A Cauchy-Davenport theorem for linear maps

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Abstract

We prove a version of the Cauchy-Davenport theorem for general linear maps. For subsets A, B of the finite field \mathbb{F}_p , the classical Cauchy-Davenport theorem gives a lower bound for the size of the sumset A+B in terms of the sizes of the sets A and B. Our theorem considers a general linear map $L: \mathbb{F}_p^n \to \mathbb{F}_p^m$, and subsets $A_1, \ldots, A_n \subseteq \mathbb{F}_p$, and gives a lower bound on the size of $L(A_1 \times A_2 \times \ldots \times A_n)$ in terms of the sizes of the sets A_1, \ldots, A_n .

Our proof uses Alon's Combinatorial Nullstellensatz and a variation of the polynomial method.

1 Introduction

Let p be a prime, and let \mathbb{F}_p denote the finite field of integers modulo p. The classical Cauchy-Davenport theorem states that if $A, B \subseteq \mathbb{F}_p$, then the sumset A+B (defined to equal $\{a+b \mid a \in A, b \in B\}$) satisfies the inequality: $|A+B| \geq |A| + |B| - 1$, provided $p \geq |A| + |B| - 1$. It is instructive to compare this with the elementary inequality $|A+B| \geq |A| + |B| - 1$ for $A, B \subseteq \mathbb{R}$ (this has a simple proof using the natural order on \mathbb{R}). The Cauchy-Davenport theorem says that this inequality continues to hold mod p, for p large enough.

The Cauchy-Davenport theorem can be seen as a statement about the size of the image of the product set $A \times B$ under the the map $+ : \mathbb{F}_p \times \mathbb{F}_p \to \mathbb{F}_p$. Here we study a similar phenomenon for general linear maps. Let $L : \mathbb{F}_p^n \to \mathbb{F}_p^m$ be an \mathbb{F}_p -linear map. For subsets $A_1, \ldots, A_n \subseteq \mathbb{F}_p$, we define

$$L(A_1, ..., A_n) = \{L(a_1, ..., a_n) \mid a_i \in A_i \text{ for each } i\}.$$

(Equivalently, this is the image of $A_1 \times A_2 \times ... \times A_n$ under L.) We are interested in a Cauchy-Davenport theorem for L: given integers $k_1, ..., k_n$,

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what is the minimum possible size, over subsets $A_i \subseteq \mathbb{F}_p$ with $|A_i| = k_i$, of $|L(A_1, \ldots, A_n)|$? This question is already interesting for the map $L^* : \mathbb{F}_p^3 \to \mathbb{F}_p^2$, given by L(x, y, z) = (x + y, x + z).

Our main theorem, Theorem 2.2, gives a lower bound on the size of $L(A_1, \ldots, A_n)$. For now we just state an interesting special case of this theorem, where all the $|A_i| = k$. While the bound itself is quite complex, the bound (surprisingly) turns out to be tight for every linear map L when m = n - 1.

Theorem 1.1. Let m < n, and let $L : \mathbb{F}_p^n \to \mathbb{F}_p^m$ be a linear map with rank m. Let v be a nonzero vector in $\ker(L)$ with minimal support, and let s be the size of its support. Let k be an integer with $p \ge 2k - 1$.

Then for every $A_1, \ldots, A_n \subseteq \mathbb{F}_p$, with $|A_i| = k$ for all $i \leq n$, we have:

$$|L(A_1, \dots, A_n)| \ge (k^s - (k-1)^s) \cdot k^{m-s+1}.$$

Some remarks about this theorem:

- If m = n 1 and $p \ge 2k 1$, this lower bound is optimal for every linear map L. See Lemma 2.3.
 - If m = n 1 and p < 2k 1, this lower bound can be violated for every linear map L.
- If our sets are taken to be subsets of \mathbb{R} instead of \mathbb{F}_p , then for m=n-1, an identical lower bound holds for every linear map $L:\mathbb{R}^n\to\mathbb{R}^m$, and this lower bound is optimal for every L. As in the case of the Cauchy Davenport theorem, the lower bound also has an elementary proof using the natural order on \mathbb{R} .
- If m is small, and k is large, then the lower bound is approximately $s \cdot k^m$.

Thus for the map $L^*: \mathbb{F}_p^3 \to \mathbb{F}_p^2$ mentioned above, if $p \geq 2k-1$, then for every three sets A_1, A_2, A_3 with $|A_i| = k$, we get that

$$|L^*(A_1, A_2, A_3)| \ge k^3 - (k-1)^3 = 3k^2 - 3k + 1,$$

and this is the best bound possible in term of k.

1.1 Proof Outline

Our proof is based on the Combinatorial Nullstellensatz [1], generalizing one of the known proofs of the Cauchy-Davenport theorem.

The Combinatorial Nullstellensatz is an algebraic statement characterizing multivariate polynomials $Q(Y_1,\ldots,Y_n)$ which vanish on a given product set $A_1\times\ldots\times A_n$ as those polynomials which lie in a certain explicitly given ideal. Let us recall the Combinatorial Nullstellensatz proof [2, 1] of the Cauchy-Davenport theorem. For given sets $A_1,A_2\subseteq \mathbb{F}_p$, one wants to prove a lower bound on the

size of the sumset $C = A_1 + A_2$. Suppose C was small. The key step of this proof is to consider the univariate polynomial $T(X) \in \mathbb{F}_p[X]$, given by:

$$T(X) = \prod_{c \in C} (X - c),$$

and the bivariate polynomial $Q(Y_1, Y_2) \in \mathbb{F}_p[Y_1, Y_2]$ given by:

$$Q(Y_1, Y_2) = T(Y_1 + Y_2) = \prod_{c \in C} (Y_1 + Y_2 - c).$$

Since C is small, T and Q are of low degree. By design, the polynomial Q vanishes on every point $(a_1,a_2)\in A_1\times A_2$. Thus, by the Combinatorial Null-stellensatz, one concludes that $Q(Y_1,Y_2)$ must lie in a certain ideal. Then, inspecting monomials and using the upper-triangular criterion for linear independence, one shows that no low-degree polynomial of the form $R(Y_1+Y_2)$ (with $R(X)\in \mathbb{F}_p[X]$) can lie this ideal. Since $Q(Y_1,Y_2)=T(Y_1+Y_2)$, this a contradiction.

Our proof will follow the same high-level strategy, but with some important differences. If $L(A_1, \ldots, A_n)$ is small, we will find a multivariate polynomial Q of low "complexity" which vanishes on $A_1 \times A_2 \times \ldots \times A_n$, and thus by the Combinatorial Nullstellensatz, it must lie in a certain ideal I. We then use some linear algebra arguments, along with the low complexity of Q, to show that Q cannot lie in I, thus deriving a contradiction.

There are two new technical ingredients that enter the proof. The first ingredient appears in the construction of the polynomial Q. Since the range of L is a high-dimensional vector space, there is no natural way of explictly giving a polynomial vanishing on $C = L(A_1, \ldots, A_n)$. Instead, we will use a dimension argument to show the existence of a suitable polynomial $T(X_1, \ldots, X_m)$ vanishing on C, and define $Q(Y_1, \ldots, Y_n)$ to be $T(L(Y_1, \ldots, Y_n))$. The second ingredient appears in the linear algebra argument showing that Q does not lie in I. In order to make this argument, we will need Q to have a very special kind of monomial structure. This monomial structure is enforced when we choose T; it is because of this requirement that we do not simply take T to be a low-degree polynomial, but instead choose T from a larger space of polynomials satisfying some constraints (this is what we have termed low complexity in the above description).

Organization of this paper In the next section we give a formal statement of our main result. In Section 3 we prove our main result. In Section 4 we discuss limitations of our methods to prove an optimal bound in the m < n-1 case. We conclude with some open problems.

Notation We use [n] to denote the set $\{1, 2, ..., n\}$. For a vector $v \in \mathbb{F}^n$, we define its support, denoted $\mathsf{supp}(v)$ to be the set of its nonzero coordinates, namely $\{i \in [n] \mid v_i \neq 0\}$. We use $\deg(h)$ to denote the total degree of a

polynomial h, and $\deg_Y(h)$ to denote the degree in the variable Y of the polynomial h. We say a monomial \mathcal{M} appears in a polynomial h if in the standard representation of h as a linear combination of monomials, \mathcal{M} has a nonzero coefficient.

2 The main result

We first state our main theorem. It gives, for every linear map $L: \mathbb{F}_p^n \to \mathbb{F}_p^m$, a lower bound on the size of $L(A_1, \ldots, A_n)$, in terms of the sizes of A_1, \ldots, A_n .

Definition 2.1. For a linear map $L: \mathbb{F}_p^n \to \mathbb{F}_p^m$, we define the support-kernel of L to be the set:

$$\mathsf{suppker}(L) = \{ S \subseteq [n] \mid \exists v \in \ker(L), v \neq 0, \ \textit{with} \ \mathsf{supp}(v) = S \}.$$

Theorem 2.2. Let p be prime. Let $n \ge 2$ be an integer. Let m < n.

Let $L: \mathbb{F}_p^n \to \mathbb{F}_p^m$ be a linear map of rank m. Let S be a minimal element of $\operatorname{suppker}(L)$. Let S' be a maximal subset of $[n] \setminus S$ such that $2^{S' \cup S} \cap \operatorname{suppker}(L) = \{S\}$.

Let $1 \leq k_1, \ldots, k_n \leq p$. Let $k_{\max} = \max_{i \in S} k_i$ and $k_{\min} = \min_{i \in S} k_i$. Suppose $p \geq k_{\max} + k_{\min} - 1$.

Define

$$\lambda = \left(\left(\prod_{i \in S} k_i \right) - \left(\prod_{i \in S} (k_i - 1) \right) \right) \cdot \left(\prod_{i \in S'} k_i \right).$$

Then for every $A_1, \ldots, A_n \subseteq \mathbb{F}_p$ with $|A_i| = k_i$, we have:

$$|L(A_1,\ldots,A_n)| \geq \lambda.$$

Taking all the k_i to equal k, and observing that S' has size m + 1 - s, we get the theorem stated in the introduction.

The following lemma shows that when m = n - 1, and $|A_1| = |A_2| = ... = |A_n|$, then the above lower bound is the best possible.

Lemma 2.3. Let p be prime. Let $n \ge 2$ be an integer. Let m = n - 1.

Let $L: \mathbb{F}_p^n \to \mathbb{F}_p^m$ be a linear map of rank m. Let $S \subseteq [n]$ be the unique element of $\operatorname{suppker}(L)$. Let $S' = [n] \backslash S$, and observe that $2^{S' \cup S} \cap \operatorname{suppker}(L) = S$. Let $k_1 = k_2 = \ldots = k_n = k$. Define

$$\lambda = \left(\left(\prod_{i \in S} k_i \right) - \left(\prod_{i \in S} (k_i - 1) \right) \right) \cdot \left(\prod_{i \in S'} k_i \right).$$

Then:

1. If $p \geq 2k-1$, there exist $A_1, \ldots, A_n \subseteq \mathbb{F}_p$ with $|A_i| = k_i$, such that:

$$|L(A_1,\ldots,A_n)|=\lambda.$$

2. If p < 2k - 1, there exist $A_1, \ldots, A_n \subseteq \mathbb{F}_p$ with $|A_i| = k_i$, such that:

$$|L(A_1,\ldots,A_n)|<\lambda.$$

3 Proof of the main theorem

For a linear map $L: \mathbb{F}_p^n \to \mathbb{F}_p^m$ and integers k_1, \dots, k_n , define:

$$\mu(L, k_1, \dots, k_n) \stackrel{\text{def}}{=} \min_{\substack{A_1, A_2, \dots, A_n \subseteq \mathbb{F}_p \\ |A_i| = k_i}} |L(A_1, \dots, A_n)|.$$

The proof of the main theorem, Theorem 2.2 has two steps. The first step performs elementary operations on the linear map L to bring it into a simple form, while preserving the value of $\mu(L, k_1, \ldots, k_n)$. The second step applies the polynomial method to give a lower bound on $\mu(L, k_1, \ldots, k_n)$ for these simple L. The allowable operations to simplify the linear map are listed in Lemma 3.1 and the lower bound for the simpler map is the subject of Theorem 3.2.

Lemma 3.1. Let $L: \mathbb{F}_p^n \to \mathbb{F}_p^m$ be a linear map, and let $1 \leq k_1, \ldots, k_n \leq p$.

- 1. Let $L': \mathbb{F}_p^m \to \mathbb{F}_p^m$ be a full rank linear transformation. Then $\mu(L, k_1, \dots, k_n) = \mu(L' \circ L, k_1, \dots, k_n)$.
- 2. Let $L'': \mathbb{F}_p^n \to \mathbb{F}_p^n$ be a linear map whose matrix is a diagonal matrix with all diagonal entries nonzero. Then $\mu(L, k_1, \ldots, k_n) = \mu(L \circ L'', k_1, \ldots, k_n)$.
- 3. Let $\pi:[n] \to [n]$ be a permutation. Let $L_{\pi}: \mathbb{F}_p^n \to \mathbb{F}_p^n$ be the linear map that permutes coordinates according to π (i.e.; $L_{\pi}(e_i) = e_{\pi(i)}$). Then $\mu(L, k_1, \ldots, k_n) = \mu(L \circ L_{\pi}, k_{\pi^{-1}(1)}, \ldots, k_{\pi^{-1}(n)})$.

Proof. 1. L' is an isomorphism, so

$$|L' \circ L(A_1, \ldots, A_n)| = |L(A_1, \ldots, A_n)|.$$

Taking the minimum over the choices of the sets A_i , $i \in [n]$, we get $\mu(L, k_1, \ldots, k_n) = \mu(L' \circ L, k_1, \ldots, k_n)$.

2. Applying L'' to (A_1, \ldots, A_n) simply scales the set A_i by a factor of $L''_{i,i}$. In particular, L'' preserves the sizes of the sets. So we have:

$$|L \circ L''(A_1, \ldots, A_n)| = |L(L''_{1,1}A_1, \ldots, L''_{n,n}A_n)| \ge \mu(L, k_1, \ldots, k_n).$$

Taking the minimum over the choices of the sets $A_i, i \in [n]$, we get $\mu(L \circ L'', k_1, \ldots, k_n) \ge \mu(L' \circ L, k_1, \ldots, k_n)$.

For the other direction, observe that any scaling is reversible by an inverse scaling:

$$|L(A_1,\ldots,A_n)| = |L \circ L''(\frac{1}{L''_{1,1}}A_1,\ldots,\frac{1}{L''_{n,n}}A_n)| \ge \mu(L \circ L'',k_1,\ldots,k_n).$$

Taking the minimum over the $A_i, i \in [n]$ gives the reverse inequality.

3. L_{π} permutes the indices of the sets, and so permutes the sizes of the sets. Taking this into account, the size of the image should remain the same:

$$|L \circ L_{\pi}(A_{\pi^{-1}(1)}, \dots, A_{\pi^{-1}(n)})| = |L(A_{1}, \dots, A_{n})| \ge \mu(L, k_{1}, \dots, k_{n}),$$

$$|L(A_{1}, \dots, A_{n})| = |L \circ L_{\pi}(A_{\pi^{-1}(1)}, \dots, A_{\pi^{-1}(n)})|$$

$$\ge \mu(L \circ L_{\pi}, k_{\pi^{-1}(1)}, \dots, k_{\pi^{-1}(n)}).$$

Taking the minimum over the $A_i, i \in [n]$ gives both directions of the inequality.

Theorem 3.2. Let p be prime. Let $m \ge 1$ be an integer.

Let $U_1, U_2, \ldots, U_m, V \subseteq F_p$ be subsets of size $|U_i| = k_i$ for $1 \le i \le m$, and $|V| = \hat{k}$. Suppose $p \ge \hat{k} + k_i - 1$ for each i. Let

$$C = \{(u_1 + v, u_2 + v, \dots, u_m + v) | u_i \in U_i \text{ for each } i, v \in V\}.$$

Then

$$|C| \ge \hat{k} \cdot \prod_{i=1}^{m} k_i - (\hat{k} - 1) \cdot \prod_{i=1}^{m} (k_i - 1).$$

Preliminaries: multivariate polynomials and Combi-3.1 natorial Nullstellensatz

In preparation for our proof of Theorem 3.2, we recall the statement of the Combinatorial Nullstellensatz, along with some important facts about reducing multivariate polynomials modulo ideals of the kind that arise in the Combinatorial Nullstellensatz.

Lemma 3.3 (Combinatorial Nullstellensatz [1]). Let \mathbb{F} be a field, and let $A_1, \ldots, A_n \subseteq$

F. For $i \in [n]$, let $P_i(T) \in \mathbb{F}[T]$ be given by $P_i(T) = \prod_{\alpha \in A_i} (T - \alpha)$.

Let $h(Y_1, \dots, Y_n) \in \mathbb{F}[Y_1, \dots, Y_n]$. Then $h(Y_1, \dots, Y_n)$ vanishes on $A_1 \times \dots \times A_n$ if and only if h lies in the ideal generated by $P_1(Y_1), P_2(Y_2), \dots, P_n(Y_n)$.

Now let $P_1(T), \ldots, P_n(T) \in \mathbb{F}[T]$ be polynomials, with $\deg(P_i) = k_i$. Let I be the ideal generated by $\langle P_i(Y_i) \rangle_{i \in [n]}$.

Given this setup, we now discuss the operation of reducing a polynomial $mod\ I$. A monomial $\prod_{i=1}^{n} Y_i^{e_i}$ is called legal for I if $e_i < k_i$ for each $i \in [n]$. Given a polynomial h, there is a canonical reduction mod I, denoted \overline{h} , with the property that $h \equiv \overline{h} \mod I$, and that every monomial appearing in the expansion of \overline{h} is legal for I (equivalently, for each i we have $\deg_{V_i}(\overline{h}) < k_i$). This canonical reduction can be obtained as follows. Reducing a polynomial mod $P_i(Y_i) = Y_i^{k_i} - \sum_{j=0}^{k_i-1} a_j Y_i^j$ is simply the act of repeatedly replacing every occurrence of $Y_i^{k_i}$ with $\sum_{j=0}^{k_i-1} a_j Y_i^j$, until the Y_i degree is less than k_i . Reducing

the polynomial $h \mod P_i(Y_i)$ in succession for each $i \in [n]$ gives the canonical reduction \overline{h} .

Here are some important (and easy to verify) points about canonical reduction:

- 1. $h \in I$ if and only if $\overline{h} = 0$.
- 2. The map $h \mapsto \overline{h}$ is \mathbb{F} -linear.

It will be important for us to understand the degrees of the monomials in \overline{h} . Let $\mathcal{M} = \prod_{i=1}^n Y_i^{e_i}$ be a monomial, and consider its reduction $\overline{\mathcal{M}}$ mod I. If

 $e_i < k_i$ for each $i \in [n]$, then we have $\overline{\mathcal{M}} = \mathcal{M}$. Furthermore, if there is some $e_i \geq k_i$, then $\deg \overline{\mathcal{M}} < \deg(\mathcal{M})$. This is because the act of replacing $Y_i^{k_i}$ with a lower degree polynomial in Y_i strictly decreases the degree. Combining these two facts, we get the following fact.

Fact 3.4. With notation as above, let $h(Y_1, ..., Y_n) \in \mathbb{F}[Y_1, ..., Y_n]$. Suppose \mathcal{M} is a monomial that (1) appears in h, (2) has $\deg(\mathcal{M}) = \deg(h)$, and (3) is legal for I.

Then \mathcal{M} appears in the canonical reduction \overline{h} .

This is because $\overline{\mathcal{M}} = \mathcal{M}$, and the canonical reductions of the other monomials will have smaller degree than \mathcal{M} , and will therefore leave \mathcal{M} untouched. Very similar considerations give us the following related fact.

Fact 3.5. With notation as above, let $h(Y_1, ..., Y_n) \in \mathbb{F}[Y_1, ..., Y_n]$. Suppose \mathcal{M} is a monomial that (1) appears in \overline{h} , (2) has $\deg(\mathcal{M}) = \deg(h)$, and (3) is legal for I.

Then \mathcal{M} appears in h.

3.2 Correlated sumsets and the polynomial method

We now prove Theorem 3.2.

Proof. We begin by defining some sets of monomials which will be useful to us. In the polynomial ring $\mathbb{F}_p[Y_1,\ldots,Y_m,Z]$, consider the following set of monomials:

$$\Gamma = \{ Y_1^{e_1} Y_2^{e_2} \cdots Y_m^{e_m} Z^e | 0 \le e_i \le k_i - 1 \text{ for each } i, \text{ and } 0 \le e \le \hat{k} - 1,$$
 and $e > 0 \Rightarrow e_i = k_i - 1 \text{ for some } i \}.$

We will also consider the polynomial ring $\mathbb{F}_p[X_1,\ldots,X_m]$. To each monomial $\mathcal{M}(Y_1,\ldots,Y_m,Z)\in\Gamma$, we associate a monomial $\phi(\mathcal{M})\in\mathbb{F}_p[X_1,\ldots,X_m]$ as

follows. If $\mathcal{M}(Y_1, \dots, Y_m, Z) = Y_1^{e_1} Y_2^{e_2} \cdots Y_m^{e_m} Z^e$, then define:

$$\phi(\mathcal{M}) = \begin{cases} \prod_{i=1}^{m} X_i^{e_i} & \text{if } e = 0\\ \left(\prod_{i=1}^{m} X_i^{e_i}\right) \cdot X_j^e & \text{if } e > 0, \text{ where } j \text{ is the first index} \\ & \text{so that } e_j = k_j - 1. \end{cases}$$

Let $\Delta = \{\phi(\mathcal{M}) \mid \mathcal{M} \in \Gamma\}$ be the set of all such monomials constructed in this way.

Note that ϕ is a bijection, and ϕ preserves the degree of each monomial. Thus, ϕ also gives a bijection when we restrict to monomials in Γ and Δ of fixed total degree. We defined ϕ so that ϕ^{-1} would have the following description: Let $X_1^{f_1}X_2^{f_2}\cdots X_m^{f_m}\in \Delta$. Let $e_i=\min\{f_i,k_i-1\}$ for each $i\in[m]$. Let $e=\sum_{i=1}^m f_i-\sum_{i=1}^m e_i$. Then

$$\phi^{-1}(X_1^{f_1}X_2^{f_2}\cdots X_m^{f_m}) = Y_1^{e_1}Y_2^{e_2}\cdots Y_m^{e_m}\cdot Z^e.$$

Note that by choice of e, ϕ^{-1} preserves degree.

With these definitions in hand, we proceed with the main parts of the proof.

Interpolating a polynomial

Suppose for contradiction that
$$|C| < \hat{k} \cdot \left(\prod_{i=1}^m k_i\right) - (\hat{k} - 1) \cdot \left(\prod_{i=1}^m (k_i - 1)\right)$$
.
Since $|\Delta| = |\Gamma| = \hat{k} \cdot \prod_{i=1}^m k_i - (\hat{k} - 1) \cdot \prod_{i=1}^m (k_i - 1)$, there is a non-zero polynomial $f(X_1, \ldots, X_m) = \sum_{K \in \Delta} c_K \mathcal{K}(X_1, \ldots, X_m)$ which vanishes on C . By the definition of C , this means that $g(Y_1, \ldots, Y_m, Z) \stackrel{\text{def}}{=} f(Y_1 + Z, \ldots, Y_m + Z)$ is a non-zero polynomial vanishing on every point $(u_1, u_2, \ldots, u_m, v) \in \prod_{i=1}^m U_i \times V$.

Application of the Combinatorial Nullstellensatz

For each
$$1 \le i \le m$$
, let $P_i(Y_i) = \prod_{a \in U_i} (Y_i - a)$. Also let $P(Z) = \prod_{a \in V} (Z - a)$.

By the Combinatorial Nullstellensatz,

$$g(Y_1, \ldots, Y_m, Z) \equiv 0 \pmod{I},$$

where I is the ideal generated by the $P_i(Y_i), i \in [m]$ and P(Z).

Explicitly, we have that:

$$\sum_{K \in \Lambda} c_K \mathcal{K}(Y_1 + Z, Y_2 + Z, \dots, Y_m + Z) \equiv 0 \pmod{I},$$

where at least one $c_{\mathcal{K}}$ is nonzero.

Consider the canonical reduction \overline{g} of $g \mod I$: since $g \in I$ we get that $\overline{g} = 0$. On the other hand, we have by linearity of canonical reduction:

$$\overline{g} = \sum_{\mathcal{K} \in \Delta} c_{\mathcal{K}} \overline{\mathcal{K}}(Y_1, Y_2, \dots, Y_m, Z),$$

where $\overline{\mathcal{K}}(Y_1, Y_2, \dots, Y_m, Z)$ is the canonical reduction mod I of $\mathcal{K}(Y_1 + Z, Y_2 + Z, \dots, Y_m + Z)$. By Fact 3.4, any monomial \mathcal{M} that appears in the expansion of $\mathcal{K}(Y_1 + Z, Y_2 + Z, \dots, Y_m + Z)$ with $\deg(\mathcal{M}) = \deg(\mathcal{K})$ and is legal for I, also appears in $\overline{\mathcal{K}}(Y_1, Y_2, \dots, Y_m, Z)$.

Arriving at a contradiction

We may now summarize the strategy for the rest of the proof. We will first find an ordering of the monomials in Δ such that:

- 1. If $\mathcal{K}, \mathcal{K}'$ are monomials in Δ with $\deg(\mathcal{K}') < \deg(\mathcal{K})$, then \mathcal{K}' is smaller than \mathcal{K} in the ordering.
- 2. For each $K \in \Delta$, there is some monomial $\mathcal{M}_{K}(Y_{1}, \ldots, Y_{m}, Z)$ with the following four properties:
 - (a) $\mathcal{M}_{\mathcal{K}}$ appears the expansion of $\mathcal{K}(Y_1 + Z, Y_2 + Z, \dots, Y_m + Z)$,
 - (b) $\deg(\mathcal{M}_{\mathcal{K}}) = \deg(\mathcal{K}),$
 - (c) $\mathcal{M}_{\mathcal{K}}$ is legal for I,
 - (d) $\mathcal{M}_{\mathcal{K}}$ does not appear in the expansion of $\mathcal{K}'(Y_1+Z,Y_2+Z,\ldots,Y_m+Z)$ for any $\mathcal{K}' \in \Delta$ smaller than \mathcal{K} in the ordering.

Once we have such an ordering, consider the largest \mathcal{K} in the ordering for which $c_{\mathcal{K}} \neq 0$. By Fact 3.4, $\mathcal{M}_{\mathcal{K}}$ appears in $\overline{\mathcal{K}}(Y_1, \ldots, Y_m, Z)$. For every other $\mathcal{K}' \in \Delta$ with $c_{\mathcal{K}'} \neq 0$, we will show that $\overline{\mathcal{K}'}(Y_1, \ldots, Y_m, Z)$ does not include the monomial $\mathcal{M}_{\mathcal{K}}$; this then shows that $\mathcal{M}_{\mathcal{K}}$ appears in \overline{g} with a nonzero coefficient, contradicting our equation $\overline{g} = 0$. This gives the desired contradiction.

Monomial $\mathcal{M}_{\mathcal{K}}$ does not appear in $\overline{\mathcal{K}'}$ (for $\mathcal{K}' \neq \mathcal{K}$ with $c_{\mathcal{K}'} \neq 0$)

Suppose $\mathcal{K}' \in \Delta$, $\mathcal{K}' \neq \mathcal{K}$ and $c_{\mathcal{K}'} \neq 0$. We will show that $\mathcal{M}_{\mathcal{K}}$ does not appear in $\overline{\mathcal{K}'}$. By choice of \mathcal{K} , we have that \mathcal{K}' is smaller than \mathcal{K} in the ordering, and hence that $\deg(\mathcal{K}') \leq \deg(\mathcal{K})$.

Suppose $\mathcal{M}_{\mathcal{K}}$ appeared in $\overline{\mathcal{K}'}$. Then the following chain of inequalities:

$$\deg(\mathcal{M}_{\mathcal{K}}) \leq \deg(\overline{\mathcal{K}'}) \leq \deg(\mathcal{K}'(Y_1 + Z, \dots, Y_m + Z) \leq \deg(\mathcal{K}') \leq \deg(\mathcal{K}) = \deg(\mathcal{M}_{\mathcal{K}}),$$

(because of the equality of the endpoints, this is a chain of equalities), shows that $\deg(\mathcal{M}_{\mathcal{K}}) = \deg(\mathcal{K}'(Y_1 + Z, \dots, Y_m + Z))$. Thus by Fact 3.5, we can conclude that $\mathcal{M}_{\mathcal{K}}$ appears in $\mathcal{K}'(Y_1 + Z, \dots, Y_m + Z)$. But this contradicts the property that $\mathcal{M}_{\mathcal{K}}$ does not appear in $\mathcal{K}'(Y_1 + Z, \dots, Y_m + Z)$ for any $\mathcal{K}' \in \Delta$ that is smaller than \mathcal{K} in the ordering. Thus $\mathcal{M}_{\mathcal{K}}$ cannot appear in $\overline{\mathcal{K}'}$.

The ordering of Δ

All that remains now is to define the ordering of Δ , and to prove the desired properties of this ordering.

Arrange the monomials in Δ in order of increasing total degree. Within each fixed total degree, order by decreasing $\deg_Z(\phi^{-1}(\mathcal{K}(X_1,\ldots,X_m)))$. Then for $\mathcal{K}(X_1,\ldots,X_m)\in\Delta$ in that ordering, set $\mathcal{M}_{\mathcal{K}}=\phi^{-1}(\mathcal{K}(X_1,\ldots,X_m))=Y_1^{e_1}Y_2^{e_2}\cdots Y_m^{e_m}Z^e$. We claim that $\mathcal{M}_{\mathcal{K}}$ satisfies the four properties listed above.

- (a) $\mathcal{M}_{\mathcal{K}}$ appears the expansion of $\mathcal{K}(Y_1+Z,Y_2+Z,\ldots,Y_m+Z)$: We show that the coefficient of $\mathcal{M}_{\mathcal{K}}$ in $\mathcal{K}(Y_1+Z,Y_2+Z,\ldots,Y_m+Z)$ is nonzero. By the definition of Δ , $\deg_{X_i}(\mathcal{K}) \leq \hat{k}+k_i-2 < p$ for $i \in [m], \mathcal{K} \in \Delta$. Also, there is at most one i such that $\deg_{X_i}(\mathcal{K}) > k_i-1$. Call this index j if it exists, and let $l = \deg_{X_j}(\mathcal{K})$. Since $\phi^{-1}(\mathcal{K})$ extracts the largest powers of Y_i in $\mathcal{K}(Y_1+Z,Y_2+Z,\ldots,Y_m+Z)$ up to k_i-1 for $i \in [m]$, we get that the coefficient of $\mathcal{M}_{\mathcal{K}}$ is 1 if j does not exist and $\binom{l}{k_j-1}$ if j exists. In both cases, the coefficient of $\mathcal{M}_{\mathcal{K}}$ is non-zero in F_p as l < p.
- (b) $\deg(\mathcal{M}_{\mathcal{K}}) = \deg(\mathcal{K})$: Recall that ϕ is a bijection from one set of monomials to another which preserves the degree of the monomials. So $\deg(\mathcal{M}_{\mathcal{K}}) = \deg \phi^{-1}(\mathcal{K}(X_1, \dots, X_m)) = \deg(\mathcal{K})$.
- (c) $\mathcal{M}_{\mathcal{K}}$ is legal for I:

Recall that writing $\mathcal{K} = X_1^{f_1} X_2^{f_2} \cdots X_m^{f_m}$, we have

$$\phi^{-1}(\mathcal{K}) = Y_1^{e_1} Y_2^{e_2} \cdots Y_m^{e_m} \cdot Z^e,$$

where $e_i = \min\{f_i, k_i - 1\}$ for each $i \in [m]$, and $e = \sum_{i=1}^m f_i - \sum_{i=1}^m e_i$. So

 $e_i \leq k_i - 1$, $\forall i \in [m]$. It remains to show that $e \leq \hat{k} - 1$. Suppose $f_i \leq k_i - 1$, $\forall i \in [m]$. Then $e_i = f_i$, $\forall i \in [m]$ and so e = 0. Otherwise, $f_i \leq k_i - 1$ for all but one $i \in [m]$, call this index j. We have $e_i = f_i$ for $i \neq j$ and $e_j = k_j - 1$. So

$$e = f_j - e_j \le \hat{k} + k_j - 2 - (k_j - 1) = \hat{k} - 1.$$

(d) $\mathcal{M}_{\mathcal{K}}$ does not appear in the expansion of $\mathcal{K}'(Y_1 + Z, Y_2 + Z, \dots, Y_m + Z)$ for any $\mathcal{K}' \in \Delta$ smaller than \mathcal{K} in the ordering:

To show that the monomials selected by ϕ^{-1} do not appear in any previous entries of the ordering, first note that the degree of $\mathcal{M}_{\mathcal{K}}$ is too large to have appeared in any previous $\mathcal{K}' \in \Delta$ of lower total degree. Next, consider the expansion of a previous $\mathcal{K}'(Y_1+Z,Y_2+Z,\ldots,Y_m+Z)$ in the ordering of the same total degree, then $\mathcal{M}_{\mathcal{K}'} = \phi^{-1}(\mathcal{K}'(X_1,\ldots,X_m)) = Y_1^{e'_1}Y_2^{e'_2}\cdots Y_m^{e'_n}Z^{e'}$ must have $e'_i < e_i \le k_i - 1$ for some $i \in [m]$, as $e' \ge e$. By the way ϕ is defined, this means that $\deg_{X_i}(\mathcal{K}') = e'_i$, so $\deg_{Y_i}(\mathcal{K}'(Y_1+Z,Y_2+Z,\ldots,Y_m+Z)) = e'_i$. But $\deg_{Y_i}(\mathcal{M}_{\mathcal{K}}) = e_i > e'_i$. So $\mathcal{M}_{\mathcal{K}}$ cannot be a monomial in the expansion of $\mathcal{K}'(Y_1+Z,Y_2+Z,\ldots,Y_m+Z)$.

This completes the proof that the ordering of Δ has the desired properties, and hence we arrive at a contradiction.

Thus we must have that
$$|C| \ge \hat{k} \cdot \left(\prod_{i=1}^{m} k_i\right) - (\hat{k} - 1) \cdot \left(\prod_{i=1}^{m} (k_i - 1)\right)$$
.

3.3 Proving the main result

We now combine Lemma 3.1 and Theorem 3.2 to prove our main theorem, Theorem 2.2.

Proof of Theorem 2.2:

By basic linear algebra, we have that $|S| \leq m+1$, and $|S \cup S'| = m+1$.

We first get rid of the coordinates in $[n] \setminus (S \cup S')$. Observe that taking away elements from any of the sets A_i cannot increase the size of the image $|L(A_1, \ldots, A_n)|$. Let $\mathbf{a} \in \prod_{i \in [n] \setminus (S \cup S')} A_i$. Fix the coordinates in $[n] \setminus (S \cup S')$ to

a and consider the resulting map $M: \mathbb{F}_p^{m+1} \to \mathbb{F}_p^m$ (i.e., $M(\mathbf{x}) = L(\mathbf{x}, \mathbf{a})$). If $L': \mathbb{F}_p^{m+1} \to \mathbb{F}_p^m$ is the linear map obtained by restricting the coordinates of

$$[n] \setminus (S \cup S')$$
 to 0, then the image $L'\left(\prod_{i \in S \cup S'} A_i\right)$ is a translate of the image of

$$M\left(\prod_{i\in S\cup S'}A_i\right)$$
. So we have:

$$|L(A_1,\ldots,A_n)| \ge \left| M\left(\prod_{i \in S \cup S'} A_i\right) \right| = L'\left(\prod_{i \in S \cup S'} A_i\right).$$

Then a lower bound on $L'\left(\prod_{i \in S \cup S'} A_i\right)$ gives a lower bound on $|L(A_1, \ldots, A_n)|$.

The next step is to use the simple transformations in Lemma 3.1 to greatly simplify our linear map L', while preserving $\mu(L, k_1, \ldots, k_n)$. The transformations allow us to apply elementary row operations on L', scale the columns of L', and rearrange the columns of L'.

As L' has rank m, $\ker(L')$ has rank 1. Consider a nonzero vector $v \in \ker(L')$. Then S must be the support of v. Let \hat{i} be the index in S that minimizes k_i , i.e. $\hat{i} = \arg\min_{i \in S} k_i$. With the above row and column operations at our disposal, we perform the following reduction of the problem. First, permute the columns so that the columns with indices in S are on the left and move column \hat{i} so that it is the first column. Then the last m columns are now linearly independent. This is because if they were linearly dependent, there would be a nonzero vector in the kernel of L whose support does not include \hat{i} . So there would be two nonzero vectors in $\ker(L)$ with different supports, which is impossible. Next, apply the sequence of elementary row operations that turns the last m columns into the identity matrix. Scale each row so that the first element is either 0 or 1. Finally, scale each of the last m columns so that they again form the identity matrix. We are left with a column of 1's and 0's followed by the m by m identity matrix. We will call this matrix \hat{L}' , the reduction of L'.

$$\hat{L}' = \left(egin{array}{cccc} 1 & & & & & \\ draingle & draingle & & & & \\ 1 & & & & & \\ 0 & & & & & \\ draingle & & & & \\ 1 & & & & & \\ 0 & & & & & \\ \end{array} \right).$$

Considering the projection P of the image of $\hat{L}'(\prod_{i \in S} A_i, \prod_{i \in S'} A_i)$ onto the first |S|-1 coordinates, we find ourselves in the setting of Theorem 3.2. Letting $U=A_{\hat{i}}$, and $\{V_1,\ldots,V_{|S|-1}\}=\{A_i\mid i\in S-\{\hat{i}\}\}$, Theorem 3.2 tells us that

$$|P| \ge \prod_{i \in S} k_i - \prod_{i \in S} (k_i - 1).$$

Finally, note that as \mathbf{a}' varies in the set $\prod_{i \in S'} A_i$, the sets $\hat{L}'(U, V_1, \dots, V_{|S|-1}, \mathbf{a}')$ are all translates of P and are disjoint (the disjointness follows from the fact that $\operatorname{suppker}(L) \cap 2^{S \cup S'} = \{S\}$). Hence, the total size of the image of \hat{L}' is at least $|P| \cdot \prod_{i \in S'} |A_i|$, which is at least:

$$\left(\left(\prod_{i \in S} k_i \right) - \left(\prod_{i \in S} (k_i - 1) \right) \right) \cdot \left(\prod_{i \in S'} k_i \right),\,$$

as desired.

Proof of Lemma 2.3:

We first provide a tight example for our lower bound when $p \geq 2k-1$. Using the same transformations as above, we produce the simple linear transformation \hat{L} from L. Lemma 3.1 implies that providing a tight example for \hat{L} implies the existence of a tight example for L. We claim that setting $A_i = \{0, \ldots, k_i - 1\}$ attains the smallest possible image size $\left(\prod_{i \in S} k_i - \prod_{i \in S} (k_i - 1)\right) \cdot \prod_{i \in S'} k_i$.

attains the smallest possible image size $\left(\prod_{i \in S} k_i - \prod_{i \in S} (k_i - 1)\right) \cdot \prod_{i \in S'} k_i$. As before, every choice of $\mathbf{a} \in \prod_{i \notin S} A_i$ yields |P| distinct points in the image of \hat{L} , where P is the projection of $\hat{L}(\prod_{i \in S} A_i, \prod_{i \in S'} A_i)$ onto the first |S| - 1 coordinates. So it suffices to show that $|P| \geq (\prod_{i \in S} k_i - \prod_{i \in S} (k_i - 1))$. This is equivalent to showing that equality is attained in Theorem 3.2 when the sets are all taken to be intervals starting from 0.

Suppose we have sets $U_i = \{0, \dots, k_i - 1\}, i \in [m] \text{ and } V = \{0, \dots, \hat{k} - 1\}.$ We want to show that $C = \{(u_1 + v, u_2 + v, \dots, u_m + v) | u_i \in U_i \text{ for each } i \in [m], v \in m\}$

V} has size exactly equal to
$$\hat{k} \cdot \prod_{i=1}^{m} k_i - (\hat{k} - 1) \cdot \prod_{i=1}^{m} (k_i - 1)$$
 as long as $p \ge \hat{k} + k_i - 1$.

In particular, this will give a tight example when the set sizes are all the same. Let $C_j = \{(u_1+j, u_2+j, \dots, u_m+j) | u_i \in U_i \text{ for each } i \in [m]\}, j=0,\dots,\hat{k}-1$

1. Then $C = \bigcup_{j=0}^{\hat{k}-1} C_j$. We start with $|C_0| = \prod_{j=0}^m k_j$, and ask how many additional elements we add when we take the union with C_1 :

$$|C_1 - C_0| = |C_1| - |C_1 \cap C_0|$$

= $\prod_{i=1}^m k_i - \prod_{i=1}^m (k_i - 1).$

Since $p \geq \hat{k} + k_i - 1$, none of the sums that we take exceed p - 1, so we

will continue to add
$$\prod_{i=1}^{m} k_i - \prod_{i=1}^{m} (k_i - 1)$$
 for each successive C_j . Total this gives $\prod_{i=1}^{m} k_i + (\hat{k} - 1) \cdot \left(\prod_{i=1}^{m} k_i - \prod_{i=1}^{m} (k_i - 1)\right)$, which is equal to $\hat{k} \cdot \prod_{i=1}^{m} k_i - (\hat{k} - 1) \cdot \prod_{i=1}^{m} (k_i - 1)$.

We now show that the lower bound is not tight when p < 2k - 1. In fact, the same example of taking the sets to be intervals will produce an image whose size is strictly smaller than our lower bound. Let $U_i = \{0, \dots, k-1\}, i \in [m]$ and $V = \{0, \dots, k-1\}$ in the statement of Theorem 3.2. We want to show that $C = \{(u_1 + v, u_2 + v, \dots, u_m + v) | u_i \in U_i \text{ for each } i \in [m], v \in V\}$ has size strictly less than $k^{m+1} - (k-1)^{m+1}$.

As before, let $C_j = \{(u_1 + j, u_2 + j, \dots, u_m + j) | u_i \in U_i \text{ for each } i \in [m]\}, j = 0$ $0, \ldots, k-1$. Then $C = \bigcup_{j=0}^{k-1} C_j$. Note that the element $(k-1+p-k+1, \ldots, k-1+p-k+1, k-1+p-k+1) = (0, \ldots, 0) \in C_{p-k+1}$ is in C_0 . But this was one of the "new" elements of C_{p-k+1} that we counted in the argument for the tight example, which was previously not in any of the C_i , for i . Hence,the number $k^{m+1} - (k-1)^{m+1}$ is a strict overcount for the number of elements in the image.

4 Linear maps of smaller rank

Our lower bound in the general case n > m-1 is not tight for every linear map. The main reason for this is that our proof strategy only uses information about the support of vectors in the kernel of L (and not the actual vectors). As the following example shows, if m < n-1 the optimal lower bound for $L(A_1, \ldots, A_n)$ may not be determined solely be the set of all supports of vectors in $\ker(L)$.

Example 4.1. Let p be a large prime, and let $k \ll p$. Consider the following 2×4 matrices over \mathbb{F}_p :

$$M = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix},$$

$$M' = \begin{bmatrix} 1 & 0 & 100 & 1 \\ 0 & 1 & 1 & 100 \end{bmatrix}.$$

Define $L: \mathbb{F}_p^4 \to \mathbb{F}_p^2$ and $L': \mathbb{F}_p^4 \to \mathbb{F}_p^2$ by L(x) = Mx and L'(x) = M'x. Observe that $\operatorname{suppker}(L)$ and $\operatorname{suppker}(L')$ both equal $\binom{[4]}{3}$.

Letting $A_1, A_2, A_3, A_4 = \{1, 2, ..., k\} \subseteq \mathbb{F}_p$, then $|L(A_1, A_2, A_3, A_4)| \le 16k^2$. In contrast, we will show in Lemma 4.3 that $|L'(A'_1, A'_2, A'_3, A'_4)| \ge 100k^2$, for any k-elements sets $A'_1, A'_2, A'_3, A'_4 \subseteq \mathbb{F}_p$,

Our analysis of this example will use some results on "sums of dilates". For a constant λ and a set A, we define the *dilate* λA denote the set $\{\lambda a \mid a \in A\}$. We will use the following result of Pontiveros [4] (which builds on a beautiful result of Bukh [3]) on sums of dilates in \mathbb{Z}_p .

Lemma 4.2. For every coprime $\lambda_1, \dots, \lambda_n \in \mathbb{Z}$, there exists a constant $\alpha > 0$ such that $|\lambda_1 X + \lambda_2 X + \dots + \lambda_n X| \ge (\sum \lambda_i) \cdot |X| - o(|X|)$, for sufficiently large prime p, and every $X \subseteq \mathbb{Z}_p$, with $|X| \le \alpha p$.

We use this estimate on the size of the sum of dilates, to construct linear maps with arbitrarily large image.

Lemma 4.3. For every positive integer constant c, there is a linear map L: $\mathbb{F}_p^4 \to \mathbb{F}_p^2$ such that for every $A_1, A_2, A_3, A_4 \subseteq \mathbb{F}_p$ with $|A_i| = k$, and any prime p sufficiently larger than k, we have:

$$L(A_1, A_2, A_3, A_4) \ge ck^2$$
.

Proof. Consider the linear map

$$L(A_1, A_2, A_3, A_4) = \{(a_1 + c \cdot a_3 + a_4, a_2 + a_3 + c \cdot a_4) | (a_1, a_2, a_3, a_4) \in A_1 \times A_2 \times A_3 \times A_4\}$$

and let $A_1, A_2, A_3, A_4 \subseteq F_p$ be any k-elements sets.

By Ruzsa triangle inequality [5],

$$|A_4 + c^2 A_4| \le \frac{|A_4 + cA_3| |cA_3 + c^2 A_4|}{|cA_3|} = \frac{|A_4 + cA_3| |A_3 + cA_4|}{|A_3|}$$

From Lemma 4.2 we know that $|A_4 + c^2 A_4| \ge c^2 |A_4|$, assuming p sufficiently larger than $k = |A_4|$.

Hence
$$|A_4 + cA_3| \cdot |A_3 + cA_4| \ge c^2 |A_4| |A_3| = c^2 k^2$$
.

Without loss of generality, assume that $|A_3 + cA_4| \ge ck$.

In particular, fixing $a_2 \in A_2$, and an element $(a_3 + c \cdot a_4) \in A_3 + cA_4$, the subset $\{(a_1 + (c \cdot a_3 + a_4), a_2 + a_3 + c \cdot a_4) | a_1 \in A_1\}$ has at least k elements, all with the same second coordinate. Therefore holding some element $a_2 \in A_2$ fixed, and letting $a_3 \in A_3$, $a_4 \in A_4$ be any elements, we obtain $|A_3 + cA_4|$ distinct second coordinates, and so

$$|\{(a_1+c\cdot a_3+a_4,a_2+a_3+c\cdot a_4)\}|a_1\in A_1,a_3\in A_3,a_4\in A_4\}|\geq ck^2$$

We conclude that
$$|L(A_1, A_2, A_3, A_4)| \ge ck^2$$
.

5 Questions

We conclude with some interesting open questions.

- 1. The main open question is to obtain the best bound for the Cauchy-Davenport problem for every linear map.
- 2. Even for the case m = n 1 and all the k_i equal to k, we do not know the optimal bound for the Cauchy-Davenport problem when p < 2k 1. Our method can be extended to give a better bound, but we believe that this is not the optimal bound.
- 3. What can be said about the "symmetric" Cauchy-Davenport problem: what is smallest possible size of L(A, A, ..., A) over all sets A with |A| = k? This seems to be closely related to the theory of sums of dilates.
- 4. Even over \mathbb{R} , finding the optimal bound for the Cauchy-Davenport problem for every linear map seems nontrivial.
- 5. It will be interesting to study analogues of other theorems of additive combinatorics in the setting of linear maps.

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