Asymptotically optimal k-step nilpotency of quadratic algebras and the Fibonacci numbers

Natalia Iyudu and Stanislav Shkarin

Abstract

It follows from the Golod–Shafarevich theorem that if $k \in \mathbb{N}$ and R is an associative algebra given by n generators and $d < \frac{n^2}{4}\cos^{-2}(\frac{\pi}{k+1})$ quadratic relations, then R is not k-step nilpotent. We show that the above estimate is asymptotically optimal. Namely, for every $k \in \mathbb{N}$, there is a sequence of algebras R_n given by n generators and d_n quadratic relations such that R_n is k-step nilpotent and $\lim\limits_{n\to\infty} \frac{d_n}{n^2} = \frac{1}{4}\cos^{-2}(\frac{\pi}{k+1})$.

MSC: 17A45, 16A22
Keywords: Quadratic algebras, Golod–Shafarevich theorem, Anick's conjecture

1 Introduction

Throughout this paper $\mathbb{K}$ is an arbitrary field, $\mathbb{Z}_+$ is the set of non-negative integers and $\mathbb{N}$ is the set of positive integers. For a set X, $\mathbb{K}\langle X\rangle$ stands for the free associative algebra over $\mathbb{K}$ generated by X. We deal with *quadratic algebras*, that is, algebras R given as $\mathbb{K}\langle X\rangle/I$, where I is the ideal in $\mathbb{K}\langle X\rangle$ generated by a collection of homogeneous elements (called relations) of degree 2.

Algebras of this class, their growth, their Hilbert series and nil/nilpotency properties have been extensively studied, see [11, 12, 13] and references therein. One of the most challenging questions in the area (see [12, 15]) is the Kurosh problem of whether there is an infinite dimensional nil algebra in this class. A version of this question dealing with algebras of finite Gelfand–Kirillov dimension was solved in [8]. The Golod–Shafarevich type lower estimates for the dimensions of the graded components of an algebra play a crucial role in the study of quadratic algebras. These estimates have many other applications, for instance, to p-groups and class field theory [5, 16].

Recall that a $\mathbb{K}$-algebra R defined by the set X of generators and a set of homogeneous relations inherits the degree grading from the free algebra $\mathbb{K}\langle X\rangle$. If X is finite, one can consider the Hilbert series of R:

$$H_R(t) = \sum_{q=0}^{\infty} (\dim_{\mathbb{K}} R_q)\, t^q,$$

where R_q is the q^{th} homogeneous component of R. The original Golod–Shafarevich theorem provides a lower estimate for the coefficients of H_R. In the case of quadratic algebras the theorem reads as follows [5, 11]. For two power series $a(t)$ and $b(t)$ with real coefficients we write $a(t) \geqslant b(t)$ if $a_j \geqslant b_j$ for any $j \in \mathbb{Z}_+$, while $|a(t)|$ stands for the power series obtained from $a(t)$ by replacing by zeros all coefficients starting from the first non-positive one.

Theorem GS. *Let, $n \in \mathbb{N}$, $0 \leqslant d \leqslant n^2$ and R be a quadratic $\mathbb{K}$-algebra with n generators and d relations. Then $H_R(t) \geqslant |(1-nt+dt^2)^{-1}|$.*

In particular, Theorem GS provides a lower estimate on the order of nilpotency of R.

Definition 1.1. A graded algebra R is called *k-step nilpotent* if $R_k = \{0\}$.

Analysing the series $K(t)=|(1-nt+dt^2)^{-1}|$ in a standard way, one can easily see that it is a polynomial of degree $<k$ if and only if

$$\frac{d}{n^2}\geqslant\varphi_k,\quad\text{where}\quad \varphi_k=\frac{1}{4}\cos^{-2}\Bigl(\frac{\pi}{k+1}\Bigr). \tag{1.1}$$

For the sake of convenience, we outline the argument. If $(1-nt+dt^2)^{-1}=\sum\limits_{m=0}^{\infty}c_mt^m$ (the Taylor series expansion), then $K(t)$ is not a polynomial of degree $<k$ precisely when $c_m>0$ for $0\leqslant m\leqslant k$. Next, if $x^2-nx+d=(x-a)(x-b)$ (a and b are complex numbers in general), then an easy computation yields that $c_m=(m+1)(n/2)^m$ if $a=b$ and $c_m=\frac{a^{m+1}-b^{m+1}}{a-b}$ otherwise for $m\in\mathbb{Z}_+$. It follows that $c_m>0$ for all $m\in\mathbb{Z}_+$ if a and b are real, which happens precisely when $d\leqslant\frac{n^2}{4}$. If $n^2\geqslant d>\frac{n^2}{4}$, then $a,b=\sqrt{d}e^{\pm i\alpha}$, where $\alpha=\arccos\frac{n}{\sqrt{d}}$. Hence $c_m=\frac{a^{m+1}-b^{m+1}}{a-b}=d^{m/2}\frac{\sin(m+1)\alpha}{\sin\alpha}$ for $m\in\mathbb{Z}_+$. Clearly c_m for $0\leqslant m\leqslant k$ are positive precisely when $(k+1)\alpha<\pi$. After plugging in $\alpha=\arccos\frac{n}{\sqrt{d}}$, (1.1) follows.

Formula (1.1) together with Theorem GS and the obvious fact that the sequence $\{\varphi_k\}$ decreases and converges to $\frac14$ implies the following corollary, which can be found in [11].

Corollary GS. *If R is a quadratic $\mathbb{K}$-algebra given by n generators and $d<\varphi_kn^2$ relations, then* $\dim R_k>0$, *where φ_k is defined in* (1.1). *That is, R is not k-step nilpotent. In particular, if $d\leqslant\frac{n^2}{4}$, then* $\dim R_k>0$ *for every $k\in\mathbb{N}$ and therefore R is infinite dimensional.*

Asymptotic optimality of the last statement in Corollary GS was proved by Wisliceny [14].

Theorem W. *For every $n\in\mathbb{N}$, there exists a quadratic $\mathbb{K}$-algebra R given by n generators and d_n relations such that R is finite dimensional and $\lim\limits_{n\to\infty}\frac{d_n}{n^2}=\frac14$.*

More specifically, Wisliceny has constructed a quadratic algebra given by n generators and $\bigl\lceil\frac{n^2+2n}{4}\bigr\rceil$ semigroup relations (that is, every relation is either a degree 2 monomial or a difference of two degree 2 monomials), which is finite dimensional. Note that here and everywhere below $\lfloor t\rfloor$ is the largest integer $\leqslant t$, while $\lceil t\rceil$ is the smallest integer $\geqslant t$, where t is a real number. The authors [7] have improved the last result by showing that the minimal number of semigroup quadratic relations needed for finite dimensionality of an algebra with n generators is exactly $\bigl\lceil\frac{n^2+n}{4}\bigr\rceil$. The number $\bigl\lceil\frac{n^2+1}{4}\bigr\rceil$ remains a conjectural answer to the same question in the class of general quadratic (not necessarily semigroup) algebras.

1.1 Results

Note that if R is k-step nilpotent, then $R_m=\{0\}$ for $m\geqslant k$ and therefore R is finite dimensional provided $|X|<\infty$, where X is the set of generators of R. Thus R is k-step nilpotent if and only if H_R is a polynomial of degree $<k$.

In this article we show that the first statement in Corollary GS is asymptotically optimal for every $k\geqslant2$. In order to formulate the exact statement, we shall introduce the following numbers. For $n\in\mathbb{N}$ and $k\geqslant2$ let

$$d_{n,k}=\min_{n=a_1+\ldots+a_{k-1}}\ \max_{1\leqslant j\leqslant k-1}(a_1+\ldots+a_j)(a_j+\ldots+a_{k-1}), \tag{1.2}$$

where a_j are assumed to be non-negative integers. It turns out that the integers $d_{n,k}$ are not too far from φ_kn^2.

Lemma 1.2. *For each $n,k\in\mathbb{N}$ with $k\geqslant2$,*

$$\varphi_kn^2\leqslant d_{n,k}\leqslant\varphi_kn^2+\tfrac{(1+\varphi_k)n}{2}+\tfrac14. \tag{1.3}$$

In particular, $\lim\limits_{n\to\infty}\frac{d_{n,k}}{\varphi_kn^2}=1$ for each $k\geqslant2$.

We have defined the numbers $d_{n,k}$ since they feature in the following theorem.

Theorem 1.3. *Let $k \geqslant 2$. Then for every $n \in \mathbb{N}$, there exists a quadratic $\mathbb{K}$-algebra R given by n generators and $d_{n,k}$ relations such that R is k-step nilpotent.*

Corollary GS, Theorem 1.3 and Lemma 1.2 imply that the first statement in Corollary GS is asymptotically optimal. Note that Anick [1, 2] conjectured that for any $n \in \mathbb{N}$ and $0 \leqslant d \leqslant n^2$, there is a quadratic $\mathbb{K}$-algebra R with n generators and d relations such that $H_R(t) = |(1-nt+dt^2)^{-1}|$. The problem whether this conjecture is true remains open. Theorem 1.3 can be considered as an affirmative solution of its natural asymptotic version. It is also worth noting that for $k=2$, the statement of Theorem 1.3 is trivial, while the case $k=3$ was done by Anick [1]. It is also worth mentioning that the asymptotic optimality of the first statement in Corollary GS for $k=4$ and for $k=5$ in the case $|\mathbb{K}|=\infty$ was earlier obtained by the authors [6] building upon the ideas set in [3] and using a completely different approach. We refer to [10] for a result on asymptotic optimality of Theorem GS in a completely different sense.

Curiously enough, for some pairs (n,k) the estimate provided by Theorem 1.3 hits the mark. We illustrate this observation by the following result dealing with the cases $k=4$ and $k=5$. Note that $\varphi_4 = \frac{3-\sqrt{5}}{2}$ and $\varphi_5 = \frac{1}{3}$. Recall that Fibonacci numbers are the members of the recurrent sequence defined by $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geqslant 2$.

Theorem 1.4. *The equality $d_{n,4} = \left\lceil \frac{3-\sqrt{5}}{2} n^2 \right\rceil$ holds if and only if n is a Fibonacci number. The equality $d_{n,5} = \left\lceil \frac{n^2}{3} \right\rceil$ holds if and only if $n \in \{1,2\}$ or n is divisible by 6.*

Note that Theorem 1.4, Theorem 1.3 and Corollary GS imply that if $k=4$ and n is a Fibonacci number or if $k=5$ and 6 divides n, then the minimal number of quadratic relations needed for the finite dimensionality of an algebra with n generators is exactly $\lceil \varphi_k n^2 \rceil$. The proof of Theorem 1.3 is based upon the following general result. We start by introducing some notation.

Definition 1.5. Let X be the union of pairwise disjoint sets $A_1, \dots, A_k$ and

$$M = M(A_1, \dots, A_k) = \bigcup_{1 \leqslant j \leqslant q \leqslant n} A_q \times A_j \subseteq X \times X. \tag{1.4}$$

We introduce the following *partial ordering on M, generated by the partition* $\{A_1, \dots, A_k\}$. Namely, for distinct elements (a,b) and (c,d) of M, we write $(a,b) \prec (c,d)$ if $(a,b) \in A_l \times A_j$ and $(c,d) \in A_m \times A_r$ with $m \geqslant r > l \geqslant j$.

Definition 1.6. For a homogeneous degree 2 polynomial g in the free algebra $\mathbb{K}\langle X \rangle$, the (uniquely determined) finite subset S of $X \times X$ such that $g = \sum\limits_{(x,y) \in S} c_{x,y} xy$ with $c_{x,y} \in \mathbb{K} \setminus \{0\}$ is called the *support* of g and is denoted $S = \mathtt{supp}\,(g)$.

The next result is one of the main tools in the proof of Theorem 1.3.

Theorem 1.7. *Let $k \in \mathbb{N}$, $\{A_1, \dots, A_k\}$ be a partition of a set X and M be the set defined in (1.4). Assume also that $\{f_\alpha\}_{\alpha \in \Lambda}$ is a family of homogeneous degree 2 elements of the free algebra $\mathbb{K}\langle X \rangle$ such that $\bigcup\limits_{\alpha \in \Lambda} \mathtt{supp}\,(f_\alpha) = M$ and each $\mathtt{supp}\,(f_\alpha)$ is a chain in M with respect to the partial ordering $\prec$ on M, generated by the partition $\{A_1, \dots, A_k\}$ as in Definition 1.5. Then the algebra $R = \mathbb{K}\langle X \rangle / I$ with $I = \mathtt{Id}\{f_\alpha : \alpha \in \Lambda\}$ is $(k+1)$-step nilpotent.*

We conclude the introduction by providing a specific example of an application of Theorem 1.7.

Example 1.8. *Let $X = \{a,b,c,p,q,x,y,z\}$ be an 8-element set partitioned into 3 subsets $A_1 = \{a,b,c\}$, $A_2 = \{p,q\}$ and $A_3 = \{x,y,z\}$. Let M and the partial ordering $\prec$ on M be as in Definition 1.5. Consider the following 25 quadratic relations:*

$$
\begin{array}{lllll}
f_1 = xc, & f_2 = xa, & f_3 = xp + ab, & f_4 = yz + qc, & f_5 = pq, \\
f_6 = yc, & f_7 = ya, & f_8 = yp + bb, & f_9 = yy + qb, & \\
f_{10} = zc, & f_{11} = za, & f_{12} = zp + cb, & f_{13} = yx + qa, & \\
f_{14} = xb, & f_{15} = xq + ac, & f_{16} = xz + pc, & f_{17} = zz + qq + ca, & \\
f_{18} = yb, & f_{19} = yq + bc, & f_{20} = xy + pb, & f_{21} = zy + qp + ba, & \\
f_{22} = zb, & f_{23} = zq + cc, & f_{24} = xx + pa, & f_{25} = zx + pp + aa. &
\end{array}
$$

It is straightforward to verify that the support of each f_j is a chain in $(M,\prec)$ and that the union of $\mathtt{supp}\,(f_j)$ for $1 \leqslant j \leqslant 25$ is M. Theorem 1.7 ensures that the algebra given by the 8-element generator set X and the relations f_j with $1 \leqslant j \leqslant 25$ is 4-step nilpotent. Incidentally, $25 = \lceil \varphi_4 \cdot 8^2 \rceil$, which means (see Corollary GS) that a quadratic algebra given by 8 generators and $\leqslant 24$ relations is never 4-step nilpotent.

2 Combinatorial lemmas

Theorem 1.7 allows us to construct k-step nilpotent quadratic algebras with few relations. In order to do this, we need an estimate on the number of relations in an algebra featuring in Theorem 1.7. Recall that the *width* $w(X,<)$ of a partially ordered set $(X,<)$ is the supremum of the cardinalities of antichains in X.

Lemma 2.1. *Let $k \in \mathbb{N}$, $\{A_1,\dots,A_k\}$ be a partition of a finite set X and $M \subseteq X^2$ be the set defined in (1.4) with the partial ordering $\prec$ introduced in Definition 1.5. For $1 \leqslant q \leqslant k$, let $B_q = \bigcup\limits_{j \geqslant q \geqslant m} A_j \times A_m$. Then $w(M,\prec) = \max\{|B_1|,\dots,|B_k|\}$.*

Proof. It is a straightforward exercise to verify that each B_q is an antichain in $(M,\prec)$ and that every antichain is contained in at least one of the sets B_q. □

We also need the following observation.

Lemma 2.2. *Let $k \geqslant 2$ and $\alpha_0, \alpha_1, \dots, \alpha_{k-1} \geqslant 0$ be defined by the fromulae $\alpha_0 = 0$, $\alpha_1 = \varphi_k$ and $\alpha_j = \frac{\varphi_k}{1-\alpha_{j-1}}$ for $2 \leqslant j \leqslant k-1$. Then*

$$0 = \alpha_0 < \alpha_1 < \ldots < \alpha_{k-1} = 1, \tag{2.1}$$

$$\alpha_j(1-\alpha_{j-1}) = \varphi_k \text{ for } 1 \leqslant j \leqslant k-1 \tag{2.2}$$

$$\text{and } \max_{1 \leqslant j \leqslant k-1} (\alpha_j - \alpha_{j-1}) = \varphi_k \text{ (attained for } j = 1 \text{ and for } j = k-1). \tag{2.3}$$

Proof. Obviously, (2.2) is a direct consequence of the definition of α_j. Next, (2.3) follows easily from (2.1). Indeed, assuming that (2.1) holds, we have $\alpha_{k-1} = 1$, which implies $\alpha_{k-2} = 1 - \varphi_k$. Since $\alpha_j - \alpha_{j-1} = \frac{\varphi_k}{1-\alpha_{j-1}} - \alpha_{j-1}$ and $0 \leqslant \alpha_{j-1} \leqslant 1 - \varphi_k$ for $1 \leqslant j \leqslant k-1$, (2.3) follows from the elementary fact that the function $\frac{\varphi_k}{1-x} - x$ on the interval $[0, 1-\varphi_k]$ attains its maximal value at the end-points.

Thus it remains to verify (2.1). For $0 < t \leqslant 1$ consider the rational function $f_t(x) = \frac{t}{1-x}$ and for $m \in \mathbb{Z}_+$ let $f_t^{[m]}$ be the m^{th} iterate of f_t: $f_t^{[0]}(x) = x$ and $f_t^{[m]} = f_t \circ \ldots \circ f_t$ m times for $m \in \mathbb{N}$. We start with an elementary observation

$$\begin{aligned}&\text{if } 0\leqslant t\leqslant \tfrac14, \text{ then the sequence } \{f_t^{[m]}(0)\}_{m\in\mathbb{Z}_+} \text{ is strictly increasing}\\ &\text{and converges to the fixed point } w_t=\tfrac{1-\sqrt{1-4t}}{2}\in\big[0,\tfrac12\big] \text{ of } f_t.\end{aligned}\tag{2.4}$$

For instance, to justify (2.4), one can use induction with respect to m to prove the chain of inequalities $0\leqslant f_t^{[m]}(0)<f_t^{[m+1]}(0)<w_t$.

Next, it is easy to verify that if $\frac14<t\leqslant 1$, then $f_t(x)>x$ for $x\in[0,1)$. Hence,

$$f_t^{[m+1]}(0)>f_t^{[m]}(0) \text{ provided } 0\leqslant f_t^{[m]}(0)<1. \tag{2.5}$$

For each $m\in\mathbb{Z}_+$, we consider the rational function $h_m(t)=f_t^{[m]}(0)$ of the variable t. Now we observe that (2.3) follows from the claim

$$\text{for every } m\in\mathbb{N},\ \varphi_{m+1} \text{ is the smallest solution of the equation } h_m(t)=1 \text{ on } \big(\tfrac14,1\big]. \tag{2.6}$$

Indeed, assume that (2.6) holds. By (2.4), $0<h_m(t)<\frac12$ for every $m\in\mathbb{N}$ and $t\in\big(0,\frac14\big]$. Since the sequence $\{\varphi_m\}$ is decreasing, $h_j(t)<1$ whenever $j\leqslant m$ and $0\leqslant t<\varphi_{m+1}$. Using (2.6) with $m=k-1$ and (2.5), we now have

$$0=f_{\varphi_k}^{[0]}(0)<f_{\varphi_k}^{[1]}(0)<\ldots<f_{\varphi_k}^{[k-1]}(0)=h_{k-1}(\varphi_k)=1.$$

On the other hand, by definition of α_j, $\alpha_j=f_{\varphi_k}^{[j]}(0)$ for $0\leqslant j\leqslant k-1$ and (2.3) follows.

Thus it remains to prove (2.6). Using the obvious recurrent relation $h_{j+1}(t)=\frac{t}{1-h_j(t)}$ together with the initial data $h_0=0$, one can use the induction with respect to m to verify that

$$h_m(t)=t\tfrac{a^m-\overline{a}^m}{a^{m+1}-\overline{a}^{m+1}} \ \text{ for } m\in\mathbb{Z}_+ \text{ and } t\in\big[\tfrac14,1\big], \text{ where } a=a(t)=\tfrac{1+i\sqrt{4t-1}}{2}.$$

Hence for $t\in\big[\frac14,1\big]$,

$$h_m(t)=1 \iff (a/\overline{a})^m=(\overline{a}-t)/(a-t) \iff e^{im\alpha(t)}=e^{i\beta(t)}, \tag{2.7}$$

where

$$\alpha(t)=2\arccos\tfrac{1}{2\sqrt{t}} \ \text{ and } \ \beta(t)=2\pi-2\arccos\big(\tfrac{1}{2t}-1\big)$$

are the arguments of the unimodular complex numbers $a/\overline{a}$ and $(\overline{a}-t)/(a-t)$. The case $m=1$ is trivial. Assuming that $m\geqslant 2$ and using (2.7), we see that the smallest $t\in\big[\frac14,\frac12\big]$ satisfying $h_m(t)=1$ must satisfy $m\alpha(t)=\beta(t)$. Since the function $m\alpha(t)-\beta(t)$ on the interval $\big[\frac14,\frac12\big]$ is strictly increasing (look at the derivative) and has values of opposite signs at the ends, there is exactly one $t_m\in\big[\frac14,\frac12\big]$ satisfying $m\alpha(t_m)=\beta(t_m)$. Then t_m is the smallest solution of the equation $h_m(t)=t$ on the interval $\big[\frac14,1\big]$. Since $\varphi_{m+1}\in\big[\frac14,\frac12\big]$, (2.6) will follow if we show that $m\alpha(\varphi_{m+1})=\beta(\varphi_{m+1})$. This is indeed true: plugging in $\varphi_{m+1}=\frac{1}{4\cos^2(\pi/(m+2))}$, we have

$$\begin{aligned}&m\alpha(\varphi_{m+1})=2m\arccos\big(\cos\big(\tfrac{\pi}{m+2}\big)\big)=\tfrac{2\pi m}{m+2};\\ &\beta(\varphi_{m+1})=2\pi-2\arccos\big(2\cos^2\big(\tfrac{\pi}{m+2}\big)-1\big)=2\pi-2\arccos\big(\cos\big(\tfrac{2\pi}{m+2}\big)\big)=2\pi-\tfrac{4\pi}{m+2}=\tfrac{2\pi m}{m+2}.\end{aligned}$$

Hence $m\alpha(\varphi_{m+1})=\beta(\varphi_{m+1})$, which completes the proof. □

3 Proof of Theorem 1.7

For $k\in\mathbb{N}$, we denote $\mathbb{N}_k=\{1,2,\ldots,k\}$. Assume the contrary. Then the set Ω of $j=(j_1,\ldots,j_{k+1})\in\mathbb{N}_k^{k+1}$ such that there are $x_1\in A_{j_1},\ \ldots,\ x_{k+1}\in A_{j_{k+1}}$ for which $x_1\ldots x_{k+1}\notin I$ is non-empty. We endow $\mathbb{N}_k^{k+1}$ with the lexicographical ordering $<$ counting from the right-hand side. That is, $j<m$

if and only if there is $l \in \mathbb{N}_{k+1}$ such that $j_l < m_l$ and $j_r = m_r$ for $r > l$. Since $<$ is a total ordering on the finite set $\mathbb{N}_k^{k+1}$ and $\Omega \subseteq \mathbb{N}_k^{k+1}$ is non-empty, Ω has a unique element j minimal with respect to $<$. Since $j \in \Omega$, there are $x_1 \in A_{j_1}, \dots, x_{k+1} \in A_{j_{k+1}}$ for which $x_1 \dots x_{k+1} \notin I$.

Now we shall construct inductively $m_1, \dots, m_{k+1} \in \mathbb{N}_k$ and monomials $u_1, \dots, u_{k+1}$ in $\mathbb{K}\langle X\rangle$ of degree $k+1$ such that

$$m_l > m_{l-1} \text{ if } l \geqslant 2; \tag{3.1}$$

$$u_l \notin I; \tag{3.2}$$

$$u_l = v_l w_l x_{l+1} x_{l+2} \dots x_{k+1}, \text{ where } w_l \in A_{m_l} \text{ and } v_l \text{ is a monomial of degree } l-1. \tag{3.3}$$

We start by setting $u_1 = x_1 \dots x_{k+1}$ and $m_1 = j_1$ and observing that (3.1–3.3) with $l = 1$ are satisfied. Assume now that $2 \leqslant l \leqslant k+1$ and that $m_1, \dots, m_{l-1}$ and $u_1, \dots, u_{l-1}$ satisfying the desired conditions are already constructed.

If $m_{l-1} < j_l$, then we set $m_l = j_l$, $w_l = x_l$, $u_l = u_{l-1}$ and $v_l = v_{l-1} w_{l-1}$. Using the induction hypothesis, we see that (3.1–3.3) are satisfied. It remains to consider the case $m_{l-1} \geqslant j_l$. In this case $w_{l-1} x_l \in M$ and therefore there is $\alpha \in \Lambda$ such that $(w_{l-1}, x_l) \in \mathtt{supp}\,(f_\alpha)$. Let $S = \mathtt{supp}\,(f_\alpha) \setminus \{(w_{l-1}, x_l)\}$. Since $f_\alpha \in I$,

$$w_{l-1} x_l = \sum_{(a,b)\in S} c_{a,b}\, ab \pmod{I} \quad \text{with } c_{a,b} \in \mathbb{K}.$$

Using (3.3) for $l-1$ and the above display, we get

$$u_{l-1} = \sum_{(a,b)\in S} c_{a,b} v_{l-1} a b x_{l+1} \dots x_{k+1} \pmod{I}.$$

Since $\mathtt{supp}\,(f_\alpha)$ is a chain in M with respect to $\prec$, for every $(a,b) \in S$, either $(a,b) \prec (w_{l-1}, x_l)$ or $(w_{l-1}, x_l) \prec (a,b)$. If $(a,b) \prec (w_{l-1}, x_l)$, b is contained in A_q with $q < j_l$. Using the definition of Ω and the minimality of j in Ω, we obtain

$$v_{l-1} a b x_{l+1} \dots x_{k+1} \in I \ \text{ if } (a,b) \in S,\ (a,b) \prec (w_{l-1}, x_l).$$

According to the last two displays

$$u_{l-1} = \sum_{\substack{(a,b)\in S \\ (w_{l-1},x_l)\prec(a,b)}} c_{a,b} v_{l-1} a b x_{l+1} \dots x_{k+1} \pmod{I}.$$

By (3.2) for $l-1$, $u_{l-1} \notin I$. Thus, using the above display, we can pick $(a,b) \in S$ such that $(w_{l-1}, x_l) \prec (a,b)$ and $v_{l-1} a b x_{l+1} \dots x_{k+1} \notin I$. Now we set $u_l = v_{l-1} a b x_{l+1} \dots x_{k+1}$, $w_l = b$, $v_l = v_{l-1} a$ and take m_l such that $w_l = b \in A_{m_l}$.

Since $w_{l-1} \in A_{m_{l-1}}$ and $(w_{l-1}, x_l) \prec (a,b) = (a, w_l)$, we have $m_l > m_{l-1}$. Thus (3.1–3.3) are satisfied. This completes the inductive procedure of constructing $m_1, \dots, m_{k+1}$ and $u_1, \dots, u_{k+1}$. By (3.1), m_j for $1 \leqslant j \leqslant k+1$ are $k+1$ pairwise distinct elements of the k-element set $\mathbb{N}_k$. We have arrived to a contradiction, which proves that R is $(k+1)$-step nilpotent.

4 Proofs of Theorem 1.3 and Lemma 1.2

Let $k \geqslant 2$, $n \in \mathbb{N}$ and $a_1, \dots, a_{k-1} \in \mathbb{Z}_+$ be such that $a_1 + \dots + a_{k-1} = n$. In order to prove Theorem 1.3, it suffices to prove that there is a quadratic $\mathbb{K}$-algebra R given by n generators and

$$d = \max_{1\leqslant j\leqslant k-1} (a_1 + \dots + a_j)(a_j + \dots + a_{k-1})$$

relations such that R is k-step nilpotent.

Let X be an n-element set of generators. Since $a_1+\ldots+a_{k-1}=n$, we can present X as the union of the pairwise disjoint sets $A_1,\ldots,A_{k-1}$ with $|A_j|=a_j$ for $1\leqslant j\leqslant k-1$. Consider the set $M\subset X^2$ defined in (1.4) and the partial ordering $\prec$ on M generated by the partition $\{A_1,\ldots,A_{k-1}\}$. For $1\leqslant j\leqslant k-1$, let $B_j=\bigcup\limits_{q\geqslant j\geqslant m} A_q\times A_m$. Clearly, $|B_j|=(a_1+\ldots+a_j)(a_j+\ldots+a_{k-1})$. Hence $d=\max\{|B_1|,\ldots,|B_{k-1}|\}$. By Lemma 2.1, $w(M,\prec)=d$. According to the Dilworth theorem (see [4] for a short inductive proof) the width of a finite partially ordered set P is precisely the minimal number of chains needed to cover P. Hence, we can write $M=\bigcup\limits_{q=1}^{d} C_q$, where each C_q is a chain in M. Now we consider the homogeneous degree 2 elements of $\mathbb{K}\langle X\rangle$ given by

$$f_q=\sum_{(a,b)\in C_q} ab \quad\text{for } 1\leqslant q\leqslant d.$$

Clearly $\texttt{supp}\,(f_q)=C_q$. Thus the union of the supports of f_q is M and each $\texttt{supp}\,(f_q)$ is a chain in M. By Theorem 1.7, the algebra R given by the relations f_q for $1\leqslant q\leqslant d$ is k-step nilpotent. This completes the proof of Theorem 1.3.

Now we shall prove Lemma 1.2. By Theorems GS and 1.3, $d_{n,k}\geqslant\varphi_k n^2$ for every $k\geqslant 2$ and $n\in\mathbb{N}$. This proves the first inequality in (1.3). It remains to prove the second one. By Lemma 2.2, there are $\alpha_0,\ldots,\alpha_{k-1}\in[0,1]$ such that $0=\alpha_0<\alpha_1<\ldots<\alpha_{k-1}=1$ and $\alpha_j(1-\alpha_{j-1})=\varphi_k$ for $1\leqslant j\leqslant k-1$. Now for $0\leqslant j\leqslant k-1$ let $b_j=\lceil n\alpha_j-\frac12\rceil$. Clearly $0=b_0\leqslant b_1\leqslant\ldots\leqslant b_{k-1}=n$. Now we set $a_j=b_j-b_{j-1}$ for $1\leqslant j\leqslant k-1$. Then $a_j\in\mathbb{Z}_+$ and $a_1+\ldots+a_{k-1}=n$. Hence

$$d_{n,k}\leqslant\max_{1\leqslant j\leqslant k-1}(a_1+\ldots+a_j)(a_j+\ldots+a_{k-1})=\max_{1\leqslant j\leqslant k-1}b_j(n-b_{j-1})=\max_{1\leqslant j\leqslant k-1}\lceil n\alpha_j-\tfrac12\rceil\cdot\lfloor n(1-\alpha_{j-1})+\tfrac12\rfloor.$$

It is easy to see that for every $\alpha,\beta\in[0,1]$,

$$\lceil n\alpha-\tfrac12\rceil\cdot\lfloor n\beta+\tfrac12\rfloor-\alpha\beta n^2\leqslant\tfrac{\alpha+\beta}{2}n+\tfrac14.$$

From the last two displays and the equalities $\alpha_j(1-\alpha_{j-1})=\varphi_k$ it follows that

$$d_{n,k}\leqslant\varphi_k n^2+\frac{n}{2}\max_{1\leqslant j\leqslant k-1}(1+\alpha_j-\alpha_{j-1})+\frac14.$$

By Lemma 2.2, the maximum in the above display equals φ_k. Thus $d_{n,k}\leqslant\varphi_k n^2+\frac{1+\varphi_k}{2}n+\frac14$, which completes the proof of Lemma 1.2.

5 4-Step nilpotency and the Fibonacci numbers

First, we derive an explicit formula for $d_{n,4}$.

Lemma 5.1. *For every $n\in\mathbb{N}$,*

$$d_{n,4}=\min\bigl\{\bigl\lceil\tfrac{\sqrt5-1}{2}n\bigr\rceil^2,n\bigl\lceil\tfrac{3-\sqrt5}{2}n\bigr\rceil\bigr\}.\tag{5.1}$$

Proof. Using (1.2) with $k=4$ and denoting $a=a_1$ and $b=a_3$, we obtain

$$d_{n,4}=\min\{\max\{na,nb,(n-a)(n-b)\}:a,b\in\mathbb{Z}_+,\ a+b\leqslant n\}.$$

An obvious symmetry consideration yields

$$d_{n,4}=\min\{\max\{na,nb,(n-a)(n-b)\}:a,b\in\mathbb{Z}_+,\ b\leqslant a,\ a+b\leqslant n\}.$$

Since $nb \leqslant na$ and $(n-a)(n-b) \geqslant (n-a)^2$ when $a, b \in \mathbb{Z}_+$ satisfy $b \leqslant a \leqslant n$, we have

$$d_{n,4} = \min\{\max\{na, (n-a)^2\} : a \in \mathbb{Z}_+,\ 2a \leqslant n\}. \tag{5.2}$$

Now, assume that $a \in \mathbb{Z}_+$ satisfies $2a \leqslant n$. Solving a quadratic inequality we see that $na \geqslant (n-a)^2$ holds precisely when $a \geqslant \varphi_4 n$. Hence (5.2) can be rewritten as

$$\begin{aligned} &d_{n,4} = \min\{a_n, b_n\}, \quad \text{where} \\ &a_n = \min\{na : a \in \mathbb{Z}_+,\ \varphi_4 n \leqslant a \leqslant n/2\} \quad \text{and} \quad b_n = \min\{(n-a)^2 : a \in \mathbb{Z}_+,\ a \leqslant \varphi_4 n\}. \end{aligned}$$

Clearly, the minimum in the definition of a_n is attained for $a = \lceil \varphi_4 n \rceil$ and the minimum in the definition of b_n is attained for $a = \lfloor \varphi_4 n \rfloor$. Hence $a_n = n\lceil \varphi_4 n \rceil$ and $b_n = \lceil (1-\varphi_4)n \rceil^2$. Using the equalities $\varphi_4 = \frac{3-\sqrt{5}}{2}$ and $1 - \varphi_4 = \frac{\sqrt{5}-1}{2}$, we see that (5.1) follows from the above display. □

Corollary 5.2. *The equality $d_{n,4} = \lceil \varphi_4 n^2 \rceil$ holds if and only if either $\lceil \varphi_4 n^2 \rceil$ is divisible by n or $\lceil \varphi_4 n^2 \rceil$ is a square of a positive integer.*

Proof. Let $m = \lceil \varphi_4 n^2 \rceil$. From Lemma 5.1 it follows that $d_{n,4}$ is always either divisible by n or is a square. Thus the equality $m = d_{n,4}$ can only hold if either m is divisible by n or m is a square.

If m is divisible by n, we can write $m = nj$ for some $j \in \mathbb{N}$. Now it is easy to see that $j = \lceil \frac{3-\sqrt{5}}{2} n \rceil$ and therefore, by Lemma 5.1, $d_{n,4} \geqslant jn = m$. On the other hand, choosing $a = j$ and using (5.2), we get $d_{n,4} \leqslant \max\{nj, (n-j)^2\} = nj$. Thus $d_{n,4} = nj = m$.

If m is a square, we can write $m = j^2$ for some $j \in \mathbb{N}$. Now it is easy to see that $j = \lceil \frac{\sqrt{5}-1}{2} n \rceil$ and therefore, by Lemma 5.1, $d_{n,4} \geqslant j^2 = m$. On the other hand, choosing $a = n - j$ and using (5.2), we get $d_{n,4} \leqslant \max\{n(n-j), j^2\} = j^2$. Thus $d_{n,4} = j^2 = m$. □

Proof of the first part of Theorem 1.4. Let $F_0, F_1, \ldots$ be the Fibonacci sequence and $\varphi = \frac{\sqrt{5}+1}{2}$ be the golden ratio number. Using the formula $F_n = \frac{\varphi^n - (-\varphi)^{-n}}{\sqrt{5}}$ together with the equality $\varphi_4 = \varphi^{-2}$, one can easily verify that $\lceil \varphi_4 F_k^2 \rceil = F_{k-1}^2$ if k is odd and $\lceil \varphi_4 F_k^2 \rceil = F_k F_{k-2}$ if k is even. Thus if n is a Fibonacci number, then $\lceil \varphi_4 n^2 \rceil$ is either divisible by n or is a square.

To show the converse, we use the following criterion of recognizing the Fibonacci numbers due to Möbius [9]. It says that a positive integer n is a Fibonacci number if and only if the interval $(\varphi n - n^{-1}, \varphi n + n^{-1})$ contains an integer. Furthermore, if m is an integer belonging to $(\varphi n - n^{-1}, \varphi n + n^{-1})$, then m is the next Fibonacci number after n.

First, assume that $n \in \mathbb{N}$ and $\lceil \varphi_4 n^2 \rceil$ is divisible by n. Then $\varphi_4 n^2 + \theta = nk$, where $k \in \mathbb{N}$ and $0 < \theta < 1$. Since $\varphi_4 = 2 - \varphi$, it follows that $\varphi n - (2n - k) = \frac{\theta}{n}$ and therefore $2n - k \in (\varphi n - n^{-1}, \varphi n + n^{-1})$. By the criterion of Möbius, n is a Fibonacci number. Finally, assume that $\lceil \varphi_4 n^2 \rceil$ is a square number. Since $\varphi_4 = \varphi^{-2}$, this means that $\frac{n^2}{\varphi^2} + \theta = k^2$, where $k \in \mathbb{N}$ and $0 < \theta < 1$. It immediately follows that $k = \lceil \frac{n}{\varphi} \rceil$. In other words $k = \frac{n}{\varphi} + \alpha$ with $0 < \alpha < 1$. Squaring the last equality, we get $k^2 = \frac{n^2}{\varphi^2} + \theta = \frac{n^2}{\varphi^2} + \frac{2n\alpha}{\varphi} + \alpha^2$. In particular, $\frac{2n\alpha}{\varphi} < \theta < 1$. Hence $\varphi\alpha < \frac{\varphi^2}{2n}$. Thus the equality $k = \frac{n}{\varphi} + \alpha$ implies $n = \varphi k - \varphi\alpha$ and

$$\varphi\alpha < \frac{\varphi^2}{2n} = \frac{\varphi^2}{2(\varphi k - \varphi\alpha)} < \frac{\varphi^2}{2(\varphi k - \varphi^2/2n)}.$$

Since $n \geqslant k$, we have

$$\varphi\alpha < \frac{\varphi^2}{2(\varphi k - \varphi^2/2k)} < \frac{1}{k},$$

where the last inequality is satisfied for $k > 2$. Now the above display and the equality $n = \varphi k - \varphi\alpha$ imply that n belongs to the interval $(\varphi k - k^{-1}, \varphi k + k^{-1})$. By the criterion of Möbius, both k and

n are Fibonacci numbers provided $k > 2$. If $k = 1$ or $k = 2$, a direct computation yields $n = 2$ or $n = 3$ respectively, which are Fibonacci numbers as well.

Thus we have proven that $\lceil \varphi_4 n^2 \rceil$ is either divisible by n or is a square number precisely when n is a Fibonacci number. By Lemma 5.2, $d_{n,4} = \lceil \varphi_4 n^2 \rceil$ if and only if n is a Fibonacci number. $\square$

6 5-Step nilpotency

In this section we prove the second part of Theorem 1.4. As in the previous section we start by simplification the formula defining $d_{n,5}$.

Lemma 6.1. *If $n \in \mathbb{N}$ is even, then $d_{n,5} = \frac{n}{2}\lceil \frac{2n}{3} \rceil$. If $n \in \mathbb{N}$ is congruent to -1 modulo 6, then $d_{n,5} = n\lceil \frac{n(n+1)}{3n+1} \rceil$. If $n \in \mathbb{N}$ is congruent to 1 or to 3 modulo 6, then $d_{n,5} = \frac{n+1}{2}\lceil \frac{2n^2}{3n+1} \rceil$.*

Proof. Using the symmetry in (1.2) with respect to reversing the order of a_j, we have

$$\begin{array}{l} d_{n,5} = \min\{S(a) : a \in \mathbb{Z}_+^4,\ a_1 + a_2 + a_3 + a_4 = n,\ a_1 \leqslant a_4\}, \quad \text{where} \\ S(a) = \max\{na_1, na_4, (a_1 + a_2)(a_2 + a_3 + a_4), (a_1 + a_2 + a_3)(a_3 + a_4)\}. \end{array} \tag{6.1}$$

It is easy to see that the minimum in (6.1) can not be attained when $a_2 = 0$ if $n > 1$ (the case $n = 1$ is trivial anyway). If $a_1 < a_4$ and $a_2 > 0$, one can easily check that $S(a') \leqslant S(a)$, where a' is obtained from a by increasing a_1 by 1 with simultaneous decreasing of a_2 by 1. Similarly, if $a_1 = a_4$ and $|a_2 - a_3| > 1$, $S(a') \leqslant S(a)$, where a' is obtained from a by increasing the smaller of a_2 and a_3 by 1 with simultaneous decreasing of the bigger one by 1. It follows that among $a \in \mathbb{Z}_+^4$ for which the minimum in (6.1) is attained there must be at least one point satisfying $a_1 = a_4$ and $|a_2 - a_3| \leqslant 1$. Thus the minimum in (6.1) is attained at a point a of the shape $a = (\alpha, \beta, \beta, \alpha)$ if n is even and it is attained at a point a of the shape $a = (\alpha, \beta + 1, \beta, \alpha)$ if n is odd. Substituting this data into (6.1), we get

$$d_{n,5} = \frac{n}{2} \min\{\max\{2a, n - a\} : a \in \mathbb{Z}_+,\ a \leqslant n/2\} \quad \text{if } n \text{ is even} \tag{6.2}$$

and

$$d_{n,5} = \min\{\max\{na, (n + 1)(n - a)/2\} : a \in \mathbb{Z}_+,\ a \leqslant n/2\} \quad \text{if } n \text{ is odd.} \tag{6.3}$$

Since $\max\{2a, n - a\} = n - a$ if $3a \leqslant n$ and $\max\{2a, n - a\} = 2a$ if $3a \geqslant n$, (6.2) implies that $d_{n,5} = \min\{n\lceil \frac{n}{3} \rceil, \frac{n}{2}\lceil \frac{2n}{3} \rceil\} = \frac{n}{2}\lceil \frac{2n}{3} \rceil$ if n is even (the two numbers in the last minimum are equal in all cases except for the numbers n congruent to -2 modulo 6 in which case the second one is less by 1).

Next, $\max\{na, (n+1)(n-a)/2\} = (n+1)(n-a)/2$ if $a \leqslant \frac{n(n+1)}{3n+1}$ and $\max\{na, (n+1)(n-a)/2\} = na$ if $a \geqslant \frac{n(n+1)}{3n+1}$. Plugging this into (6.3), we get $d_{n,5} = \min\{n\lceil \frac{n(n+1)}{3n+1} \rceil, \frac{n+1}{2}\lceil \frac{2n^2}{3n+1} \rceil\}$. Considering the cases of n being 1, 3 and -1 modulo 6 separately, we see that $d_{n,5} = n\lceil \frac{n(n+1)}{3n+1} \rceil$ if n is congruent to -1 modulo 6 and $d_n = \frac{n+1}{2}\lceil \frac{2n^2}{3n+1} \rceil$ ff $n \in \mathbb{N}$ is congruent to 1 or to 3 modulo 6. $\square$

From Lemma 6.1 it immediately follows that $d_{n,5} = \frac{n^2}{3} = \varphi_5 n^2$ if 6 is a factor of n. Considering the exact formula provided by Lemma 6.1 and treating the possible remainders for the division of n by 6 as separate cases, one easily sees that $d_{n,5} - \frac{n^2}{3} \geqslant 1$ and therefore $d_{n,5} > \lceil \varphi_5 n^2 \rceil$ if n is not divisible by 6 and $n \geqslant 3$. It is easy to verify that the equality $d_{n,5} = \lceil \varphi_5 n^2 \rceil$ holds for $n = 1$ and for $n = 2$. This completes the Proof of Theorem 1.4.

We conclude by reminding that the following particular cases of the Anick's conjecture [1] remain unproved.

Conjecture 6.2. *There is a k-step nilpotent $\mathbb{K}$-algebra given by n generators and d quadratic relations whenever $d \geqslant \varphi_k n^2$.*

Conjecture 6.3. *There is a finite dimensional $\mathbb{K}$-algebra given by n generators and d quadratic relations whenever $d > \frac{n^2}{4}$.*

Acknowledgements

We are grateful to the Max-Planck-Institute for Mathematics in Bonn and to IHES, where parts of this research have been done, for hospitality, support, and excellent research atmosphere. This work is funded by the ERC grant 320974, and partially supported by the project PUT9038. We also would like to thank a referee for valuable suggestions, which helped to improve the presentation.

References

[1] D. Anick, *Generic algebras and CW complexes,* Algebraic topology and algebraic K-theory (Princeton, N.J., 1983), 247–321, Ann. of Math. Stud. **113**, Princeton Univ. Press, Princeton, NJ, 1987

[2] D. Anick, *Noncommutative graded algebras and their Hilbert series,* J. Algebra **78** (1982), 120–140

[3] P. Cameron and N. Iyudu, *Graphs of relations and Hilbert series,* J. Symbolic Comput. **42** (2007), 1066–1078

[4] F. Galvin, *A proof of Dilworth's chain decomposition theorem,* Amer. Math. Monthly **101** (1994), 352–353

[5] E. Golod and I. Shafarevich, *On the class field tower* (Russian), Izv. Akad. Nauk SSSR Ser. Mat. **28** 1964, 261–272

[6] N. Iyudu and S. Shkarin, *The Golod-Shafarevich inequality for Hilbert series of quadratic algebras and the Anick conjecture,* Proc. Roy. Soc. Edinburgh **A141** (2011), 609–629

[7] N. Iyudu and S. Shkarin, *Finite dimensional semigroup quadratic algebras with minimal number of relations,* Monats. Math. [to appear]

[8] T. Lenagan and A. Smoktunowicz, *An infinite dimensional affine nil algebra with finite Gelfand–Kirillov dimension,* J. Amer. Math. Soc. **20**, 989–1001 (2007)

[9] M. Möbius, *Wie erkennt Man eine Fibonacci Zahl?,* Math. Semesterber **45** (1998), 243–246

[10] M. Newman, C. Schneider and A. Shalev, *The entropy of graded algebras,* J. Algebra **223** (2000), 85–100

[11] A. Polishchuk and L. Positselski, *Quadratic algebras,* University Lecture Series **37** American Mathematical Society, Providence, RI, 2005

[12] A. Smoktunowicz, *Some results in non-commutative ring theory,* International Congress of Mathematicians II, Eur. Math. Soc., Zurich, 259–269 (2006)

[13] V. Ufnarovskii, *Combinatorial and asymptotic methods in algebra* (Russian) Current problems in mathematics. Fundamental directions **57** 5–177, Moscow, 1990

[14] I. Wisliceny, *Konstruktion nilpotenter associativer Algebren mit wenig Relationen,* Math. Nachr. **147**(1990), 75–82

[15] E. Zelmanov, *Some open problems in the theory of infinite dimensional algebras,* J. Korean Math. Soc. **44**, 1185–1195 (2007)

[16] E. Zelmanov, *Infinite algebras and pro-p groups,* In: *Infinite groups: geometric, combinatorial and dynamical aspects,* Progress in Mathematics **248** , Birkhäuser, Basel, 403–413 (2005)

Natalia Iyudu
School of Mathematics
The University of Edinburgh
James Clerk Maxwell Building
The King's Buildings
Mayfield Road
Edinburgh

Scotland EH9 3JZ
E-mail address: niyudu@staffmail.ed.ac.uk
and
Stanislav Shkarin
Queens's University Belfast
Pure Mathematics Research Centre
University road, Belfast, BT7 1NN, UK
E-mail address: s.shkarin@qub.ac.uk