

CLUSTERED COLOURING IN MINOR-CLOSED CLASSES

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Abstract. The *clustered chromatic number* of a class of graphs is the minimum integer k such that for some integer c every graph in the class is k -colourable with monochromatic components of size at most c . We prove that for every graph H , the clustered chromatic number of the class of H -minor-free graphs is tied to the tree-depth of H . In particular, if H is connected with tree-depth t then every H -minor-free graph is $(2^{t+1} - 4)$ -colourable with monochromatic components of size at most $c(H)$. This provides the first evidence for a conjecture of Ossona de Mendez, Oum and Wood (2016) about defective colouring of H -minor-free graphs. If $t = 3$ then we prove that 4 colours suffice, which is best possible. We also determine those minor-closed graph classes with clustered chromatic number 2. Finally, we develop a conjecture for the clustered chromatic number of an arbitrary minor-closed class.

1 Introduction

In a vertex-coloured graph, a *monochromatic component* is a connected component of the subgraph induced by all the vertices of one colour. A graph G is *k -colourable with clustering c* if each vertex can be assigned one of k colours such that each monochromatic component has at most c vertices. We shall consider such colourings, where the first priority is to minimise the number of colours, with small clustering as a secondary goal. With this viewpoint the following definition arises. The *clustered chromatic number* of a graph class \mathcal{G} , denoted by $\chi_*(\mathcal{G})$, is the minimum integer k such that, for some integer c , every graph in \mathcal{G} has a k -colouring with clustering c . See [24] for a survey on clustered graph colouring.

This paper studies clustered colouring in minor-closed classes of graphs. A graph H is a *minor* of a graph G if a graph isomorphic to H can be obtained from some subgraph of G by contracting edges. A class of graphs \mathcal{M} is *minor-closed* if for every graph $G \in \mathcal{M}$ every minor of G is in \mathcal{M} , and some graph is not in \mathcal{M} . For a graph H , let \mathcal{M}_H be the class of H -minor-free graphs (that is, not containing H as a minor). Note that we only consider simple finite graphs.

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As a starting point, consider Hadwiger's Conjecture, which states that every graph containing no K_t -minor is properly $(t - 1)$ -colourable. This conjecture is easy for $t \leq 4$, is equivalent to the 4-colour theorem for $t = 5$, is true for $t = 6$ [18], and is open for $t \geq 7$. The best known upper bound on the chromatic number is $O(t\sqrt{\log t})$, independently due to Kostochka [9, 10] and Thomason [20, 21]. This conjecture is widely considered to be one of the most important open problems in graph theory; see [19] for a survey.

Clustered colourings of K_t -minor-free graphs provide an avenue for attacking Hadwiger's Conjecture. Kawarabayashi and Mohar [8] first proved a $O(t)$ upper bound on $\chi_*(\mathcal{M}_{K_t})$. In particular, they proved that every K_t -minor-free graph is $\lceil \frac{31}{2}t \rceil$ -colourable with clustering $f(t)$, for some function f . The number of colours in this result was improved to $\lceil \frac{7t-3}{2} \rceil$ by Wood [23], to $4t - 4$ by Edwards, Kang, Kim, Oum, and Seymour [5], to $3t - 3$ by Liu and Oum [12], and to $2t - 2$ by Norin [14]. Thus $\chi_*(\mathcal{M}_{K_t}) \leq 2t - 2$. See [6, 7] for analogous results for graphs excluding odd minors. For all of these results, the function $f(t)$ is very large, often depending on constants from the Graph Minor Structure Theorem. Van den Heuvel and Wood [22] proved the first such result with $f(t)$ explicit. In particular, they proved that every K_t -minor-free graph is $(2t - 2)$ -colourable with clustering $\lceil \frac{t-2}{2} \rceil$. The result of Edwards et al. [5] mentioned below implies that $\chi_*(\mathcal{M}_{K_t}) \geq t - 1$. Dvořák and Norin [4] have announced a proof that $\chi_*(\mathcal{M}_{K_t}) = t - 1$.

Now consider the class \mathcal{M}_H of H -minor-free graphs for an arbitrary graph H . The maximum chromatic number of a graph in \mathcal{M}_H is at most $O(|V(H)|\sqrt{\log |V(H)|})$ and is at least $|V(H)| - 1$ (since $K_{|V(H)|-1}$ is H -minor-free), and Hadwiger's Conjecture would imply that $|V(H)| - 1$ is the answer. However, for clustered colourings, fewer colours often suffice. For example, Dvořák and Norin [4] proved that graphs embeddable on any fixed surface are 4-colourable with bounded clustering, whereas the chromatic number is $\Theta(\sqrt{g})$ for surfaces of Euler genus g . Van den Heuvel and Wood [22] proved that $K_{2,t}$ -minor-free graphs are 3-colourable with clustering $t - 1$, and that $K_{3,t}$ -minor-free graphs are 6-colourable with clustering $2t$. These results show that $\chi_*(\mathcal{M}_H)$ depends on the structure of H , unlike the usual chromatic number which only depends on $|V(H)|$.

At the heart of this paper is the following question: what property of H determines $\chi_*(\mathcal{M}_H)$? The following definitions help to answer this question. Let T be a rooted tree. The *depth* of T is the maximum number of vertices on a root-to-leaf path in T . The *closure* of T is obtained from T by adding an edge between every ancestor and descendent in T . The *connected tree-depth* of a graph H , denoted by $\overline{\text{td}}(H)$, is the minimum depth of a rooted tree T such that H is a subgraph of the closure of T . This definition is a variant of the more commonly used definition of the *tree-depth* of H , denoted by $\text{td}(H)$, which equals the maximum connected tree-depth of the connected components of H . See [13] for background on tree-depth. If H is connected, then $\text{td}(H) = \overline{\text{td}}(H)$. In fact, $\text{td}(H) = \overline{\text{td}}(H)$ unless H has two connected components H_1 and H_2 with $\text{td}(H_1) = \text{td}(H_2) = \text{td}(H)$, in which case $\overline{\text{td}}(H) = \text{td}(H) + 1$. We choose to work with connected tree-depth to avoid this distinction.

The following result is the primary contribution of this paper; it is proved in Section 2.

Theorem 1. *For every graph H , $\chi_\star(\mathcal{M}_H)$ is tied to the (connected) tree-depth of H . In particular,*

$$\overline{\text{td}}(H) - 1 \leq \chi_\star(\mathcal{M}_H) \leq 2^{\overline{\text{td}}(H)+1} - 4.$$

The upper bound in Theorem 1 gives evidence for, and was inspired by, a conjecture of Ossona de Mendez, Oum, and Wood [15], which we now introduce. A graph G is k -colourable with defect d if each vertex of G can be assigned one of k colours so that each vertex is adjacent to at most d neighbours of the same colour; that is, each monochromatic component has maximum degree at most d . The *defective chromatic number* of a graph class \mathcal{G} , denoted by $\chi_\Delta(\mathcal{G})$, is the minimum integer k such that, for some integer d , every graph in \mathcal{G} is k -colourable with defect d . Every colouring of a graph with clustering c has defect $c - 1$. Thus the defective chromatic number of a graph class is at most its clustered chromatic number. Ossona de Mendez et al. [15] conjectured the following behaviour for the defective chromatic number of \mathcal{M}_H .

Conjecture 2 ([15]). *For every graph H ,*

$$\chi_\Delta(\mathcal{M}_H) = \overline{\text{td}}(H) - 1.$$

Ossona de Mendez et al. [15] proved the lower bound, $\chi_\Delta(\mathcal{M}_H) \geq \overline{\text{td}}(H) - 1$, in Conjecture 2. This follows from the observation that the closure of the rooted complete c -ary tree of depth k is not $(k-1)$ -colourable with clustering c . The lower bound in Theorem 1 follows since $\chi_\Delta \leq \chi_\star$ for every class. The upper bound in Conjecture 2 is known to hold in some special cases. Edwards et al. [5] proved it if $H = K_t$; that is, $\chi_\Delta(\mathcal{M}_{K_t}) = t - 1$, which can be thought of as a defective version of Hadwiger's Conjecture. Ossona de Mendez et al. [15] proved the upper bound in Conjecture 2 if $\overline{\text{td}}(H) \leq 3$ or if H is a complete bipartite graph. In particular, $\chi_\Delta(\mathcal{M}_{K_{s,t}}) = \min\{s, t\}$.

Theorem 1 provides some evidence for Conjecture 2 by showing that $\chi_\Delta(\mathcal{M}_H)$ and $\chi_\star(\mathcal{M}_H)$ are bounded from above by some function of $\overline{\text{td}}(H)$. This was previously not known to be true.

While it is conjectured that $\chi_\Delta(\mathcal{M}_H) = \overline{\text{td}}(H) - 1$, the following lower bound, proved in Section 2.3, shows that $\chi_\star(\mathcal{M}_H)$ might be larger, thus providing some distinction between defective and clustered colourings.

Theorem 3. *For each $k \geq 2$, there is a graph H_k with $\overline{\text{td}}(H_k) = \text{td}(H_k) = k$ such that*

$$\chi_\star(\mathcal{M}_{H_k}) \geq 2k - 2.$$

We conjecture an analogous upper bound:

Conjecture 4. *For every graph H ,*

$$\chi_\star(\mathcal{M}_H) \leq 2\overline{\text{td}}(H) - 2.$$

A further contribution of the paper is to precisely determine the minor-closed graph classes with clustered chromatic number 2. This result is introduced and proved in Section 3. Section 4 studies clustered colourings of graph classes excluding so-called fat stars as a minor. This leads to a proof of Conjecture 4 in the $\overline{\text{td}}(H) = 3$ case. We conclude in Section 5 with a conjecture about the clustered chromatic number of an arbitrary minor-closed class that generalises Conjecture 4.

2 Tree-depth Bounds

The main goal of this section is to prove that $\chi_*(\mathcal{M}_H)$ is bounded from above by some function of $\overline{\text{td}}(H)$. We actually provide two proofs. The first proof depends on deep results from graph structure theory and gives no explicit bound on the clustering. The second proof is self-contained, but gives a worse upper bound on the number of colours. Both proofs have their own merits, so we include both.

Let $C\langle h, k \rangle$ be the closure of the rooted complete k -ary tree of depth h . (Here each non-leaf node has exactly k children.)

If r is a vertex in a connected graph G and $V_i := \{v \in V(G) : \text{dist}_G(v, r) = i\}$ for $i \geq 0$, then V_0, V_1, \dots is called the *BFS layering* of G starting at r .

2.1 First Proof

The first proof depends on the following Erdős-Pósa Theorem by Robertson and Seymour [17]. For a graph H and integer $p \geq 1$, let pH be the disjoint union of p copies of H .

Theorem 5 ([17]; see [16, Lemma 3.10]). *For every non-empty graph H with c connected components and for all integers $p, w \geq 1$, for every graph G with treewidth at most w and containing no pH minor, there is a set $X \subseteq V(G)$ of size at most pwc such that $G - X$ has no H minor.*

The next lemma is the heart of our proof.

Lemma 6. *For all integers $h, k, w \geq 1$, every $C\langle h, k \rangle$ -minor-free graph G of treewidth at most w is $(2^h - 2)$ -colourable with clustering kw .*

Proof. We proceed by induction on $h \geq 1$, with w and k fixed. The case $h = 1$ is trivial since $C\langle 1, k \rangle$ is the 1-vertex graph, so only the empty graph has no $C\langle 1, k \rangle$ minor, and the empty graph is 0-colourable with clustering 0. Now assume that $h \geq 2$, the claim holds for $h - 1$, and G is a $C\langle h, k \rangle$ -minor-free graph with treewidth at most w . Let V_0, V_1, \dots be the BFS layering of G starting at some vertex r .

Fix $i \geq 1$. Then $G[V_i]$ contains no $kC\langle h - 1, k \rangle$ as a minor, as otherwise contracting $V_0 \cup \dots \cup V_{i-1}$ to a single vertex gives a $C\langle h, k \rangle$ minor (since every vertex in V_i has a neighbour in V_{i-1}). Since G has treewidth at most w , so does $G[V_i]$. By Theorem 5 with $H = C\langle h - 1, k \rangle$ and $c = 1$, there is a set $X_i \subseteq V_i$ of size at most kw , such that $G[V_i \setminus X_i]$ has no $C\langle h - 1, k \rangle$ minor. By induction, $G[V_i \setminus X_i]$ is $(2^{h-1} - 2)$ -colourable with clustering kw . Use one new colour for X_i . Thus $G[V_i]$ is $(2^{h-1} - 1)$ -colourable with clustering kw .

Use disjoint sets of colours for even and odd i , and colour r by one of the colours used for even i . No edge joins V_i with V_j for $j \geq i + 2$. Thus G is $(2^h - 2)$ -coloured with clustering kw . \square

To drop the assumption of bounded treewidth, we use the following result of DeVos, Ding, Oporowski, Sanders, Reed, Seymour, and Vertigan [3], the proof of which depends on the graph minor structure theorem.

Theorem 7 ([3]). *For every graph H there is an integer w such that for every graph G containing no H -minor, there is a partition V_1, V_2 of $V(G)$ such that $G[V_i]$ has treewidth at most w , for $i \in \{1, 2\}$.*

Lemma 6 and Theorem 7 imply:

Lemma 8. *For all integers $h, k \geq 1$, there is an integer $g(h, k)$, such that every $C\langle h, k \rangle$ -minor-free graph G is $(2^{h+1} - 4)$ -colourable with clustering at most $g(h, k)$.*

Fix a graph H . By definition, H is a subgraph of $C\langle \text{td}(H), |V(H)| \rangle$. Thus every H -minor-free graph contains no $C\langle \text{td}(H), |V(H)| \rangle$ -minor. Hence, Lemma 8 implies

$$\chi_*(\mathcal{M}_H) \leq 2^{\text{td}(H)+1} - 4,$$

which is the upper bound in Theorem 1.

Note Theorem 26 below improves the $h = 3$ case in Lemma 6, which leads to a small constant-factor improvement in Theorem 1 for $h \geq 3$.

2.2 Second Proof

We now present our second proof that $\chi_*(\mathcal{M}_H)$ is bounded from above by some function of $\text{td}(H)$. This proof is self-contained (not using Theorems 5 and 7).

Let T be a rooted tree. Recall that the *closure* of T is the graph G with vertex set $V(T)$, where two vertices are adjacent in G if one is an ancestor of the other in T . The *weak closure* of T is the graph G with vertex set $V(T)$, where two vertices are adjacent in G if one is a leaf and the other is one of its ancestors. For $h, k \geq 1$, let $T\langle h, k \rangle$ be the rooted complete k -ary tree of depth h . Let $W\langle h, k \rangle$ be the weak closure of $T\langle h, k \rangle$.

Lemma 9. *For $h, k \geq 2$, the graph $W\langle h, k \rangle$ contains $C\langle h, k - 1 \rangle$ as a minor.*

Proof. Let r be the root vertex. Colour r blue. For each non-leaf vertex v , colour $k - 1$ children of v blue and colour the other child of v red. Let X be the set of blue vertices v in $T\langle h, k \rangle$, such that every ancestor of v is blue. Note that X induces a copy of $T\langle h, k - 1 \rangle$ in $T\langle h, k \rangle$. Let v be a non-leaf vertex in X . Let w be the red child of v , and let T_w be the subtree of $T\langle h, k \rangle$ rooted at w . Then every leaf of T_w is adjacent in $W\langle h, k \rangle$ to v and to every ancestor of v . Contract T_w and the edge vw into v . Now v is adjacent to every ancestor of v in X . Do this for each non-leaf vertex in X . Note that T_u and T_v are disjoint for distinct non-leaf vertices $u, v \in X$. Thus, we obtain $C\langle h, k - 1 \rangle$ as a minor of $W\langle h, k \rangle$. \square

A *model* of a graph H in a graph G is a collection $\{J_x : x \in V(H)\}$ of pairwise disjoint subtrees of G such that for every $xy \in E(H)$ there is an edge of G with one end in $V(J_x)$ and the other end in $V(J_y)$. Observe that a graph contains H as a minor if and only if it contains a model of H .

Lemma 10. *For $h \geq 2$ and $k \geq 1$, if a graph G contains $W\langle h, 6k \rangle$ as a minor, then G contains subgraphs G' and G'' , both containing $W\langle h, k \rangle$ as a minor, such that $|V(G') \cap V(G'')| \leq 1$.*

Proof. Let $\{J_x : x \in V(W\langle h, 6k \rangle)\}$ be a model of $W\langle h, 6k \rangle$ in G . Let r be the root vertex of $W\langle h, 6k \rangle$. We may assume that for each leaf vertex x of $T\langle h, 6k \rangle$, there is exactly one edge between J_x and J_r .

Let Q be a tree obtained from J_r by splitting vertices, where:

- Q has maximum degree at most 3,
- J_r is a minor of Q ; let $\{Q_v : v \in V(J_r)\}$ be the model of J_r in Q , so each edge vw of J_r corresponds to an edge of Q between Q_v and Q_w ,
- there is a set L of leaf vertices in Q , and a bijection ϕ from L to the set of leaves of $T\langle h, 6k \rangle$, such that for each leaf x of $T\langle h, 6k \rangle$, if the edge between J_x and J_r in G is incident with vertex v in J_r , then $\phi^{-1}(x)$ is a vertex z in $L \cap Q_v$, in which case we say x and z are *associated*.

Let $L' \subseteq L$. Apply the following ‘propagation’ process in $T\langle h, 6k \rangle$. Initially, say that the vertices in $\phi(L')$ are *alive* with respect to L' . For each parent vertex y of leaves in $T\langle h, 6k \rangle$, if at least $2k$ of its $6k$ children are alive with respect to L' , then y is also alive with respect to L' . Now propagate up $T\langle h, 6k \rangle$, so that a non-leaf vertex y of $T\langle h, 6k \rangle$ is *alive* if and only if at least $2k$ of its children are alive with respect to L' . Say L' is *good* if r is alive with respect to L' .

For an edge vw of Q let L_{vw} be the set of vertices in L in the subtree of $Q - vw$ containing v , and let L_{wv} be the set of vertices in L in the subtree of $Q - vw$ containing w . Since L is the disjoint union of L_{vw} and L_{wv} , every leaf vertex of $T\langle h, 6k \rangle$ is in exactly one of $\phi(L_{vw})$ or $\phi(L_{wv})$. By induction, every vertex in $T\langle h, 6k \rangle$ is alive with respect to L_{vw} or L_{wv} (possibly both). In particular, L_{vw} or L_{wv} is good (possibly both).

Suppose that both L_{vw} and L_{wv} are good. Then at least $2k$ children of r are alive with respect to L_{vw} , and at least $2k$ children of r are alive with respect to L_{wv} . Thus there are disjoint sets A and B , each consisting of k children of r , where every vertex in A is alive with respect to L_{vw} , and every vertex in B is alive with respect to L_{wv} . We now define a set of vertices, said to be *chosen* by v , all of which are alive with respect to L_{vw} . First, each vertex in A is *chosen* by v . Then for each non-leaf vertex z chosen by v , choose k children of z that are also alive with respect to L_{vw} , and say they are *chosen* by v . Continue this process down to the leaves of $T\langle h, 6k \rangle$. We now define the graph G' , which is initially empty. For each vertex z chosen by v , add the subgraph

J_z to G' . Furthermore, for each leaf vertex z of $T\langle h, 6k \rangle$ chosen by v and for each ancestor y of z chosen by v , add the edge in G between J_z and J_y to G' . Define G'' analogously with respect to B and L_{wv} . At this point, G' and G'' are disjoint.

The edge vw in Q either corresponds to an edge or a vertex of J_r . First suppose that vw corresponds to an edge ab of J_r , where v is in Q_a and w is in Q_b . Let J_r^1 be the subtree of $J_r - ab$ containing a . Add J_r^1 to G' , plus the edge in G between J_r^1 and J_z for each leaf z of $T\langle h, 6k \rangle$ chosen by v . Similarly, let J_r^2 be the subtree of $J_r - ab$ containing b , and add J_r^2 to G'' , plus the edge in G between J_r^2 and J_z for each leaf z of $T\langle h, 6k \rangle$ chosen by w . Observe that G' and G'' are disjoint, and they both contain $W\langle h, k \rangle$ as a minor, as desired.

Now consider the case in which vw corresponds to a vertex z in J_r ; that is, v and w are both in Q_z . Let J_r^1 be the subtree of J_r corresponding to the subtree of $Q - vw$ containing v (which includes z). Add J_r^1 to G' , plus the edge in G between J_r^1 and J_z for each leaf z of $T\langle h, 6k \rangle$ chosen by v . Similarly, let J_r^2 be the subtree of J_r corresponding to the subtree of $Q - vw$ containing w (which includes z). Add J_r^2 to G'' , plus the edge in G between J_r^2 and J_z for each leaf z of $T\langle h, 6k \rangle$ chosen by w . Observe that both G' and G'' contain $W\langle h, k \rangle$ as a minor, and $V(G_1) \cap V(G_2) = \{z\}$, as desired.

We may therefore assume that for each edge vw of Q , exactly one of L_{vw} and L_{wv} is good. Orient vw towards v if L_{vw} is good, and towards w if L_{wv} is good. Since at most one leaf of $T\langle h, 6k \rangle$ is associated with each leaf of Q , each edge incident with a leaf of Q is oriented away from the leaf. Since Q is a tree, Q contains a sink vertex v , which is therefore not a leaf. Let w_1, w_2 and possibly w_3 be the neighbours of v in Q . Let L_i be the set of vertices in L in the subtree of $Q - vw_i$ containing w_i . Since vw_i is oriented towards v , with respect to vw_i , the set L_i is not good. Since no leaf of $T\langle h, 6k \rangle$ is associated with v , the sets $\phi(L_1)$, $\phi(L_2)$ and $\phi(L_3)$ partition the leaves of $T\langle h, 6k \rangle$. Since each non-leaf vertex y in $T\langle h, 6k \rangle$ has $6k$ children, y is alive with respect to at least one of L_1, L_2 or L_3 . In particular, at least one of L_1, L_2 or L_3 is good. This is a contradiction. \square

Theorem 11. *Let $f(h) := \frac{1}{6}(4^h - 4)$ for every $h \geq 1$. Then there is a function $g : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ such that for every $k \geq 1$, every graph either contains $W\langle h, k \rangle$ as a minor or is $f(h)$ -colourable with clustering $g(h, k)$.*

Proof. We proceed by induction on $h \geq 1$. In the base case, $h = 1$, since $W\langle 1, k \rangle$ is the 1-vertex graph, the result holds with $f(1) = g(1, k) = 0$. Now assume that $h \geq 2$ and the result holds for $h - 1$ and all k .

Let G be a graph, which we may assume is connected. Let V_0, V_1, \dots be a BFS layering of G .

Fix $i \geq 1$. Let s be the maximum integer such that $G[V_i]$ contains s disjoint subgraphs G_1, \dots, G_s each containing a $W\langle h - 1, \max\{1, 6^{k-s}\}k \rangle$ minor. First suppose that $s \geq k$. Then $G[V_i]$ contains k disjoint subgraphs each containing a $W\langle h - 1, k \rangle$ minor. Contracting $V_0 \cup \dots \cup V_{i-1}$ to a single vertex gives a $W\langle h, k \rangle$ minor (since every vertex in V_i has a neighbour in V_{i-1}), and we are done. Now assume that $s \leq k - 1$.

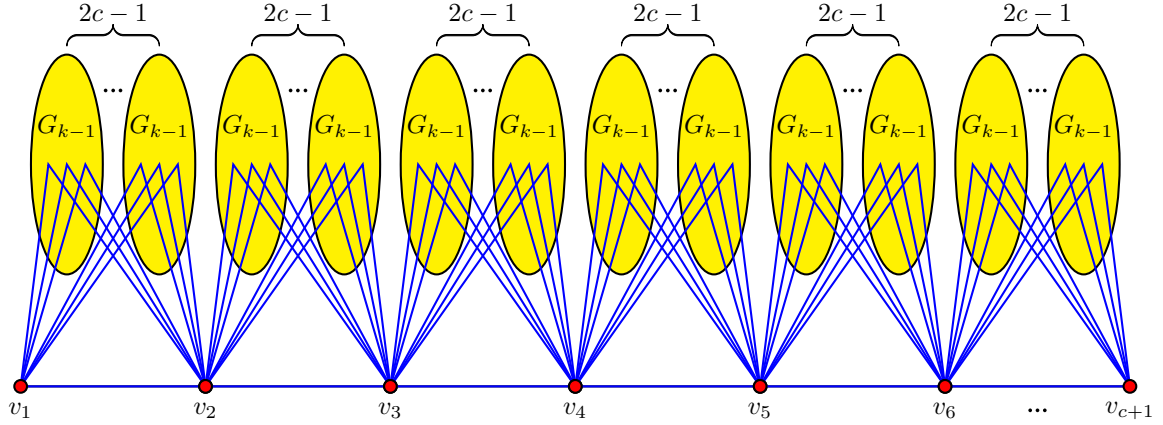


Figure 1: Construction of G_k .

For the base case $k = 2$, let G_2 be the path on $c + 1$ vertices. Then G_2 has no $C\langle 2, 3 \rangle = K_{1,3}$ minor, and G_2 has no 1-colouring with clustering c .

Assume G_{k-1} is defined for some $k \geq 3$, that G_{k-1} has no $(2k - 5)$ -colouring with clustering c , and $C\langle k - 1, 3 \rangle$ is not a minor of G_{k-1} . As illustrated in Figure 1, let G_k be obtained from a path (v_1, \dots, v_{c+1}) as follows: for $i \in \{1, \dots, c\}$ add $2c - 1$ pairwise disjoint copies of G_{k-1} complete to $\{v_i, v_{i+1}\}$.

Suppose that G_k has a $(2k - 3)$ -colouring with clustering c . Then v_i and v_{i+1} receive distinct colours for some $i \in \{1, \dots, c\}$. Consider the $2c - 1$ copies of G_{k-1} complete to $\{v_i, v_{i+1}\}$. At most $c - 1$ such copies contain a vertex assigned the same colour as v_i , and at most $c - 1$ such copies contain a vertex assigned the same colour as v_{i+1} . Thus some copy avoids both colours. Hence G_{k-1} is $(2k - 5)$ -coloured with clustering c , which is a contradiction. Therefore G_k has no $(2k - 3)$ -colouring with clustering c .

It remains to show that $C\langle k, 3 \rangle$ is not a minor of G_k . Suppose that G_k contains a model $\{J_x : x \in V(C\langle k, 3 \rangle)\}$ of $C\langle k, 3 \rangle$. Let r be the root vertex in $C\langle k, 3 \rangle$. Choose the $C\langle k, 3 \rangle$ -model to minimise $\sum_{x \in V(C\langle k, 3 \rangle)} |V(J_x)|$. Since $\{v_1, \dots, v_{c+1}\}$ induces a connected dominating subgraph in G_k , by the minimality of the model, J_r is a connected subgraph of (v_1, \dots, v_{c+1}) . Say $J_r = (v_i, \dots, v_j)$. Note that $C\langle k, 3 \rangle - r$ consists of three pairwise disjoint copies of $C\langle k - 1, 3 \rangle$. The model X of one such copy avoids v_{i-1} and v_{j+1} (if these vertices are defined). Since $C\langle k - 1, 3 \rangle$ is connected, X is contained in a component of $G_k - \{v_{i-1}, \dots, v_{j+1}\}$ and is adjacent to (v_i, \dots, v_j) . Each such component is a copy of G_{k-1} . Thus $C\langle k - 1, 3 \rangle$ is a minor of G_{k-1} , which is a contradiction. Thus $C\langle k, 3 \rangle$ is not a minor of G_k . \square

3 2-Colouring with Bounded Clustering

This section considers the following question: which minor-closed graph classes have clustered chromatic number 2? To answer this question we introduce three classes of graphs that are not

2-colourable with bounded clustering, as illustrated in Figure 2.

The first example is the n -fan, which is the graph obtained from the n -vertex path by adding one dominant vertex. If the n -fan is 2-colourable with clustering c , then the underlying path contains at most $c - 1$ vertices of the same colour as the dominant vertex, implying that the other colour has at most c monochromatic components each with at most c vertices, and $n \leq c^2 + c - 1$. That is, if $n \geq c^2 + c$ then the n -fan is not 2-colourable with clustering c .

The second example is the n -fat star, which is the graph obtained from the n -star (the star with n leaves) as follows: for each edge vw in the n -star, add n degree-2 vertices adjacent to v and w . Note that the n -fat star is $C\langle 3, n \rangle$. Suppose that the n -fat star has a 2-colouring with clustering $c \leq n$. Deleting the dominant vertex in the n -fat star gives n disjoint n -stars. Since $n \geq c$, in at least one of these n -stars, no vertex receives the same colour as the dominant vertex, implying there is a monochromatic component on $n + 1 \geq c + 1$ vertices. Thus, for $n \geq c$ there is no 2-colouring of the n -fat star with clustering c .

The third example is the n -fat path, which is the graph obtained from the n -vertex path as follows: for each edge vw of the n -vertex path, add n degree-2 vertices adjacent to v and w . If $n \geq 2c - 1$ then in every 2-colouring of the n -fat path with clustering c , adjacent vertices in the underlying path receive the same colour, implying that the underlying path is contained in a monochromatic component with more than c vertices. Thus, for $n \geq 2c - 1$ there is no 2-colouring of the n -fat path with clustering c .

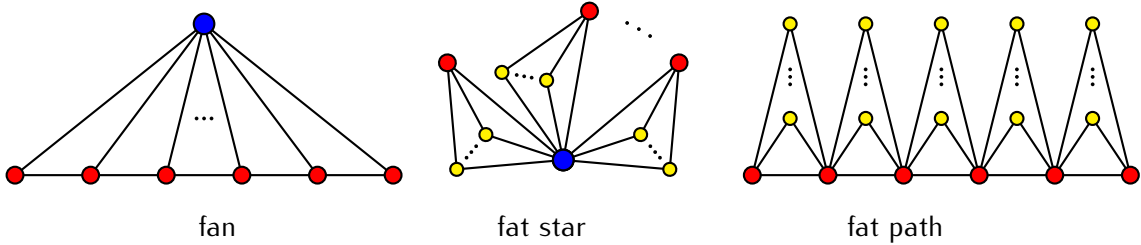


Figure 2: Graph classes that are not 2-colourable with bounded clustering.

These three examples all need three colours in a colouring with bounded clustering. The main result of this section is the following converse result.

Theorem 14. *Let \mathcal{G} be a minor-closed graph class. Then $\chi_*(\mathcal{G}) \leq 2$ if and only if for some integer $k \geq 2$, the k -fan, the k -fat path, and the k -fat star are not in \mathcal{G} .*

Lemma 24 below shows that every graph containing no k -fan minor, no k -fat path minor, and no k -fat star minor is 2-colourable with clustering $f(k)$ for some explicit function f . Along with the above discussion, this implies Theorem 14. We assume $k \geq 2$ for the remainder of this section.

The following definition is a key to the proof. For an h -vertex graph H with vertex set $\{v_1, \dots, v_h\}$, a k -strong H -model in a graph G consists of h pairwise disjoint connected subgraphs X_1, \dots, X_h

in G , such that for each edge $v_i v_j$ of H there are at least k vertices in $V(G) \setminus \bigcup_{i=1}^h V(X_i)$ adjacent to both X_i and X_j . Note that a vertex in $V(G) \setminus \bigcup_{i=1}^h V(X_i)$ might count towards this set of k vertices for distinct edges of H . This definition leads to the following sufficient condition for a graph to contain a k -fat star or k -fat path

Lemma 15. *If a graph G contains a $k(k+1)$ -strong H -model for some connected graph H with k^k edges, then G contains a k -fat star or a k -fat path as a minor.*

Proof. Use the notation introduced in the definition of k -strong H -model. Since H is connected with k^k edges, H contains a k -vertex path or a k -leaf star as a subgraph. Suppose that (v_1, \dots, v_k) is a k -vertex path in H . For $i = 1, 2, \dots, k-1$, let N_i be a set of $k+1$ vertices in

$$\left(V(G) \setminus \bigcup_{j=1}^h V(X_j) \right) \setminus \bigcup_{j=1}^{i-1} N_j,$$

each of which is adjacent to both X_i and X_{i+1} . Such a set exists since X_i and X_{i+1} have at least $k(k+1)$ common neighbours in $V(G) \setminus \bigcup_{j=1}^h V(X_j)$. For $i \in [1, k-1]$, contract one vertex of N_i into X_i . Then contract each of X_1, \dots, X_h into a single vertex. We obtain the k -fat path as a minor in G . The case of a k -leaf star is analogous. \square

Lemma 16. *If a connected graph G contains a $(k+2c-2)$ -strong H -model, for some graph H with c connected components, then G contains a k -strong H' -model for some connected graph H' with $|E(H')| = |E(H)|$.*

Proof. We proceed by induction on $c \geq 1$. The case $c = 1$ is vacuous. Assume $c \geq 2$, and the result holds for $c-1$. Let H_1, \dots, H_c be the components of H . We may assume that H has no isolated vertices. Say X_1, \dots, X_h is a $(k+2c-2)$ -strong H -model in G . For each edge $v_i v_j$ in H , let N_{ij} be a set of $k+2c-2$ common neighbours of X_i and X_j . For each component H_a of H , note that $(\bigcup_{v_i \in V(H_a)} V(X_i)) \cup (\bigcup_{v_i v_j \in E(H_a)} N_{ij})$ induces a connected subgraph in G , which we denote by G_a . Since G is connected, there is a path P between G_a and G_b , for some distinct $a, b \in [1, c]$, such that no internal vertex of P is in $G_1 \cup \dots \cup G_c$. Note that P might be a single vertex. For some edge $v_i v_{i'}$ in H_a and some edge $v_j v_{j'}$ in H_b , without loss of generality, P joins some vertex x in $V(X_i) \cup N_{ii'}$ and some vertex y in $V(X_j) \cup N_{jj'}$. Let H' be the graph obtained from H by identifying v_i and v_j into a new vertex v_0 . Now H' has $c-1$ components and $|E(H')| = |E(H)|$. Define $X_0 := X_i \cup X_j \cup P$. If $x \notin V(X_i)$ then add the edge between x and X_i to X_0 . Similarly, if $y \notin V(X_j)$ then add the edge between y and X_j to X_0 . Remove x and/or y from $N_{\alpha\beta}$ for each edge $v_\alpha v_\beta$ of H' . Now $|N_{\alpha\beta}| \geq k+2(c-1)-2$. We obtain a $(k+2(c-1)-2)$ -strong H' -model in G . By induction, G contains a k -strong H'' -model for some connected graph H'' with $|E(H'')| = |E(H)|$. \square

Lemma 17. *If a connected graph G contains a $3k^k$ -strong H -model for some graph H with at least k^k edges, then G contains a k -fat star or a k -fat path as a minor.*

Proof. We may assume that H has exactly k^k edges and has no isolated vertices. Say H has c connected components. Then $c \leq k^k$ and $3k^k \geq k^2 + k + 2c - 2$. Hence G contains a $(k^2 + k + 2c - 2)$ -strong H -model. The result then follows from Lemmas 15 and 16. \square

Lemma 18. *Let G be a connected graph such that $\deg_G(v) \geq 2\ell k$ for some non-cut-vertex v and integers $k, \ell \geq 1$. Then G contains a k -fan as a minor, or G contains a connected subgraph X and v has ℓ neighbours not in X and all adjacent to X (thus contracting X gives a $K_{2,\ell}$ minor).*

Proof. Let r be a vertex of $G - v$. For each $w \in N_G(v)$, let P_w be a wr -path in $G - v$. If $|P_w \cap N_G(v)| \geq k$ for some $w \in N_G(v)$, then G contains a k -fan minor. Now assume that $|P_w \cap N_G(v)| \leq k - 1$ for each $w \in N_G(v)$. Let H be the digraph with vertex set $N_G(v)$, where $N_H^+(w) := V(P_w) \cap N_G(v)$ for each vertex w . Thus H has maximum outdegree at most $k - 1$, and the underlying undirected graph of H has average degree at most $2k - 2$. Since $|V(H)| \geq 2\ell k$, by Turán's Theorem, H contains a stable set S of size ℓ . Let $X := \bigcup\{P_w : w \in S\} - S$, which is connected since S is stable. Each vertex in S is adjacent to v and to X , as desired. \square

Lemma 19. *Let G be a graph with distinct vertices v_1, \dots, v_k , such that $C := G - \{v_1, \dots, v_k\}$ is connected and $\deg_C(v_i) \geq k^3$ for each $i \in [1, k]$. Then G contains a k -fan or k -fat star as a minor.*

Proof. The idea of the proof is to attempt to build a k -fan model by constructing a subtree X such that each v_i is adjacent to a subset S_i of k leaves of X (where the S_i are disjoint). We construct X and the S_i by adding, one at a time, paths to some neighbour w of some v_i to increase the size of S_i . We always choose a neighbour at maximal distance from some root vertex, among all neighbours of all v_i for which S_i is not yet large enough: this ensures that later paths will not pass through the sets S_i that have been previously constructed.

We now formalise this idea. Let r be a vertex in C . Let V_0, V_1, \dots, V_n be a BFS layering of C starting at r . Initialise $t := n$ and $X := \{r\}$ and $S_i := \emptyset$ for $i \in [1, k]$ and $S := \emptyset$. The following properties trivially hold:

- (0) $S = \bigcup_{i \in [1, k]} S_i$ and $S \subseteq V_t \cup V_{t+1} \cup \dots \cup V_n$.
- (1) X is a (connected) subtree of C rooted at r with (non-root) leaf set S .
- (2) $S_i \cap S_j = \emptyset$ for distinct $i, j \in [1, k]$.
- (3) S_i is a set of at most $k + 1$ neighbours of v_i for $i \in [1, k]$ (and so $|S| \leq k(k + 1)$).
- (4) $|N_{C-V(X)}(v_i)| \geq k^3 - 1 - (k - 1)|S| > 0$ for $i \in [1, k]$.

Now execute the following algorithm, which maintains properties (0) – (4). Think of V_t as the ‘current’ layer.

While $|S_i| \leq k$ for some $i \in [1, k]$ repeat the following: If $V_t \cap N_{C-V(X)}(v_i) = \emptyset$ for all $i \in [1, k]$ with $|S_i| \leq k$, then let $t := t - 1$. Properties (0) – (4) are trivially maintained. Otherwise, let w be a vertex in $V_t \cap N_{C-V(X)}(v_i)$ for some $i \in [1, k]$ with $|S_i| \leq k$. Since V_0, V_1, \dots, V_n is a BFS layering of C rooted at r and r is in X , there is a path P from w to X consisting of at most one vertex from each of V_0, \dots, V_t , and with no internal vertices in X . By (0) and since $w \notin S$, P avoids S . By (1), the endpoint of P in X is not a leaf of X . If P contains at least k vertices in $N_C(v_j)$ for some $j \in [1, k]$, then G contains a k -fan minor and we are done. Now assume that P contains at most $k - 1$ vertices in $N_C(v_j)$ for each $j \in [1, k]$. Let $S_i := S_i \cup \{w\}$ and $S := S \cup \{w\}$

and $X := X \cup P$. Now w is a leaf of X , and property (1) is maintained. Properties (0), (2) and (3) are maintained by construction. Property (4) is maintained since $|S|$ increases by 1 and P contains at most $k - 1$ vertices in $N_C(v_j)$ for each $j \in [1, k]$.

The algorithm terminates when $|S_i| = k + 1$ for each $i \in [1, k]$. Delete $C - V(X)$. Contract $X - S$ (which is connected by (1)) to a single vertex z . Since S is the set of leaves of X , each vertex in S_i is adjacent to both v_i and z . Contract one edge between v_i and S_i for each $i \in [1, k]$. We obtain the k -fat star as a minor. \square

Lemma 20. *Let G be a bipartite graph with bipartition A, B , such that at least p vertices in A have degree at least $k|A|$, and every vertex in B has degree at least 2. Then G contains a k -strong H -model for some graph H with at least $p/2$ edges.*

Proof. Let H be the graph with $V(H) := A$ where $vw \in E(H)$ whenever $|N_G(v) \cap N_G(w)| \geq k$. Since every vertex in B has degree at least 2, every vertex in A with degree at least $k|A|$ is incident with some edge in H . Thus H has at least $p/2$ edges. By construction, G contains a k -strong H -model. \square

For the remainder of this section, let $d := (k + 2)k^k(18k^{2k+1} + 1)$. A vertex v is *high-degree* if $\deg(v) \geq d$, otherwise v is *low-degree*.

Lemma 21. *If a 2-connected graph G has at least $(k + 2)k^k$ high-degree vertices, then G contains a k -fat path, a k -fat star, or a k -fan as a minor.*

Proof. Let A be a set of exactly $(k + 2)k^k$ high-degree vertices in G . Let C_1, \dots, C_p be the components of $G - A$. Say (v, C_j) is a *heavy pair* if $v \in A$ and v has at least $6k^{k+1}$ neighbours in C_j . Since $6k^{k+1} \geq k^3$, by Lemma 19, if some C_j is in at least k heavy pairs, then G contains a k -fan or k -fat star as a minor, and we are done. Now assume that each C_j is in fewer than k heavy pairs. Let h be the total number of heavy pairs. Then there is a set P of at least h/k heavy pairs containing at most one heavy pair for each component C_j . For each such heavy pair (v, C_j) , by Lemma 18 with $\ell = 3k^k$, $G[V(C_j) \cup \{v\}]$ contains a k -fan as a minor (and we are done) or a $K_{2,3k^k}$ minor, where $G[\{v\}]$ is the subgraph corresponding to one of the vertices in the colour class of size 2 in $K_{2,3k^k}$. We obtain a $3k^k$ -strong H -model for some graph H , where $|E(H)| = |P| \geq h/k$. If $h/k \geq k^k$, then we are done by Lemma 17. Now assume that $h < k^{k+1}$. In particular, the number of vertices in A that are in a heavy pair is less than k^{k+1} . Let A' be the set of vertices in A in no heavy pair; thus $|A'| \geq 2k^k$. Let H be the bipartite graph with bipartition A, B , where there is one vertex w_j in B for each component C_j , and $v \in A$ is adjacent to $w_j \in B$ if and only if v is adjacent to some vertex in C_j . In H , every vertex in A' has degree at least $(d - |A|)/6k^{k+1}$, which is at least $3k^k|A|$. (Note that d is defined so that this property holds.) Since G is 2-connected, each C_j is adjacent to at least two vertices in A . Thus every vertex in B has degree at least 2 in H . By Lemma 20, H contains a $3k^k$ -strong model of a graph with at least $|A'|/2 \geq k^k$ edges. By Lemma 17 we are done. \square

Lemma 22. *Let V_0, V_1, \dots be a BFS layering in a connected graph G . If $G[V_i \cup V_{i+1} \cup \dots \cup V_{i+c}]$ contains a path on at least k^{c+1} vertices for some $i, c \geq 0$, then G contains a k -fan minor.*

Proof. We proceed by induction on c . Let P be a path in $G[V_i \cup V_{i+1} \cup \dots \cup V_{i+c}]$ on k^{c+1} vertices. First suppose that P contains k vertices v_1, \dots, v_k in V_i (which must happen in the base case $c = 0$). Each vertex v_i has a neighbour in V_{i-1} . Thus, contracting $G[V_0 \cup \dots \cup V_{i-1}]$ into a single vertex and contracting P between v_i and v_{i+1} to an edge (for $i \in [1, k-1]$) gives a k -fan minor. Now assume that P contains at most $k-1$ vertices in V_i and $c \geq 1$. Thus $P - V_i$ has at least $k^{c+1} - (k-1)$ vertices and at most k components. Thus some component of $P - V_i$ has at least $\lceil (k^{c+1} - k + 1)/k \rceil = k^c$ vertices and is contained in $G[V_{i+1} \cup V_{i+2} \cup \dots \cup V_{i+c}]$. By induction, G contains a k -fan minor. \square

Say a vertex v in a coloured graph is *properly* coloured if no neighbour of v gets the same colour as v .

Lemma 23. *Let G be a 2-connected graph containing no k -fan, k -fat star or k -fat path as a minor. Let h be the number of high-degree vertices in G . Let r be a vertex in G . Then G is 2-colourable with clustering at most $d^{k^3(k+2)k^k}$. Moreover, if $h = 0$ then we can additionally demand that r is properly coloured.*

Proof. Let V_0, V_1, \dots be the BFS layering of G starting at r .

First suppose that $h = 0$. Colour each vertex $v \in V_i$ by $i \bmod 2$. Then r is properly coloured. Every monochromatic component is contained in some V_i . Suppose that some component X of $G[V_i]$ has at least d^k vertices. Thus $i \geq 1$. Since G and thus X has maximum degree at most d , X contains a path of k vertices. Contracting $G[V_0 \cup \dots \cup V_{i-1}]$ into a single vertex gives a k -fan minor. This contradiction shows that the 2-colouring has clustering at most d^k .

Now assume that $h \geq 1$. By Lemma 21, $h \leq (k+2)k^k$. Colour all the high-degree vertices black. Let I be the set of integers $i \geq 0$ such that V_i contains a high-degree vertex. Colour all the low-degree vertices in $\bigcup \{V_i : i \in I\}$ white.

Let $V_i, V_{i+1}, \dots, V_{i+c}$ be a maximal sequence of layers with no high-degree vertices, where $c \geq 0$. Thus V_{i-1} is empty or contains a high-degree vertex. Similarly, V_{i+c+1} is empty or contains a high-degree vertex. If c is even, then colour $V_i \cup V_{i+2} \cup \dots \cup V_{i+c}$ white and colour $V_{i+1} \cup V_{i+3} \cup \dots \cup V_{i+c-1}$ black. If c is odd, then colour $V_i \cup V_{i+2} \cup \dots \cup V_{i+c-1}$ and V_{i+c} white, and colour $V_{i+1} \cup V_{i+3} \cup \dots \cup V_{i+c-2}$ black. Note that if $c \geq 2$ then at least one of $V_{i+1}, \dots, V_{i+c-1}$ is black.

We now show that each black component X has bounded size. If X contains some high-degree vertex, then every vertex in X is high-degree and $|X| \leq h \leq (k+2)k^k$. Now assume that X contains no high-degree vertices. Say X intersects V_j . Since each black layer is preceded by and followed by a white layer, X is contained in V_j . Every vertex in X has degree at most d in G . Thus if X has at least d^k vertices, then X contains a path of length k , and contracting $V_0 \cup \dots \cup V_{j-1}$ to a single vertex gives a k -fan. Hence X has at most d^k vertices.

Finally, let X be a white component. Then X is contained within at most $3h \leq 3(k+2)k^k$ consecutive layers (since in the notation above, if all of $V_i, V_{i+1}, \dots, V_{i+c}$ are white, then $c \leq 1$). Suppose that $|X| \geq d^{k^{3(k+2)k^k}}$. Since X has maximum degree at most d , X contains a path of length $k^{3(k+2)k^k}$. Thus, Lemma 22 with $c+1 = 3(k+2)k^k$ implies that G contains a k -fan minor. Hence $|X| \leq d^{k^{3(k+2)k^k}}$. \square

We now complete the proof of Theorem 14.

Lemma 24. *Let G be a graph containing no k -fan, no k -fat path, and no k -fat star as a minor. Then G is 2-colourable with clustering $kd^{k^{3(k+2)k^k}}$.*

Proof. We may assume that G is connected. Let r be a vertex of G . If B is a block of G containing r , then consider B to be rooted at r . If B is a block of G not containing r , then consider B to be rooted at the unique vertex in B that separates B from r . Say (B, v) is a *high-degree pair* if B is a block of G and v has high-degree in B . Note that one vertex might be in several high-degree pairs.

Suppose that some vertex v is in at least k high-degree pairs with blocks B_1, \dots, B_k . Since $d \geq 2k(k+1)$, by Lemma 18 with $\ell = k+1$, for $i \in [k]$, there is a connected subgraph X_i in $B_i - v$ and there is a set $N_i \subseteq N_{B_i}(v) \setminus V(X_i)$ of size $k+1$, such that each vertex in N_i is adjacent to X_i . For $i \in [1, k]$, contract X_i into a single vertex, and contract one edge between v and N_i . We obtain a k -fat star as a minor. Now assume that each vertex is in fewer than k high-degree pairs.

Colour each block B in non-decreasing order of the distance in G from r to the root of B . Let B be a block of G rooted at v (possibly equal to r). Then v is already coloured in the parent block of B . Let h_B be the number of high-degree pairs involving B . By Lemma 23, B is 2-colourable with clustering at most $d^{k^{3(k+2)k^k}}$, such that if $h_B = 0$ then v is properly coloured. Permute the colours in B so that the colour assigned to v matches the colour assigned to v by the parent block. Then the monochromatic component containing v is contained within the parent block of B along with those blocks rooted at v that form a high-degree pair with v . As shown above, there are at most k such blocks. Thus each monochromatic component has at most $kd^{k^{3(k+2)k^k}}$ vertices. \square

4 Excluding a Fat Star

This section considers colourings of graphs excluding a fat star. We need the following more general lemma.

Lemma 25. *For every planar graph H ,*

$$\chi_*(\mathcal{M}_H) \leq 2\chi_\Delta(\mathcal{M}_H).$$

Proof. The grid minor theorem of Robertson and Seymour [17] says that every graph in \mathcal{M}_H has tree-width at most some function $w(H)$. (Chekuri and Chuzhoy [2] recently showed that w can be

taken to be polynomial in $|V(H)|$.) Alon, Ding, Oporowski, and Vertigan [1] observed that every graph with tree-width w and maximum degree Δ is 2-colourable with clustering $24w\Delta$. Let $k := \chi_\Delta(\mathcal{M}_H)$. That is, every H -minor-free graph G is k -colourable with monochromatic components of maximum degree at most some function $d(H)$. Apply the above result of Alon et al. [1] to each monochromatic component. Thus G is $2k$ -colourable with clustering $24w(H)d(H)$. Hence $\chi_\star(\mathcal{M}_H) \leq 2k$. \square

A variant of Lemma 25 holds for arbitrary graphs H with “2” replaced by “3”. The proof uses a result of Liu and Oum [12] in place of the result of Alon et al. [1]; see [5, 22].

Theorem 26. *For $k \geq 3$, the clustered chromatic number of the class of graphs containing no k -fat star minor equals 4.*

Proof. As illustrated in Figure 2, the k -fat star is planar. Ossona de Mendez et al. [15] proved that graphs containing no k -fat star minor are 2-colourable with defect $O(k^{13})$. Thus, Lemma 25 implies that the clustered chromatic number of the class of graphs containing no k -fat star is at most 4. To obtain a bound on the clustering, note that a result of Leaf and Seymour [11] implies that every graph containing no k -fat star minor has tree-width $O(k^2)$. It follows from the proof of Lemma 25 that every graph containing no k -fat star minor is 4-colourable with clustering $O(k^{15})$. Since the 3-fat star is $C\langle 3, 3 \rangle$, Lemma 13 implies that for $k \geq 3$, the clustered chromatic number of the class of graphs containing no k -fat star minor is at least 4. \square

Every graph H with $\overline{\text{td}}(H) \leq 3$ is a subgraph of the k -fat star for some $k \leq |V(H)|$. Thus Theorem 26 implies Conjecture 4 in the case of connected tree-depth 3.

Corollary 27. *For every graph H with $\overline{\text{td}}(H) \leq 3$,*

$$\chi_\star(\mathcal{M}_H) \leq 4.$$

We can push this result further.

Theorem 28. *For every graph H with $\text{td}(H) \leq 3$,*

$$\chi_\star(\mathcal{M}_H) \leq 5.$$

Proof. Say H has p components. Each component of H is a subgraph of the k -fat star for some $k \leq |V(H)|$. Let H' consist of p pairwise disjoint copies of the k -fat star. Let G be an H -minor-free graph. Thus G is also H' -minor-free. By the Grid Minor Theorem of Robertson and Seymour [17] and since H' is planar, G has treewidth at most $w = w(H')$. By Theorem 5, there is a set X of at most $(p-1)(w-1)$ vertices in G , such that $G - X$ contains no k -fat star as a minor. By Theorem 26, $G - X$ is 4-colourable with clustering at most some function of H . Assign vertices in X a fifth colour. Thus G is 5-colourable with clustering at most some function of H . \square

5 A Conjecture about Clustered Colouring

We now formulate a conjecture about the clustered chromatic number of an arbitrary minor-closed class of graphs. Consider the following recursively defined class of graphs. Let $\mathcal{X}_{1,c} := \{P_{c+1}, K_{1,c}\}$. Here P_{c+1} is the path with $c + 1$ vertices, and $K_{1,c}$ is the star with c leaves. As illustrated in Figure 3, for $k \geq 2$, let $\mathcal{X}_{k,c}$ be the set of graphs obtained by the following three operations. For the first two operations, consider an arbitrary graph $G \in \mathcal{X}_{k-1,c}$.

- Let G' be the graph obtained from c disjoint copies of G by adding one dominant vertex. Then G' is in $\mathcal{X}_{k,c}$.
- Let G^+ be the graph obtained from G as follows: for each k -clique D in G , add a stable set of $k(c-1) + 1$ vertices complete to D . Then G^+ is in $\mathcal{X}_{k,c}$.
- If $k \geq 3$ and $G \in \mathcal{X}_{k-2,c}$, then let G^{++} be the graph obtained from G as follows: for each $(k-1)$ -clique D in G , add a path of $(c^2 - 1)(k-1) + (c+1)$ vertices complete to D . Then G^{++} is in $\mathcal{X}_{k,c}$.

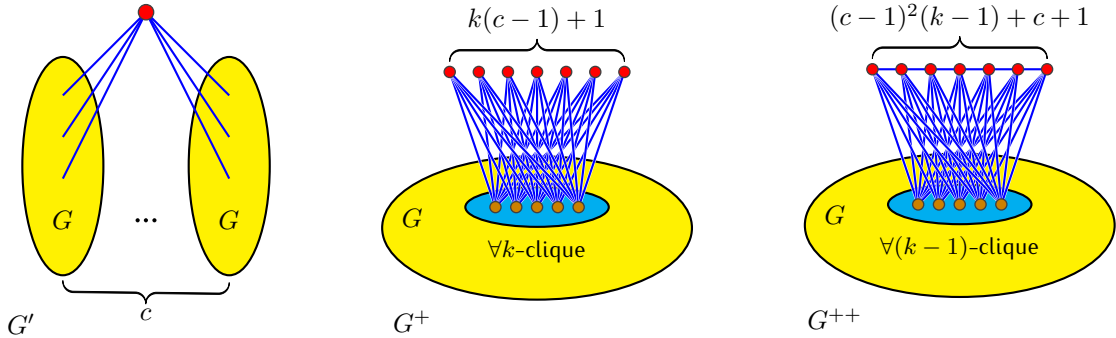


Figure 3: Construction of $\mathcal{X}_{k,c}$.

A vertex-coloured graph is *rainbow* if every vertex receives a distinct colour.

Lemma 29. *For every $c \geq 1$ and $k \geq 2$, for every graph $G \in \mathcal{X}_{k,c}$, every colouring of G with clustering c contains a rainbow K_{k+1} . In particular, no graph in $\mathcal{X}_{k,c}$ is k -colourable with clustering c .*

Proof. We proceed by induction on $k \geq 1$. In the case $k = 1$, every colouring of P_{c+1} or $K_{1,c}$ with clustering c contains an edge whose endpoints receive distinct colours, and we are done. Now assume the claim for $k - 1$ and for $k - 2$ (if $k \geq 3$).

Let $G \in \mathcal{X}_{k-1,c}$. Consider a colouring of G' with clustering c . Say the dominant vertex v is blue. At most $c - 1$ copies of G contain a blue vertex. Thus, some copy of G has no blue vertex. By induction, this copy of G contains a rainbow K_k . With v we obtain a rainbow K_{k+1} .

Now consider a colouring of G^+ with clustering c . By induction, the copy of G in G^+ contains a clique w_1, \dots, w_k receiving distinct colours. Let S be the set of $k(c-1) + 1$ vertices adjacent to w_1, \dots, w_k in G^+ . At most $c-1$ vertices in S receive the same colour as w_i . Thus some vertex in S receives a colour distinct from the colours assigned to w_1, \dots, w_k . Hence G^+ contains a rainbow K_{k+1} .

Now suppose $k \geq 3$ and $G \in \mathcal{X}_{k-2,c}$. Consider a colouring of G^{++} with clustering c . By induction, the copy of G in G^{++} contains a clique w_1, \dots, w_{k-1} receiving distinct colours. Let P be the path of $(c^2 - 1)(k-1) + (c+1)$ vertices in G^{++} complete to w_1, \dots, w_{k-1} . Let X_i be the set of vertices in P assigned the same colour as w_i , and let $X := \bigcup_i X_i$. Thus $|X_i| \leq c-1$ and $|X| \leq (c-1)(k-1)$. Hence $P - X$ has at most $(c-1)(k-1) + 1$ components, and $|V(P - X)| \geq (c^2 - 1)(k-1) + (c+1) - (c-1)(k-1) = c((c-1)(k-1) + 1) + 1$. Some component of $P - X$ has at least $c+1$ vertices, and therefore contains a bichromatic edge xy . Then $\{w_1, \dots, w_{k-1}\} \cup \{x, y\}$ induces a rainbow K_{k+1} in G^{++} . \square

We conjecture that a minor-closed class that excludes every graph in $\mathcal{X}_{k,c}$ for some c is k -colourable with bounded clustering. More precisely:

Conjecture 30. *For every minor-closed class \mathcal{M} of graphs,*

$$\chi_\star(\mathcal{M}) = \min\{k : \exists c \mathcal{M} \cap \mathcal{X}_{k,c} = \emptyset\}.$$

Several comments about Conjecture 30 are in order (see Appendix A for proofs of the following claims):

- To prove the lower bound in Conjecture 30, let k be the minimum integer such that $\mathcal{M} \cap \mathcal{X}_{k,c} = \emptyset$ for some integer c . Thus for every integer c some graph $G \in \mathcal{X}_{k-1,c}$ is in \mathcal{M} . By Lemma 29, G has no $(k-1)$ -colouring with clustering c . Thus $\chi_\star(\mathcal{M}) \geq k$.
- Note that the $k = 1$ case of Conjecture 30 is trivial: a graph is 1-colourable with bounded clustering if and only if each component has bounded size, which holds if and only if every path has bounded length and every vertex has bounded degree.
- We note that Theorem 14 implies Conjecture 30 with $k = 2$. If $G = P_{c+1}$, then G' is contained in the $c(c+1)$ -fan and G^+ is contained in the $(2c-1)$ -fat path. If $G = K_{1,c}$, then G' is the c -fat star and G^+ is contained in the $(2c-1)$ -fat star. It follows that if a minor-closed class \mathcal{M} excludes every graph in $\mathcal{X}_{2,c}$ for some c , then \mathcal{M} excludes the $c(c+1)$ -fan, the $(2c-1)$ -fat path, and the $(2c-1)$ -fat star. Then $\chi_\star(\mathcal{M}) \leq 2$ by Theorem 14.
- We now relate Conjectures 4 and 30. Fix a graph H . Conjecture 30 says that the clustered chromatic number of \mathcal{M}_H equals the minimum integer k such that for some integer c , every graph in $\mathcal{X}_{k,c}$ contains H as a minor. Let $k := \overline{\text{td}}(H) \geq 2$. An easy inductive argument shows that every graph in $\mathcal{X}_{2k-2,c}$ contains a $C\langle k, c \rangle$ minor. Thus, for a suitable value of c , every graph in $\mathcal{X}_{2k-2,c}$ contains H as a minor. Hence, Conjecture 30 implies Conjecture 4.

- Consider the case of excluding the complete bipartite graph $K_{s,t}$ as a minor for $s \leq t$. Van den Heuvel and Wood [22] proved the lower bound, $\chi_*(\mathcal{M}_{K_{s,t}}) \geq s + 1$ for $t \geq \max\{s, 3\}$. Their construction is a special case of the construction above. We claim that Conjecture 30 asserts that $\chi_*(\mathcal{M}_{K_{s,t}}) = s + 1$ for $t \geq \max\{s, 3\}$. To see this, first note that an easy inductive argument shows that every graph in $\mathcal{X}_{s+1,t}$ contains a $K_{s,t}$ subgraph; thus $\mathcal{M}_{K_{s,t}} \cap \mathcal{X}_{s+1,t} = \emptyset$. Furthermore, another easy inductive argument shows that for all $s, c \geq 1$, there is a graph in $\mathcal{X}_{s,c}$ containing no $K_{s, \max\{s, 3\}}$ minor. This implies that $\mathcal{M}_{K_{s,t}} \cap \mathcal{X}_{s,c} \neq \emptyset$ for all $t \geq \max\{s, 3\}$. Together these observations show that $\min\{k : \exists c \mathcal{M}_{s,t} \cap \mathcal{X}_{k,c} = \emptyset\} = s + 1$ for $t \geq \max\{s, 3\}$. That is, Conjecture 30 asserts that $\chi_*(\mathcal{M}_{K_{s,t}}) = s + 1$ for $t \geq \max\{s, 3\}$. Van den Heuvel and Wood [22] proved the upper bound, $\chi_*(\mathcal{M}_{K_{s,t}}) \leq 3s$ for $t \geq s$, which was improved to $2s + 2$ by Dvořák and Norin [4].

6 An Alternative View

We conclude the paper with alternative versions of Conjectures 2 and 30 that shift the focus to characterising minimal minor-closed classes of given defective and clustered chromatic number.

We start with some definitions. Let H and G be two vertex-disjoint graphs, and let $S \subseteq V(G)$. Let G' be obtained from $G \cup H$ by joining every vertex of S to every vertex of H by an edge. Then we say that G' is obtained from G by *taking a join with H along S* . Let \mathcal{H} be a class of graphs. We say that a graph G' is an \mathcal{H} -*decoration* of a graph G , if G' is obtained from G by repeatedly taking joins with graphs in \mathcal{H} along cliques of G . For a class of graphs \mathcal{G} , let $\mathcal{G} \wedge \mathcal{H}$ denote the class of all minors of \mathcal{H} -decorations of graphs in \mathcal{G} . One can routinely verify that the \wedge operation is associative. The examples below show that it is not always commutative.

First, we introduce notation for some minor-closed classes that will serve as the basis for our constructions. Let \mathcal{I} denote the class of graphs on at most one vertex, let \mathcal{O} denote the class of edgeless graphs, and let \mathcal{P} denote the class of linear forests (that is, subgraphs of paths). Let \mathcal{T}_d denote the class of all graphs of tree-depth at most d . Then \mathcal{T}_1 is a class of all edgeless graphs. It follows from the alternative definition of tree-depth given in [13, Section 6.1] that $\mathcal{T}_{d+1} = \mathcal{O} \wedge \mathcal{T}_d$.

The operations used in Conjecture 30 can be described as follows.

- Adding a vertex adjacent to several copies of graphs in the class \mathcal{G} (and taking all possible minors) produces the class $\mathcal{I} \wedge \mathcal{G}$.
- Adding stable sets complete to cliques in graphs in \mathcal{G} produces the class $\mathcal{G} \wedge \mathcal{I}$.
- Adding paths complete to cliques in graphs in \mathcal{G} produces the class $\mathcal{G} \wedge \mathcal{P}$.

Note that by definition $\mathcal{G} \wedge \mathcal{H}$ is a minor-closed class for any pair of minor-closed classes \mathcal{G} and \mathcal{H} .

We next present an analogue of Lemma 29 using the notions introduced above. A class of graphs \mathcal{G} is *k-cluster rainbow* (respectively, *k-defect rainbow*) if for every c there exists $G \in \mathcal{G}$ such that every colouring of G with clustering (respectively, defect) at most c contains a rainbow clique of size k . For example, \mathcal{I} is 1-cluster rainbow and 1-defect rainbow, \mathcal{P} is 2-cluster rainbow, but not 2-defect rainbow. Note that if a class of graphs \mathcal{G} is *k-cluster rainbow*, then clearly $\chi_*(\mathcal{G}) \geq k$. Similarly, if \mathcal{G} is *k-defect rainbow*, then $\chi_\Delta(\mathcal{G}) \geq k$.

The proof of the following lemma parallels the proof of Lemma 29; we present it for completeness.

Lemma 31. *Let \mathcal{G}, \mathcal{H} be graph classes, such that \mathcal{G} is *k-cluster rainbow* and \mathcal{H} is *l-cluster rainbow*. Then $\mathcal{G} \wedge \mathcal{H}$ is $(k + l)$ -cluster rainbow.*

Proof. Fix c , and let $G \in \mathcal{G}$ and $H \in \mathcal{H}$ be such that every colouring of G with clustering at most c contains a rainbow clique of size k , and every colouring of H with clustering at most c contains a rainbow clique of size ℓ . Let J be obtained from G by taking a join of G with H , $(c - 1)k + 1$ times along every clique S of G . Then $J \in \mathcal{G} \wedge \mathcal{H}$ by definition. It remains to show that every colouring $\phi : V(J) \rightarrow C$ of J for some set of colours C with clustering at most c contains a rainbow clique of size $k + \ell$. By the choice of J there exists a clique S in G of size k , which is rainbow in ϕ . Let H_1, H_2, \dots, H_r be copies of H glued along S to G . By the choice of H , for every i there exists a clique S_i of size ℓ in H_i that is rainbow in ϕ . Suppose for a contradiction that $S \cup S_i$ is not rainbow for any i . Then there exists $s \in S$ with a neighbour of the same colour in S_i for at least c choices of i . Thus s belongs to a monochromatic component of size at least $c + 1$ in ϕ , a contradiction. \square

Note that an analogue of Lemma 31 also holds for defective colourings. The proof is identical.

Let \mathcal{G} be a graph class obtained by taking a wedge-product of v copies of \mathcal{I} and p copies of \mathcal{P} in some order such that $v + 2p = k + 1$. Then we say that \mathcal{G} is *k-cluster critical*. By Lemma 31 the clustered chromatic number of a *k-cluster critical* class is at least $k + 1$. (In fact, it is not difficult to see that equality holds.) For example, the class $\mathcal{I} \wedge \mathcal{P}$ of minors of fans, the class $\mathcal{I} \wedge \mathcal{I} \wedge \mathcal{I}$ of minors of fat stars, and the class $\mathcal{P} \wedge \mathcal{I}$ of minors of fat paths are all possible 2-cluster critical classes. Thus Theorem 14 is equivalent to the statement that $\chi_*(\mathcal{G}) \leq 2$ if and only if \mathcal{G} contains no 2-cluster critical class.

The discussion above implies that for all k and c every graph in $\mathcal{X}_{k,c}$ is a member of some *k-cluster critical* class. Conversely, for all n, k there exists c such that for every graph $G \in \mathcal{X}_{k,c}$ there exists a *k-cluster critical* class \mathcal{G} such that $\mathcal{X}_{k,c}$ contains as minors all graphs in \mathcal{G} on at most n vertices. Thus Conjecture 30 can be reformulated as follows.

Conjecture 32. *Let \mathcal{M} be a minor-closed class of graphs and $k \geq 0$ an integer. Then $\chi_*(\mathcal{G}) \geq k + 1$ if and only if $\mathcal{G} \not\subseteq \mathcal{M}$ for some *k-cluster critical* class \mathcal{G} .*

Similarly, note that the k -term \wedge -product $\wedge^k \mathcal{I} = \mathcal{I} \wedge \mathcal{I} \wedge \dots \wedge \mathcal{I}$ is the class of minors of connected graphs of tree-depth k and therefore the following conjecture is equivalent to Conjecture 2.

Conjecture 33. *Let \mathcal{M} be a minor-closed class of graphs and $k \geq 0$ an integer. Then $\chi_{\Delta}(\mathcal{G}) \geq k+1$ if and only if $\wedge^{k+1} \mathcal{G} \notin \mathcal{M}$.*

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A Proofs of Claims in Section 5

This appendix includes proofs of several claims made in the comments after Conjecture 30.

Lemma 34. *For every graph G in $\mathcal{X}_{k,c}$, every clique in G is contained in a $(k+1)$ -clique in G .*

Proof. We proceed by induction on $k \geq 1$. The case $k = 1$ holds by construction. Now assume that $k \geq 2$ and the claim holds for smaller values of k .

First, consider a clique C in a graph $G' \in \mathcal{X}_{k,c}$ for some $G \in \mathcal{X}_{k-1,c}$. Let v be the dominant vertex in G' . If C is contained in some copy of G , then by induction, C is contained in a k -clique

in G , and adding v , we find that C is contained in a $(k+1)$ -clique in G' . Otherwise C includes v . By induction, $C - v$ is contained in a k -clique in G , and adding v , again C is contained in a $(k+1)$ -clique in G' .

Now consider a clique C in a graph $G^+ \in \mathcal{X}_{k,c}$ for some $G \in \mathcal{X}_{k-1,c}$. If C is contained in G , then by induction, C is contained in a k -clique in G . By construction, C is contained in a $(k+1)$ -clique in G^+ . Otherwise C includes a vertex v in G^+ that is not in G . By construction, the neighbourhood of v is a k -clique, and C is contained in the neighbourhood of v . Thus C is contained in a $(k+1)$ -clique in G^{++} .

Finally consider a clique C in a graph $G^{++} \in \mathcal{X}_{k,c}$ for some $G \in \mathcal{X}_{k-2,c}$. If C is contained in G , then by induction, C is contained in a $(k-1)$ -clique D in G . Including two consecutive vertices in the path complete to D , we find that C is contained in a $(k+1)$ -clique in G^{++} . Otherwise C includes a vertex v in G^{++} that is not in G . By construction, the neighbourhood of v contains a k -clique, and C is contained in the neighbourhood of v . Thus C is contained in a $(k+1)$ -clique in G^{++} . \square

Lemma 35. *For $k \geq 2$, every graph in $\mathcal{X}_{2k-2,c}$ contains a $C\langle k, c \rangle$ subgraph.*

Proof. We proceed by induction on $k \geq 2$. First consider the base case, $k = 2$. Consider a graph G' , G^+ or G^{++} in $\mathcal{X}_{2,c}$. By construction, G' contains $K_{1,c}$, which is isomorphic to $C\langle 2, c \rangle$. By construction, G^+ contains $K_{2,2c-1}$, which contains $C\langle 2, c \rangle$. Note that G^{++} does not apply in the $k = 2$ case. Now assume that $k \geq 3$, and for $\ell \leq k-1$, every graph in $\mathcal{X}_{2\ell-2,c}$ contains a $C\langle \ell, c \rangle$ subgraph. Consider a graph G' , G^+ or G^{++} in $\mathcal{X}_{2k-2,c}$.

First consider G' for some $G \in \mathcal{X}_{2k-3,c}$. By construction, G contains some graph in $\mathcal{X}_{2k-4,c}$ as a subgraph. By induction, G contains a $C\langle k-1, c \rangle$ subgraph. By construction, G' contains a $C\langle k, c \rangle$ subgraph.

Now consider G^+ for some $G \in \mathcal{X}_{2k-3,c}$. By construction, G contains some graph in $\mathcal{X}_{2k-4,c}$ as a subgraph. By induction, G contains a $C\langle k-1, c \rangle$ subgraph. Say r is the root vertex in $C\langle k-1, c \rangle$. For each leaf vertex v , the vr -path in the tree induces a $(k-1)$ -clique in $C\langle k-1, c \rangle$, which is contained in a $(2k-2)$ -clique D_v in G by Lemma 34. By construction, in G^+ there is a set S_v of $k(c-1) + 1 \geq c$ vertices complete to D'_v . Moreover, $S_v \cap S_w = \emptyset$ for distinct leaves v, w . It follows that G^+ contains a $C\langle k, c \rangle$ subgraph.

Finally, consider G^{++} for some $G \in \mathcal{X}_{2k-4,c}$. By induction, G contains a $C\langle k-1, c \rangle$ subgraph. Say r is the root vertex in $C\langle k-1, c \rangle$. For each leaf vertex v , the vr -path induces a $(k-1)$ -clique in $C\langle k-1, c \rangle$, which is contained in a $(2k-3)$ -clique D_v in G by Lemma 34. By construction, in G^{++} there is a set S_v of $(c^2-1)(k-1) + (c+1) \geq c$ vertices complete to D_v . Moreover, $S_v \cap S_w = \emptyset$ for distinct leaves v, w . It follows that G^{++} contains a $C\langle k, c \rangle$ subgraph. \square

Lemma 36. *For $s \geq 1$, every graph in $\mathcal{X}_{s+1,t}$ contains a $K_{s,t}$ subgraph.*

Proof. We proceed by induction on $s \geq 1$. Let $k = s+1$ and $c = t$. Consider G' , G^+ or G^{++} in $\mathcal{X}_{k,c}$ for some $G \in \mathcal{X}_{k-1,c}$. Since $G \in \mathcal{X}_{k-1,c}$, by Lemma 29, G contains K_{s+1} . Since

$k(c-1)+1 \geq t$, by construction, G^+ contains a $K_{s+1,t}$ subgraph. Since $(c^2-1)(k-1)+(c+1) \geq t$, by construction, G^{++} contains a $K_{s,t}$ subgraph. Now consider G' for some $G \in \mathcal{X}_{s,t}$. If $s = 2$ then G' contains $K_{1,t}$ since $|V(G)| \geq t$. Now assume that $s \geq 3$. By induction, G contains a $K_{s-1,t}$ subgraph. By construction, G' contains a $K_{s,t}$ subgraph. This completes the proof. \square

Lemma 37. *For $s, c \geq 1$, if $t(s) = \max\{s, 3\}$ then there is a $K_{s,t(s)}$ -minor-free graph in $\mathcal{X}_{s,c}$.*

Proof. We proceed by induction on $s \geq 1$. In the case $s = 1$, note that P_{c+1} is in $\mathcal{X}_{1,c}$ and P_{c+1} contains no $K_{1,3}$ minor. In the case $s = 2$, if $G = P_{c+1}$ then G' is in $\mathcal{X}_{2,c}$ and G' contains no $K_{2,3}$ minor (since G' is outerplanar). In the case $s = 3$, if $G'' = (G')'$, then G' is in $\mathcal{X}_{3,c}$ and G'' contains no $K_{3,3}$ minor (since G'' is planar). Now assume that $s \geq 4$ and there is a $K_{s-1,t(s-1)}$ -minor-free graph G in $\mathcal{X}_{s-1,c}$. Consider G' in $\mathcal{X}_{s,c}$. Suppose that G' contains a $K_{s,t(s)}$ minor. Then deleting the dominant vertex from G' , we find that G contains $K_{s-1,t(s)}$ or $K_{s,t(s)-1}$ as a minor. Since $t(s) = t(s-1) + 1$, in both cases, G contains $K_{s-1,t(s-1)}$ as a minor. This contradiction shows that G' contains no $K_{s,t(s)}$ minor. \square