

# COLORFUL COVERINGS OF POLYTOPES AND PIERCING NUMBERS OF COLORFUL $d$ -INTERVALS

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ABSTRACT. We prove a common strengthening of Bárány’s colorful Carathéodory theorem and the KKMS theorem. In fact, our main result is a colorful polytopal KKMS theorem, which extends a colorful KKMS theorem due to Shih and Lee [*Math. Ann.* 296 (1993), no. 1, 35–61] as well as a polytopal KKMS theorem due to Komiya [*Econ. Theory* 4 (1994), no. 3, 463–466]. The (seemingly unrelated) colorful Carathéodory theorem is a special case as well. We apply our theorem to establish an upper bound on the piercing number of colorful  $d$ -interval hypergraphs, extending earlier results of Tardos [*Combinatorica* 15 (1995), no. 1, 123–134] and Kaiser [*Discrete Comput. Geom.* 18 (1997), no. 2, 195–203].

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## 1. INTRODUCTION

The KKM theorem of Knaster, Kuratowski, and Mazurkiewicz [11] is a set covering variant of Brouwer’s fixed point theorem. It states that for any covering of the  $k$ -simplex  $\Delta_k$  on vertex set  $[k+1]$  with closed sets  $A_1, \dots, A_{k+1}$  such that the face spanned by vertices in  $S$  is contained in  $\bigcup_{i \in S} A_i$  for every  $S \subset [k+1]$ , the intersection  $\bigcap_{i \in [k+1]} A_i$  is nonempty.

The KKM theorem has inspired many extensions and variants, some of which we will briefly survey in Section 2. Important strengthenings include a colorful extension of the KKM theorem due to Gale [9] that deals with  $k+1$  possibly distinct coverings of the  $k$ -simplex and the KKMS theorem of Shapley [16], where the sets in the covering are associated to faces of the  $k$ -simplex instead of its vertices. Further generalizations of the KKMS theorem are a polytopal version due to Komiya [12] and the colorful KKMS theorem of Shih and Lee [17].

In this note we prove a colorful polytopal KKMS theorem, extending all results above. This result is finally sufficiently general to also specialize to Bárány’s celebrated colorful Carathéodory theorem [5] from 1982, which asserts that if  $X_1, \dots, X_{k+1}$  are subsets of  $\mathbb{R}^k$  with  $0 \in \text{conv } X_i$  for every  $i \in [k+1]$ , then there exists a choice of points  $x_1 \in X_1, \dots, x_{k+1} \in X_{k+1}$  such that  $0 \in \text{conv}\{x_1, \dots, x_{k+1}\}$ . Carathéodory’s classical result is the case  $X_1 = X_2 = \dots = X_{k+1}$ . We deduce the colorful Carathéodory theorem from our main result in Section 3.

For a set  $\sigma \subset \mathbb{R}^k$  we denote by  $C_\sigma$  the *cone of*  $\sigma$ , that is, the union of all rays emanating from the origin that intersect  $\sigma$ . Our main result is the following:

**Theorem 1.1.** *Let  $P$  be a  $k$ -dimensional polytope with  $0 \in P$ . Suppose for every nonempty, proper face  $\sigma$  of  $P$  we are given  $k + 1$  points  $y_\sigma^{(1)}, \dots, y_\sigma^{(k+1)} \in C_\sigma$  and  $k + 1$  closed sets  $A_\sigma^{(1)}, \dots, A_\sigma^{(k+1)} \subset P$ . If  $\sigma \subset \bigcup_{\tau \subset \sigma} A_\tau^{(j)}$  for every face  $\sigma$  of  $P$  and every  $j \in [k + 1]$ , then there exist faces  $\sigma_1, \dots, \sigma_{k+1}$  of  $P$  such that  $0 \in \text{conv}\{y_{\sigma_1}^{(1)}, \dots, y_{\sigma_{k+1}}^{(k+1)}\}$  and  $\bigcap_{i=1}^{k+1} A_{\sigma_i}^{(i)} \neq \emptyset$ .*

Our proof of this result relies on a topological mapping degree argument. As such, it is entirely different from Bárány's proof of the colorful Carathéodory theorem, and thus provides a new topological route to prove this theorem. Our argument is also less involved than the topological proof given recently by Meunier, Mulzer, Sarrabezolles, and Stein [14] to show that algorithmically finding the configuration whose existence is guaranteed by the colorful Carathéodory theorem is in PPAD (that is, informally speaking, it can be found by a path-following algorithm). Our method, however, involves a limiting argument and thus does not have immediate algorithmic consequences. Finally, our proof of Theorem 1.1 exhibits a surprisingly simple way to prove KKMS-type results and their polytopal and colorful extensions.

As an application of Theorem 1.1 we prove a bound on the piercing numbers of colorful  $d$ -interval hypergraphs. A  $d$ -interval is a union of at most  $d$  disjoint closed intervals on  $\mathbb{R}$ . A  $d$ -interval  $h$  is *separated* if it consists of  $d$  disjoint interval components  $h = h^1 \cup \dots \cup h^d$  with  $h^{i+1} \subset (i, i+1)$  for  $i \in \{0, \dots, d-1\}$ . A *hypergraph of (separated)  $d$ -intervals* is a hypergraph  $H$  whose vertex set is  $\mathbb{R}$  and whose edge set is a finite family of (separated)  $d$ -intervals.

A *matching* in a hypergraph  $H = (V, E)$  with vertex set  $V$  and edge set  $E$  is a set of disjoint edges. A *cover* is a subset of  $V$  intersecting all edges. The *matching number*  $\nu(H)$  is the maximal size of a matching, and the *covering number* (or *piercing number*)  $\tau(H)$  is the minimal size of a cover. Tardos [19] and Kaiser [10] proved the following bound on the covering number in hypergraphs of  $d$ -intervals:

**Theorem 1.2** (Tardos [19], Kaiser [10]). *In every hypergraph  $H$  of  $d$ -intervals we have  $\tau(H) \leq (d^2 - d + 1)\nu(H)$ . Moreover, if  $H$  is a hypergraph of separated  $d$ -intervals then  $\tau(H) \leq (d^2 - d)\nu(H)$ .*

Matoušek [13] showed that this bound is not far from the truth: There are examples of hypergraphs of  $d$ -intervals in which  $\tau = \Omega(\frac{d^2}{\log d}\nu)$ . Aharoni, Kaiser and Zerbib [1] gave a proof of Theorem 1.2 that used the KKMS theorem and Komiyama's polytopal extension, Theorem 2.1. Using Theorem 1.1 we prove here a colorful generalization of Theorem 1.2:

**Theorem 1.3.** *1. Let  $\mathcal{F}_i$ ,  $i \in [k+1]$ , be  $k+1$  hypergraphs of  $d$ -intervals and let  $\mathcal{F} = \bigcup_{i=1}^{k+1} \mathcal{F}_i$ . If  $\tau(\mathcal{F}_i) > k$  for all  $i \in [k+1]$ , then there exists a collection  $\mathcal{M}$  of pairwise disjoint  $d$ -intervals in  $\mathcal{F}$  of size  $|\mathcal{M}| \geq \frac{k+1}{d^2-d+1}$ , with  $|\mathcal{M} \cap \mathcal{F}_i| \leq 1$ .*

*2. Let  $\mathcal{F}_i$ ,  $i \in [k+1]$ , be  $kd+1$  hypergraphs of separated  $d$ -intervals and let  $\mathcal{F} = \bigcup_{i=1}^{k+1} \mathcal{F}_i$ . If  $\tau(\mathcal{F}_i) > kd$  for all  $i \in [k+1]$ , then there exists a collection  $\mathcal{M}$  of pairwise disjoint separated  $d$ -intervals in  $\mathcal{F}$  of size  $|\mathcal{M}| \geq \frac{k+1}{d-1}$ , with  $|\mathcal{M} \cap \mathcal{F}_i| \leq 1$ .*

Note that Theorem 1.2 is the case where all the hypergraphs  $\mathcal{F}_i$  are the same. In Section 2 we introduce some notation and, as an introduction to our methods, provide a new simple proof of Komiya's theorem. Then, in Section 3, we prove Theorem 1.1 and use it to derive Bárány's colorful Carathéodory theorem. Section 4 is devoted to the proof of Theorem 1.3.

## 2. COVERINGS OF POLYTOPES AND KOMIYA'S THEOREM

Let  $\Delta_k$  be the  $k$ -dimensional simplex with vertex set  $[k+1]$  realized in  $\mathbb{R}^{k+1}$  as  $\{x \in \mathbb{R}_{\geq 0}^{k+1} : \sum_{i=1}^{k+1} x_i = 0\}$ . For every  $S \subset [k+1]$  let  $\Delta^S$  be the face of  $\Delta_k$  spanned by the vertices in  $S$ . Recall that the KKM theorem asserts that if  $A_1, \dots, A_{k+1}$  are closed sets covering  $\Delta_k$  so that  $\Delta^S \subset \bigcup_{i \in S} A_i$  for every  $S \subset [k+1]$ , then the intersection of all the sets  $A_i$  is non-empty. We will refer to covers  $A_1, \dots, A_{k+1}$  as above as *KKM cover*.

A generalization of this result, known as the KKMS theorem, was proven by Shapley [16] in 1973. Now we have a cover of  $\Delta_k$  by closed sets  $A_T$ ,  $T \subset [k+1]$ , so that  $\Delta^S \subset \bigcup_{T \subset S} A_T$  for every  $S \subset [k+1]$ . Such a collection of sets  $A_T$  is called *KKMS cover*. The conclusion of the KKMS theorem is that there exists a balanced collection of  $T_1, \dots, T_m$  of subsets of  $[k+1]$  for which  $\bigcap_{i=1}^m A_{T_i} \neq \emptyset$ . Here  $T_1, \dots, T_m$  form a balanced collection if the barycenters of the corresponding faces  $\Delta_{T_1}, \dots, \Delta_{T_m}$  contain the barycenter of  $\Delta_k$  in their convex hull.

A different generalization of the KKM theorem is a colorful version due to Gale [9]. It states that given  $k+1$  KKM covers  $A_1^{(i)}, \dots, A_{k+1}^{(i)}$ ,  $i \in [k+1]$ , of the  $k$ -simplex  $\Delta_k$ , there is a permutation  $\pi$  of  $[k+1]$  such that  $\bigcap_{i \in [k+1]} A_{\pi(i)}^{(i)}$  is nonempty. This theorem is colorful in the sense that we think of each KKM cover as having a different color; the theorem then asserts that there is an intersection of  $k+1$  sets of pairwise distinct colors associated to pairwise distinct vertices. Asada et al. [2] showed that one can additionally prescribe  $\pi(1)$ .

In 1993 Shih and Lee [17] proved a common generalization of the KKMS theorem and Gale's colorful KKM theorem: Given  $k+1$  such KKMS covers  $A_T^i$ ,  $T \subset [k+1]$ ,  $i \in [k+1]$ , of  $\Delta_k$ , there exists a balanced collection of  $T_1, \dots, T_{k+1}$  of subsets of  $[k+1]$  for which we have  $\bigcap_{i=1}^m A_{T_i}^i \neq \emptyset$ .

Another far reaching extension of the KKMS theorem to general polytopes is due to Komiya [12] from 1994. Komiya proved that the simplex  $\Delta_k$  in the KKMS theorem can be replaced by any  $k$ -dimensional polytope  $P$ , and that the barycenters of the faces can be replaced by any points  $y_\sigma$  in the face  $\sigma$ :

**Theorem 2.1** (Komiya's theorem [12]). *Let  $P$  be a polytope, and for every nonempty face  $\sigma$  of  $P$  choose a point  $y_\sigma \in \sigma$  and a closed set  $A_\sigma \subset P$ . If  $\sigma \subset \bigcup_{\tau \subset \sigma} A_\tau$  for every face  $\sigma$  of  $P$ , then there are faces  $\sigma_1, \dots, \sigma_m$  of  $P$  such that  $y_P \in \text{conv}\{y_{\sigma_1}, \dots, y_{\sigma_m}\}$  and  $\bigcap_{i=1}^m A_{\sigma_i} \neq \emptyset$ .*

This specializes to the KKMS theorem if  $P$  is the simplex and each point  $y_\sigma$  is the barycenter of the face  $\sigma$ . Moreover, there are quantitative versions of the KKM theorem due to De Loera, Peterson, and Su [6] as well as Asada et al. [2] and KKM theorems for general pairs of spaces due to Musin [15].

To set the stage we will first present a simple proof of Komiya's theorem. Recall that the KKM theorem can be easily deduced from Sperner's lemma on vertex labelings of triangulations of a simplex. Our proof of Komiya's theorem – just as Shapley's original proof of the KKMS theorem – first establishes an equivalent Sperner-type version. A *Sperner–Shapley labeling* of a triangulation  $T$  of a polytope  $P$  is a map  $f: V(T) \rightarrow \{\sigma : \sigma \text{ a nonempty face of } P\}$  from the vertex set  $V(T)$  of  $T$  to the set of nonempty faces of  $P$  such that  $f(v) \subset \text{supp}(v)$ , where  $\text{supp}(v)$  is the minimal face of  $P$  containing  $v$ . We prove the following polytopal Sperner–Shapley theorem that will imply Theorem 2.1 by a limiting and compactness argument:

**Theorem 2.2.** *Let  $T$  be a triangulation of the polytope  $P \subset \mathbb{R}^k$ , and let  $f: V(T) \rightarrow \{\sigma : \sigma \text{ a nonempty face of } P\}$  be a Sperner–Shapley labeling of  $T$ . For every nonempty face  $\sigma$  of  $P$  choose a point  $y_\sigma \in \sigma$ . Then there is a face  $\tau$  of  $T$  such that  $y_P \in \text{conv}\{y_{f(v)} : v \text{ vertex of } \tau\}$ .*

*Proof.* The Sperner–Shapley labeling  $f$  maps vertices of the triangulation  $T$  of  $P$  to faces of  $P$ ; thus mapping vertex  $v$  to the chosen point  $y_{f(v)}$  in the face  $f(v)$  and extending linearly onto faces of  $T$  defines a continuous map  $F: P \rightarrow P$ . By the Sperner–Shapley condition for every face  $\sigma$  of  $P$  we have that  $F(\sigma) \subset \sigma$ . This implies that  $F$  is homotopic to the identity on  $\partial P$ , and thus  $F|_{\partial P}$  has degree one. Then  $F$  is surjective and we can find a point  $x \in P$  such that  $F(x) = y_P$ . Let  $\tau$  be the smallest face of  $T$  containing  $x$ . By definition of  $F$  the image  $F(\tau)$  is equal to the convex hull  $\text{conv}\{y_{f(v)} : v \text{ vertex of } \tau\}$ .  $\square$

*Proof of Theorem 2.1.* Let  $\varepsilon > 0$ , and let  $T$  be a triangulation of  $P$  such that every face of  $T$  has diameter at most  $\varepsilon$ . Given a cover  $\{A_\sigma : \sigma \text{ a nonempty face of } P\}$  that satisfies the covering condition of the theorem we define a Sperner–Shapley labeling in the following way: For a vertex  $v$  of  $T$ , label  $v$  by a face  $\sigma \subset \text{supp}(v)$  such that  $v \in A_\sigma$ . Such a face  $\sigma$  exists since  $v \in \text{supp}(v) \subset \bigcup_{\sigma \subset \text{supp}(v)} A_\sigma$ . Thus by Theorem 2.2 there is a face  $\tau$  of  $T$  whose vertices are labeled by faces  $\sigma_1, \dots, \sigma_m$  of  $P$  such that  $y_P \in \text{conv}\{y_{\sigma_1}, \dots, y_{\sigma_m}\}$ . In particular, the  $\varepsilon$ -neighborhoods of the sets  $A_{\sigma_i}$ ,  $i \in [m]$ , intersect. Now let  $\varepsilon$  tend to zero. As there are only finitely many collections of faces of  $P$ , one collection  $\sigma_1, \dots, \sigma_m$  must appear infinitely many times. By compactness of  $P$  the sets  $A_{\sigma_i}$ ,  $i \in [m]$ , then all intersect since they are closed.  $\square$

Note that Theorem 2.1 is true also if all the sets  $A_\sigma$  are open in  $P$ . Indeed, given an open cover  $\{A_\sigma : \sigma \text{ a nonempty face of } P\}$  of  $P$  as in Theorem 2.1, we can find closed sets  $B_\sigma \subset A_\sigma$  that have the same nerve as  $A_\sigma$  (namely, any collection of sets  $\{B_{\sigma_i} : i \in I\}$  intersects if and only if the corresponding collection  $\{A_{\sigma_i} : i \in I\}$  intersects) and still satisfy  $\sigma \subset \bigcup_{\tau \subset \sigma} B_\tau$  for every face  $\sigma$  of  $P$ .

### 3. A COLORFUL KOMIYA THEOREM

Recall that the colorful KKMS theorem of Shih and Lee [17] states the following: If for every  $i \in [k + 1]$  the collection  $\{A_\sigma^i : \sigma \text{ a nonempty face of } \Delta_k\}$  forms a KKMS cover of  $\Delta_k$ , then

there exists a balanced collection of faces  $\sigma_1, \dots, \sigma_{k+1}$  so that  $\bigcap_{i=1}^{k+1} A_{\sigma_i}^i \neq \emptyset$ . Theorem 1.1, proved in this section, is a colorful extension of Theorem 2.1, and thus generalizes the colorful KKMS theorem to any polytope.

Let  $P$  be a  $k$ -dimensional polytope. Suppose that for every nonempty face  $\sigma$  of  $P$  we choose  $k + 1$  points  $y_\sigma^{(1)}, \dots, y_\sigma^{(k+1)} \in \sigma$  and  $k + 1$  closed sets  $A_\sigma^{(1)}, \dots, A_\sigma^{(k+1)} \subset P$ , so that  $\sigma \subset \bigcup_{\tau \subset \sigma} A_\tau^{(j)}$  for every face  $\sigma$  of  $P$  and every  $j \in [k + 1]$ . Theorem 2.1 now guarantees that for every fixed  $j \in [k + 1]$  there are faces  $\sigma_1^{(j)}, \dots, \sigma_{m_j}^{(j)}$  of  $P$  such that  $y_P^{(j)} \in \text{conv}\{y_{\sigma_1}^{(j)}, \dots, y_{\sigma_{m_j}}^{(j)}\}$  and  $\bigcap_{i=1}^{m_j} A_{\sigma_i}^{(j)}$  is nonempty. Now let us choose  $y_P^{(1)} = y_P^{(2)} = \dots = y_P^{(k+1)}$  and denote this point by  $y_P$ . The colorful Carathéodory theorem implies the existence of points  $z_j \in \{y_{\sigma_1}^{(j)}, \dots, y_{\sigma_{m_j}}^{(j)}\}$ ,  $j \in [k + 1]$ , such that  $y_P \in \text{conv}\{z_1, \dots, z_{k+1}\}$ . Theorem 1.1 shows that this implication can be realized simultaneously with the existence of sets  $B_j \in \{A_{\sigma_1}^{(j)}, \dots, A_{\sigma_{m_j}}^{(j)}\}$ ,  $j \in [k + 1]$ , such that  $\bigcap_{j=1}^{k+1} B_j$  is nonempty. We prove Theorem 1.1 by applying the Sperner–Shapley version of Komiya’s theorem – Theorem 2.2 – to a labeling of the barycentric subdivision of a triangulation of  $P$ . The same idea was used by Su [18] to prove a colorful Sperner’s lemma. For related Sperner-type results for multiple Sperner labelings see Babson [3], Bapat [4], and Frick, Houston-Edwards, and Meunier [7].

*Proof of Theorem 1.1.* Let  $\varepsilon > 0$ , and let  $T$  be a triangulation of  $P$  such that every face of  $T$  has diameter at most  $\varepsilon$ . We will also assume that the chosen points  $y_\sigma^{(1)}, \dots, y_\sigma^{(k+1)}$  are contained in  $\sigma$ . This assumption does not restrict the generality of our proof since  $0 \in \text{conv}\{x_1, \dots, x_{k+1}\}$  for vectors  $x_1, \dots, x_{k+1} \in \mathbb{R}^k$  if and only if  $0 \in \text{conv}\{\alpha_1 x_1, \dots, \alpha_{k+1} x_{k+1}\}$  with arbitrary coefficients  $\alpha_i > 0$ . Denote by  $T'$  the barycentric subdivision of  $T$ . We now define a Sperner–Shapley labeling of the vertices of  $T'$ : For  $v \in V(T')$  let  $\sigma_v$  be the face of  $T$  so that  $v$  lies at the barycenter of  $\sigma_v$ , let  $\ell = \dim \sigma_v$ , and let  $\sigma$  be the minimal supporting face of  $P$  containing  $\sigma_v$ . By the conditions of the theorem,  $v$  is contained in a set  $A_\tau^{(\ell+1)}$  where  $\tau \subset \sigma$ . We label  $v$  by  $\tau$ . Thus by Theorem 2.2 there exists a face  $\tau$  of  $T'$  (without loss of generality  $\tau$  is a facet) whose vertices are labeled by faces  $\sigma_1, \dots, \sigma_{k+1}$  of  $P$  such that  $0 \in \text{conv}\{y_{\sigma_1}^{(1)}, \dots, y_{\sigma_{k+1}}^{(k+1)}\}$ . In particular, the  $\varepsilon$ -neighborhoods of the sets  $A_{\sigma_i}^{(i)}$ ,  $i \in [k + 1]$ , intersect. Now use a limiting argument as before.  $\square$

Note that by the same argument as before, Theorem 1.1 is true also if all the sets  $A_\sigma^{(i)}$  are open.

For a point  $x \neq 0$  in  $\mathbb{R}^k$  let  $H(x) = \{y \in \mathbb{R}^k : \langle x, y \rangle = 0\}$  be the hyperplane perpendicular to  $x$  and let  $H^+(x) = \{y \in \mathbb{R}^k : \langle x, y \rangle \geq 0\}$  be the closed halfspace with boundary  $H(x)$  containing  $x$ . Let us now show that Bárány’s colorful Carathéodory theorem is a special case of Theorem 1.1.

**Theorem 3.1** (Colorful Carathéodory theorem, Bárány [5]). *Let  $X_1, \dots, X_{k+1}$  be finite subsets of  $\mathbb{R}^k$  with  $0 \in \text{conv} X_i$  for every  $i \in [k + 1]$ . Then there are  $x_1 \in X_1, \dots, x_{k+1} \in X_{k+1}$  such that  $0 \in \text{conv}\{x_1, \dots, x_{k+1}\}$ .*

*Proof.* We will assume that 0 is not contained in any of the sets  $X_1, \dots, X_{k+1}$ , for otherwise we are done. Let  $P \subset \mathbb{R}^k$  be a polytope containing 0 in its interior, such that if points  $x$  and  $y$  belong to the same face of  $P$  then  $\langle x, y \rangle \geq 0$ . For example, a sufficiently fine subdivision of any polytope that contains 0 in its interior (slightly perturbed to be a strictly convex polytope) satisfies this condition. We can assume that any ray emanating from the origin intersects each  $X_i$  in at most one point by arbitrarily deleting any additional points from  $X_i$ . This will not affect the property that  $0 \in \text{conv } X_i$ . Furthermore, we can choose  $P$  in such a way that for each face  $\sigma$  and  $i \in [k+1]$  the intersection  $C_\sigma \cap X_i$  contains at most one point.

For every  $i \in [k+1]$  let  $y_P^{(i)} = 0$  and  $A_P^{(i)} = \emptyset$ . Now for each nonempty, proper face  $\sigma$  of  $P$  choose points  $y_\sigma^{(i)}$  and sets  $A_\sigma^{(i)}$  in the following way: If there exists  $x \in C_\sigma \cap X_i$  then let  $y_\sigma^{(i)} = x$  and  $A_\sigma^{(i)} = \{y \in P : \langle y, x \rangle \geq 0\} = P \cap H^+(x)$ ; otherwise let  $y_\sigma^{(i)}$  be some point in  $\sigma$  and let  $A_\sigma^{(i)} = \sigma$ .

Suppose the statement of the theorem was incorrect; then in particular, we can slightly perturb the vertices of  $P$  and those points  $y_\sigma^{(i)}$  that were chosen arbitrarily in  $\sigma$ , to make sure that for any collection of points  $y_{\sigma_1}^{(1)}, \dots, y_{\sigma_{k+1}}^{(k+1)}$  and any subset  $S$  of this collection of size at most  $k$ ,  $0 \notin \text{conv } S$ .

Let us now check that with these definitions the conditions of Theorem 1.1 hold. Clearly, all the sets  $A_\sigma^{(i)}$  are closed. The fact that  $P$  is covered by the sets  $A_\sigma^{(i)}$  for every fixed  $i$  follows from the condition  $0 \in \text{conv } X_i$ . Indeed, this condition implies that for every  $p \in P$  there exists a point  $x \in X_i$  with  $\langle p, x \rangle \geq 0$ , and therefore, for the face  $\sigma$  of  $P$  for which  $x \in C_\sigma$  we have  $p \in A_\sigma^{(i)}$ .

Now fix a proper face  $\sigma$  of  $P$ . We claim that  $\sigma \subset A_\sigma^{(i)}$  for every  $i$ . Indeed, either  $X_i \cap C_\sigma = \emptyset$  in which case  $A_\sigma^{(i)} = \sigma$ , or otherwise, pick  $x \in X_i \cap C_\sigma$  and let  $\lambda > 0$  such that  $\lambda x \in \sigma$ ; then for every  $p \in \sigma$  we have  $\langle p, \lambda x \rangle \geq 0$  by our assumption on  $P$ , and thus  $\langle p, x \rangle \geq 0$ , or equivalently  $p \in A_\sigma^{(i)}$ .

Thus by Theorem 1.1 there exist faces  $\sigma_1, \dots, \sigma_{k+1}$  of  $P$  such that  $\bigcap_{i=1}^{k+1} A_{\sigma_i}^{(i)} \neq \emptyset$  and  $0 \in \text{conv}\{y_{\sigma_1}^{(1)}, \dots, y_{\sigma_{k+1}}^{(k+1)}\}$ . We claim that  $\bigcap_{i=1}^{k+1} A_{\sigma_i}^{(i)}$  can contain only the origin. Indeed, suppose that  $0 \neq x_0 \in \bigcap_{i=1}^{k+1} A_{\sigma_i}^{(i)}$ . Fix  $i \in [k+1]$ . If  $y_{\sigma_i}^{(i)} \in C_{\sigma_i} \cap X_i$ , then since  $x_0 \in A_{\sigma_i}^{(i)}$  we have  $y_{\sigma_i}^{(i)} \in H^+(x_0)$  by definition. Otherwise  $x_0 \in A_{\sigma_i}^{(i)} = \sigma_i$  and  $y_{\sigma_i}^{(i)} \in \sigma_i$ , so by our choice of  $P$  we obtain again that  $y_{\sigma_i}^{(i)} \in H^+(x_0)$ . Thus all the points  $y_{\sigma_1}^{(1)}, \dots, y_{\sigma_{k+1}}^{(k+1)}$  are in  $H^+(x_0)$ . But since  $0 \in \text{conv}\{y_{\sigma_1}^{(1)}, \dots, y_{\sigma_{k+1}}^{(k+1)}\}$  this implies that the convex hull of the points in  $\{y_{\sigma_1}^{(1)}, \dots, y_{\sigma_{k+1}}^{(k+1)}\} \cap H(x_0)$  contains the origin. Now, the dimension of  $H(x_0)$  is  $k-1$ , and thus by Carathéodory's theorem there exists a set  $S$  of at most  $k$  of the points in  $y_{\sigma_1}^{(1)}, \dots, y_{\sigma_{k+1}}^{(k+1)}$  with  $0 \in \text{conv } S$ , in contradiction to our general position assumption.

We have shown that  $\bigcap_{i=1}^{k+1} A_{\sigma_i}^{(i)} = \{0\}$ , and thus in particular,  $A_{\sigma_i}^{(i)} \neq \sigma_i$  for all  $i$ . By our definitions, this implies  $y_{\sigma_i}^{(i)} \in X_i$  for all  $i$ , concluding the proof of the theorem.  $\square$

*Remark 3.2.* Note that we could have avoided the usage of Carathéodory's theorem in the proof of Theorem 3.1 by taking a more restrictive assumption on the polytope  $P$ , namely,

that  $\langle x, y \rangle > 0$  whenever the points  $x$  and  $y$  belong to the same face of  $P$ . Therefore, in particular, Theorem 3.1 specializes to Carathéodory's theorem in the case where all the sets  $X_i$  are the same.

#### 4. A COLORFUL $d$ -INTERVAL THEOREM

Recall that a *fractional matching* in a hypergraph  $H = (V, E)$  is a function  $f: E \rightarrow \mathbb{R}_{\geq 0}$  satisfying  $\sum_{e: e \ni v} f(e) \leq 1$  for all  $v \in V$ . A *fractional cover* is a function  $g: V \rightarrow \mathbb{R}_{\geq 0}$  satisfying  $\sum_{v: v \in e} g(v) \geq 1$  for all  $e \in E$ . The *fractional matching number*  $\nu^*(H)$  is the maximum of  $\sum_{e \in E} f(e)$  over all fractional matchings  $f$  of  $H$ , and the *fractional covering number*  $\tau^*(H)$  is the minimum of  $\sum_{v \in V} g(v)$  over all fractional covers  $g$ . By linear programming duality,  $\nu \leq \nu^* = \tau^* \leq \tau$ . A *perfect fractional matching* in  $H$  is a fractional matching  $f$  in which  $\sum_{e: v \in e} f(e) = 1$  for every  $v \in V$ . It is a simple observation that a collection of sets  $\mathcal{I} \subset 2^{[k+1]}$  is balanced if and only if the hypergraph  $H = ([k+1], \mathcal{I})$  has a perfect fractional matching (see e.g., [1]). The *rank* of a hypergraph  $H = (V, E)$  is the maximal size of an edge in  $H$ .  $H$  is  $d$ -partite if there exists a partition  $V_1, \dots, V_d$  of  $V$  such that  $|e \cap V_i| = 1$  for every  $e \in E$  and  $i \in [d]$ .

For the proof of Theorem 1.3 we will use the following theorem by Füredi.

**Theorem 4.1** (Füredi [8]). *If  $H$  is a hypergraph of rank  $d \geq 2$ , then  $\nu(H) \geq \frac{\nu^*(H)}{d-1+\frac{1}{d}}$ . If, in addition,  $H$  is  $d$ -partite, then  $\nu(H) \geq \frac{\nu^*(H)}{d-1}$ .*

We will also need the following simple counting argument.

**Lemma 4.2.** *If a hypergraph  $H = (V, E)$  of rank  $d$  has a perfect fractional matching, then  $\nu^*(H) \geq \frac{|V|}{d}$ .*

*Proof.* Let  $f: E \rightarrow \mathbb{R}_{\geq 0}$  be a perfect fractional matching of  $H$ . Then  $\sum_{v \in V} \sum_{e: v \in e} f(e) = \sum_{v \in V} 1 = |V|$ . Since  $f(e)$  was counted  $|e| \leq d$  times in this equation for every edge  $e \in E$ , we have that  $\nu^*(H) \geq \sum_{e \in E} f(e) \geq \frac{|V|}{d}$ .  $\square$

We are now ready to prove Theorem 1.3. The proof is an adaption of the methods in [1]. For the first part we need the simplex version of Theorem 1.1, which was already proven by Shih and Lee [17], while the second part requires our more general polytopal extension.

*Proof of Theorem 1.3.* For a point  $\vec{x} = (x_1, \dots, x_{k+1}) \in \Delta_k$  let  $p_{\vec{x}}(j) = \sum_{t=1}^j x_t \in [0, 1]$ . Since  $\mathcal{F}$  is finite, by rescaling  $\mathbb{R}$  we may assume that  $\mathcal{F} \subset (0, 1)$ . For every  $T \subset [k+1]$  let  $A_T^i$  be the set consisting of all  $\vec{x} \in \Delta_k$  for which there exists a  $d$ -interval  $f \in \mathcal{F}_i$  satisfying:

- (a)  $f \subset \bigcup_{j \in T} (p_{\vec{x}}(j-1), p_{\vec{x}}(j))$ , and
- (b)  $f \cap (p_{\vec{x}}(j-1), p_{\vec{x}}(j)) \neq \emptyset$  for each  $j \in T$ .

Note that  $A_T^i = \emptyset$  whenever  $|T| > d$ .

Clearly, the sets  $A_T^i$  are open. The assumption  $\tau(\mathcal{F}_i) > k$  implies that for every  $\vec{x} = (x_1, \dots, x_{k+1}) \in \Delta_k$ , the set  $P(\vec{x}) = \{p_{\vec{x}}(j) : j \in [k]\}$  is not a cover of  $\mathcal{F}_i$ , meaning that

there exists  $f \in \mathcal{F}_i$  not containing any  $p_{\vec{x}}(j)$ . This, in turn, means that  $\vec{x} \in A_T^i$  for some  $T \subseteq [k+1]$ , and thus the sets  $A_T^i$  form a cover of  $\Delta_k$  for every  $i \in [k+1]$ .

To show that this is a KKMS cover, let  $\Delta^S$  be a face of  $\Delta_k$  for some  $S \subset [k+1]$ . If  $\vec{x} \in \Delta^S$  then  $(p_{\vec{x}}(j-1), p_{\vec{x}}(j)) = \emptyset$  for  $j \notin S$ , and hence it is impossible to have  $f \cap (p_{\vec{x}}(j-1), p_{\vec{x}}(j)) \neq \emptyset$ . Thus  $\vec{x} \in A_T^i$  for some  $T \subseteq S$ . This proves that  $\Delta^S \subseteq \bigcup_{T \subseteq S} A_T^i$  for all  $i \in [k+1]$ .

By Theorem 1.1 there exists a balanced collection of sets  $\mathcal{T} = \{T_1, \dots, T_{k+1}\}$  of subsets of  $[k+1]$ , satisfying  $\bigcap_{i=1}^{k+1} A_{T_i}^i \neq \emptyset$ . In particular,  $|T_i| \leq d$  for all  $i$ . (Recall that we think of a collection of sets  $\mathcal{I} \subset 2^{[k+1]}$  as faces of the  $k$ -dimensional simplex to apply the earlier geometric definition of balancedness.) Then by the observation mentioned above, the hypergraph  $H = ([k+1], \mathcal{T})$  of rank  $d$  has a perfect fractional matching, and thus by Lemma 4.2 we have  $\nu^*(H) \geq \frac{k+1}{d}$ . Therefore, by Theorem 4.1,  $\nu(H) \geq \frac{\nu^*(H)}{d-1+\frac{1}{d}} \geq \frac{k+1}{d^2-d+1}$ .

Let  $M$  be a matching in  $H$  of size  $m \geq \frac{k+1}{d^2-d+1}$ . Let  $\vec{x} \in \bigcap_{i=1}^{k+1} A_{T_i}^i$ . For every  $i \in [k+1]$  let  $f(T_i)$  be the  $d$ -interval of  $\mathcal{F}_i$  witnessing the fact that  $\vec{x} \in A_{T_i}^i$ . Then the set  $\mathcal{M} = \{f(T_i) \mid T_i \in M\}$  is a matching of size  $m$  in  $\mathcal{F}$  with  $|\mathcal{M} \cap \mathcal{F}_i| \leq 1$ . This proves the first assertion of the theorem.

Now suppose that  $\mathcal{F}_i$  is a hypergraph of separated  $d$ -intervals for all  $i \in [kd+1]$ . For  $f \in \mathcal{F}$  let  $f^t \subset (t-1, t)$  be the  $t$ -th interval component of  $f$ . We can assume without loss of generality that  $f^t$  is nonempty. Let  $P = (\Delta_k)^d$ . For a  $d$ -tuple  $T = (j_1, \dots, j_d) \subset [k+1]^d$  let  $A_T^i$  consist of all  $\vec{X} = \vec{x}^1 \times \dots \times \vec{x}^d \in P$  for which there exists  $f \in \mathcal{F}_i$  satisfying  $f^t \subset (t-1 + p_{\vec{x}^t}(j_t-1), t-1 + p_{\vec{x}^t}(j_t))$  for all  $t \in [d]$ .

Since  $\tau(\mathcal{F}_i) > kd$ , the points  $t-1 + p_{\vec{x}^t}(j)$ ,  $t \in [d]$ ,  $j \in [k]$ , do not form a cover of  $\mathcal{F}_i$ . Therefore, by the same argument as before, the sets  $A_T^i$  are open and satisfy the covering condition of Theorem 1.1. Thus, by Theorem 1.1, there exists a set  $\mathcal{T} = \{T_1, \dots, T_{kd+1}\}$  of  $d$ -tuples in  $[k+1]^d$  containing the point  $(\frac{1}{k+1}, \dots, \frac{1}{k+1}) \times \dots \times (\frac{1}{k+1}, \dots, \frac{1}{k+1}) \in P$  in its convex hull and satisfying  $\bigcap_{i \in [kd+1]} A_{T_i}^i \neq \emptyset$ . Then the  $d$ -partite hypergraph  $H = (\bigcup_{i=1}^d V_i, \mathcal{T})$ , where  $V_i = [k+1]$  for all  $i$ , has a perfect fractional matching, and hence by Lemma 4.2 we have  $\nu^*(H) \geq k+1$ . By Theorem 4.1, this implies  $\nu(H) \geq \frac{\nu^*(H)}{d-1} \geq \frac{k+1}{d-1}$ . Now, by the same argument as before, by taking  $\vec{X} \in \bigcap_{i \in [kd+1]} A_{T_i}^i$  we obtain a matching in  $\mathcal{F}$  of the same size as a maximal matching in  $H$ , concluding the proof of the theorem.  $\square$

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## REFERENCES

1. R. Aharoni, T. Kaiser, and S. Zerbib, *Fractional covers and matchings in families of weighted  $d$ -intervals*, *Combinatorica* **37** (2017), no. 4, 555–572.
2. M. Asada, F. Frick, V. Pisharody, M. Polevy, D. Stoner, L. Tsang, and Z. Wellner, *Fair division and generalizations of Sperner- and KKM-type results*, *SIAM J. Discrete Math.* **32**(1), 591–610 (2018).
3. E. Babson, *Meunier conjecture*, arXiv preprint arXiv:1209.0102 (2012).
4. R. B. Bapat, *A constructive proof of a permutation-based generalization of Sperner’s lemma*, *Math. Program.* **44** (1989), no. 1-3, 113–120.
5. I. Bárány, *A generalization of Carathéodory’s theorem*, *Discrete Math.* **40** (1982), no. 2-3, 141–152.
6. J. A. De Loera, E. Peterson, and F. E. Su, *A polytopal generalization of Sperner’s lemma*, *J. Combin. Theory, Ser. A* **100** (2002), no. 1, 1–26.
7. F. Frick, K. Houston-Edwards, and F. Meunier, *Achieving rental harmony with a secretive roommate*, *Amer. Math. Monthly*, to appear.
8. Z. Füredi, *Maximum degree and fractional matchings in uniform hypergraphs*, *Combinatorica* **1** (1981), no. 2, 155–162.
9. D. Gale, *Equilibrium in a discrete exchange economy with money*, *Int. J. Game Theory* **13** (1984), no. 1, 61–64.
10. T. Kaiser, *Transversals of  $d$ -intervals*, *Discrete Comput. Geom.* **18** (1997), no. 2, 195–203.
11. B. Knaster, C. Kuratowski, and S. Mazurkiewicz, *Ein Beweis des Fixpunktsatzes für  $n$ -dimensionale Simplexe*, *Fund. Math.* **14** (1929), no. 1, 132–137.
12. H. Komiyama, *A simple proof of KKMS theorem*, *Econ. Theory* **4** (1994), no. 3, 463–466.
13. J. Matoušek, *Lower bounds on the transversal numbers of  $d$ -intervals*, *Discrete Comput. Geom.* **26** (2001), no. 3, 283–287.
14. F. Meunier, W. Mulzer, P. Sarrabezolles, and Y. Stein, *The rainbow at the end of the line: a PPAD formulation of the colorful Carathéodory theorem with applications*, *Proceedings of the Twenty-Eighth Annual ACM-SIAM Symposium on Discrete Algorithms*, Society for Industrial and Applied Mathematics, 2017, pp. 1342–1351.
15. O. R. Musin, *KKM type theorems with boundary conditions*, *J. Fixed Point Theory Appl.* **19** (2017), no. 3, 2037–2049.
16. L. S. Shapley, *On balanced games without side payments*, *Math. Program., Math. Res. Center Publ.* (T. C. Hu and S. M. Robinson, eds.), vol. 30, Academic Press, New York, 1973, pp. 261–290.
17. M. Shih and S. Lee, *Combinatorial formulae for multiple set-valued labellings*, *Math. Ann.* **296** (1993), no. 1, 35–61.
18. F. E. Su, *Rental harmony: Sperner’s lemma in fair division*, *Amer. Math. Monthly* **106** (1999), no. 10, 930–942.
19. G. Tardos, *Transversals of 2-intervals, a topological approach*, *Combinatorica* **15** (1995), no. 1, 123–134.

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