

# A remark on the paper “Properties of intersecting families of ordered sets” by O. Einstein

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## Abstract

O. Einstein (2008) proved Bollobás-type theorems on intersecting families of ordered sets of finite sets and subspaces. Unfortunately, we report that the proof of a theorem on ordered sets of subspaces had a mistake. We prove two weaker variants.

## 1 Introduction

The following theorem generalizing the theorem of Bollobás [2] is well known and proved by using the wedge product method (see [1]).

**Theorem 1** (Lovász [6]; skew version). *Let  $a, b$  be positive integers. Let  $U_1, U_2, \dots, U_m, V_1, V_2, \dots, V_m$  be subspaces satisfying the following:*

- (i)  $\dim U_i \leq a$  and  $\dim V_i \leq b$  for all  $i = 1, 2, \dots, m$ .
- (ii)  $U_i \cap V_i = \{0\}$  for all  $i = 1, 2, \dots, m$ .
- (iii)  $U_i \cap V_j \neq \{0\}$  for all  $1 \leq i < j \leq m$ .

Then  $m \leq \binom{a+b}{a}$ .

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Ori Einstein [3] published a paper on a generalization of the above theorem and its consequence on finite sets by Frankl [4]. We will show that his proof of Theorem 2.7 in [3] is incorrect and so we state it as a conjecture.

**Conjecture 1** (Theorem 2.7 of [3]). *Let  $\ell_1, \ell_2, \dots, \ell_k$  be positive integers. Let  $U$  be a linear space over a field  $\mathbb{F}$ . Consider the following matrix of subspaces:*

$$\begin{array}{cccc} U_{11} & U_{12} & \cdots & U_{1k} \\ U_{21} & U_{22} & \cdots & U_{2k} \\ \cdots & \cdots & \vdots & \cdots \\ U_{m1} & U_{m2} & \cdots & U_{mk} \end{array}$$

If these subspaces satisfy:

- (i) for every  $1 \leq j \leq k$ ,  $1 \leq i \leq m$ ,  $\dim U_{ij} \leq \ell_j$ ;
- (ii) for every fixed  $i$ , all subspaces  $U_{ij}$  are pairwise disjoint;
- (iii) for each  $i < i'$ , there exist some  $j < j'$  such that  $U_{ij} \cap U_{i'j'} \neq \{0\}$ ;

then

$$m \leq \frac{(\sum_{r=1}^k \ell_r)!}{\prod_{r=1}^k \ell_r!}.$$

Here is the overview of this note. In the next section, we will sketch the reason why the proof of Theorem 2.7 in [3] is incorrect and present a weaker theorem (Theorem 3) obtained by tightening condition (ii). In Section 3, we prove another weaker theorem (Theorem 4), by providing a weaker upper bound for  $m$  instead of modifying any assumptions. Section 4 will discuss the threshold versions.

## 2 The mistake and its first remedy

Let us first point out the mistake in the proof of Conjecture 1 in [3]. As it is typical in the wedge product method, we take  $v_i = \bigwedge_{j=1}^{k-1} \wedge T_j(U_{ij})$  and  $w_i = \bigwedge_{j=1}^{k-1} \bigwedge_{r=j+1}^k \wedge T_j(U_{ir})$  for some linear transformations  $T_1, T_2, \dots, T_{k-1}$ . Then the following claim is made:

**Claim** (Page 41 in [3]). *For every  $i \leq i'$ ,  $v_i \wedge w_{i'} \neq 0$  if and only if  $i = i'$ .*

This claim is false in general. For instance, if  $U_{11} \cap (U_{12} + U_{13} + \cdots + U_{1,k-1}) \neq \{0\}$ , then  $\wedge T_1(U_{11}) \wedge \bigwedge_{r=2}^k \wedge T_1(U_{1r}) = 0$  and therefore  $v_1 \wedge w'_1 = 0$ .

The crucial mistake is that condition (ii) in Conjecture 1 does not imply that  $\dim(U_{i1} + U_{i2} + \cdots + U_{ik}) = \sum_{j=1}^k \dim U_{ij}$ . (For instance the spans of  $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ ,  $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ , and  $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$  are pairwise disjoint and yet their sum has dimension 2 only.)

If  $\dim(U_{i1} + U_{i2} + \cdots + U_{ik}) = \sum_{j=1}^k \dim U_{ij}$ , then the claim is true and so we can recover the following weaker theorem by the proof in [3].

**Theorem 2.** *Let  $\ell_1, \ell_2, \dots, \ell_k$  be positive integers. Let  $U$  be a linear space over a field  $\mathbb{F}$ . Consider the following matrix of subspaces:*

$$\begin{array}{cccc} U_{11} & U_{12} & \cdots & U_{1k} \\ U_{21} & U_{22} & \cdots & U_{2k} \\ \cdots & \cdots & \vdots & \cdots \\ U_{m1} & U_{m2} & \cdots & U_{mk} \end{array}$$

If these subspaces satisfy:

- (i) for every  $1 \leq j \leq k$ ,  $1 \leq i \leq m$ ,  $\dim U_{ij} \leq \ell_j$ ;
- (ii) for every fixed  $i$ ,  $\dim(\sum_{j=1}^k U_{ij}) = \sum_{j=1}^k \dim U_{ij}$ ;
- (iii) for each  $i < i'$ , there exist some  $j < j'$  such that  $U_{ij} \cap U_{i'j'} \neq \{0\}$ ;

then

$$m \leq \frac{(\sum_{r=1}^k \ell_r)!}{\prod_{r=1}^k \ell_r!}.$$

Though Theorem 2 is weaker than Conjecture 1, it allows us to recover Theorem 2.8 of [3].

**Theorem 3** (Theorem 2.8 of [3]). *Let  $\ell_1, \ell_2, \dots, \ell_k$  be positive integers. Consider the following matrix of sets:*

$$\begin{array}{cccc} A_{11} & A_{12} & \cdots & A_{1k} \\ A_{21} & A_{22} & \cdots & A_{2k} \\ \cdots & \cdots & \vdots & \cdots \\ A_{m1} & A_{m2} & \cdots & A_{mk} \end{array}$$

If these sets satisfy:

- (i) for every  $1 \leq j \leq k$ ,  $1 \leq i \leq m$ ,  $|A_{ij}| \leq \ell_j$ ;
- (ii) for every fixed  $i$ , all sets  $A_{ij}$  are pairwise disjoint;
- (iii) for each  $i < i'$ , there exist some  $j < j'$  such that  $A_{ij} \cap A_{i'j'} \neq \emptyset$ ;

then

$$m \leq \frac{(\sum_{r=1}^k \ell_r)!}{\prod_{r=1}^k \ell_r!}.$$

Note that Theorem 3 implies that Conjecture 1 is true when  $\ell_1 = \ell_2 = \dots = \ell_k = 1$ .

### 3 Second remedy

Naturally we ask whether Conjecture 1 can be proven with some upper bound on  $m$ . Here we show that this is possible, while generalizing Theorem 1.

**Theorem 4.** *Under the same assumptions of Conjecture 1, we have*

$$m \leq \frac{\prod_{1 \leq a < b \leq k} (\ell_a + \ell_b)!}{(\prod_{r=1}^k \ell_r!)^{k-1}}.$$

*Proof.* We may assume that  $\dim U_{ij} = \ell_j$  for all  $i, j$  and  $\mathbb{F}$  is infinite. Let  $V = V_{1,2} \oplus V_{1,3} \oplus \dots \oplus V_{1,k} \oplus \dots \oplus V_{2,3} \oplus \dots \oplus V_{k-1,k} = \bigoplus_{a=1}^{k-1} \bigoplus_{b=a+1}^k V_{a,b}$  be a  $\sum_{a=1}^{k-1} \sum_{b=a+1}^k (\ell_a + \ell_b)$ -dimensional vector space over  $\mathbb{F}$ , decomposed into the direct sum of subspaces  $V_{a,b}$ , each of dimension  $\ell_a + \ell_b$ . By Corollary 3.14 of [1], for all  $i < j$ , there exists a linear transformation  $T_{ab} : U \rightarrow V_{a,b}$  such that for all  $1 \leq i \leq m$ ,  $\dim T_{ab}(U_{ia}) = \ell_a$ ,  $\dim T_{ab}(U_{ib}) = \ell_b$ , and  $\dim T_{ab}(U_{ia}) \cap T_{ab}(U_{jb}) = \dim U_{ia} \cap U_{jb}$  for all  $1 \leq i, j \leq m$ . Finally, for each  $1 \leq i \leq m$ , let  $v_i = \bigwedge_{a=1}^{k-1} \bigwedge_{b=a+1}^k \wedge T_{ab}(U_{ia})$  and  $w_i = \bigwedge_{a=1}^{k-1} \bigwedge_{b=a+1}^k \wedge T_{ab}(U_{ib})$ .

We claim that for  $i \leq i'$ ,  $v_i \wedge w_{i'} \neq 0$  if and only if  $i = i'$ . If  $i < i'$ , then there exist  $1 \leq j < j' \leq k$  such that  $U_{ij} \cap U_{i'j'} \neq \{0\}$ . By the choice of  $T_{jj'}$ ,  $T_{jj'}(U_{ij}) \cap T_{jj'}(U_{i'j'}) \neq \{0\}$  and so  $(\wedge T_{jj'}(U_{ij})) \wedge (\wedge T_{jj'}(U_{i'j'})) = 0$ , which implies that  $v_i \wedge w_{i'} = 0$ . If  $i = i'$ , then  $v_i \wedge w_{i'}$  is the wedge product of disjoint subspaces and so  $v_i \wedge w_{i'} \neq 0$ .

Therefore  $v_1, v_2, \dots, v_m$  are linearly independent in the space  $\bigwedge_{a=1}^{k-1} \bigwedge_{b=a+1}^k \bigwedge^{\ell_a} V_{a,b}$ , whose dimension is  $\prod_{a=1}^{k-1} \prod_{b=a+1}^k \binom{\ell_a + \ell_b}{\ell_a} = \frac{\prod_{1 \leq a < b \leq k} (\ell_a + \ell_b)!}{(\prod_{i=1}^k \ell_i!)^{k-1}}$ . This proves that  $m \leq \frac{\prod_{1 \leq a < b \leq k} (\ell_a + \ell_b)!}{(\prod_{i=1}^k \ell_i!)^{k-1}}$ .  $\square$

### 4 Threshold versions

The paper [3] uses Conjecture 1 to deduce the threshold versions (Lemma 2.9 and Theorem 2.10) to generalize a result of Füredi [5]. We do not know

how to prove Lemma 2.9 and Theorem 2.10 of [3] and so we leave them as conjectures. It is not clear how one can relax conditions in Lemma 2.9 and Theorem 2.10 of [3], while avoiding ugly conditions from (ii) of Theorem 3. (A necessary condition  $\ell_i \geq t$  was missing in [3].)

**Conjecture 2** (Lemma 2.9 of [3]). *Let  $\ell_1, \ell_2, \dots, \ell_k$  be positive integers such that  $\ell_i \geq t$  for all  $i$ . Let  $U$  be a linear space over a field  $\mathbb{F}$ . Consider the following matrix of subspaces:*

$$\begin{array}{cccc} U_{11} & U_{12} & \cdots & U_{1k} \\ U_{21} & U_{22} & \cdots & U_{2k} \\ \cdots & \cdots & \vdots & \cdots \\ U_{m1} & U_{m2} & \cdots & U_{mk} \end{array}$$

*If these subspaces satisfy:*

- (i) *for every  $1 \leq j \leq k$ ,  $1 \leq i \leq m$ ,  $\dim U_{ij} \leq \ell_j$ ;*
  - (ii) *for every fixed  $i$ ,  $\dim(U_{ij} \cap U_{ij'}) \leq t$ ;*
  - (iii) *for each  $i < i'$ , there exists some  $j < j'$  such that  $\dim(U_{ij} \cap U_{i'j'}) > t$ ;*
- then*

$$m \leq \frac{[(\sum_{r=1}^k \ell_r) - kt]!}{\prod_{r=1}^k (\ell_r - t)!}.$$

**Conjecture 3** (Theorem 2.10 of [3]). *Let  $\ell_1, \ell_2, \dots, \ell_k$  be positive integers such that  $\ell_i \geq t$  for all  $i$ . Consider the following matrix of sets:*

$$\begin{array}{cccc} A_{11} & A_{12} & \cdots & A_{1k} \\ A_{21} & A_{22} & \cdots & A_{2k} \\ \cdots & \cdots & \vdots & \cdots \\ A_{m1} & A_{m2} & \cdots & A_{mk} \end{array}$$

*If these sets satisfy:*

- (i) *for every  $1 \leq j \leq k$ ,  $1 \leq i \leq m$ ,  $|A_{ij}| \leq \ell_j$ ;*
  - (ii) *for every  $i, j$  and  $j'$ ,  $|A_{ij} \cap A_{ij'}| \leq t$ ;*
  - (iii) *for each  $i < i'$ , there exists some  $j < j'$  such that  $|A_{ij} \cap A_{i'j'}| > t$ ;*
- then*

$$m \leq \frac{[(\sum_{r=1}^k \ell_r) - kt]!}{\prod_{r=1}^k (\ell_r - t)!}.$$

By using Theorem 4, we can prove the following weaker variants of Conjectures 2 and 3 by the same reduction in [3].

**Theorem 5.** *Under the same assumptions of Conjecture 2, we have*

$$m \leq \frac{\prod_{1 \leq a < b \leq k} (\ell_a + \ell_b - 2t)!}{(\prod_{r=1}^k (\ell_r - t)!)^{k-1}}.$$

**Theorem 6.** *Under the same assumptions of Conjecture 3, we have*

$$m \leq \frac{\prod_{1 \leq a < b \leq k} (\ell_a + \ell_b - 2t)!}{(\prod_{r=1}^k (\ell_r - t)!)^{k-1}}.$$

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