# Optimal Littlewood-Offord inequalities in groups 

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#### Abstract

We prove several Littlewood-Offord type inequalities for arbitrary groups. In groups having elements of finite order the worst case scenario is provided by the simple random walk on a cyclic subgroup. The inequalities we obtain are optimal if the underlying group contains an element of a certain order. It turns out that for torsion-free groups Erdős's bound still holds. Our results strengthen and generalize some very recent results by Tiep and Vu.


## 1 Introduction

Let $V_{n}=\left\{g_{1}, \ldots, g_{n}\right\}$ be a multiset of non-identify elements of an arbitrary group $G$. Consider a collection of independent random variables $X_{i}$ that are each distributed on a two point set $\left\{g_{i}^{-1}, g_{i}\right\}$ and define the quantity

$$
\rho\left(V_{n}\right)=\sup _{g \in G} \mathbb{P}\left(X_{1} * \cdots * X_{n}=g\right) .
$$

In the case $G=\mathbb{R}$ the latter quantity is the maximum probability of the sum $X_{1}+\ldots+X_{n}$. Whenever $G=\mathbb{R}, \mathbb{Z}_{m}$ and $g_{i}=1$ we shall adopt the convention to write $\varepsilon_{i}$ instead of the random variable $X_{i}$.

Investigating random polynomials Littlewood and Offord 9] proved an almost optimal bound for the probability that a sum of random signs with non-zero weights hits a point. To be more precise, using harmonic analysis they proved that in the case $G=\mathbb{R}$ we have

$$
\rho\left(V_{n}\right)=O\left(n^{-1 / 2} \log n\right) .
$$

Erdős [5], using Sperner's theorem from finite set combinatorics, showed that, actually,

$$
\rho\left(V_{n}\right) \leq \frac{\binom{n}{\lfloor n / 2\rfloor}}{2^{n}} .
$$

[^0]This bound is optimal as can be seen by taking $g_{i}=1$ in $V_{n}$. In this case we have

$$
\rho\left(V_{n}\right)=\mathbb{P}\left(\varepsilon_{1}+\cdots+\varepsilon_{n} \in\{0,1\}\right)=\frac{\binom{n}{n / 2\rfloor}}{2^{n}} .
$$

Answering a question of Erdős, Kleitman [7] used an ingenious induction to show that the latter bound still holds for $g_{i}$ lying in an arbitrary normed space. See also [2] for a very nice exposition of Kleitman's beautiful argument. Griggs used a similar approach in [6] as in Erdős's seminal paper [5] to obtain the best possible result in $\mathbb{Z}_{m}$.

More recently Tiep and Vu [10] investigated the same question for certain matrix groups and obtained results that are sharp up to a constant factor. To be more precise, let $m, k, n \geq 2$ be integers and $G=G L_{k}(\mathbb{C})$. Let $V_{n}=\left\{g_{1}, \ldots, g_{n}\right\}$ be a multiset of elements in $G$, each of which has order at least $m$. In this case they have obtained the bound

$$
\begin{equation*}
\rho\left(V_{n}\right) \leq 141 \max \left\{\frac{1}{m}, \frac{1}{\sqrt{n}}\right\} . \tag{1}
\end{equation*}
$$

Furthermore, they have also established the same bound for $G L_{k}(p)$.
Let us explain the meaning of the two terms in the upper bound given in (1). Take some element $g$ in $G$ of order $m$ and consider the multiset $V_{n}=\{g, \ldots, g\}$. Let us for the simplicity assume that $m$ is odd. In this setup the random variable $S_{k}=X_{1} * \cdots * X_{k}$ is just the simple random walk on a subgroup of $G$ that is isomorphic to $\mathbb{Z}_{m}$. It is a well known fact that the distribution of $S_{n}$ is asymptotically uniform, which accounts for the $\frac{1}{m}$ term in (11). For $n<m$ the point masses of $S_{n}$ are just the usual binomial probabilities $\binom{n}{\lfloor n / 2\rfloor} / 2^{n}$. Therefore in this regime $\mathbb{P}\left(S_{n}=g\right) \leq\binom{ n}{\lfloor n / 2\rfloor} / 2^{n} \sim \frac{1}{\sqrt{n}}$. This shows that the inequality (11) cannot be improved apart from the constant factor. It is also very natural that the term $\frac{1}{m}$ is dominant for $n \geq m^{2}$, exactly above the mixing time of $S_{n}$, that is known to be of magnitude $m^{2}$ (see [8], page 96).

In this paper we shall prove an optimal upper bound for $\rho\left(V_{n}\right)$, where the elements of the multiset $V_{n}$ lie in an arbitrary group. It turns out that a bound as in (1) holds for arbitrary groups. Furthermore, for groups with elements having odd or infinite order we shall establish an optimal inequality for $\mathbb{P}\left(X_{1} * \cdots * X_{n}=x\right)$ without the requirement that the random variables $X_{i}$ are two-valued.

Let us remind the reader that we denote by $\varepsilon$ (usually supplied with a subscript) a uniform random variable on $\{-1,1\}$. Sometimes it will be important to stress that these random variables are defined on $\mathbb{Z}_{m}$ instead of $\mathbb{R}$ and we shall do so on each occasion. We denote by $(a, b]_{m}$ and $[a, b]_{m}$ the set of integers in the intervals $(a, b]$ and $[a, b]$ modulo $m$. Given a natural number $m$, we shall write $\tilde{m}$ for the smallest even number such that $\tilde{m} \geq m$. That is, we have $\tilde{m}=2\left\lceil\frac{m}{2}\right\rceil$.
Theorem 1. Let $g_{1}, \ldots, g_{n}$ be elements of some group $G$ such that $\left|g_{i}\right| \geq m \geq 2$. Let $X_{1}, \ldots, X_{n}$ be independent random variables so that each $X_{i}$ has the uniform distribution on the two point set $\left\{g_{i}^{-1}, g_{i}\right\}$. Then for any $A \subset G$ with $|A|=k$ we have

$$
\begin{equation*}
\mathbb{P}\left(X_{1} * \cdots * X_{n} \in A\right) \leq \mathbb{P}\left(\varepsilon_{1}+\cdots+\varepsilon_{n} \in(-k, k]_{\tilde{m}}\right) \tag{2}
\end{equation*}
$$

where $\varepsilon_{i}$ are independent uniform random variables on the set $\{-1,1\} \subset \mathbb{Z}_{\tilde{m}}$.

Note that Theorem $\mathbb{1}$ is optimal in the sense that if $G$ contains an element of order $\tilde{m}$, the bound in (2) can be attained. For instance, in the case $G=G L_{k}(\mathbb{C})$ the upper bound in (2) is achieved by taking two point distributions concentrated on the diagonal matrix $\mathrm{e}^{\frac{2 \pi i}{m}} \mathbb{I}_{k}$ and its inverse. Theorem $\mathbb{1}$ implies an inequality of the same type as the one by Tiep and Vu, but with a much better constant.

Corollary 1. Let $V_{n}=\left\{g_{1}, \ldots, g_{n}\right\}$ be elements in some group $G$ satisfying $\left|g_{i}\right| \geq m \geq 2$. Then

$$
\begin{equation*}
\rho\left(V_{n}\right) \leq \frac{2}{\tilde{m}}+\sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{n}} \leq 3 \max \left\{\frac{1}{m}, \frac{1}{\sqrt{n}}\right\} . \tag{3}
\end{equation*}
$$

The sequence of sums appearing on the right hand side of (2) is a periodic Markov chain and so does not converge to a limit as $n \rightarrow \infty$. Nonetheless, it is well known that it does converge to a limit if we restrict the parity of $n$. Let us now express the quantity in the right hand side of (2) in the case $|A|=1$ in asymptotic terms.

Proposition 2. Let $m \in \mathbb{N}$ and assume that $n \rightarrow \infty$. Then for any $l \in \mathbb{Z}_{\tilde{m}}$ of the same parity as $n$ we have

$$
\mathbb{P}\left(\varepsilon_{1}+\cdots+\varepsilon_{n}=l\right)=\frac{2}{\tilde{m}}+o(1)
$$

The $o(1)$ term is actually exponentially small in terms of $n$. For such sharp quantitative estimates see [3] pages 124-125. Note that Proposition 2 implies that in (3) the constant after the last inequality cannot be smaller than 2 . Let us also note that both constants in the expression $\frac{2}{\tilde{m}}+\sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{n}}$ are sharp. The term $\frac{2}{\tilde{m}}$ is dominant in the case $m, n \rightarrow \infty$ and $n \gg m^{2}$ and so Proposition 2 shows that the constant 2 cannot be reduced. In the case $m, n \rightarrow \infty$ and $n<m$ the therm $\sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{n}}$ is dominating. For $V_{n}=\{g, \ldots, g\}$ for some element $g$ of order $\tilde{m}$ we have

$$
\rho\left(V_{n}\right)=\mathbb{P}\left(\varepsilon_{1}+\cdots+\varepsilon_{n} \in(-1,1]_{\tilde{m}}\right)=\frac{\binom{n}{\lfloor n / 2\rfloor}}{2^{n}}=(1+o(1)) \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{n}} .
$$

The simple random walk on $\mathbb{Z}_{m}$ for $m$ odd converges to the uniform distribution on $\mathbb{Z}_{m}$ and so all probabilities converge to $\frac{1}{m}$. It should now be unsurprising that the simple random walk on $\mathbb{Z}_{m+1}$ is a much better "candidate" for a maximizer of the left hand side in (2), as by Proposition 2 we gain an extra factor of 2 asymptotically.

From this point our prime focus will be on the particular case $G=\mathbb{Z}_{m}^{l}$ for $m$ odd. In this case Theorem $\square$ does not provide the optimal bound. The approach we have for
this case also works for certain groups other than $\mathbb{Z}_{m}^{l}$ and therefore we will state it in a general form. For $k \geq 1$ we define

$$
I_{n, k}^{m}=\left[\left\lceil\frac{n-k+1}{2}\right\rceil, \ldots,\left\lceil\frac{n+k-1}{2}\right\rceil\right]_{m} .
$$

The latter set is an interval of $k$ points in $\mathbb{Z}_{m}$. We shall use the convention that $I_{n, 0}^{m}=\emptyset$.
Theorem 3. Let $X_{1}, \ldots, X_{n}$ be independent discrete random variables taking values in some group $G$ such that for each $i$ we have

$$
\begin{equation*}
\sup _{g \in G} \mathbb{P}\left(X_{i}=g\right) \leq \frac{1}{2} . \tag{4}
\end{equation*}
$$

Furthermore, assume that all non-identity elements in $G$ have odd or infinite order and that the minimal such order is at least some odd number $m \geq 3$. Then for any set $A \subset G$ of cardinality $k$ we have

$$
\mathbb{P}\left(X_{1} * \cdots * X_{n} \in A\right) \leq \mathbb{P}\left(\tau_{1}+\cdots+\tau_{n} \in I_{n, k}^{m}\right)
$$

where $\tau_{i}$ are independent uniform random variables on the set $\{0,1\} \subset \mathbb{Z}_{m}$.
The distribution of $\tau_{1}+\cdots+\tau_{n}$ is asymptotically uniform in $\mathbb{Z}_{m}$ and thus we have $\mathbb{P}\left(X_{1} * \cdots * X_{n}=g\right) \leq \frac{1}{m}+o(1)$.

Remark 1. Note that

$$
\mathbb{P}\left(\tau_{1}+\cdots+\tau_{n} \in I_{n, k}^{m}\right)=\mathbb{P}\left(\varepsilon_{1}+\cdots+\varepsilon_{n} \in 2 I_{n, k}^{m}-n\right) .
$$

We formulated the result in terms of $\{0,1\}$ random variables $\tau_{i}$ for the sake of convenience only - in this formulation the set of maximum probability is an interval. As one notices, it is not so in formulating it in terms of $\{-1,1\}$ distributions $\varepsilon_{i}$.

Remark 2. The reason we restrict the elements to have odd order in Theorem 3 is as follows. If there is an element of even order in the underlying group, then the group contains an element of order 2 , say $h$. Then by taking independent uniform random variables $X_{i}$ on the set $\{1, h\}$ we obtain $\sup _{g \in G} \mathbb{P}\left(X_{1} * \cdots * X_{n}=g\right)=\frac{1}{2}$.

In the case when $G$ is torsion-free we can actually prove that Erdős's bound still holds even in this general setting.

Proposition 4. Under the notation of Theorem 3 and assuming that $G$ is torsion-free for any set $A \subset G$ of cardinality $k$ we have

$$
\mathbb{P}\left(X_{1} * \cdots * X_{n} \in A\right) \leq \mathbb{P}\left(\varepsilon_{1}+\cdots+\varepsilon_{n} \in(-k, k]\right)
$$

where $\varepsilon_{i}$ are independent. In particular, for any $g \in G$ we have

$$
\mathbb{P}\left(X_{1} * \cdots * X_{n}=g\right) \leq \frac{\binom{n}{\lfloor n / 2\rfloor}}{2^{n}} .
$$

The latter proposition immediately follows by taking $m$ large enough in Theorem 3 so that $\tau_{1}+\cdots+\tau_{n}$ is concentrated in a proper subset of $\mathbb{Z}_{m}$. For instance, assume that $m=n+2$. In this case the latter sum is strictly contained in $\mathbb{Z}_{m}$ and its probabilities are exactly the largest $k$ probabilities of $\varepsilon_{1}+\cdots+\varepsilon_{n}$ and we are done.

Our proofs are similar in spirit to Kleitman's approach in his solution of the LittlewoodOfford problem in all dimensions. Actually, it is closer to a simplification of Kleitman's proof in dimension 1 obtained in 4. The proofs thus proceed by induction on dimension, taking into account a certain recurrence relation satisfied by the worst-case random walk.

## 2 An open problem

Theorem 1 gives an optimal inequality if an element with a given order exists. To be more precise, if an element of order $\tilde{m}$ exists. For groups in which all elements have odd or infinite order, Theorem 3 gives the best possible result. It is thus natural to ask what happens if we have full knowledge of the orders of the elements of the underlying group $G$ and we are not in the aforementioned cases. The asymptotics of the cases when we do know the exact answer suggest the following guess.

Conjecture. Let $G$ be any group and fix an odd integer $m \geq 3$. Suppose that all possible even orders of elements in $G$ greater than $m$ are given by the sequence $S=$ $\left\{m_{1}, m_{2}, \ldots\right\}$ in increasing order. Consider a collection of independent random variables $X_{1}, \ldots, X_{n}$ in $G$ such that each $X_{i}$ is concentrated on a two point set $\left\{g_{i}, g_{i}^{-1}\right\}$ and $\left|g_{i}\right| \geq m$. Then if $m_{1}<2 m$ for any $A \subset G$ with $|A|=k$ we have

$$
\mathbb{P}\left(X_{1} * \cdots * X_{n} \in A\right) \leq \mathbb{P}\left(\varepsilon_{1}+\cdots+\varepsilon_{n} \in(-k, k]_{m_{1}}\right)
$$

where $\varepsilon_{i}$ are independent uniform random variables on the set $\{-1,1\} \subset \mathbb{Z}_{m_{1}}$. On the other hand, if $m_{1} \geq 2 m$ we have

$$
\mathbb{P}\left(X_{1} * \cdots * X_{n} \in A\right) \leq \mathbb{P}\left(\tau_{1}+\cdots+\tau_{n} \in I_{n, k}^{m}\right)
$$

where $\tau_{i}$ are independent uniform random variables on the set $\{0,1\} \subset \mathbb{Z}_{m}$.
If true, the latter conjecture would settle the remaining cases.

## 3 Proofs

In order to prove Theorems 173, we shall require a simple group theoretic statement contained in the following lemma.

Lemma 1. Let $G$ be a group and $g \in G$ be an element of order greater then or equal to $m \geq 2$. Then for any finite set $A \subset G$ and a positive integer such that $s<\frac{m}{|A|}$ we have $A \neq A g^{s}$.

Proof of Lemma 1. Suppose there is a nonempty set $A \subset G$ and a positive integer $s$ such that $|A|=k<\frac{m}{s}$ and $A=A g^{s}$. Take some $a \in A$ and consider elements $a g^{s i}, i=0 \ldots k$. All these $k+1$ elements are in the set $A$ hence at least two of them must be equal. Let us say $a g^{s i}=a g^{s j}$ for some integers $0 \leq i<j \leq k$. But this immediately gives a contradiction since then $g^{s(j-i)}$ is equal to the group identity element and $m \leq s(j-i) \leq s k$.

Proof of Theorem 11. If $n=1$ the inequality (21) is trivial. For $k \geq \frac{m}{2}$ and all $n$ the right hand side of (22) becomes 1 since in this case $(-k, k]_{\tilde{m}}$ covers the support of the sum $\varepsilon_{1}+\cdots+\varepsilon_{n}$ and so there is nothing to prove. We shall henceforth assume that $n>1$ and $k<\frac{m}{2}$.
By Lemma $\rceil$ we have that $A g_{n} \neq A g_{n}^{-1}$. Take some $h \in A g_{n} \backslash A g_{n}^{-1}$ and define $B=A g_{n} \backslash\{h\}$ and $C=A g_{n}^{-1} \cup\{h\}$. We then have

$$
\begin{align*}
& 2 \mathbb{P}\left(X_{1} * \cdots * X_{n} \in A\right)=\mathbb{P}\left(X_{1} * \cdots * X_{n-1} \in A g_{n}\right)+\mathbb{P}\left(X_{1} * \cdots * X_{n-1} \in A g_{n}^{-1}\right) \\
= & \mathbb{P}\left(X_{1} * \cdots * X_{n-1} \in B\right)+\mathbb{P}\left(X_{1} * \cdots * X_{n-1} \in C\right)  \tag{5}\\
\leq & \mathbb{P}\left(\varepsilon_{1}+\cdots+\varepsilon_{n-1} \in(-k-1, k+1]_{\tilde{m}}\right)+\mathbb{P}\left(\varepsilon_{1}+\cdots+\varepsilon_{n-1} \in(-k+1, k-1]_{\tilde{m}}\right)  \tag{6}\\
= & \mathbb{P}\left(\varepsilon_{1}+\cdots+\varepsilon_{n-1} \in(-k-1, k-1]_{\tilde{m}}\right)+\mathbb{P}\left(\varepsilon_{1}+\cdots+\varepsilon_{n-1} \in(-k+1, k+1]_{\tilde{m}}\right)  \tag{7}\\
= & 2 \mathbb{P}\left(\varepsilon_{1}+\cdots+\varepsilon_{n} \in(-k, k]_{\tilde{m}}\right) .
\end{align*}
$$

This completes the proof.
Remark 3. Note that in (6)-(7) we used the fact that for $k<\frac{m}{2}$ the sets $(-k+1, k-1]_{\tilde{m}}$ and $(k-1, k+1]_{\tilde{m}}$ are disjoint in $\mathbb{Z}_{\tilde{m}}$.

In the proof of Theorem 3 we shall make use of the following simple lemma which will allow us to switch from general distributions satisfying the condition (4) to two-point distributions.

Lemma 2. Let $X$ be a random variable on some group $G$ that takes only finitely many values, say $x_{1}, \ldots, x_{n}$. Suppose that $p_{i}=\mathbb{P}\left(X=x_{i}\right)$ are rational numbers and that $p_{i} \leq \frac{1}{2}$. Then we can express the distribution of $X$ as a convex combination of distributions that are uniform on some two point set.

Proof of Lemma 2. Denote by $\mu$ the distribution of $X$. Since the $p_{i}$ 's are all rational, we have $p_{i}=\frac{k_{i}}{K_{i}}$ for some $k_{i}, K_{i} \in \mathbb{Z}$. We shall now view $\mu$ as a distribution on a multiset $M$ made from the elements $x_{i}$ in the following way - take $x_{i}$ exactly $2 k_{i} \prod_{j \neq i} K_{i}$ times into $M$. This way $\mu$ has the uniform distribution on $M$. We thus have that $M=\left\{y_{1}, \ldots, y_{2 N}\right\}$ for the appropriate $N$. Construct a graph on the elements on $M$ by joining two of them by an edge if and only if they are distinct. Since we had $p_{i} \leq \frac{1}{2}$, each vertex of this graph
has degree at least $N$. Thus by Dirac's Theorem, our graph contains a Hamiltonian cycle, and, consequently - a perfect matching. Let $\mu_{i}$ be the uniform distribution on two vertices of the latter matching $(i=1,2, \ldots, N)$. We have

$$
\mu=\frac{1}{N} \sum_{i=1}^{N} \mu_{i}
$$

Proof of Theorem 3. We shall argue by induction. First notice that the claim of the Theorem is true for $n=1$. Furthermore, it is also true for $k \geq m$ since in that case the bound for the probability in question becomes 1 . We therefore shall from now on assume that $n>1$ and $1 \leq k \leq m-1$. Denote by $\mu_{i}$ the distribution of the random variable $X_{i}$. We can without loss of generality assume that each $X_{i}$ is concentrated on finitely many points and that for each $g \in G$ we have $\mathbb{P}\left(X_{i}=g\right) \in \mathbb{Q}$. By Lemma 2, each $\mu_{i}$ can be written as a convex combination of distributions that are uniform on some two-point set. Define the random variable $f_{i}\left(X_{i}\right)=\mathbb{E}_{i} 1\left\{X_{1} * \cdots * X_{n} \in A\right\}$, where $\mathbb{E}_{i}$ stands for integration with respect to all underlying random variables except $X_{i}$. Then for each $i$ we have

$$
\begin{equation*}
\mathbb{P}\left(X_{1} * \cdots * X_{n} \in A\right)=\mathbb{E} f_{i}\left(X_{i}\right) \tag{8}
\end{equation*}
$$

The latter expectation is linear with respect to the distribution of $X_{i}$. Therefore we can assume that it will be maximized by some choice of two-point distributions coming from the decomposition of $\mu_{i}$. We shall therefore from this point assume that $X_{n}$ takes only two values, say $h_{1}$ and $h_{2}$, with equal probabilities.

Note that the intervals $I_{n, k}^{m}$ have recursive structure. Namely, if $1 \leq k \leq m-1$ and we regard them as multisets, we have the relation $I_{n, k}^{m} \cup\left(I_{n, k}^{m}-1\right)=I_{n-1, k-1}^{m} \cup I_{n-1, k+1}^{m}$. The pairs on intervals appearing on both sides of the latter equality heavily overlap. This means that we can take one endpoint of $I_{n-1, k+1}^{m}$ that does not belong to $I_{n-1, k-1}^{m}$ and move it to this shorter interval. The resulting intervals are both of length $k$ and are exactly the intervals $I_{n, k}^{m}$ and $I_{n, k}^{m}-1$. We shall use this after the inductive step.

Take a finite set $A \subset G$ with $k$ elements. Note that the element $h_{2}^{-1} h_{1} \neq 1_{G}$ and so it has order at least $m$. By Lemma 1 we have that $A h_{1}^{-1} \neq A h_{2}^{-1}$ as $A \neq A h_{2}^{-1} h_{1}$. Take some $h \in A h_{1}^{-1} \backslash A h_{2}^{-1}$ and define $B=A h_{1}^{-1} \backslash\{h\}$ and $C=A h_{2}^{-1} \cup\{h\}$. We have

$$
\begin{aligned}
2 \mathbb{P}\left(X_{1} * \cdots * X_{n} \in A\right) & =\mathbb{P}\left(X_{1} * \cdots * X_{n-1} \in A h_{1}^{-1}\right)+\mathbb{P}\left(X_{1} * \cdots * X_{n-1} \in A h_{2}^{-1}\right) \\
& =\mathbb{P}\left(X_{1} * \cdots * X_{n-1} \in B\right)+\mathbb{P}\left(X_{1} * \cdots * X_{n-1} \in C\right) \\
& \leq \mathbb{P}\left(\tau_{1}+\cdots+\tau_{n-1} \in I_{n-1, k-1}^{m}\right)+\mathbb{P}\left(\tau_{1}+\cdots+\tau_{n-1} \in I_{n-1, k+1}^{m}\right) \\
& =\mathbb{P}\left(\tau_{1}+\cdots+\tau_{n-1} \in I_{n, k}^{m}-1\right)+\mathbb{P}\left(\tau_{1}+\cdots+\tau_{n-1} \in I_{n, k}^{m}\right) \\
& =2 \mathbb{P}\left(\tau_{1}+\cdots+\tau_{n} \in I_{n, k}^{m}\right) .
\end{aligned}
$$

This completes the proof.

Proof of Corollary 1. We shall use an identity on evenly spaced binomial coefficients proved in [1]:

$$
\begin{equation*}
\binom{n}{t}+\binom{n}{t+s}+\binom{n}{t+2 s}+\cdots=\frac{1}{s} \sum_{j=0}^{s-1}\left(2 \cos \frac{i \pi}{s}\right)^{n} \cos \frac{\pi(n-2 t) j}{s} . \tag{9}
\end{equation*}
$$

By Theorem 1 we have

$$
\begin{equation*}
\rho\left(V_{n}\right) \leq \mathbb{P}\left(\varepsilon_{1}+\cdots+\varepsilon_{n} \in(-1,1]_{\tilde{m}}\right) . \tag{10}
\end{equation*}
$$

The right hand of the equation (10) is the sum of binomial probabilities $\binom{n}{i} / 2^{n}$, where $i$ is such that $2 i-n$ is congruent to $1_{\{n \in 2 \mathbb{Z}+1\}}$ modulo $\tilde{m}$. Let $t$ be the residue of $\left(n-1_{\{n \in 2 \mathbb{Z}+1\}}\right) / 2$ modulo $\frac{\tilde{m}}{2}$.

Using the identity (19) and the elementary inequalities $\cos x \leq \exp \left(-x^{2} / 2\right)$ for $x \in\left[0, \frac{\pi}{2}\right]$ and $\int_{0}^{\infty} \mathrm{e}^{\frac{-x^{2}}{2 \sigma^{2}}} d x \leq \frac{\sigma \sqrt{2 \pi}}{2}$ we obtain

$$
\begin{align*}
\mathbb{P}\left(\varepsilon_{1}+\cdots+\varepsilon_{n} \in(-1,1]_{\tilde{m}}\right) & =\frac{\binom{n}{t}+\binom{n}{t+\tilde{m} / 2}+\binom{n}{t+2 \tilde{m} / 2}+\ldots}{2^{n}} \\
& =\frac{2}{\tilde{m}} \sum_{j=0}^{\frac{\tilde{m}}{2}-1}\left(2 \cos \frac{2 i \pi}{\tilde{m}}\right)^{n} \cos \frac{2 \pi(n-2 t) j}{\tilde{m}} \\
& \leq \frac{2}{\tilde{m}}+\frac{2}{\tilde{m}} \sum_{j=1}^{\frac{\tilde{m}}{2}-1}\left|\cos \frac{2 j \pi}{\tilde{m}}\right|^{n}  \tag{11}\\
& \leq \frac{2}{\tilde{m}}+\frac{4}{\tilde{m}} \sum_{j=1}^{\left\lfloor\frac{\tilde{m}}{4}\right\rfloor}\left|\cos \frac{2 j \pi}{\tilde{m}}\right|^{n} \\
& \leq \frac{2}{\tilde{m}}+\frac{4}{\tilde{m}} \sum_{j=1}^{\left\lfloor\frac{\tilde{m}}{4}\right\rfloor} \mathrm{e}^{-2 \pi^{2} j^{2} n / \tilde{m}^{2}} \\
& <\frac{2}{\tilde{m}}+\frac{4}{\tilde{m}} \int_{0}^{\infty} \mathrm{e}^{-2 \pi^{2} x^{2} n / \tilde{m}^{2}} d x \\
& \leq \frac{2}{\tilde{m}}+\sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{n}} \leq \frac{2}{m}+\sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{n}} .
\end{align*}
$$

Note that in (11) we replaced $\left|\cos \frac{2 \pi j}{\tilde{m}}\right|$ by $\left|\cos \frac{\pi(\tilde{m}-2 j)}{\tilde{m}}\right|$ when $j>\frac{\tilde{m}}{4}$. This completes the proof.

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