# Optimal Littlewood-Offord inequalities in groups

T. Juškevičius<sup>1</sup>, G. Šemetulskis<sup>2</sup>

#### **Abstract**

We prove several Littlewood-Offord type inequalities for arbitrary groups. In groups having elements of finite order the worst case scenario is provided by the simple random walk on a cyclic subgroup. The inequalities we obtain are optimal if the underlying group contains an element of a certain order. It turns out that for torsion-free groups Erdős's bound still holds. Our results strengthen and generalize some very recent results by Tiep and Vu.

## 1 Introduction

Let  $V_n = \{g_1, \ldots, g_n\}$  be a multiset of non-identify elements of an arbitrary group G. Consider a collection of independent random variables  $X_i$  that are each distributed on a two point set  $\{g_i^{-1}, g_i\}$  and define the quantity

$$\rho(V_n) = \sup_{g \in G} \mathbb{P}(X_1 * \dots * X_n = g).$$

In the case  $G = \mathbb{R}$  the latter quantity is the maximum probability of the sum  $X_1 + \ldots + X_n$ . Whenever  $G = \mathbb{R}$ ,  $\mathbb{Z}_m$  and  $g_i = 1$  we shall adopt the convention to write  $\varepsilon_i$  instead of the random variable  $X_i$ .

Investigating random polynomials Littlewood and Offord [9] proved an almost optimal bound for the probability that a sum of random signs with non-zero weights hits a point. To be more precise, using harmonic analysis they proved that in the case  $G = \mathbb{R}$  we have

$$\rho(V_n) = O(n^{-1/2} \log n).$$

Erdős [5], using Sperner's theorem from finite set combinatorics, showed that, actually,

$$\rho(V_n) \le \frac{\binom{n}{\lfloor n/2 \rfloor}}{2^n}.$$

 $<sup>^1{\</sup>mbox{Vilnius}}$  University, Institute of Mathematics and Informatics, Vilnius, Lithuania, email -tomas.juskevicius@gmail.com.

<sup>&</sup>lt;sup>2</sup>University of Vilnius, Vilnius, Lithuania, email - grazvydas.semetulskis@gmail.com.

This bound is optimal as can be seen by taking  $g_i = 1$  in  $V_n$ . In this case we have

$$\rho(V_n) = \mathbb{P}(\varepsilon_1 + \dots + \varepsilon_n \in \{0, 1\}) = \frac{\binom{n}{\lfloor n/2 \rfloor}}{2^n}.$$

Answering a question of Erdős, Kleitman [7] used an ingenious induction to show that the latter bound still holds for  $g_i$  lying in an arbitrary normed space. See also [2] for a very nice exposition of Kleitman's beautiful argument. Griggs used a similar approach in [6] as in Erdős's seminal paper [5] to obtain the best possible result in  $\mathbb{Z}_m$ .

More recently Tiep and Vu [10] investigated the same question for certain matrix groups and obtained results that are sharp up to a constant factor. To be more precise, let  $m, k, n \geq 2$  be integers and  $G = GL_k(\mathbb{C})$ . Let  $V_n = \{g_1, \ldots, g_n\}$  be a multiset of elements in G, each of which has order at least m. In this case they have obtained the bound

$$\rho(V_n) \le 141 \max\{\frac{1}{m}, \frac{1}{\sqrt{n}}\}. \tag{1}$$

Furthermore, they have also established the same bound for  $GL_k(p)$ .

Let us explain the meaning of the two terms in the upper bound given in (1). Take some element g in G of order m and consider the multiset  $V_n = \{g, \ldots, g\}$ . Let us for the simplicity assume that m is odd. In this setup the random variable  $S_k = X_1 * \cdots * X_k$  is just the simple random walk on a subgroup of G that is isomorphic to  $\mathbb{Z}_m$ . It is a well known fact that the distribution of  $S_n$  is asymptotically uniform, which accounts for the  $\frac{1}{m}$  term in (1). For n < m the point masses of  $S_n$  are just the usual binomial probabilities  $\binom{n}{\lfloor n/2 \rfloor}/2^n$ . Therefore in this regime  $\mathbb{P}(S_n = g) \leq \binom{n}{\lfloor n/2 \rfloor}/2^n \sim \frac{1}{\sqrt{n}}$ . This shows that the inequality (1) cannot be improved apart from the constant factor. It is also very natural that the term  $\frac{1}{m}$  is dominant for  $n \geq m^2$ , exactly above the mixing time of  $S_n$ , that is known to be of magnitude  $m^2$  (see [8], page 96).

In this paper we shall prove an optimal upper bound for  $\rho(V_n)$ , where the elements of the multiset  $V_n$  lie in an arbitrary group. It turns out that a bound as in (1) holds for arbitrary groups. Furthermore, for groups with elements having odd or infinite order we shall establish an optimal inequality for  $\mathbb{P}(X_1 * \cdots * X_n = x)$  without the requirement that the random variables  $X_i$  are two-valued.

Let us remind the reader that we denote by  $\varepsilon$  (usually supplied with a subscript) a uniform random variable on  $\{-1,1\}$ . Sometimes it will be important to stress that these random variables are defined on  $\mathbb{Z}_m$  instead of  $\mathbb{R}$  and we shall do so on each occasion. We denote by  $(a,b]_m$  and  $[a,b]_m$  the set of integers in the intervals (a,b] and [a,b] modulo m. Given a natural number m, we shall write  $\tilde{m}$  for the smallest even number such that  $\tilde{m} \geq m$ . That is, we have  $\tilde{m} = 2\lceil \frac{m}{2} \rceil$ .

**Theorem 1.** Let  $g_1, \ldots, g_n$  be elements of some group G such that  $|g_i| \ge m \ge 2$ . Let  $X_1, \ldots, X_n$  be independent random variables so that each  $X_i$  has the uniform distribution on the two point set  $\{g_i^{-1}, g_i\}$ . Then for any  $A \subset G$  with |A| = k we have

$$\mathbb{P}\left(X_1 * \dots * X_n \in A\right) \le \mathbb{P}\left(\varepsilon_1 + \dots + \varepsilon_n \in (-k, k|_{\tilde{m}}),\right) \tag{2}$$

where  $\varepsilon_i$  are independent uniform random variables on the set  $\{-1,1\} \subset \mathbb{Z}_{\tilde{m}}$ .

Note that Theorem 1 is optimal in the sense that if G contains an element of order  $\tilde{m}$ , the bound in (2) can be attained. For instance, in the case  $G = GL_k(\mathbb{C})$  the upper bound in (2) is achieved by taking two point distributions concentrated on the diagonal matrix  $e^{\frac{2\pi i}{m}} \mathbb{I}_k$  and its inverse. Theorem 1 implies an inequality of the same type as the one by Tiep and Vu, but with a much better constant.

Corollary 1. Let  $V_n = \{g_1, \ldots, g_n\}$  be elements in some group G satisfying  $|g_i| \ge m \ge 2$ . Then

$$\rho(V_n) \le \frac{2}{\tilde{m}} + \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{n}} \le 3 \max\{\frac{1}{m}, \frac{1}{\sqrt{n}}\}. \tag{3}$$

The sequence of sums appearing on the right hand side of (2) is a periodic Markov chain and so does not converge to a limit as  $n \to \infty$ . Nonetheless, it is well known that it does converge to a limit if we restrict the parity of n. Let us now express the quantity in the right hand side of (2) in the case |A| = 1 in asymptotic terms.

**Proposition 2.** Let  $m \in \mathbb{N}$  and assume that  $n \to \infty$ . Then for any  $l \in \mathbb{Z}_{\tilde{m}}$  of the same parity as n we have

$$\mathbb{P}\left(\varepsilon_1 + \dots + \varepsilon_n = l\right) = \frac{2}{\tilde{m}} + o(1).$$

The o(1) term is actually exponentially small in terms of n. For such sharp quantitative estimates see [3] pages 124-125. Note that Proposition 2 implies that in (3) the constant after the last inequality cannot be smaller than 2. Let us also note that both constants in the expression  $\frac{2}{\tilde{m}} + \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{n}}$  are sharp. The term  $\frac{2}{\tilde{m}}$  is dominant in the case  $m, n \to \infty$  and  $n \gg m^2$  and so Proposition 2 shows that the constant 2 cannot be reduced. In the case  $m, n \to \infty$  and n < m the therm  $\sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{n}}$  is dominating. For  $V_n = \{g, \ldots, g\}$  for some element g of order  $\tilde{m}$  we have

$$\rho(V_n) = \mathbb{P}\left(\varepsilon_1 + \dots + \varepsilon_n \in (-1, 1]_{\tilde{m}}\right) = \frac{\binom{n}{\lfloor n/2 \rfloor}}{2^n} = (1 + o(1))\sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{n}}.$$

The simple random walk on  $\mathbb{Z}_m$  for m odd converges to the uniform distribution on  $\mathbb{Z}_m$  and so all probabilities converge to  $\frac{1}{m}$ . It should now be unsurprising that the simple random walk on  $\mathbb{Z}_{m+1}$  is a much better "candidate" for a maximizer of the left hand side in (2), as by Proposition 2 we gain an extra factor of 2 asymptotically.

From this point our prime focus will be on the particular case  $G = \mathbb{Z}_m^l$  for m odd. In this case Theorem 1 does not provide the optimal bound. The approach we have for this case also works for certain groups other than  $\mathbb{Z}_m^l$  and therefore we will state it in a general form. For  $k \geq 1$  we define

$$I_{n,k}^m = \left[ \left\lceil \frac{n-k+1}{2} \right\rceil, \dots, \left\lceil \frac{n+k-1}{2} \right\rceil \right]_m.$$

The latter set is an interval of k points in  $\mathbb{Z}_m$ . We shall use the convention that  $I_{n,0}^m = \emptyset$ .

**Theorem 3.** Let  $X_1, \ldots, X_n$  be independent discrete random variables taking values in some group G such that for each i we have

$$\sup_{g \in G} \mathbb{P}\left(X_i = g\right) \le \frac{1}{2}.\tag{4}$$

Furthermore, assume that all non-identity elements in G have odd or infinite order and that the minimal such order is at least some odd number  $m \geq 3$ . Then for any set  $A \subset G$  of cardinality k we have

$$\mathbb{P}\left(X_1 * \cdots * X_n \in A\right) \leq \mathbb{P}\left(\tau_1 + \cdots + \tau_n \in I_{n,k}^m\right),\,$$

where  $\tau_i$  are independent uniform random variables on the set  $\{0,1\} \subset \mathbb{Z}_m$ .

The distribution of  $\tau_1 + \cdots + \tau_n$  is asymptotically uniform in  $\mathbb{Z}_m$  and thus we have  $\mathbb{P}(X_1 * \cdots * X_n = g) \leq \frac{1}{m} + o(1)$ .

Remark 1. Note that

$$\mathbb{P}\left(\tau_1 + \dots + \tau_n \in I_{n,k}^m\right) = \mathbb{P}\left(\varepsilon_1 + \dots + \varepsilon_n \in 2I_{n,k}^m - n\right).$$

We formulated the result in terms of  $\{0,1\}$  random variables  $\tau_i$  for the sake of convenience only - in this formulation the set of maximum probability is an interval. As one notices, it is not so in formulating it in terms of  $\{-1,1\}$  distributions  $\varepsilon_i$ .

**Remark 2.** The reason we restrict the elements to have odd order in Theorem 3 is as follows. If there is an element of even order in the underlying group, then the group contains an element of order 2, say h. Then by taking independent uniform random variables  $X_i$  on the set  $\{1, h\}$  we obtain  $\sup_{g \in G} \mathbb{P}(X_1 * \cdots * X_n = g) = \frac{1}{2}$ .

In the case when G is torsion-free we can actually prove that Erdős's bound still holds even in this general setting.

**Proposition 4.** Under the notation of Theorem 3 and assuming that G is torsion-free for any set  $A \subset G$  of cardinality k we have

$$\mathbb{P}\left(X_{1} * \cdots * X_{n} \in A\right) \leq \mathbb{P}\left(\varepsilon_{1} + \cdots + \varepsilon_{n} \in \left(-k, k\right]\right),\,$$

where  $\varepsilon_i$  are independent. In particular, for any  $g \in G$  we have

$$\mathbb{P}\left(X_1 * \dots * X_n = g\right) \le \frac{\binom{n}{\lfloor n/2 \rfloor}}{2^n}.$$

The latter proposition immediately follows by taking m large enough in Theorem 3 so that  $\tau_1 + \cdots + \tau_n$  is concentrated in a proper subset of  $\mathbb{Z}_m$ . For instance, assume that m = n + 2. In this case the latter sum is strictly contained in  $\mathbb{Z}_m$  and its probabilities are exactly the largest k probabilities of  $\varepsilon_1 + \cdots + \varepsilon_n$  and we are done.

Our proofs are similar in spirit to Kleitman's approach in his solution of the Littlewood-Offord problem in all dimensions. Actually, it is closer to a simplification of Kleitman's proof in dimension 1 obtained in [4]. The proofs thus proceed by induction on dimension, taking into account a certain recurrence relation satisfied by the worst-case random walk.

# 2 An open problem

Theorem 1 gives an optimal inequality if an element with a given order exists. To be more precise, if an element of order  $\tilde{m}$  exists. For groups in which all elements have odd or infinite order, Theorem 3 gives the best possible result. It is thus natural to ask what happens if we have full knowledge of the orders of the elements of the underlying group G and we are not in the aforementioned cases. The asymptotics of the cases when we do know the exact answer suggest the following guess.

**Conjecture**. Let G be any group and fix an odd integer  $m \geq 3$ . Suppose that all possible even orders of elements in G greater than m are given by the sequence  $S = \{m_1, m_2, \ldots\}$  in increasing order. Consider a collection of independent random variables  $X_1, \ldots, X_n$  in G such that each  $X_i$  is concentrated on a two point set  $\{g_i, g_i^{-1}\}$  and  $|g_i| \geq m$ . Then if  $m_1 < 2m$  for any  $A \subset G$  with |A| = k we have

$$\mathbb{P}\left(X_1 * \cdots * X_n \in A\right) \leq \mathbb{P}\left(\varepsilon_1 + \cdots + \varepsilon_n \in (-k, k]_{m_1}\right),$$

where  $\varepsilon_i$  are independent uniform random variables on the set  $\{-1,1\} \subset \mathbb{Z}_{m_1}$ . On the other hand, if  $m_1 \geq 2m$  we have

$$\mathbb{P}\left(X_{1} * \cdots * X_{n} \in A\right) \leq \mathbb{P}\left(\tau_{1} + \cdots + \tau_{n} \in I_{n,k}^{m}\right),\,$$

where  $\tau_i$  are independent uniform random variables on the set  $\{0,1\} \subset \mathbb{Z}_m$ .

If true, the latter conjecture would settle the remaining cases.

# 3 Proofs

In order to prove Theorems 1-3, we shall require a simple group theoretic statement contained in the following lemma.

**Lemma 1.** Let G be a group and  $g \in G$  be an element of order greater then or equal to  $m \geq 2$ . Then for any finite set  $A \subset G$  and a positive integer s such that  $s < \frac{m}{|A|}$  we have  $A \neq Ag^s$ .

**Proof of Lemma 1.** Suppose there is a nonempty set  $A \subset G$  and a positive integer s such that  $|A| = k < \frac{m}{s}$  and  $A = Ag^s$ . Take some  $a \in A$  and consider elements  $ag^{si}$ , i = 0...k. All these k+1 elements are in the set A hence at least two of them must be equal. Let us say  $ag^{si} = ag^{sj}$  for some integers  $0 \le i < j \le k$ . But this immediately gives a contradiction since then  $g^{s(j-i)}$  is equal to the group identity element and  $m \le s(j-i) \le sk$ .

**Proof of Theorem 1.** If n=1 the inequality (2) is trivial. For  $k \geq \frac{m}{2}$  and all n the right hand side of (2) becomes 1 since in this case  $(-k, k]_{\tilde{m}}$  covers the support of the sum  $\varepsilon_1 + \cdots + \varepsilon_n$  and so there is nothing to prove. We shall henceforth assume that n > 1 and  $k < \frac{m}{2}$ .

By Lemma 1 we have that  $Ag_n \neq Ag_n^{-1}$ . Take some  $h \in Ag_n \setminus Ag_n^{-1}$  and define  $B = Ag_n \setminus \{h\}$  and  $C = Ag_n^{-1} \cup \{h\}$ . We then have

$$2\mathbb{P}(X_{1} * \cdots * X_{n} \in A) = \mathbb{P}(X_{1} * \cdots * X_{n-1} \in Ag_{n}) + \mathbb{P}(X_{1} * \cdots * X_{n-1} \in Ag_{n}^{-1})$$

$$= \mathbb{P}(X_{1} * \cdots * X_{n-1} \in B) + \mathbb{P}(X_{1} * \cdots * X_{n-1} \in C)$$

$$\leq \mathbb{P}(\varepsilon_{1} + \cdots + \varepsilon_{n-1} \in (-k-1, k+1]_{\tilde{m}}) + \mathbb{P}(\varepsilon_{1} + \cdots + \varepsilon_{n-1} \in (-k+1, k-1]_{\tilde{m}})$$

$$= \mathbb{P}(\varepsilon_{1} + \cdots + \varepsilon_{n-1} \in (-k-1, k-1]_{\tilde{m}}) + \mathbb{P}(\varepsilon_{1} + \cdots + \varepsilon_{n-1} \in (-k+1, k+1]_{\tilde{m}})$$

$$= 2\mathbb{P}(\varepsilon_{1} + \cdots + \varepsilon_{n} \in (-k, k]_{\tilde{m}}).$$

$$(5)$$

This completes the proof.

**Remark 3.** Note that in (6)-(7) we used the fact that for  $k < \frac{m}{2}$  the sets  $(-k+1, k-1]_{\tilde{m}}$  and  $(k-1, k+1]_{\tilde{m}}$  are disjoint in  $\mathbb{Z}_{\tilde{m}}$ .

In the proof of Theorem 3 we shall make use of the following simple lemma which will allow us to switch from general distributions satisfying the condition (4) to two-point distributions.

**Lemma 2.** Let X be a random variable on some group G that takes only finitely many values, say  $x_1, \ldots, x_n$ . Suppose that  $p_i = \mathbb{P}(X = x_i)$  are rational numbers and that  $p_i \leq \frac{1}{2}$ . Then we can express the distribution of X as a convex combination of distributions that are uniform on some two point set.

**Proof of Lemma 2.** Denote by  $\mu$  the distribution of X. Since the  $p_i$ 's are all rational, we have  $p_i = \frac{k_i}{K_i}$  for some  $k_i, K_i \in \mathbb{Z}$ . We shall now view  $\mu$  as a distribution on a multiset M made from the elements  $x_i$  in the following way - take  $x_i$  exactly  $2k_i \prod_{j \neq i} K_i$  times into M. This way  $\mu$  has the uniform distribution on M. We thus have that  $M = \{y_1, \ldots, y_{2N}\}$  for the appropriate N. Construct a graph on the elements on M by joining two of them by an edge if and only if they are distinct. Since we had  $p_i \leq \frac{1}{2}$ , each vertex of this graph

has degree at least N. Thus by Dirac's Theorem, our graph contains a Hamiltonian cycle, and, consequently - a perfect matching. Let  $\mu_i$  be the uniform distribution on two vertices of the latter matching (i = 1, 2, ..., N). We have

$$\mu = \frac{1}{N} \sum_{i=1}^{N} \mu_i.$$

**Proof of Theorem 3.** We shall argue by induction. First notice that the claim of the Theorem is true for n=1. Furthermore, it is also true for  $k \geq m$  since in that case the bound for the probability in question becomes 1. We therefore shall from now on assume that n>1 and  $1\leq k\leq m-1$ . Denote by  $\mu_i$  the distribution of the random variable  $X_i$ . We can without loss of generality assume that each  $X_i$  is concentrated on finitely many points and that for each  $g\in G$  we have  $\mathbb{P}(X_i=g)\in\mathbb{Q}$ . By Lemma 2, each  $\mu_i$  can be written as a convex combination of distributions that are uniform on some two-point set. Define the random variable  $f_i(X_i) = \mathbb{E}_i 1\{X_1 * \cdots * X_n \in A\}$ , where  $\mathbb{E}_i$  stands for integration with respect to all underlying random variables except  $X_i$ . Then for each i we have

$$\mathbb{P}(X_1 * \dots * X_n \in A) = \mathbb{E}f_i(X_i). \tag{8}$$

The latter expectation is linear with respect to the distribution of  $X_i$ . Therefore we can assume that it will be maximized by some choice of two-point distributions coming from the decomposition of  $\mu_i$ . We shall therefore from this point assume that  $X_n$  takes only two values, say  $h_1$  and  $h_2$ , with equal probabilities.

Note that the intervals  $I^m_{n,k}$  have recursive structure. Namely, if  $1 \le k \le m-1$  and we regard them as multisets, we have the relation  $I^m_{n,k} \cup (I^m_{n,k}-1) = I^m_{n-1,k-1} \cup I^m_{n-1,k+1}$ . The pairs on intervals appearing on both sides of the latter equality heavily overlap. This means that we can take one endpoint of  $I^m_{n-1,k+1}$  that does not belong to  $I^m_{n-1,k-1}$  and move it to this shorter interval. The resulting intervals are both of length k and are exactly the intervals  $I^m_{n,k}$  and  $I^m_{n,k}-1$ . We shall use this after the inductive step.

Take a finite set  $A \subset G$  with k elements. Note that the element  $h_2^{-1}h_1 \neq 1_G$  and so it has order at least m. By Lemma 1 we have that  $Ah_1^{-1} \neq Ah_2^{-1}$  as  $A \neq Ah_2^{-1}h_1$ . Take some  $h \in Ah_1^{-1} \backslash Ah_2^{-1}$  and define  $B = Ah_1^{-1} \backslash \{h\}$  and  $C = Ah_2^{-1} \cup \{h\}$ . We have

$$2\mathbb{P}(X_{1}*\cdots*X_{n}\in A) = \mathbb{P}(X_{1}*\cdots*X_{n-1}\in Ah_{1}^{-1}) + \mathbb{P}(X_{1}*\cdots*X_{n-1}\in Ah_{2}^{-1})$$

$$= \mathbb{P}(X_{1}*\cdots*X_{n-1}\in B) + \mathbb{P}(X_{1}*\cdots*X_{n-1}\in C)$$

$$\leq \mathbb{P}(\tau_{1}+\cdots+\tau_{n-1}\in I_{n-1,k-1}^{m}) + \mathbb{P}(\tau_{1}+\cdots+\tau_{n-1}\in I_{n-1,k+1}^{m})$$

$$= \mathbb{P}(\tau_{1}+\cdots+\tau_{n-1}\in I_{n,k}^{m}-1) + \mathbb{P}(\tau_{1}+\cdots+\tau_{n-1}\in I_{n,k}^{m})$$

$$= 2\mathbb{P}(\tau_{1}+\cdots+\tau_{n}\in I_{n,k}^{m}).$$

This completes the proof.

**Proof of Corollary 1.** We shall use an identity on evenly spaced binomial coefficients proved in [1]:

$$\binom{n}{t} + \binom{n}{t+s} + \binom{n}{t+2s} + \dots = \frac{1}{s} \sum_{j=0}^{s-1} \left( 2\cos\frac{i\pi}{s} \right)^n \cos\frac{\pi(n-2t)j}{s}. \tag{9}$$

By Theorem 1 we have

$$\rho(V_n) \le \mathbb{P}\left(\varepsilon_1 + \dots + \varepsilon_n \in (-1, 1]_{\tilde{m}}\right). \tag{10}$$

The right hand of the equation (10) is the sum of binomial probabilities  $\binom{n}{i}/2^n$ , where i is such that 2i-n is congruent to  $1_{\{n\in 2\mathbb{Z}+1\}}$  modulo  $\tilde{m}$ . Let t be the residue of  $(n-1_{\{n\in 2\mathbb{Z}+1\}})/2$  modulo  $\frac{\tilde{m}}{2}$ .

Using the identity (9) and the elementary inequalities  $\cos x \leq \exp(-x^2/2)$  for  $x \in [0, \frac{\pi}{2}]$  and  $\int_0^\infty e^{\frac{-x^2}{2\sigma^2}} dx \leq \frac{\sigma\sqrt{2\pi}}{2}$  we obtain

$$\mathbb{P}(\varepsilon_{1} + \dots + \varepsilon_{n} \in (-1, 1]_{\tilde{m}}) = \frac{\binom{n}{t} + \binom{n}{t + \tilde{m}/2} + \binom{n}{t + 2\tilde{m}/2} + \dots}{2^{n}}$$

$$= \frac{2}{\tilde{m}} \sum_{j=0}^{\frac{\tilde{m}}{2} - 1} \left( 2 \cos \frac{2i\pi}{\tilde{m}} \right)^{n} \cos \frac{2\pi (n - 2t)j}{\tilde{m}}$$

$$\leq \frac{2}{\tilde{m}} + \frac{2}{\tilde{m}} \sum_{j=1}^{\frac{\tilde{m}}{2} - 1} \left| \cos \frac{2j\pi}{\tilde{m}} \right|^{n}$$

$$\leq \frac{2}{\tilde{m}} + \frac{4}{\tilde{m}} \sum_{j=1}^{\lfloor \frac{\tilde{m}}{4} \rfloor} \left| \cos \frac{2j\pi}{\tilde{m}} \right|^{n}$$

$$\leq \frac{2}{\tilde{m}} + \frac{4}{\tilde{m}} \sum_{j=1}^{\lfloor \frac{\tilde{m}}{4} \rfloor} e^{-2\pi^{2}j^{2}n/\tilde{m}^{2}}$$

$$\leq \frac{2}{\tilde{m}} + \frac{4}{\tilde{m}} \int_{0}^{\infty} e^{-2\pi^{2}x^{2}n/\tilde{m}^{2}} dx$$

$$\leq \frac{2}{\tilde{m}} + \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{n}} \leq \frac{2}{m} + \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{n}}.$$

Note that in (11) we replaced  $|\cos \frac{2\pi j}{\tilde{m}}|$  by  $|\cos \frac{\pi(\tilde{m}-2j)}{\tilde{m}}|$  when  $j > \frac{\tilde{m}}{4}$ . This completes the proof.

### References

- [1] A. Benjamin, B. Chen, and K. Kindred, Sums of Evenly Spaced Binomial Coefficients, Mathematics Magazine 83 (2010), 370–373.
- [2] B. Bollobás, *Combinatorics*, Cambridge University Press, Cambridge, 1986, Set systems, hypergraphs, families of vectors and combinatorial probability.
- [3] P. Diaconis, Random walks on groups: characters and geometry, London Mathematical Society Lecture Note Series, vol. 1, pp. 120–142, Cambridge University Press, 2003.
- [4] D. Dzindzalieta, T. Juškevičius, and M. Šileikis, Optimal probability inequalities for random walks related to problems in extremal combinatorics, SIAM J. Discrete Math. 26 (2012), no. 2, 828–837.
- [5] P. Erdős, On a lemma of Littlewood and Offord, Bull. Amer. Math. Soc. **51** (1945), 898–902.
- [6] J. R. Griggs, On the distribution of sums of residues, Bull. Amer. Math. Soc. (N.S.) **28** (1993), no. 2, 329–333.
- [7] D. J. Kleitman, On a lemma of Littlewood and Offord on the distributions of linear combinations of vectors, Advances in Math. 5 (1970), 155–157.
- [8] D. A. Levin, Y. Peres, and E. L. Wilmer, *Markov chains and mixing times*, American Mathematical Society, 2006.
- [9] J. E. Littlewood and A. C. Offord, On the number of real roots of a random algebraic equation. III, Rec. Math. [Mat. Sbornik] N.S. 12 (1943), no. 53, 277–286.
- [10] P. H. Tiep and V. H. Vu, Non-abelian Littlewood-Offord inequalities, Advances in Mathematics **302** (2016), 1233–1250.