s \in S\}
$$

Note that the Cayley graph $\operatorname{Cay}(G, S)$ is $|S|$-regular. If the generating set $S$ is conjugationinvariant, i.e. is a union of conjugacy classes of $G$, the Cayley graph $\operatorname{Cay}(G, S)$ is said to be a normal Cayley graph.

The connection between normal Cayley graphs and the Fourier transform arises from the following fundamental theorem, which states that for any normal Cayley graph, the eigenspaces of its adjacency matrix are in 1-1 correspondence with the isomorphism classes of irreducible representations of the group.
Theorem 7 (Frobenius / Schur / Diaconis-Shahshahani). Let $G$ be a finite group, let $S \subset G$ be an inverse-closed, conjugation-invariant subset of $G$, let $\Gamma$ be the Cayley graph on $G$ with generating set $S$, and let $A$ be the adjacency matrix of $\Gamma$. Let $\left\{\rho_{1}, \ldots, \rho_{k}\right\}$ be a complete set of non-isomorphic irreducible representations of $G-$ i.e., containing one representative from each isomorphism class of irreducible representations of $G$. Let $U_{\rho_{i}}$ denote the subspace of $\mathbb{C}[G]$ consisting of functions whose Fourier transform is supported on $\left[\rho_{i}\right]$. Then we have

$$
\mathbb{C}[G]=\bigoplus_{i=1}^{k} U_{\rho_{i}}
$$

and each $U_{\rho_{i}}$ is an eigenspace of $A$ with dimension $\operatorname{dim}\left(\rho_{i}\right)^{2}$ and eigenvalue

$$
\begin{equation*}
\lambda_{i}=\frac{1}{\operatorname{dim}\left(\rho_{i}\right)} \sum_{g \in S} \chi_{i}(g) \tag{40}
\end{equation*}
$$

where $\chi_{i}(\sigma)=\operatorname{Trace}\left(\rho_{i}(\sigma)\right)$ denotes the character of the irreducible representation $\rho_{i}$.

### 4.1 Large intersecting families in $S_{n}$

In this section, we will apply Theorem1to give a new proof of a conjecture of Cameron and Ku on the structure of large intersecting families in $S_{n}$.

The following definition was introduced by Deza and Frankl [7.

Definition. We say that a family $\mathcal{A} \subset S_{n}$ is intersecting if any two permutations in $\mathcal{A}$ agree on some point - i.e., for any $\sigma, \pi \in \mathcal{A}$, there exists $i \in[n]$ such that $\sigma(i)=\pi(i)$.

Deza and Frankl [7] proved the following analogue of the Erdős-Ko-Rado theorem [16] for permutations.

Theorem 8 (Deza-Frankl). If $\mathcal{A} \subset S_{n}$ is intersecting, then $|\mathcal{A}| \leq(n-1)$ !.
Proof. We reproduce the original proof of Deza and Frankl. Let $\rho \in S_{n}$ be an $n$-cycle, and let $H$ be the cyclic group of order $n$ generated by $\rho$. For any left coset $\sigma H$ of $H$, any two distinct permutations in $\sigma H$ disagree at every point, and therefore $\sigma H$ contains at most one member of $\mathcal{A}$. Since the left cosets of $H$ partition $S_{n}$, it follows that $|\mathcal{A}| \leq(n-1)$ !.

Note that equality holds in Theorem 8 if $\mathcal{A}$ is a 1-coset. Deza and Frankl conjectured that equality holds only for the 1-cosets. This turned out to be much harder to prove than is usual with equality statements for Erdős-Ko-Rado type theorems; it was eventually proved by Cameron and Ku [6].

Theorem 9 (Cameron-Ku). If $\mathcal{A} \subset S_{n}$ is intersecting with $|\mathcal{A}|=(n-1)$ !, then $\mathcal{A}$ is a 1-coset.

Larose and Malvenuto [29] independently found a different proof of Theorem 9 , More recently, Wang and Zhang [35] gave a shorter proof. All three proofs were combinatorial; none are straightforward, all requiring a certain amount of ingenuity. In [20, Godsil and Meagher gave an algebraic proof. In [15, a proof quite similar to that of 20] is presented.

We say that an intersecting family $\mathcal{A} \subset S_{n}$ is centred if there exist $i, j \in[n]$ such that every permutation in $\mathcal{A}$ maps $i$ to $j$, i.e. $\mathcal{A}$ is contained within a 1-coset. Cameron and Ku asked how large a non-centred intersecting family can be. Experimentation suggests that the further an intersecting family is from being centred, the smaller it must be. The following are natural candidates for large, non-centred intersecting families:

- $\mathcal{B}=\left\{\sigma \in S_{n}: \sigma\right.$ fixes at least two elements of $\left.\{1,2,3\}\right\}$.

This has size $3(n-2)!-2(n-3)$ !.
It requires the removal of $(n-2)!-(n-3)$ ! permutations to make it centred.

- $\mathcal{C}=\left\{\sigma: \sigma(1)=1, \sigma\right.$ intersects $\left(\begin{array}{ll}1 & 2)\end{array}\right\} \cup\left\{\left(\begin{array}{ll}1 & 2)\end{array}\right.\right.$.

Claim: $|\mathcal{C}|=(1-1 / e+o(1))(n-1)$ !.
Proof of Claim: Let $\mathcal{D}_{n}=\left\{\sigma \in S_{n}: \sigma(i) \neq i \forall i \in[n]\right\}$ be the set of derangements of $[n]$ (permutations without fixed points); let $d_{n}=\left|\mathcal{D}_{n}\right|$ be the number of derangements of $[n]$. By the inclusion-exclusion formula,

$$
d_{n}=\sum_{i=0}^{n}(-1)^{i}\binom{n}{i}(n-i)!=n!\sum_{i=0}^{n} \frac{(-1)^{i}}{i!}=n!(1 / e+o(1))
$$

Note that a permutation which fixes 1 intersects (12) if and only if it has a fixed point greater than 2 . The number of permutations fixing 1 alone is clearly $d_{n-1}$; the number of permutations fixing 1 and 2 alone is clearly $d_{n-2}$, so the number of permutations fixing 1 and some other point greater than 2 is $(n-1)!-d_{n-1}-d_{n-2}$. Hence,

$$
|\mathcal{C}|=(n-1)!-d_{n-1}-d_{n-2}=(1-1 / e+o(1))(n-1)!,
$$

as required.
Note that $\mathcal{C}$ can be made centred just by removing (12).

For $n \leq 5, \mathcal{B}$ and $\mathcal{C}$ have the same size; for $n \geq 6, \mathcal{C}$ is larger. Cameron and Ku 6] made the following conjecture.

Conjecture 10. If $n \geq 6$, and $\mathcal{A} \subset S_{n}$ is a non-centred intersecting family, then $|\mathcal{A}| \leq|\mathcal{C}|$. Equality holds only if $\mathcal{A}$ is a 'double translate' of $\mathcal{C}$, meaning that there exist permutations $\sigma, \pi \in S_{n}$ such that $\mathcal{A}=\sigma \mathcal{C} \pi$.

This was proved for all sufficiently large $n$ by the first author in [10]. To prove it, he first shows that if $\mathcal{A} \subset S_{n}$ is an intersecting family with $|\mathcal{A}|=\Omega((n-1)!)$, then the Fourier transform of $\mathbf{1}_{\mathcal{A}}$ is highly concentrated on the first two irreducible representations of $S_{n}$. Secondly, he appeals to a weak version of Theorem 1 namely that if $\mathcal{A} \subset S_{n}$ has $|\mathcal{A}|=\Omega\left((n-1)\right.$ !), and the Fourier transform of $\mathbf{1}_{\mathcal{A}}$ is highly concentrated on the first two irreducible representations, then there exists a 1-coset $T_{i j}$ such that $\left|\mathcal{A} \cap T_{i j}\right| \geq \omega((n-2)!)$. Thirdly, he uses the fact that $\mathcal{A}$ is intersecting to 'bootstrap' this weak statement, showing that in fact, $\left|\mathcal{A} \cap T_{i j}\right| \geq \Omega((n-1)$ !). This is done by showing that if $\left|\mathcal{A} \cap T_{i j}\right|$ is somewhat large, then $\left|\mathcal{A} \cap T_{i k}\right|$ must be very small for each $k \neq j$, using an extremal result on the products of the sizes of cross-intersecting families of permutations. Fourthly, he uses a combinatorial stability argument to deduce that almost all of $T_{i j}$ is contained within $\mathcal{A}$. Note that applying Theorem 11 leads to a slicker proof, as it allows us to conclude straight away that $\left|\mathcal{A} \cap T_{i j}\right| \geq \Omega((n-1)!)$, eliminating the third stage of the argument.

Cameron and Ku also made the following weaker conjecture.
Conjecture 2. There exists $\delta>0$ such that for all $n$, if $\mathcal{A} \subset S_{n}$ is an intersecting family with $|\mathcal{A}| \geq(1-\delta)(n-1)$ !, then $\mathcal{A}$ is contained within a 1-coset of $S_{n}$.

Note that Conjecture 2 follows immediately from Conjecture 10. Again, the most natural proof of Conjecture 2 is via Theorem 1 . We give this (new) proof below.

First, we give some background on eigenvalues techniques for studying intersecting families in $S_{n}$.

Let $\Gamma_{n}$ denote the derangement graph on $S_{n}$, where two permutations are joined iff they disagree everywhere. This is simply the Cayley graph on $S_{n}$ generated by the set of derangements of $[n]$. As above, we let $\mathcal{D}_{n}$ denote the set of derangements of $[n]$, and we let $d_{n}=\left|\mathcal{D}_{n}\right|$; then $\Gamma_{n}$ is $d_{n}$-regular. Observe that an intersecting family in $S_{n}$ is precisely an independent set in $\Gamma_{n}$.

The following theorem of Hoffman gives an upper bound on the size of an independent set in a regular graph in terms of the eigenvalues of the adjacency matrix of the graph.

Theorem 11 (Hoffman's theorem). Let $G=(V, E)$ be a d-regular graph, whose adjacency matrix $A$ has eigenvalues $d=\lambda_{1} \geq \lambda_{2} \geq \ldots \geq \lambda_{|V|}$. Let $X \subset V(G)$ be an independent set, and let $\alpha=|X| /|V|$. Then

$$
|X| \leq \frac{-\lambda_{|V|}}{d-\lambda_{|V|}}|V|
$$

If equality holds, then $\mathbf{1}_{X}-\alpha \mathbf{f} \in E_{A}\left(\lambda_{|V|}\right)$, the $\lambda_{|V|}$-eigenspace of $A$, where $\mathbf{f}$ denotes the all-1's vector.

Note that $\mathcal{D}_{n}$ is a union of conjugacy classes of $S_{n}$, so $\Gamma_{n}$ is a normal Cayley graph, and therefore Theorem 7 can be used to calculate the eigenvalues of its adjacency matrix.

Using Theorem 7 together with an analysis of symmetric functions, Renteln 32 proved the following.

Theorem 12 (Renteln). The minimum eigenvalue of $\Gamma_{n}$ is $-d_{n} /(n-1)$.
Plugging the value $\lambda_{|V|}=-d_{n} /(n-1)$ into Theorem 11yields an alternative (much longer!) proof of Theorem 8 .

In [10, a different proof of Theorem 12 (avoiding symmetric functions) is given; this proof shows in addition that the $-d_{n} /(n-1)$ eigenspace is precisely $U_{(n-1,1)}$. This can be used to give an alternative proof of Theorem 9 essentially the one presented in 15 and 20 .

Proof of Theorem 9. Let $\mathcal{A} \subset S_{n}$ be an intersecting family of permutations with $|\mathcal{A}|=$ $(n-1)$ !. It follows from the equality part of Hoffman's theorem that $\mathbf{1}_{\mathcal{A}}-(|\mathcal{A}| / n!) \mathbf{f} \in$ $U_{(n-1,1)}$. Since $U_{(n)}$ is the space of constant functions, it follows that $\mathbf{1}_{\mathcal{A}} \in U_{(n)} \oplus$ $U_{(n-1,1)}=U_{1}$. Theorem 4 then implies that $\mathcal{A}$ must be a disjoint union of 1-cosets of $S_{n}$. Since $\mathcal{A}$ is intersecting, it must be a single 1-coset of $S_{n}$.

To prove Conjecture 2, we need the following 'stability version' of Hoffman's bound, proved in [10, Lemma 3.2].

Lemma 13. Let $G=(V, E)$ be a d-regular graph, whose adjacency matrix $A$ has eigenvalues $d=\lambda_{1} \geq \lambda_{2} \geq \ldots \geq \lambda_{|V|}$. Let $K=\max \left\{i: \lambda_{i}>\lambda_{|V|}\right\}$. Let $X \subset V(G)$ be an independent set; let $\alpha=|X| /|V|$. Let $U=\operatorname{Span}\{\mathbf{f}\} \oplus E\left(\lambda_{|V|}\right)$ be the direct sum of the subspace of constant vectors and the $\lambda_{|V|}$-eigenspace of $A$. Let $P_{U}$ denote orthogonal projection onto the subspace $U$. Then

$$
\left\|\mathbf{1}_{X}-P_{U}\left(\mathbf{1}_{X}\right)\right\|_{2}^{2} \leq \frac{(1-\alpha)\left|\lambda_{|V|}\right|-d \alpha}{\left|\lambda_{|V|}\right|-\left|\lambda_{K}\right|} \alpha
$$

Recall the following fact from [10]:
Fact. The derangement graph $\Gamma_{n}$ has $\left|\lambda_{K}\right|=O\left(d_{n} / n^{2}\right)$.

Substituting this into Lemma 13 shows that a large intersecting family in $S_{n}$ must have its characteristic vector close to $U_{1}$ :

Lemma 14. If $\mathcal{A} \subset S_{n}$ is an intersecting family of permutations with $|\mathcal{A}|=\alpha n$ !, then

$$
\left\|\mathbf{1}_{\mathcal{A}}-P_{U_{1}}\left(\mathbf{1}_{\mathcal{A}}\right)\right\|_{2}^{2} \leq(1-\alpha n)(1+O(1 / n)) \alpha
$$

Proof. Let $\mathcal{A} \subset S_{n}$ be an intersecting family with $|\mathcal{A}|=\alpha n$ !. Applying Lemma 13 with $G=\Gamma_{n}$ and $U=U_{1}=U_{(n)} \oplus U_{(n-1,1)}$ yields:

$$
\begin{aligned}
\left\|\mathbf{1}_{\mathcal{A}}-P_{U_{1}}\left(\mathbf{1}_{\mathcal{A}}\right)\right\|_{2}^{2} & \leq \frac{(1-\alpha) d_{n} /(n-1)-d_{n} \alpha}{d_{n} /(n-1)-\left|\lambda_{K}\right|} \alpha \\
& =\frac{1-\alpha-\alpha(n-1)}{1-(n-1)\left|\lambda_{K}\right| / d_{n}} \alpha \\
& =\frac{1-\alpha n}{1-O(1 / n)} \alpha \\
& =(1-\alpha n)(1+O(1 / n)) \alpha
\end{aligned}
$$

proving the lemma.
We now combine Lemma 14 with Theorem 1 to show that a large intersecting family in $S_{n}$ must be close to a 1-coset, an intermediate step towards proving Conjecture 2

Proposition 15. Given $\phi>0$, there exists $\delta=\delta(\phi)>0$ such that the following holds. If $\mathcal{A} \subset S_{n}$ is an intersecting family of permutations with $|\mathcal{A}| \geq(1-\delta)(n-1)$ !, then there exists a 1-coset $T_{i j}$ such that

$$
\left|\mathcal{A} \triangle T_{i j}\right| \leq \phi(n-1)!
$$

Proof. Let $\delta>0$ to be chosen later, with $\delta<1 / 2$. Let $\mathcal{A} \subset S_{n}$ be an intersecting family of permutations with $|\mathcal{A}|=\left(1-\delta_{1}\right)(n-1)$ !, where $\delta_{1} \leq \delta$. Note that, by Theorem 9 we may assume that $n \geq n_{0}$ for some fixed $n_{0} \in \mathbb{N}$, by making $\delta$ smaller if necessary.

Let $f=\mathbf{1}_{\mathcal{A}}$ denote the characteristic function of $\mathcal{A}$, and let $f_{1}=P_{U_{1}}\left(\mathbf{1}_{\mathcal{A}}\right)$. Then, applying Lemma 14 we have

$$
\left\|f-f_{1}\right\|_{2}^{2} \leq \delta_{1}(1+O(1 / n))\left(1-\delta_{1}\right) / n
$$

Hence, by Theorem there exists a family $\mathcal{C} \subset S_{n}$ such that $\mathcal{C}=T_{i j}$ is a 1-coset, and such that

$$
|\mathcal{A} \triangle \mathcal{C}| \leq C_{0}\left(1-\delta_{1}\right)^{2}\left(\delta_{1}^{1 / 2}+O(1 / n)\right)(n-1)!
$$

Provided $\delta$ is sufficiently small depending on $C_{0}$ and $\phi$, and $n$ is sufficiently large depending on $C_{0}$ and $\phi$, we have

$$
C_{0}\left(1-\delta_{1}\right)^{2}\left(\delta_{1}^{1 / 2}+O(1 / n)\right) \leq \phi
$$

so

$$
\left|\mathcal{A} \triangle T_{i j}\right| \leq \phi(n-1)!
$$

proving the proposition.

We can now give our new proof of Conjecture 2,
New proof of Conjecture 圆. Choose any $\phi$ such that $0<\phi<1 / e$, and let $\delta=\delta(\phi)$ be as given by Proposition 15, Let $\mathcal{A} \subset S_{n}$ be an intersecting family of permutations with $|\mathcal{A}| \geq(1-\delta)(n-1)$ !. Note that, by Theorem 9 we may assume throughout that $n \geq n_{0}$, for any fixed $n_{0} \in \mathbb{N}$, by making $\delta$ smaller if necessary.

By Proposition 15, there exists a 1 -coset $T_{i j}$ such that

$$
\begin{equation*}
\left|\mathcal{A} \triangle T_{i j}\right| \leq \phi(n-1)! \tag{41}
\end{equation*}
$$

We will show that this implies $\mathcal{A} \subset T_{i j}$, provided $n$ is sufficiently large. Suppose for a contradiction that $\mathcal{A} \nsubseteq T_{i j}$. Then there exists a permutation $\tau \in \mathcal{A}$ such that $\tau(i) \neq j$. Any permutation in $\mathcal{A} \cap T_{i j}$ must agree with $\tau$ at some point. But for any $i, j \in[n]$ and any $\tau \in S_{n}$ such that $\tau(i) \neq j$, the number of permutations in $S_{n}$ which map $i$ to $j$ and agree with $\tau$ at some point is

$$
(n-1)!-d_{n-1}-d_{n-2}=(1-1 / e-o(1))(n-1)!
$$

(By double translation, we may assume that $i=j=1$ and $\tau=\binom{1}{2}$; we observed above that the number of permutations fixing 1 and intersecting $(12)$ is $(n-1)$ ! -$d_{n-1}-d_{n-2}$.)

Therefore, we must have

$$
\left|\mathcal{A} \cap T_{i j}\right| \leq(1-1 / e+o(1))(n-1)!
$$

so

$$
\left|T_{i j} \backslash \mathcal{A}\right| \geq(1 / e-o(1))(n-1)!>\phi(n-1)!
$$

provided $n$ is sufficiently large depending on $\phi$, contradicting (41). This completes the proof of Conjecture 2

### 4.2 Almost isoperimetric subsets of the transposition graph

In this section, we will apply Theorem 1 to investigate the structure of subsets of $S_{n}$ with small edge-boundary in the transposition graph. The transposition graph $T_{n}$ is the Cayley graph on $S_{n}$ generated by the transpositions; equivalently, two permutations are joined if, as sequences, one can be obtained from the other by transposing two elements.

In this section, we study edge-isoperimetric inequalities for $T_{n}$. First, let us give some brief background on edge-isoperimetric inequalities for graphs. If $G$ is any graph, and $S, T \subset V(G)$, we write $E_{G}(S, T)$ for the set of edges of $G$ between $S$ and $T$, and we write $e_{G}(S, T)=\left|E_{G}(S, T)\right|$. We write $\partial_{G} S=E_{G}\left(S, S^{c}\right)$ for the set of edges of $G$ between $S$ and its complement; this is called the edge-boundary of $S$ in $G$. An edge-isoperimetric inequality for $G$ gives a lower bound on the minimum size of the edge-boundary of a set of size $k$, for each integer $k$. If $\mathcal{A} \subset S_{n}$, we write $\partial \mathcal{A}=\partial_{T_{n}} \mathcal{A}$ for the edge-boundary of $\mathcal{A}$ in the transposition graph.

It would be of great interest to prove an edge-isoperimetric inequality for the transposition graph which is sharp for all set-sizes. Ben Efraim [3] conjectures that initial segments of the lexicographic order on $S_{n}$ have the smallest edge-boundary of all sets of the same size.

Definition. If $\sigma, \pi \in S_{n}$, we say that $\sigma<\pi$ in the lexicographic order on $S_{n}$ if $\sigma(j)<\pi(j)$, where $j=\min \{i \in[n]: \sigma(i) \neq \pi(i)\}$. The initial segment of size $k$ of the lexicographic order on $S_{n}$ simply means the smallest $k$ elements of $S_{n}$ in the lexicographic order.

Conjecture 16 (Ben Efraim). For any $\mathcal{A} \subset S_{n},|\partial \mathcal{A}| \geq|\partial \mathcal{C}|$, where $\mathcal{C}$ denotes the initial segment of the lexicographic order on $S_{n}$ of size $|\mathcal{A}|$.

This is a beautiful conjecture; it may be compared to the edge-isoperimetric inequality in $\{0,1\}^{n}$, due to Harper [21], Lindsey [30], Bernstein [4] and Hart [22], stating that among all subsets of $\{0,1\}^{n}$ of size $k$, the first $k$ elements of the binary ordering on $\{0,1\}^{n}$ has the smallest edge boundary. (Recall that if $x, y \in\{0,1\}^{n}$, we say that $x<y$ in the binary ordering if $x_{j}=0$ and $y_{j}=1$, where $j=\min \left\{i \in[n]: x_{i} \neq y_{i}\right\}$.)

To date, Conjecture 16 is only known to hold for sets of size $c(n-1)$ ! where $c \in\{1,2, \ldots, n\}$; this is a consequence of the work of Diaconis and Shahshahani [8. (The authors have also verified it for sets of size $(n-t)$ !, where $n$ is large depending on $t$; this will appear in a subsequent work, [14.) Diaconis and Shahshahani proved the following isoperimetric inequality.

Theorem 17 (Diaconis, Shahshahani). If $\mathcal{A} \subset S_{n}$, then

$$
\begin{equation*}
|\partial \mathcal{A}| \geq \frac{|\mathcal{A}|(n!-|\mathcal{A}|)}{(n-1)!} \tag{42}
\end{equation*}
$$

Remark 4. Equality holds if and only if $\mathcal{A}$ is a disjoint union of 1-cosets of $S_{n}$ (a dictatorship).

Our aim in this section is to obtain a description of subsets of $S_{n}$ of size $c(n-1)$ !, whose edge-boundary is close to the minimum possible size (42), when $c$ is small. We will prove the following.

Theorem 3. For each $c \in \mathbb{N}$, there exists $n_{0}(c) \in \mathbb{N}$ and $\delta_{0}(c)>0$ such that the following holds. Let $\mathcal{A} \subset S_{n}$ with $|\mathcal{A}|=c(n-1)$ !, and with

$$
|\partial A| \leq \frac{|\mathcal{A}|(n!-|\mathcal{A}|)}{(n-1)!}+\delta n|\mathcal{A}|
$$

where $n \geq n_{0}(c)$ and $\delta \leq \delta_{0}(c)$. Then there exists a family $\mathcal{B} \subset S_{n}$ such that $\mathcal{B}$ is a union of c 1-cosets of $S_{n}$, and

$$
|\mathcal{A} \backslash \mathcal{B}| \leq O(c \delta)(n-1)!+O\left(c^{2}\right)(n-2)!
$$

(We may take $\delta_{0}(c)=\Omega\left(c^{-4}\right)$ and $n_{0}(c)=O\left(c^{2}\right)$.)
Remark 5. Theorem 3 is sharp up to an absolute constant factor when $\delta=\Omega(c / n)$; this can be seen by considering the set

$$
\begin{aligned}
& \mathcal{A}=T_{1,1} \cup T_{1,2} \cup \ldots \cup T_{1, c} \cup\left(T_{1, c+1} \cap\left(T_{2, n} \cup T_{2, n-1} \cup \ldots \cup T_{2, n-k+1}\right)\right) \\
& \quad \backslash\left(T_{1, c} \cap\left(T_{2, n} \cup T_{2, n-1} \cup \ldots \cup T_{2, n-k+1}\right)\right),
\end{aligned}
$$

where $n / 2 \geq k=\Omega\left(c^{2}\right)$.

Note that Theorem 3 is not a 'genuine' stability result; as with Theorem 1 we may call it a 'quasi-stability' result. A 'genuine' stability result would say that if $\mathcal{A}$ satisfies the hypothesis of Theorem 3 i.e. if it has edge-boundary close to the minimum possible size, then $\mathcal{A}$ is close to an extremal family - and, by Remark 4 the extremal families are precisely the dictatorships, i.e. disjoint unions of 1-cosets. However, such a statement is false when $c=2$. To see this, let

$$
\mathcal{A}=T_{11} \cup T_{22} \cup\left(T_{12} \cap T_{21}\right) ;
$$

then $|\mathcal{A}|=2(n-1)$ ! and

$$
|\partial \mathcal{A}|=2 n(n-2)(n-2)!=(1+1 / n) \frac{|\mathcal{A}|(n!-|\mathcal{A}|)}{(n-1)!}=\frac{|\mathcal{A}|(n!-|\mathcal{A}|)}{(n-1)!}+\delta n|\mathcal{A}|
$$

where $\delta=(n-2) / n^{2}$, so $\mathcal{A}$ has edge-boundary very close to the minimum possible size. However, we have

$$
|\mathcal{A} \Delta \mathcal{C}| \geq(n-1)!-(n-2)!=\left(\frac{1}{2}-\frac{1}{2(n-1)}\right)|\mathcal{A}|
$$

whenever $\mathcal{C}$ is a dictatorship, i.e. $\mathcal{A}$ is far (in symmetric difference) from any disjoint union of cosets. On the other hand, $\mathcal{A}$ is close to the union of (non-disjoint) cosets $T_{11} \cup T_{22}$; indeed,

$$
\left|\mathcal{A} \backslash\left(T_{11} \cup T_{22}\right)\right|=(n-2)!=\frac{2}{n-1}|\mathcal{A}| .
$$

This is consistent with Theorem 3.
To prove Theorem 3 we will first use an eigenvalue stability argument to show that if $\mathcal{A}$ satisfies the hypothesis of the theorem (i.e. its edge-boundary has size close to the minimum possible size), then its characteristic vector $\mathbf{1}_{\mathcal{A}}$ must be close in Euclidean distance to the subspace $U_{1}$. We will then use Theorem 1 to deduce that $\mathcal{A}$ must be somewhat close (in symmetric difference) to a family $\mathcal{B} \subset S_{n}$ of the same size, which is a union of 1-cosets. Finally, we will use a combinatorial stability argument to deduce that $\mathcal{A}$ must be very close to $\mathcal{B}$, completing the proof of the theorem.

We proceed to give the necessary background for our initial eigenvalue stability argument. Along the way, we will show how to prove Theorem 17, essentially reproducing the original proof of Diaconis and Shahshahani).

Recall that if $G=(V, E)$ is a graph, the adjacency matrix of $G$ is the $|V| \times|V|$ matrix $A$ with rows and columns indexed by $V$, and with

$$
A_{u, v}= \begin{cases}1 & \text { if } u v \in E(G) \\ 0 & \text { if } u v \notin E(G)\end{cases}
$$

The Laplacian matrix $L$ of $G$ may be defined by

$$
L=D-A
$$

where $D$ is the diagonal $|V| \times|V|$ matrix with rows and columns indexed by $V$, and with

$$
D_{u, v}=\left\{\begin{array}{cl}
\operatorname{deg}(v) & \text { if } u=v \\
0 & \text { if } u \neq v
\end{array}\right.
$$

The following theorem supplies an edge-isoperimetric inequality for a graph $G$ in terms of the eigenvalues of its Laplacian matrix.

Theorem 18 (Dodziuk [9, Alon-Milman [2]). If $G=(V, E)$ is any graph, $L$ is the Laplacian matrix of $G$, and $0=\mu_{1} \leq \mu_{2} \leq \ldots \leq \mu_{|V|}$ are the eigenvalues of $L$ (repeated with their multiplicities), then for any set $S \subset V(G)$,

$$
e\left(S, S^{c}\right) \geq \mu_{2} \frac{|S|\left|S^{c}\right|}{|V|}
$$

If equality holds, then the characteristic vector $\mathbf{1}_{S}$ of $S$ satisfies

$$
\mathbf{1}_{S}-\frac{|S|}{|G|} \mathbf{f} \in \operatorname{ker}\left(L-\mu_{2} I\right)
$$

where $\mathbf{f}$ denotes the all-1's vector.
We will show below (Lemma (20) how to calculate the value of $\mu_{2}$ for the transposition graph; plugging this value into Theorem 18 will yield a proof of Theorem 17.

To investigate the structure of subsets with small edge-boundary in the transposition graph, we will need the following 'stability version' of Theorem 18

Lemma 19. Let $G=(V, E)$ be a graph, let $L$ be the Laplacian matrix of $G$, and let $0=\mu_{1} \leq \mu_{2} \leq \ldots \leq \mu_{|V|}$ be the eigenvalues of $L$ (repeated with their multiplicities).
Let $S \subset V(G)$ with

$$
e\left(S, S^{c}\right) \leq \mu_{2} \frac{|S|\left|S^{c}\right|}{|V|}+\gamma|S|
$$

where $\gamma \geq 0$. Equip $\mathbb{R}^{V}$ with the inner product

$$
\langle f, g\rangle=\frac{1}{|V|} \sum_{v \in V} f(v) g(v)
$$

and let

$$
\|f\|_{2}=\sqrt{\frac{1}{|V|} \sum_{v \in V} f(v)^{2}}
$$

denote the induced Euclidean norm. Let $M=\min \left\{i: \mu_{i}>\mu_{2}\right\}$. Let $U$ denote the direct sum of the $\mu_{1}$ and $\mu_{2}$ eigenspaces of $L$, and let $P_{U}$ denote orthogonal projection onto $U$. Then we have

$$
\left\|\mathbf{1}_{S}-P_{U}\left(\mathbf{1}_{S}\right)\right\|_{2}^{2} \leq \frac{\gamma}{\mu_{M}-\mu_{2}} \frac{|S|}{|V|}
$$

Proof. Recall that for any vector $x \in \mathbb{R}^{V}$, we have

$$
\langle x, L x\rangle=\frac{1}{|V|} \sum_{i j \in E(G)}\left(x_{i}-x_{j}\right)^{2}
$$

Hence, in particular,

$$
\left\langle\mathbf{1}_{S}, L \mathbf{1}_{S}\right\rangle=\frac{e\left(S, S^{c}\right)}{|V|}
$$

Let $u_{1}, u_{2}, \ldots, u_{|V|}$ denote an orthonormal basis of eigenvectors of $L$ corresponding to the eigenvalues $\mu_{1}, \mu_{2}, \ldots, \mu_{|V|}$, where $u_{1}=\mathbf{f}$ is the all- 1 's vector. Write $\mathbf{1}_{S}$ as a linear combination of these basis vectors:

$$
\mathbf{1}_{S}=\sum_{i=1}^{|V|} \xi_{i} u_{i}
$$

Let $\alpha=|S| /|V|$ denote the measure of $S$. Observe that

$$
\xi_{1}=\alpha, \quad \sum_{i=1}^{|V|} \xi_{i}^{2}=\alpha, \quad\left\|\mathbf{1}_{S}-P_{U}\left(\mathbf{1}_{S}\right)\right\|_{2}^{2}=\sum_{i \geq M} \xi_{i}^{2}
$$

Write

$$
\phi=\left\|\mathbf{1}_{S}-P_{U}\left(\mathbf{1}_{S}\right)\right\|_{2}^{2}
$$

We have
$\frac{e\left(S, S^{c}\right)}{|V|}=\left\langle\mathbf{1}_{S}, L \mathbf{1}_{S}\right\rangle=\sum_{i=1}^{|V|} \mu_{i} \xi_{i}^{2} \geq \mu_{2}\left(\alpha-\alpha^{2}-\phi\right)+\mu_{M} \phi=\mu_{2} \alpha(1-\alpha)+\phi\left(\mu_{M}-\mu_{2}\right)$.
Hence, if $e\left(S, S^{c}\right) \leq \mu_{2} \frac{|S|\left|S^{c}\right|}{|V|}+\gamma|S|$, then

$$
\mu_{2} \alpha(1-\alpha)+\phi\left(\mu_{M}-\mu_{2}\right) \leq \frac{e\left(S, S^{c}\right)}{|V|} \leq \mu_{2} \alpha(1-\alpha)+\gamma \alpha
$$

Rearranging yields

$$
\phi \leq \frac{\gamma}{\mu_{M}-\mu_{2}} \alpha
$$

as required.
We now proceed to calculate $\mu_{2}$ and $\mu_{M}$ for the transposition graph.
Lemma 20. The transposition graph on $S_{n}$ has $\mu_{2}=n$ (for all $n \geq 2$ ) and $\mu_{M}=2 n-2$ (for all $n \geq 4$ ). The 0 -eigenspace of its Laplacian is $U_{(n)}$, and provided $n \geq 4$, the $n$-eigenspace is $U_{(n-1,1)}$.

Proof. If $G=(V, E)$ is a $d$-regular graph, then its Laplacian matrix is given by $L=$ $d I-A$. Therefore, if the eigenvalues of its adjacency matrix are

$$
d=\lambda_{1} \geq \lambda_{2} \geq \ldots \geq \lambda_{|V|},
$$

then $\mu_{i}=d-\lambda_{i}$ for each $i$. The transposition graph on $S_{n}$ is $\binom{n}{2}$-regular, so it has $\mu_{i}=\binom{n}{2}-\lambda_{i}$ for each $i$.

Note that the transposition graph is a normal Cayley graph, and therefore Theorem 7 applies to its adjacency matrix. Recall from section 2.1 that there is an explicit 1-1 correspondence between isomorphism classes of irreducible representations of $S_{n}$ and partitions of $n$; given a partition $\alpha$, we write $\chi_{\alpha}$ for the character of the corresponding irreducible representation of $S_{n}$.

Frobenius gave the following formula for the value of $\chi_{\alpha}$ at a transposition.

$$
\chi_{\alpha}\left(\left(\begin{array}{ll}
1 & 2)
\end{array}\right)=\frac{\operatorname{dim}\left(\rho_{\alpha}\right)}{\binom{n}{2}} \frac{1}{2} \sum_{j=1}^{l}\left(\left(\alpha_{j}-j\right)\left(\alpha_{j}-j+1\right)-j(j-1)\right) \quad\left(\alpha=\left(\alpha_{1}, \ldots, \alpha_{l}\right) \vdash n\right) .\right.
$$

Combining this with (40) on page 22 yields the following formula for the eigenvalues of the adjacency matrix of the transposition graph:

$$
\lambda_{\alpha}=\frac{1}{2} \sum_{j=1}^{l}\left(\left(\alpha_{j}-j\right)\left(\alpha_{j}-j+1\right)-j(j-1)\right) \quad(\alpha \vdash n) .
$$

Note that $\lambda_{(n)}=\binom{n}{2}$ and $\lambda_{(n-1,1)}=\binom{n}{2}-n$. Diaconis and Shashahani [8] verify that if $\alpha$ and $\alpha^{\prime}$ are two partitions of $n$ with $\alpha \unrhd \alpha^{\prime}$, then $\lambda_{\alpha} \geq \lambda_{\alpha^{\prime}}$. Since ( $n-$ $1,1) \unrhd \alpha$ for all $\alpha \neq(n)$, we have $\lambda_{\alpha} \leq \lambda_{(n-1,1)}$ for all $\alpha \neq(n)$, and therefore $\lambda_{2}=\binom{n}{2}-n$. Hence, the 0-eigenspace of the Laplacian is precisely $U_{(n)}$, the space of constant functions, and we have $\mu_{2}=n$.

Note also that $\lambda_{(n-2,2)}=\binom{n}{2}-2 n+2$. Since $(n-2,2) \unrhd \alpha$ for all $\alpha \neq(n),(n-1,1)$, we have $\lambda_{\alpha} \leq\binom{ n}{2}-2 n+2$ for all $\alpha \neq(n),(n-1,1)$. Provided $n \geq 4$, we have

$$
\binom{n}{2}-\lambda_{(n-1,1)}=n<2 n-2=\binom{n}{2}-\lambda_{(n-2,2)} \leq\binom{ n}{2}-\lambda_{\alpha} \quad \forall \alpha \neq(n),(n-1,1),
$$

and therefore $\mu_{M}=2 n-2$, and the $n$-eigenspace of $L$ is precisely $U_{(n-1,1)}$.
Theorem 17 follows immediately by plugging $\mu_{2}=n$ into Theorem 18. We can also use the equality part of Theorem 18 to deduce Remark 4

Corollary 21. Equality holds in Theorem 17 only if $\mathcal{A}$ is a disjoint union of 1 -cosets of $S_{n}$ (a dictatorship).

Proof. It is easy to see that the corollary holds for all $n \leq 3$, so we may assume that $n \geq 4$. If equality holds in (42) for $\mathcal{A}$, then by the equality part of Theorem 18 $\mathbf{1}_{\mathcal{A}}-(|\mathcal{A}| / n!) \mathbf{f}$ lies in the $\mu_{2}$-eigenspace of $L$, which by Lemma 20 is precisely $U_{(n-1,1)}$. Therefore, $\mathbf{1}_{\mathcal{A}} \in U_{(n)} \oplus U_{(n-1,1)}=U_{1}$. It follows from Theorem 4 that $\mathcal{A}$ is a disjoint union of 1-cosets of $S_{n}$.

We now use Lemmas 19 and 20 to show that a subset of $S_{n}$ with small edgeboundary in the transposition graph, has characteristic vector which is close to $U_{1}$.

Lemma 22. Let $n \geq 4$, and let $\mathcal{A} \subset S_{n}$ with $|\mathcal{A}|=c(n-1)$ !, and with

$$
|\partial A| \leq \frac{|\mathcal{A}|(n!-|\mathcal{A}|)}{(n-1)!}+\delta n|\mathcal{A}|
$$

Then

$$
\left\|\mathbf{1}_{\mathcal{A}}-P_{U_{1}}\left(\mathbf{1}_{\mathcal{A}}\right)\right\|_{2}^{2} \leq \frac{\delta n}{n-2} \frac{c}{n}
$$

Proof. By Lemma [20, the transposition graph has $\mu_{2}=n$ and $\mu_{M}=2 n-2$, the 0 -eigenspace of its Laplacian is $U_{(n)}$, and the $n$-eigenspace is $U_{(n-1,1)}$, so the subspace $U$ in Lemma 19 is $U_{1}$. If $\mathcal{A}$ is as in the statement of the lemma, then we may apply Lemma 19 with $\gamma=\delta n$, giving

$$
\left\|\mathbf{1}_{\mathcal{A}}-P_{U_{1}}\left(\mathbf{1}_{\mathcal{A}}\right)\right\|_{2}^{2} \leq \frac{\delta n}{n-2} \frac{c}{n}
$$

as required.
We now combine Lemma 22 and Theorem to give the following very rough structural description of subsets of $S_{n}$ with small edge-boundary in the transposition graph.

Proposition 23. There exists $\delta_{0}>0$ such that for each $c \in \mathbb{N}$, the following holds. Let $\mathcal{A} \subset S_{n}$ with $|\mathcal{A}|=c(n-1)$ !, and with

$$
|\partial A| \leq \frac{|\mathcal{A}|(n!-|\mathcal{A}|)}{(n-1)!}+\delta n|\mathcal{A}| .
$$

If $\delta \leq \delta_{0}$, then there exists a family $\mathcal{B} \subset S_{n}$ such that $\mathcal{B}$ is a union of $c 1$-cosets of $S_{n}$, and

$$
|\mathcal{A} \triangle \mathcal{B}| \leq C_{1} c^{2}\left(\delta^{1 / 2}+1 / n\right)(n-1)!
$$

where $C_{1}>0$ is an absolute constant.
Proof. Suppose that $\mathcal{A} \subset S_{n}$ with $|\mathcal{A}|=c(n-1)$ ! for some $c \in \mathbb{N}$, and with

$$
|\partial \mathcal{A}| \leq \frac{|\mathcal{A}|(n!-|\mathcal{A}|)}{(n-1)!}+\delta n|\mathcal{A}|
$$

Our aim is to show that $\mathcal{A}$ must be close to a union of $c 1$-cosets of $S_{n}$. Note that we may assume that $n \geq \frac{1}{2} C_{1} c$, otherwise we have $C_{1} c^{2}(n-1)!/ n \geq 2 c(n-1)!$, so the conclusion of the proposition holds trivially whenever $|\mathcal{A}|=|\mathcal{B}|=c(n-1)$ !.

It follows from Lemma 22 that

$$
\left\|\mathbf{1}_{\mathcal{A}}-P_{U_{1}}\left(\mathbf{1}_{\mathcal{A}}\right)\right\|_{2}^{2} \leq \frac{\delta n}{n-2} \frac{c}{n}
$$

i.e. $\mathbf{1}_{\mathcal{A}}$ is close to $U_{1}$. Let $f=\mathbf{1}_{\mathcal{A}}$, and let $f_{1}=P_{U_{1}}\left(\mathbf{1}_{\mathcal{A}}\right)$; then we have

$$
\mathbb{E}\left[\left(f-f_{1}\right)^{2}\right]=\left\|f-f_{1}\right\|_{2}^{2} \leq \frac{\delta n}{n-2} \frac{c}{n}
$$

so $\mathcal{A}$ satisfies the hypotheses of Theorem 1 with $\epsilon=\delta \frac{n}{n-2}$. Therefore, by Theorem 1 there exists a family $\mathcal{B} \subset S_{n}$ which is a union of $c 1$-cosets of $S_{n}$, and

$$
|\mathcal{A} \triangle \mathcal{B}| \leq C_{0} c^{2}\left((\delta n /(n-2))^{1 / 2}+1 / n\right)(n-1)!\leq \sqrt{3} C_{0} c^{2}\left(\delta^{1 / 2}+1 / n\right)(n-1)!
$$

using the fact that $n \geq 3$. This proves the proposition.
We will now use a combinatorial stability argument to strengthen the bounds in the conclusion of Proposition 23) proving Theorem 3.

Proof of Theorem [3: Suppose that $\mathcal{A} \subset S_{n}$ with $|\mathcal{A}|=c(n-1)$ ! for some $c \in \mathbb{N}$, and with

$$
|\partial A| \leq \frac{|\mathcal{A}|(n!-|\mathcal{A}|)}{(n-1)!}+\delta n|\mathcal{A}|
$$

Then by Proposition 23, there exists a family $\mathcal{B} \subset S_{n}$ such that $\mathcal{B}$ is a union of $c$ 1-cosets of $S_{n}$, and

$$
|\mathcal{A} \backslash \mathcal{B}|=\psi(n-1)!,
$$

where

$$
\psi \leq C_{1} c^{2}\left(\delta^{1 / 2}+1 / n\right)<1 / 3
$$

provided $\delta \leq O\left(c^{-4}\right)$ and $n \geq \Omega\left(c^{2}\right)$. We proceed to obtain a better upper bound on $\psi$ in terms of $\delta$.

Let $\mathcal{E}=\mathcal{A} \backslash \mathcal{B}$; then $|\mathcal{E}|=\psi(n-1)!$. Write $B=B_{1} \cup B_{2} \cup \ldots \cup B_{c}$, where the $B_{i}$ 's are 1-cosets of $S_{n}$, let $\mathcal{M}=\mathcal{B} \backslash \mathcal{A}$, and let $\mathcal{M}_{i}=B_{i} \backslash \mathcal{A}$ denote the set of permutations in $B_{i}$ which are missing from $\mathcal{A}$. Let $\mathcal{N}_{i}=\mathcal{M}_{i} \backslash\left(\cup_{j \neq i} B_{j}\right)$, and write $\left|\mathcal{N}_{i}\right|=\nu_{i}(n-1)$ !.

We now give a lower bound on $|\partial \mathcal{A}|$ in terms of $\psi$. Observe that

$$
\begin{aligned}
|\partial \mathcal{A}| & =|\partial \mathcal{B}|+|\partial \mathcal{E}|-2 e(\mathcal{E}, \mathcal{B})-e\left(\mathcal{M}, S_{n} \backslash(\mathcal{B} \cup \mathcal{E})\right)+e(\mathcal{M}, \mathcal{B} \backslash \mathcal{M})+e(\mathcal{E}, \mathcal{M}) \\
& =|\partial \mathcal{B}|+|\partial \mathcal{E}|-2 e(\mathcal{E}, \mathcal{B})-e\left(\mathcal{M}, S_{n} \backslash \mathcal{B}\right)+e(\mathcal{M}, \mathcal{B} \backslash \mathcal{M})+2 e(\mathcal{E}, \mathcal{M}) \\
& \geq|\partial \mathcal{B}|+|\partial \mathcal{E}|-2 e(\mathcal{E}, \mathcal{B})-e\left(\mathcal{M}, S_{n} \backslash \mathcal{B}\right)+e(\mathcal{M}, \mathcal{B} \backslash \mathcal{M})
\end{aligned}
$$

By definition, we have $\mathcal{E} \cap B_{i}=\emptyset$ for each $i \in[c]$. If $B_{i}=T_{p q}$ and $\sigma \in \mathcal{E}$ then $\sigma(p) \neq q$, and so the only neighbour of $\sigma$ in $B_{i}$ is $\sigma\left(p \sigma^{-1}(q)\right)$. It follows that $e\left(\mathcal{E}, B_{i}\right) \leq|\mathcal{E}|$ for each $i$. Summing over all $i$, we obtain:

$$
e(\mathcal{E}, \mathcal{B}) \leq \sum_{i=1}^{c} e\left(\mathcal{E}, B_{i}\right) \leq c|\mathcal{E}|=c \psi(n-1)!
$$

Similarly, each $\sigma \in B_{i}$ has at most $n-1$ neighbours in $S_{n} \backslash B_{i}$. Indeed, if $B_{i}=T_{p q}$, then the neighbours of $\sigma$ in $S_{n} \backslash B_{i}$ are $\{\sigma(p r): r \neq p\}$. It follows that

$$
e\left(\mathcal{M}, S_{n} \backslash \mathcal{B}\right) \leq(n-1)|\mathcal{M}|
$$

By Theorem 17, we have

$$
|\partial \mathcal{E}| \geq \psi(n-1)!(n-\psi)
$$

Since $\mathcal{B}$ is a union of $c 1$-cosets of $S_{n}$, it is easy to see that

$$
|\mathcal{B}| \geq c(n-1)!-\binom{c}{2}(n-2)!
$$

and so Theorem 17 implies

$$
|\partial \mathcal{B}| \geq c\left(1-\frac{c-1}{2(n-1)}\right)(n-1)!(n-c) \geq c(n-1)!(n-c)-O\left(c^{2}\right)(n-1)!
$$

using $c<n / 2$.

Finally, it remains to bound $e(\mathcal{M}, \mathcal{B} \backslash \mathcal{M})$ from below. To do this, note first that $B_{i} \backslash \mathcal{M}_{i}=\left(B_{i} \backslash \mathcal{N}_{i}\right) \backslash \cup_{j \neq i} B_{j}$ for each $i$, and therefore

$$
\left.\left\{E\left(\mathcal{N}_{i},\left(B_{i} \backslash \mathcal{N}_{i}\right) \backslash \cup_{j \neq i} B_{j}\right)\right): i \in[c]\right\}
$$

are pairwise disjoint subsets of $E(\mathcal{M}, \mathcal{B} \backslash \mathcal{M})$. Observe that for each $i$, we have

$$
e\left(\mathcal{N}_{i}, B_{i} \cap B_{j}\right) \leq\left|\mathcal{N}_{i}\right| \quad \forall j \neq i
$$

since $\mathcal{N}_{i} \cap B_{j}=\emptyset$ for each $j \neq i$. Hence, we have

$$
e\left(\mathcal{N}_{i}, B_{i} \cap \cup_{j \neq i} B_{j}\right) \leq(c-1)\left|\mathcal{N}_{i}\right|
$$

It follows that

$$
e\left(\mathcal{N}_{i},\left(B_{i} \backslash \mathcal{N}_{i}\right) \backslash \cup_{j \neq i} B_{j}\right) \geq e\left(\mathcal{N}_{i}, B_{i} \backslash \mathcal{N}_{i}\right)-(c-1)\left|\mathcal{N}_{i}\right|
$$

Note that $T_{n}\left[B_{i}\right]$ is isomorphic to $T_{n-1}$, and therefore we may apply Theorem 17 in $S_{n-1}$ to give:

$$
e\left(\mathcal{N}_{i}, B_{i} \backslash \mathcal{N}_{i}\right) \geq \nu_{i}(n-1)!\left(1-\nu_{i}\right)(n-1)
$$

We obtain

$$
e(\mathcal{M}, \mathcal{B} \backslash \mathcal{M}) \geq \sum_{i=1}^{c} \nu_{i}\left(1-\nu_{i}\right)(n-1)(n-1)!-(c-1)(n-1)!\sum_{i=1}^{c} \nu_{i}
$$

Since $|\mathcal{A}|=c(n-1)$ ! and $c(n-1)!-\binom{c}{2}(n-2)$ ! $\leq|\mathcal{B}| \leq c(n-1)$ !, we have $|\mathcal{E}|-\binom{c}{2}(n-2)!\leq|\mathcal{M}| \leq|\mathcal{E}|$. Since the $\mathcal{N}_{i}$ 's are pairwise disjoint subsets of $\mathcal{M}$, we have

$$
\begin{equation*}
\sum_{i=1}^{c}\left|\mathcal{N}_{i}\right| \leq|\mathcal{M}| \leq|\mathcal{E}| \tag{43}
\end{equation*}
$$

Note also that

$$
\mathcal{M} \backslash\left(\bigcup_{i=1}^{c} \mathcal{N}_{i}\right) \subset \bigcup_{i \neq j}\left(B_{i} \cap B_{j}\right)
$$

and therefore

$$
\left|\mathcal{M} \backslash\left(\bigcup_{i=1}^{c} \mathcal{N}_{i}\right)\right| \leq\binom{ c}{2}(n-2)!.
$$

Hence, we have

$$
\left|\bigcup_{i=1}^{c} \mathcal{N}_{i}\right| \geq|\mathcal{E}|-2\binom{c}{2}(n-2)!
$$

and therefore

$$
\begin{equation*}
\sum_{i=1}^{c}\left|\mathcal{N}_{i}\right| \geq|\mathcal{E}|-2\binom{c}{2}(n-2)! \tag{44}
\end{equation*}
$$

Combining (43) and (44) yields

$$
\begin{equation*}
\psi-2\binom{c}{2} /(n-1) \leq \sum_{i=1}^{c} \nu_{i} \leq \psi \tag{45}
\end{equation*}
$$

Putting everything together, we obtain

$$
\begin{aligned}
|\partial \mathcal{A}| & \geq c(n-1)!(n-c)-O\left(c^{2}\right)(n-1)!+\psi(n-1)!(n-\psi)-2 c \psi(n-1)! \\
& -(n-1)|\mathcal{M}|+\sum_{i=1}^{c} \nu_{i}\left(1-\nu_{i}\right)(n-1)(n-1)!-(c-1)(n-1)!\sum_{i=1}^{c} \nu_{i} \\
& \geq c(n-1)!(n-c)-O\left(c^{2}\right)(n-1)!+\psi(n-1)!(n-\psi)-2 c \psi(n-1)! \\
& -(n-1) \psi(n-1)!+\sum_{i=1}^{c} \nu_{i}\left(1-\nu_{i}\right)(n-1)(n-1)!-(c-1)(n-1)!\psi \\
& \geq c(n-1)!(n-c)-O\left(c^{2}\right)(n-1)!-\psi(n-1)!(3 c+\psi-2) \\
& +(1-1 / n) n!\sum_{i=1}^{c} \nu_{i}\left(1-\nu_{i}\right) \\
& \geq c(n-1)!(n-c)+(1-1 / n) n!\psi(1-\psi)-O\left(c^{2}\right)(n-1)! \\
& -\psi(n-1)!(3 c+\psi-2) \\
& \geq c(n-1)!(n-c-1)+(1-2 / n) n!\psi(1-\psi)-O\left(c^{2}\right)(n-1)!
\end{aligned}
$$

using $\sum_{i=1}^{c} \nu_{i} \leq \psi<1 / 3$, and the fact that $y \mapsto y(1-y)$ is concave for $y \in[0,1]$.
Hence, we have

$$
\begin{array}{r}
c(n-1)!(n-c-1)+(1-2 / n) n!\psi(1-\psi)-O\left(c^{2}\right)(n-1)! \\
\leq|\partial A| \leq c(n-1)!(n-c)+\delta n|\mathcal{A}|=c(n-1)!(n-c)+c n!\delta
\end{array}
$$

It follows that

$$
\psi(1-\psi) \leq \frac{c \delta+c / n+O\left(c^{2} / n\right)}{1-2 / n} \leq 3 c \delta+O\left(c^{2} / n\right)
$$

provided $n \geq 3$. Solving for $\psi$, we obtain

$$
\begin{equation*}
\psi \geq \frac{1}{2}(1+\sqrt{1-12 c \delta})-O\left(c^{2} / n\right) \tag{46}
\end{equation*}
$$

or

$$
\begin{equation*}
\psi \leq \frac{1}{2}(1-\sqrt{1-12 c \delta})+O\left(c^{2} / n\right) \leq 6 c \delta+O\left(c^{2} / n\right) \tag{47}
\end{equation*}
$$

using the inequality $1-\sqrt{1-x} \leq x$ for $x \in[0,1]$. Provided $n=\Omega\left(c^{2}\right)$, (46) cannot hold (since $\psi<1 / 3$ ), and therefore (47) must hold. Hence,

$$
|\mathcal{A} \Delta \mathcal{B}|=2|\mathcal{A} \backslash \mathcal{B}|=2 \psi(n-1)!\leq\left(12 c \delta+O\left(c^{2} / n\right)\right)(n-1)!,
$$

proving the theorem.

## 5 Conclusion and open problems

Note that the conclusion of Theorem is non-trivial only when $n$ is sufficiently large, and $\delta$ sufficiently small, depending on $c$. We believe these restrictions to be artefacts of our method of proof, and we conjecture the following strengthening of Theorem 1 .

Conjecture 24. There exists an absolute constant $C_{0}>0$ such that the following holds. Let $\mathcal{A} \subset S_{n}$ with $|\mathcal{A}|=c(n-1)$ !, where $0 \leq c \leq n$, and let $f=\mathbf{1}_{\mathcal{A}}: S_{n} \rightarrow\{0,1\}$ be the characteristic function of $\mathcal{A}$, so that $\mathbb{E}[f]=c / n$. Let $f_{1}$ denote orthogonal projection of $f$ onto $U_{1}=U_{(n)} \oplus U_{(n-1,1)}$. If $\mathbb{E}\left[\left(f-f_{1}\right)^{2}\right] \leq \epsilon c / n$, then there exists a Boolean function $h$ such that

$$
\mathbb{E}\left[(f-h)^{2}\right] \leq C_{0} \epsilon c / n
$$

and $h$ is the characteristic function of a union of round (c) 1-cosets of $S_{n}$. Furthermore, $|c-\operatorname{round}(c)| \leq C_{0} \epsilon$.
Note that this would be informative for all $c \leq n$.
Likewise, we conjecture the following strengthening of Theorem 3,
Conjecture 25. There exists an absolute constant $C_{1}>0$ such that the following holds. Let $\mathcal{A} \subset S_{n}$ with $|\mathcal{A}|=c(n-1)$ ! for some $c \in \mathbb{N}$, and with

$$
|\partial A| \leq \frac{|\mathcal{A}|(n!-|\mathcal{A}|)}{(n-1)!}+\delta n|\mathcal{A}|
$$

Then there exists a family $\mathcal{B} \subset S_{n}$ such that $\mathcal{B}$ is a union of c 1-cosets of $S_{n}$, and

$$
|\mathcal{A} \Delta \mathcal{B}| \leq C_{1} c \delta(n-1)!.
$$

Ben Efraim's conjecture (Conjecture (16) remains one of the most natural open problems in the area. If this could be proved, it is likely that analogues of Theorem 3 could be obtained for other set-sizes.

## Acknowledgment

We wish to thank Gil Kalai for many useful conversations. We also wish to thank two anonymous referees for their careful reading of the paper and their helpful suggestions.

## References

[1] N. Alon, I. Dinur, E. Friedgut, B. Sudakov, 'Graph products, Fourier analysis and spectral techniques', Geometric and Functional Analysis Volume 14 (2004), pp. 913-940.
[2] N. Alon, V. D. Milman, ' $\lambda_{1}$, isoperimetric inequalities for graphs, and superconcentrators', Journal of Combinatorial Theory, Series B, 38 (1985), pp. 73-88.
[3] L. Ben Efraim, Isoperimetric inequalities, Poincaré inequalities and concentration inequalities on graphs, Doctoral thesis, Hebrew University of Jerusalem, 2009.
[4] A. J. Bernstein, 'Maximally connected arrays on the $n$-cube', SIAM Journal on Applied Mathematics 15 (1967), pp. 1485-1489.
[5] J. Bourgain, 'On the distribution of the Fourier spectrum of boolean functions', Israel Journal of Mathematics 131 (2002), pp. 269-276.
[6] P.J. Cameron, C.Y. Ku, 'Intersecting Families of Permutations', European Journal of Combinatorics 24 (2003) pp. 881-890.
[7] M. Deza, P. Frankl, 'On the maximum number of permutations with given maximal or minimal distance', Journal of Combinatorial Theory, Series A 22 (1977), pp. 352-360.
[8] P. Diaconis, M. Shahshahani, 'Generating a random permutation with random transpositions', Z. Wahrsch. Verw. Gebeite, Volume 57, Issue 2 (1981), pp. 159179.
[9] J. Dodziuk, 'Difference equations, isoperimetric inequality and transience of certain random walks', Transactions of the American Mathematical Society 284 (1984), pp. 787-794.
[10] D. Ellis, 'A Proof of the Cameron-Ku Conjecture', Journal of the London Mathematical Society 85 (2012), pp. 165-190.
[11] D. Ellis, Stability for $t$-intersecting families of permutations, Journal of Combinatorial Theory, Series A 118 (2011), pp. 208-227.
[12] D. Ellis, Y. Filmus, E. Friedgut, 'Triangle-intersecting families of graphs', Journal of the European Mathematical Society 14 (2012), pp. 841-885.
[13] D. Ellis, Y. Filmus, E. Friedgut, 'A stability result for balanced dictatorships in $S_{n}{ }^{\prime}$, Random Structures and Algorithms 46 (2015), pp. 494-530.
[14] D. Ellis, Y. Filmus, E. Friedgut, Low-degree Boolean functions on $S_{n}$, with an application to isoperimetry, submitted. arXiv:1511.08694.
[15] D. Ellis, E. Friedgut and H. Pilpel, 'Intersecting Families of Permutations', Journal of the American Mathematical Society 24 (2011), pp. 649-682.
[16] P. Erdős, C. Ko and R. Rado, 'An Intersection Theorem for Systems of Finite Sets', Quart. J. Math. Oxford, Ser. 2, Volume 12 (1961), pp. 313-320.
[17] Y. Filmus, A comment on 'Intersecting Families of Permutations', manuscript, available at http://www.cs.toronto.edu/~yuvalf/EFP-comment.pdf.
[18] E. Friedgut, 'Boolean Functions with Low Average Sensitivity Depend on Few Coordinates', Combinatorica 18 (1998), pp. 27-36.
[19] E. Friedgut, G. Kalai, A. Naor, 'Boolean functions whose Fourier transform is concentrated on the first two levels', Advances in Applied Mathematics 29 (2002), pp. 427-437.
[20] C. Godsil, K. Meagher, 'A new proof of the Erdős-Ko-Rado theorem for intersecting families of permutations', European Journal of Combinatorics 30 (2009), pp. 404-414.
[21] L. H. Harper, 'Optimal assignments of numbers to vertices', SIAM Journal on Applied Mathematics 12 (1964), pp. 131-135.
[22] S. Hart, 'A note on the edges of the $n$-cube', Discrete Mathematics 14 (1976), pp. 157-163.
[23] H. Hatami, M. Ghandehari, 'Fourier analysis and large independent sets in powers of complete graphs', Journal of Combinatorial Theory, Series B 98 (2008), pp. 164-172.
[24] A.J.W. Hilton, E.C. Milner, 'Some intersection theorems for systems of finite sets', Quart. J. Math. Oxford Series 218 (1967), pp. 369-384.
[25] A. J. Hoffman, 'On eigenvalues and colourings of graphs', Graph Theory and its Applications, 1969.
[26] G. Kalai, 'A Fourier-Theoretic Perspective for the Condorcet Paradox and Arrow's theorem', Advances in Applied Mathematics 29 (2002), pp. 412-426
[27] G. Kindler, R. O'Donnell, 'Gaussian noise sensitivity and Fourier tails', 27th Annual Conference on Computational Complexity, 2012.
[28] G. Kindler, S. Safra, 'Noise resistant Boolean functions are juntas', online manuscript, available at http://www.cs.huji.ac.il/ gkindler/papers/noise-stable-r-juntas.ps.
[29] B. Larose and C. Malvenuto, 'Stable sets of maximal size in Kneser-type graphs', European Journal of Combinatorics 25 (2004), pp. 657-673.
[30] J. H. Lindsey, II, 'Assignment of numbers to vertices', American Mathematical Monthly 71 (1964), pp. 508-516.
[31] Noam Nisan and Mario Szegedy, 'On the degree of Boolean functions as real polynomials', Computational Complexity 4 (1994), pp. 301-313.
[32] P. Renteln, 'On the Spectrum of the Derangement Graph', Electronic Journal of Combinatorics 14 (2007), R82.
[33] J.-P. Serre, Linear Representations of Finite Groups, Graduate Texts in Mathematics, Volume 42, Springer-Verlag.
[34] B. E. Sagan, The Symmetric Group: Representations, Combinatorial Algorithms and Symmetric Functions, Springer-Verlag, New York, 1991. [2nd revised printing, 2001.]
[35] J. Wang, S. J. Zhang, 'An Erdős-Ko-Rado type theorem in Coxeter groups', European Journal of Combinatorics 29 (2008), pp. 1111-1115.


[^0]:    *Supported by the Canadian Friends of the Hebrew University / University of Toronto Permanent Endowment.
    ${ }^{\dagger}$ Supported in part by I.S.F. grant 0398246, and BSF grant 2010247.

[^1]:    ${ }^{1}$ Alternatively, consider the derivative of the objective function, which is $3 \theta(1-\theta)(\delta-(H-L))(L+$ $H-(1-2 \theta) \delta)$.

