# Inverses of Bipartite Graphs

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#### Abstract

Let G be a bipartite graph and its adjacency matrix  $\mathbb{A}$ . If G has a unique perfect matching, then  $\mathbb{A}$  has an inverse  $\mathbb{A}^{-1}$  which is a symmetric integral matrix, and hence the adjacency matrix of a multigraph. The inverses of bipartite graphs with unique perfect matchings have a strong connection to Möbius functions of posets. In this note, we characterize all bipartite graphs with a unique perfect matching whose adjacency matrices have inverses diagonally similar to non-negative matrices, which settles an open problem of Godsil on inverses of bipartite graphs in [Godsil, Inverses of Trees, Combinatorica 5 (1985) 33-39].

#### 1 Introduction

Throughout the paper, a graph means a simple graph (no loops and parallel edges). If parallel edges and loops are allowed, we use multigraph instead. Let G be a bipartite graph with bipartition (R, C). The adjacency matrix  $\mathbb{A}$  of G is defined such that the ij-entry  $(\mathbb{A})_{ij} = 1$  if  $ij \in E(G)$ , and 0 otherwise. The bipartite adjacency matrix  $\mathbb{B}$  of G is defined as the ij-entry  $(\mathbb{B})_{ij} = (\mathbb{A})_{ij} = 1$  for  $i \in R$  and  $j \in C$ . So  $\mathbb{B}$  is an  $|R| \times |C|$ -matrix and

$$\mathbb{A} = \begin{bmatrix} 0 & \mathbb{B} \\ \mathbb{B}^{\mathsf{T}} & 0 \end{bmatrix}.$$

A perfect matching M of G is a set of disjoint edges covering all vertices of G. If a bipartite graph G has a perfect matching, then its bipartite adjacency matrix  $\mathbb{B}$  is a square matrix. Godsil proved that if a bipartite graph G has a unique perfect matching, then  $\mathbb{B}$  is similar to a lower triangular matrix with all diagonal entries equal to 1 by permuting rows and columns ([5], see also [15]). So in the following, we always assume that the bipartite adjacency matrix of a bipartite graph with a unique perfect matching is a lower triangular matrix. Clearly,  $\mathbb{B}$  is invertible and its inverse is an integral matrix (cf. [5, 17]). If  $\mathbb{B}^{-1}$  is non-negative (i.e. all entries are non-negative), then it is the bipartite adjacency matrix of another bipartite multigraph: the ij-entry is the number of edges joining the vertices i and j. However, the adjacency matrix of a graph G has a non-negative inverse if and only if the graph G is the disjoint union of  $K_2$ 's and  $K_1$ 's (cf. Lemma 1.1 in [13], and [9]).

The inverse of  $\mathbb{B}$  is diagonally similar to a non-negative integral matrix  $\mathbb{B}^+$  if there exists a diagonal matrix  $\mathbb{D}$  with -1 and 1 on its diagonal such that  $\mathbb{DB}^{-1}\mathbb{D} = \mathbb{B}^+$ . So  $\mathbb{B}^+$  is a bipartite adjacency matrix

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of a bipartite multigraph that is called the inverse of the bipartite graph G in [5] (a broad definition of graph inverse is given in the next section). The following is a problem raised by Godsil in [5] which is still open [6].

**Problem 1.1** (Godsil, [5]). Characterize the bipartite graphs with unique perfect matchings such that  $\mathbb{B}^{-1}$  is diagonally similar to a non-negative matrix.

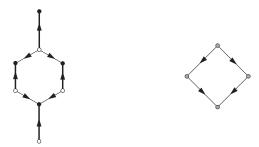


Figure 1: A bipartite graph with a unique perfect matching (left) and its corresponding digraph (right).

The bipartite graphs with unique perfect matchings are of particular interest because of the combinatorial interest of their inverses (cf. [5, 12]). Let G be a bipartite graph with a unique perfect matching M, and (R, C) be the bipartition of G. Let D be the digraph obtained from G by orienting all edges from G to G and then contracting all edges in G. Simion and Cao proved that the digraph G is acyclic ([15]). For example, see Figure 1. The acyclic digraph G corresponds to a poset G, such that for G, G, and G is a directed path from G is an G in G if and only if G is defined as follows (cf. Chapter 4 in [1])

$$(\mathbb{Z})_{ij} := \begin{cases} 1 & \text{if } a_i \le a_j; \\ 0 & \text{otherwise,} \end{cases}$$

The modified Zeta matrix  $\mathbb{Z}(x)$  of  $\mathcal{P}$  is obtained by replacing the entry 1 by a variable x for a comparable pair of  $\mathcal{P}$  which is not an arc of the digraph D. Then  $\mathbb{Z}(1) = \mathbb{Z}$  the Zeta matrix of  $\mathcal{P}$ , and  $\mathbb{Z}(0) = \mathbb{B}$  the bipartite adjacency matrix of G. Note that  $\mathbb{Z}(0)$  is the adjacency matrix of D and  $\mathbb{Z}(1)$  is the adjacency matrix of the transitive closure of D. The Möbius function on the interval of  $[a_i, a_j]$  in  $\mathcal{P}$  is  $\mu(a_i, a_j) = (\mathbb{Z}^{-1})_{ij}$  (see Ex. 22 in Chapter 2 of Lovász on page 216 in [11]), and  $\mathbb{Z}^{-1}$  is the Möbius matrix of  $(\mathcal{P}, \leq)$ . On the other hand, the Zeta matrix of a poset  $(\mathcal{P}, \leq)$  is a lower triangular matrix, corresponding to a bipartite adjacency matrix of a bipartite graph with a unique perfect matching. This sets up a connection between inverses of bipartite graphs with unique perfect matchings and Möbius functions of posets.

As observed in [5], if  $(\mathcal{P}, \leq)$  is a geometric lattice (a finite matroid lattice [16]) or the face-lattice of a convex polytope [2]), then the Möbius matrix of  $\mathcal{P}$  is diagonally similar to a non-negative matrix (cf. Corollary 4.34 in [1]). Godsil [5] proved that if G is a tree with a perfect matching, then the inverse of its adjacency matrix is diagonally similar to a non-negative matrix. Further, it has been observed that if G and H are two bipartite graphs with the property stated in Problem 1.1, then the Kronecker product  $G \otimes H$  is again a bipartite graph with the property [5]. The following is a partial solution to Problem 1.1.

**Theorem 1.2** (Godsil, [5]). Let G be a bipartite graph with a unique perfect matching M such that G/M is bipartite. Then  $\mathbb{B}^{-1}$  is diagonally similar to a non-negative matrix.

Godsil's result was generalized to weighted bipartite graphs with unique perfect matchings by Panda and Pati in [14]. In this paper, we provide a solution to Problem 1.1 as follows.

**Theorem 1.3.** Let G be a bipartite graph with a unique perfect matching M. Then  $\mathbb{B}^{-1}$  is diagonally similar to a non-negative matrix if and only if G does not contain an odd flower as a subgraph.

To define odd flower, we need more notation. Let G be a bipartite multigraph with a perfect matching M. A path P of G is M-alternating if  $E(P) \cap M$  is a perfect matching of P. For two vertices i and j of G, let  $\tau(i,j)$  be the number of M-alternating paths of G joining i and j. Further, let  $\tau_o(i,j)$  be the number of M-alternating paths P of G joining i and j such that  $|E(P) \setminus M|$  is odd, and  $\tau_e(i,j)$  be the number of M-alternating paths P joining i and j such that  $|E(P) \setminus M|$  is even. For a subset  $S = \{x_1, x_2, ...x_k\}$  of V(G), the M-span of S is defined as a subgraph of G consisting of all M-alternating paths joining  $x_i$  and  $x_j$  for any  $x_i, x_j \in S$ , denoted by  $\operatorname{Span}_M(S)$ . An M-span is called a flower if the vertices of S can be ordered such that  $\tau_o(x_i, x_j) \neq \tau_e(x_i, x_j)$  if and only if  $|i - j| \equiv 1 \pmod{k}$ . A flower is odd if there is an odd number of vertex pairs  $\{x_i, x_j\}$  with  $\tau_o(x_i, x_j) > \tau_e(x_i, x_j)$ . For example, see Figure 2. In Section 4, it will be shown that the existence of an odd flower means simply that the vertices in S induce a cycle with an odd number of negative edges in the inverse of G.

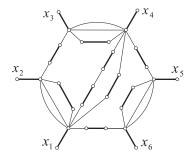


Figure 2: An odd flower: thick edges form a perfect matching M.

### 2 Inverses of weighted graphs

A weighted multigraph (G, w) is a multigraph with a weight-function  $w : E(G) \to \mathbb{F} \setminus \{0\}$  where  $\mathbb{F}$  is a field. We always assume that a weighted multigraph has no parallel edges since all parallel edges  $e_1, ..., e_k$  joining a pair of vertices i and j can be replaced by one edge ij with weight  $w(ij) = \sum w(e_i)$ . The adjacency matrix of a weighted multigraph (G, w), denoted by  $\mathbb{A}_w$ , is defined as

$$(\mathbb{A}_w)_{ij} := \begin{cases} w(ij) & \text{if } ij \in E(G); \\ 0 & \text{otherwise} \end{cases}$$

where loops, with  $w(ii) \neq 0$ , are allowed. A weighted multigraph (G, w) is invertible over  $\mathbb{F}$  if its adjacency matrix  $\mathbb{A}_w$  is invertible over  $\mathbb{F}$ . Note that  $\mathbb{A}_w$  is a symmetric matrix. Its inverse  $\mathbb{A}_w^{-1}$  is also symmetric and therefore is the adjacency matrix of some weighted graph, which is called the inverse of (G, w). The inverse of (G, w) is defined as a weighted graph  $(G^{-1}, w^{-1})$  whose vertex set is  $V(G^{-1}) = V(G)$  and whose edge set is  $E(G^{-1}) = \{ij \mid (\mathbb{A}_w^{-1})_{ij} \neq 0\}$ , and whose weight function is  $w^{-1}(ij) = (\mathbb{A}_w^{-1})_{ij}$ . Note that this definition of graph inverse is different from the definitions given in [5] and [12].

Let G be a graph. A Sachs subgraph of G is a spanning subgraph with only copies of  $K_2$  and cycles (including loops) as components. For example, a perfect matching M of G is a Sachs subgraph. For convenience, a Sachs subgraph is denoted by  $S = \mathcal{C} \cup M$  where  $\mathcal{C}$  consists of the cycles of S (including loops), and M consists of all components of S isomorphic to  $K_2$ . The following result shows how to compute the determinant of the adjacency matrix of a graph.

**Theorem 2.1** (Harary, [8]). Let G be a graph and  $\mathbb{A}$  be the adjacency matrix of G. Then

$$\det(\mathbb{A}) = \sum_{S} 2^{|\mathcal{C}|} (-1)^{|\mathcal{C}| + |E(S)|},$$

where  $S = \mathcal{C} \cup M$  is a Sachs subgraph.

If G is a bipartite graph with a Sachs subgraph  $S = \mathcal{C} \cup M$ , then every cycle C in  $\mathcal{C}$  is of even size and hence its edge set can be decomposed into two disjoint perfect matchings of C. Therefore, G has at least  $2^{|\mathcal{C}|}$  perfect matchings. So if G is a bipartite graph with a unique perfect matching M, then M is the unique Sachs subgraph of G. Hence we have the following corollary of the above result, which can also be derived easily from a result of Godsil (Lemma 2.1 in [5]).

Corollary 2.2. Let G be a bipartite graph with a unique perfect matching M. Then

$$\det(\mathbb{A}) = (-1)^{|M|},$$

where  $\mathbb{A}$  is the adjacency matrix of G.

By Corollary 2.2, the determinant of the adjacency matrix of a bipartite graph G with a unique perfect matching is either 1 or -1. So a bipartite graph with a unique perfect matching is always invertible. The inverse of a graph can be characterized in terms of its Sachs subgraphs as shown in the following theorem, which was originally proved in [17]. However, to make the paper self-contained, we include the proof here as well.

**Theorem 2.3** ([17]). Let G be a graph with adjacency matrix  $\mathbb{A}$ , and

 $\mathcal{P}_{ij} = \{P | P \text{ is a path joining } i \text{ and } j \neq i \text{ such that } G \setminus V(P) \text{ has a Sachs subgraph } S\}.$ 

If G has an inverse  $(G^{-1}, w)$ , then

$$w(ij) = \begin{cases} \frac{1}{\det(\mathbb{A})} \sum_{P \in \mathcal{P}_{ij}} \sum_{S} 2^{|\mathcal{C}|} (-1)^{|\mathcal{C}| + |E(S) \cup E(P)|} & \text{if } i \neq j; \\ \frac{1}{\det(\mathbb{A})} \det(\mathbb{A}_{i,i}) & \text{otherwise} \end{cases}$$

where  $S = \mathcal{C} \cup M$  is a Sachs subgraph of  $G \setminus V(P)$  and  $\mathbb{A}_{i,i}$  is the matrix obtained by deleting i-th row and i-th column from  $\mathbb{A}$ .

*Proof.* Let G be an invertible graph and  $(G^{-1}, w)$  be its inverse. Assume G has n vertices and  $V(G) = \{1, 2, ..., n\}$ . By the definition of the inverse of a graph,  $w(ij) = (\mathbb{A}^{-1})_{ij}$ .

Note that  $\mathbb{A}$  is symmetric and hence  $\mathbb{A}^{-1}$  is also symmetric. By Cramer's rule,

$$(\mathbb{A}^{-1})_{ij} = (\mathbb{A}^{-1})_{ji} = \frac{c_{ij}}{\det(\mathbb{A})}$$

where  $c_{ij} = (-1)^{i+j} \det(\mathbb{A}_{i,j})$  where  $\mathbb{A}_{i,j}$  is the matrix obtained from  $\mathbb{A}$  by deleting *i*-th row and *j*-th column. Let  $\mathbb{M}_{i,j}$  be the matrix obtained from  $\mathbb{A}$  by replacing the (i,j)-entry by 1 and all other entries in the *i*-th row and *j*-th column by 0. Then by the Laplace expansion,  $c_{ij} = \det(\mathbb{M}_{i,j})$ 

If i = j, then  $\det(\mathbb{M}_{i,i}) = \det(\mathbb{A}_{i,i})$ . So  $w(ii) = (\mathbb{A}^{-1})_{ii} = \frac{c_{ii}}{\det(\mathbb{A})} = \frac{\det(\mathbb{M}_{i,i})}{\det(\mathbb{A})} = \frac{\det(\mathbb{A}_{i,i})}{\det(\mathbb{A})}$ . So the theorem holds for i = j. In the following, assume that  $i \neq j$ .

Let  $m_{kl}$  be the (k,l)-entry of  $\mathbb{M}_{i,j}$ . Recall that the Leibniz formula for the determinant of  $\mathbb{M}_{i,j}$  is

$$\det(\mathbb{M}_{i,j}) = \sum_{\pi \in S_n} \operatorname{sgn}(\pi) \prod m_{k\pi(k)},$$

where the sum is computed over all permutations  $\pi$  of the set  $V(G) := \{1, 2, ..., n\}$ . Since all (i, l)-entries  $(l \neq j)$  of  $\mathbb{M}_{i,j}$  are equal to 0 but the (i, j)-entry is 1, only permutations  $\pi$  such that  $\pi(i) = j$  contribute to the determinant of  $\mathbb{M}_{i,j}$ . Let  $\Pi_{i\to j}$  be the family of all permutations on  $V(G) = \{1, 2, ..., n\}$  such that  $\pi(i) = j$ . Denote the cycle of  $\pi$  permuting i to j by  $\pi_{ij}$ . For convenience,  $\pi_{ij}$  is also used to denote the set of vertices which corresponds to the elements in the permutation cycle  $\pi_{ij}$ , for example,  $V(G) \setminus \pi_{ij}$  denotes the set of vertices in V(G) but not in  $\pi_{ij}$ . Denote the permutation of  $\pi$  restricted on  $V(G) \setminus \pi_{ij}$  by  $\pi \setminus \pi_{ij}$ . Then

$$\det(\mathbb{M}_{i,j}) = \sum_{\pi \in \Pi_{i \to j}} \operatorname{sgn}(\pi) \prod_{k \in V(G) \setminus \{i\}} m_{k\pi(k)}$$
$$= \sum_{\pi \in \Pi_{i \to j}} \left( \operatorname{sgn}(\pi_{ij}) \prod_{k \in \pi_{ij} \setminus \{i\}} m_{k\pi(k)} \right) \left( \operatorname{sgn}(\pi \setminus \pi_{ij}) \prod_{k \in V(G) \setminus \pi_{ij}} m_{k\pi(k)} \right).$$

By the definition of  $\mathbb{M}_{i,j}$ , if  $k \neq i$  or  $l \neq j$ , then  $m_{kl} = (\mathbb{A})_{kl}$ , the (k,l)-entry of  $\mathbb{A}$ .

If the permutation cycle  $\pi_{ij}$  does not correspond to a cycle of G, then for some  $k \in \pi_{ij}$ ,  $k\pi(k)$  is not an edge of G and hence  $m_{k\pi(k)} = 0$ . So  $\operatorname{sgn}(\pi_{ij}) \prod_{k \in \pi_{ij} \setminus \{i\}} m_{k\pi(k)} = 0$ . If the permutation cycle  $\pi_{ij}$  does correspond to a cycle in the graph G, let P be the path from j to i following the permutation order in  $\pi_{ij}$ . Then  $\operatorname{sgn}(\pi_{ij}) \prod_{k \in \pi_{ij} \setminus \{i\}} m_{k\pi(k)} = (-1)^{|E(P)|}$ . Note that  $\operatorname{sgn}(\pi \setminus \pi_{ij}) \prod_{k \in V(G) \setminus \pi} m_{k\pi(k)}$  is the determinant of the adjacency matrix of the graph  $G \setminus V(P)$ . By Theorem 2.1, it follows that

$$\operatorname{sgn}(\pi \backslash \pi_{ij}) \prod_{k \in V(G) \backslash \pi} m_{k\pi(k)} = \sum_{S} 2^{|\mathcal{C}|} (-1)^{|\mathcal{C}| + |E(S)|},$$

where  $S = \mathcal{C} \cup M$  is a Sachs subgraph of  $G \setminus V(P)$ . For the case that  $G \setminus V(P)$  has no Sachs subgraphs, then  $\left(\operatorname{sgn}(\pi_{ij})\prod_{k \in \pi_{ij} \setminus \{i\}} m_{k,\pi(k)}\right) \left(\operatorname{sgn}(\pi \setminus \pi_{ij})\prod_{k \in V(G) \setminus \pi_{ij}} m_{k,\pi(k)}\right) = 0$ . Hence,

$$\det(\mathbb{M}_{i,j}) = \sum_{P \in \mathcal{P}_{ij}} (-1)^{|E(P)|} \big(\sum_{S} 2^{|\mathcal{C}|} (-1)^{|\mathcal{C}| + |E(S)|} \big) = \sum_{P \in \mathcal{P}_{ij}} \sum_{S} 2^{|\mathcal{C}|} (-1)^{|\mathcal{C}| + |E(S) \cup E(P)|},$$

where  $S = \mathcal{C} \cup M$  is a Sachs subgraph of  $G \setminus V(P)$ . The theorem follows immediately from  $w(ij) = \frac{\det(\mathbb{M}_{i,j})}{\det(\mathbb{A})}$ . This completes the proof.

For a bipartite graph G with a unique perfect matching, the weight function of its inverse  $(G^{-1}, w)$  can be simplified as shown below.

**Theorem 2.4.** Let G be a bipartite graph with a unique perfect matching M, and let

$$\mathcal{P}_{ij} = \{P | P \text{ is an } M \text{-alternating path joining } i \text{ and } j\}.$$

Then G has an inverse  $(G^{-1}, w)$  such that

$$w(ij) = \begin{cases} \sum_{P \in \mathcal{P}_{ij}} (-1)^{|E(P)\backslash M|} & \text{if } i \neq j; \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* Let G be a bipartite graph with a unique perfect matching M. By Corollary 2.2, G has an inverse which is a weighted graph  $(G^{-1}, w)$ .

For any two vertices i and j, let P be a path joining i and j.

**Claim:**  $G \setminus V(P)$  has a Sachs subgraph if and only if P is an M-alternating path.

*Proof of Claim:* If P is an M-alternating path, then  $G \setminus V(P)$  has a perfect matching. So  $G \setminus V(P)$  has a Sachs subgraph.

Now assume that  $G\backslash V(P)$  has a Sachs subgraph. Note that  $G\backslash V(P)$  is a bipartite graph. Every cycle of a Sachs subgraph of  $G\backslash V(P)$  is of even size. So  $G\backslash V(P)$  has a perfect matching M'. Therefore, P is a path with even number of vertices and has a perfect matching M''. Hence  $M'\cup M''$  is a perfect matching of G. Since G has a unique perfect matching, it follows that  $M=M'\cup M''$ . So P is an M-alternating path. This completes the proof of Claim.

Let P be a path in  $\mathcal{P}_{ij}$ . Then  $G\backslash V(P)$  has a unique perfect matching  $M\backslash E(P)$ , which is also its unique Sachs subgraph. By Claim and Theorem 2.3, for  $i \neq j$ , we have

$$w(ij) = (-1)^{|M|} \sum_{P \in \mathcal{P}_{ij}} (-1)^{|(M \setminus E(P)) \cup E(P)|} = \sum_{P \in \mathcal{P}_{ij}} (-1)^{|E(P) \setminus M|}.$$

If i = j, then  $G \setminus \{i\}$  has no perfect matching and hence no Sachs subgraph. By Theorem 2.1,  $\det(\mathbb{A}_{i,i}) = 0$ . By Theorem 2.3, it follows that w(ii) = 0. This completes the proof.

## 3 Balanced weighted graphs

Let (G, w) be a weighted graph. An edge ij of a weighted graph (G, w) is positive if w(ij) > 0 and negative if w(ij) < 0. A cycle C of (G, w) is negative if  $w(C) = \prod_{ij \in E(C)} w(ij) < 0$ . A signed graph  $(G, \sigma)$  is a special weighted graph with a weight function  $\sigma : E(G) \to \{-1, +1\}$ , where  $\sigma$  is called the signature of G (see [7]). Signed graphs are well-studied combinatorial structures due to their applications in combinatorics, geometry and matroid theory (cf. [18, 20]).

A switching function of a weighted graph (G, w) is a function  $\zeta : V(G) \to \{-1, +1\}$ , and the switched weight-function of w defined by  $\zeta$  is  $w^{\zeta}(ij) := \zeta(i)w(ij)\zeta(j)$ . Two weight-functions  $w_1$  and  $w_2$  of a graph G are equivalent to each other if there exists a switching function  $\zeta$  such that  $w_1 = w_2^{\zeta}$ . A weighted graph (G, w) is balanced if there exists a switching function  $\zeta$  such that  $w^{\zeta}(ij) > 0$  for any edge  $ij \in E(G)$ . The following is a characterization of balanced signed graphs obtained by Harary [7].

**Proposition 3.1** ([7]). Let  $(G, \sigma)$  be a signed graph. Then  $(G, \sigma)$  is balanced if and only if V(G) has a bipartition  $V_1$  and  $V_2$  such that  $E(V_1, V_2) = \{e \mid e \in E(G) \text{ and } \sigma(e) = -1\}.$ 

For a weighted graph (G, w), define a signed graph  $(G, \sigma)$  such that  $\sigma(ij)w(ij) > 0$  for any edge  $ij \in E(G)$ . Then (G, w) is balanced if and only if  $(G, \sigma)$  is balanced. Therefore, the above result can be easily extended to weighted graphs (G, w) as follows.

**Proposition 3.2.** Let (G, w) be a weighted graph. Then (G, w) is balanced if and only if V(G) has a bipartition  $V_1$  and  $V_2$  such that  $E(V_1, V_2) = \{e \mid e \in E(G) \text{ and } w(e) < 0\}.$ 

**Remark.** Let (G, w) be a weighted graph such that G is connected, and let  $E^+ := \{e \mid w(e) > 0\}$ . Let  $G/E^+$  be the graph obtained from G by contracting all edges in  $E^+$  and deleting all loops. Then by Theorem 3.2, (G, w) is balanced if and only if  $G/E^+$  is a bipartite multigraph. Therefore, it takes O(m) steps to determine whether a weighted graph is balanced or not, where m is the total number of edges of G.

A direct corollary of the above theorem is the following result.

**Corollary 3.3.** Let (G, w) be a weighted graph. Then (G, w) is balanced if and only if it does not contain a negative cycle.

Let (G, w) be a weighted graph and  $\mathbb{A}_w$  be its adjacency matrix. For a switching function  $\zeta : V(G) \to \{-1, +1\}$ , define  $\mathbb{D}_{\zeta}$  to be a diagonal matrix with  $(\mathbb{D}_{\zeta})_{ii} = \zeta(i)$ . Then  $(G, w_1)$  is equivalent to  $(G, w_2)$  if and only if  $\mathbb{A}_{w_1} = \mathbb{D}_{\zeta} \mathbb{A}_{w_2} \mathbb{D}_{\zeta}$  for some switching function  $\zeta$ . So the adjacency matrices of two equivalent weighted graphs are diagonally similar to each other.

**Lemma 3.4.** Let G be a bipartite graph with a unique perfect matching M. Then  $\mathbb{B}^{-1}$  is diagonally similar to a non-negative matrix if and only if the inverse of G is a balanced weighted graph.

*Proof.* Since G is invertible, let  $(G^{-1}, w)$  be the inverse of G by Theorem 2.4. Let  $\mathbb{A}$  be the adjacency matrix of G such that

$$\mathbb{A} = \begin{bmatrix} 0 & \mathbb{B} \\ \mathbb{B}^{\mathsf{T}} & 0 \end{bmatrix},$$

where  $\mathbb{B}$  is the bipartite adjacency matrix of G, which we assume without loss of generality to be a lower triangular matrix with 1 on the diagonal. Then the inverse of  $\mathbb{A}$  is the adjacency matrix of  $(G^{-1}, w)$  as follows,

$$\mathbb{A}^{-1} = \begin{bmatrix} 0 & (\mathbb{B}^{\mathsf{T}})^{-1} \\ \mathbb{B}^{-1} & 0 \end{bmatrix}.$$

Note that  $\mathbb{B}^{-1}$  is diagonally similar to a non-negative matrix if and only if  $\mathbb{A}^{-1}$  is diagonally similar to a non-negative matrix. In other words, if and only if there exists a diagonal matrix  $\mathbb{D}$  with  $(\mathbb{D})_{ii} \in \{-1, +1\}$  such that  $\mathbb{D}\mathbb{A}^{-1}\mathbb{D}$  is non-negative. Define a switching function  $\zeta: V \to \{-1, +1\}$  such that  $\zeta(i) = (\mathbb{D}_{ii})$ . Note that

$$w^{\zeta}(ij) = \zeta(i)w(ij)\zeta(j) = \zeta(i)(\mathbb{A}^{-1})_{ij}\zeta(j) = (\mathbb{D}\mathbb{A}^{-1}\mathbb{D})_{ij}.$$

Hence  $\mathbb{A}^{-1}$  is diagonally similar to a non-negative matrix if and only if there exists a switching function  $\zeta$  such that  $w^{\zeta}: E(G^{-1}) \to \mathbb{R}^+$ . Let  $V_1 = \{v \in V \mid \zeta(v) = 1\}$  and  $V_2 = \{v \in V \mid \zeta(v) = -1\}$ . So the existence of the switching function  $\zeta$  is equivalent to the existence of a bipartition  $V_1$  and  $V_2$  of V such that  $E(V_1, V_2) = \{e \mid w(e) < 0\}$ . By Proposition 3.2, it follows that  $\mathbb{B}^{-1}$  is diagonally similar to a non-negative matrix if and only if  $(G^{-1}, w)$  is balanced.

By Lemma 3.4, Godsil's problem is equivalent to ask which bipartite graphs with unique perfect matchings have a balanced weighted graph as its inverse.

#### 4 Proof of Theorem 1.3

Now, we are ready to prove our main result.

**Theorem 1.3.** Let G be a bipartite graph with a unique perfect matching M. Then  $\mathbb{B}^{-1}$  is diagonally similar to a non-negative matrix if and only if G does not contain an odd flower as a subgraph.

*Proof.* Let G be a bipartite graph with a unique perfect matching M and  $\mathbb{B}$  the bipartite adjacency matrix of G. For any two vertices i and j of G, let

$$\mathcal{P}_{ij} = \{P \mid P \text{ is an } M\text{-alternating path joining } i \text{ and } j\}.$$

 $\Rightarrow$ : Assume that  $\mathbb{B}^{-1}$  is diagonally similar to a non-negative matrix. We need to show that G does not contain an odd flower. Suppose on the contrary that G does contain a vertex subset  $S = \{x_1, ..., x_k\}$  such that  $\operatorname{Span}_M(S)$  is an odd flower. Then all paths in  $\mathcal{P}_{x_i x_{i+1}}$  belong to  $\operatorname{Span}_M(S)$ . By Theorem 2.4, G has an inverse  $(G^{-1}, w)$  where,

$$w(x_i x_{i+1}) = \sum_{P \in \mathcal{P}_{x_i x_{i+1}}} (-1)^{|E(P) \setminus M|}.$$

So  $w(x_ix_{i+1}) \in \mathbb{Z}\setminus\{0\}$  and  $w(x_ix_{i+1}) < 0$  if and only if  $\tau_o(x_i, x_{i+1}) > \tau_e(x_i, x_{i+1})$ . Note that  $\operatorname{Span}_M(S)$  is an odd flower. So  $C = x_1 \cdots x_k x_1$  is a negative cycle in  $(G^{-1}, w)$ . By Corollary 3.3,  $(G^{-1}, w)$  is not balanced. Hence  $\mathbb{B}^{-1}$  is not diagonally similar to a non-negative matrix by Lemma 3.4, a contradiction.

 $\Leftarrow$ : Assume that G does not contain an odd flower as a subgraph. We need to show that  $\mathbb{B}^{-1}$  is diagonally similar to a non-negative matrix. Suppose on the contrary that  $\mathbb{B}^{-1}$  is not diagonally similar to a non-negative matrix. Then by Lemma 3.4, its inverse  $(G^{-1}, w)$  is not balanced, and hence contains a negative cycle by Corollary 3.3. Choose a shortest negative cycle  $C := x_k x_1 \cdots x_k$  (i.e., k is as small as possible). Then  $w(x_i x_{i+1}) \neq 0$  as  $x_i x_{i+1}$  is an edge of  $G^{-1}$  (subscripts modulo k). Hence  $\tau_o(x_i, x_{i+1}) \neq \tau_e(x_i, x_{i+1})$  (subscripts modulo k). Let  $S = \{x_1, ..., x_k\}$ . In the following, we are going to prove  $\mathrm{Span}_M(S)$  is an odd flower.

Since C is a smallest negative cycle of  $(G^{-1}, w)$ , it follows that C has no chord, which implies that  $\tau_o(x_i, x_j) = \tau_e(x_i, x_j)$  if  $x_i$  and  $x_j$  are not consecutive on C. In other words,  $\tau_o(x_i, x_j) \neq \tau_e(x_i, x_j)$  if and only if  $|i - j| \equiv 1 \pmod{k}$ . Note that C is a negative cycle. So C contains an odd number of negative edges. Hence, there is an odd number of vertex pairs  $\{x_i, x_{i+1}\}$  such that  $\tau_o(x_i, x_j) > \tau_e(x_i, x_j)$ . Hence  $\mathrm{Span}_M(S)$  is an odd flower, a contradiction. This completes the proof.

**Remark.** For a matrix  $\mathbb{B}$ , its inverse can be found in  $O(n^3)$  steps. Note that it takes  $O(n^2)$  steps to determine whether the inverse  $(G^{-1}, w)$  of G is balanced or not. Hence, it can be determined in  $O(n^3)$  whether G has a balanced weighted graph as inverse or not.

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