

# An Asymptotically Tight Bound on the Number of Relevant Variables in a Bounded Degree Boolean Function

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## Abstract

We prove that there is a constant  $C \leq 6.614$  such that every Boolean function of degree at most  $d$  (as a polynomial over  $\mathbb{R}$ ) is a  $C \cdot 2^d$ -junta, i.e. it depends on at most  $C \cdot 2^d$  variables. This improves the  $d \cdot 2^{d-1}$  upper bound of Nisan and Szegedy [Computational Complexity 4 (1994)].

The bound of  $C \cdot 2^d$  is tight up to the constant  $C$  as a lower bound of  $2^d - 1$  is achieved by a read-once decision tree of depth  $d$ . We slightly improve the lower bound by constructing, for each positive integer  $d$ , a function of degree  $d$  with  $3 \cdot 2^{d-1} - 2$  relevant variables. A similar construction was independently observed by Shinkar and Tal.

## 1 Introduction

The *degree* of a Boolean function  $f : \{0, 1\}^n \rightarrow \{0, 1\}$ , denoted  $\deg(f)$ , is the minimum degree of a polynomial in  $\mathbb{R}[x_1, \dots, x_n]$  that agrees with  $f$  on all inputs from  $\{0, 1\}^n$ . (It is well known that every Boolean function has a unique representation over the reals, called the *multilinear representation*, of the form  $\sum_{S \subseteq [n]} a_S \prod_{i \in S} x_i$ , and that  $\deg(f)$  is the degree of the multilinear representation of  $f$ .) Minsky and Papert [MP88] initiated the study of combinatorial and computational properties of Boolean functions based on their representation by polynomials. We refer the reader to the excellent book of O’Donnell [O’D14] on analysis of Boolean functions, and surveys [BDW02, HKP11] discussing relations between various complexity measures of Boolean functions.

An input variable  $x_i$  is *relevant* for a Boolean function  $f$  if it appears in a monomial of the multilinear representation of  $f$  with nonzero coefficient. Let  $R(f)$  denote the number of relevant variables of  $f$ . We say that  $f$  is a  $t$ -junta if  $R(f) \leq t$ . Nisan and Szegedy [NS94], proved that  $R(f)$  is at most at most  $\deg(f) \cdot 2^{\deg(f)-1}$ .

Let  $R_d$  denote the maximum of  $R(f)$  over Boolean functions  $f$  of degree at most  $d$ , and let  $C_d = R_d 2^{-d}$ . By the result of Nisan and Szegedy,  $C_d \leq d/2$ . On the other hand,  $R_d \geq 2R_{d-1} + 1$ , since if  $f$  is a degree  $d - 1$  Boolean function with  $R_{d-1}$  relevant variables, and  $g$  is a copy of  $f$

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on disjoint variables, and  $z$  is a new variable then  $zf + (1 - z)g$  is a degree  $d$  Boolean function with  $2R_{d-1} + 1$  relevant variables. Thus  $C_d \geq C_{d-1} + 2^{-d}$ , and so  $C_d \geq 1 - 2^{-d}$ . Since  $C^d$  is an increasing function of  $d$  it approaches a (possibly infinite) limit  $C^* \geq 1$ .

In this paper we prove:

**Theorem 1.1.** *There is a positive constant  $C$  so that  $R(f)2^{-\deg(f)} \leq C$  for all Boolean functions  $f$ , and thus  $C_d \leq C$  for all  $d \geq 0$ . In particular  $C^*$  is finite.*

Throughout this paper we use  $[n] = \{1, \dots, n\}$  for the index set of the variables to Boolean function  $f$ . A *maxonomial* of  $f$  is a set  $S \subseteq [n]$  of size  $\deg(f)$  for which  $\prod_{i \in S} x_i$  has a nonzero coefficient in the multilinear representation of  $f$ . A *maxonomial hitting set* is a subset  $H \subseteq [n]$  that intersects every maxonomial. Let  $h(f)$  denote the minimum size of a maxonomial hitting set for  $f$  and let  $h_d$  denote the maximum of  $h(f)$  over Boolean functions of degree  $d$ . Our key lemma, proved in [Section 2](#) is:

**Lemma 1.2.**

$$C_d - C_{d-1} \leq h_d 2^{-d},$$

which immediately implies  $C_d \leq \sum_{i=1}^d h_i 2^{-i}$ .

We also have:

**Lemma 1.3.** *For any boolean function  $f$ ,  $h(f) \leq d(f)^3$  and so for all  $i \geq 1$   $h_i \leq i^3$ .*

This is proved in [Section 4](#). As explained there the  $h(f) \leq 2d(f)^3$  is implicit in previous work, and an additional argument eliminates the factor of 2.

Using [Lemma 1.3](#), the summation in the upper bound of [Lemma 1.2](#) converges and [Theorem 1.1](#) follows.

Once we establish that  $C^*$  is finite, it is interesting to obtain upper and lower bounds on  $C^*$ . The best bounds we know are  $3/2 \leq C^* \leq 6.614$ . We discuss these bounds in [Section 3](#).

Filmus and Ihringer [[FI18](#)] recently considered an analog of the parameter  $R(f)$  for the family of *level  $k$  slice functions* which are Boolean functions whose domain is restricted to the set of inputs of Hamming weight exactly  $k$ . They showed that, provided that  $\min(k, n - k)$  is sufficiently large (at least  $B^d$  for some fixed constant  $B$ ) then every level  $k$  slice function on  $n$ -variables of degree at most  $d$  depends on at most  $R_d$  variables. (See [[FI18](#)] for precise definitions and details.) Thus our improved upper bound on  $R_d$  applies also to the number of relevant variables of slice functions.

## 2 Proof of [Lemma 1.2](#)

Similar to Nisan and Szegedy, we upper bound  $R(f)$  by assigning a weight to each variable, and bounding the total weight of all variables. The weight of a variable used by Nisan and Szegedy is its *influence* on  $f$ ; the novelty of our approach is to use a different weight function.

For a variable  $x_i$ , let  $\deg_i(f)$  be the maximum degree among all monomials that contain  $x_i$  and have nonzero coefficient in the multilinear representation of  $f$ . Let  $w_i(f) := 2^{-\deg_i(f)}$ . The weight of  $f$ ,  $W(f)$  is  $\sum_i w_i(f)$  and  $W_d$  denotes the maximum of  $W(f)$  over all Boolean functions  $f$  of degree at most  $d$ .

**Proposition 2.1.** For all  $d \geq 1$ ,  $C_d = W_d$ .

*Proof.* For any function  $f$  of degree  $d$ , we have  $W(f) \geq R(f)2^{-d}$ . Thus  $W_d \geq C_d$ .

To prove the reverse inequality, let  $f$  be a function of degree at most  $d$  with  $R(f)$  as large as possible subject to  $W(f) = W_d$ . We claim that  $\deg_i(f) = d$  for all relevant variables. Suppose for contradiction  $\deg_i(f) < d$  for some  $x_i$ . Let  $g$  be the function obtained by replacing  $x_i$  in  $f$  by the AND of two new variables  $y_i \wedge z_i$ . Then  $\deg(g) = \deg(f) \leq d$  and  $W(g) = W(f)$  and  $R(g) = R(f) + 1$ , contradicting the choice of  $f$ . Since  $\deg_i(f) = d$  for all  $i$ , we have  $W(f) = R(f)2^{-d} = C_d$ .  $\square$

Therefore to prove [Lemma 1.2](#) it suffices to prove that  $W_d - h_d 2^{-d} \leq W_{d-1}$ .

Let  $H$  be a maxonomial hitting set for  $f$  of minimum size. Note that  $\deg_i(f) = d$  for all  $i \in H$  (otherwise  $H - \{i\}$  is a smaller maxonomial hitting set). We have:

$$W(f) = \sum_i w_i(f) = 2^{-d}|H| + \sum_{i \notin H} w_i(f). \quad (1)$$

A *partial assignment* is a mapping  $\alpha : [n] \rightarrow \{0, 1, *\}$ , and  $Fixed(\alpha)$  is the set  $\{i : \alpha(i) \in \{0, 1\}\}$ . For  $J \subseteq [n]$ ,  $PA(J)$  is the set of partial assignments  $\alpha$  with  $Fixed(\alpha) = J$ . The *restriction* of  $f$  by  $\alpha$ ,  $f_\alpha$ , is the function on variable set  $\{x_i : i \in [n] - Fixed(\alpha)\}$  obtained by setting  $x_i = \alpha_i$  for each  $i \in Fixed(\alpha)$ .

**Claim 2.2.** Let  $J \subseteq [n]$ . For any  $i \notin J$ .

$$w_i(f) \leq 2^{-|J|} \sum_{\alpha \in PA(J)} w_i(f_\alpha)$$

*Proof.* Let  $j \in J$  and write  $f_0$  for the restriction of  $f$  by  $x_j = 0$  and  $f_1$  be the restriction of  $f$  by  $x_j = 1$ . Then  $f = (1 - x_j)f_0 + x_j f_1$ .

We proceed by induction on  $|J|$ . For the basis case  $|J| = 1$  we have  $J = \{j\}$

- If  $f_0$  does not depend on  $x_i$ , then  $w_i(f) = w_i(f_1)/2$ .
- If  $f_1$  does not depend on  $x_i$ , then  $w_i(f) = w_i(f_0)/2$ .
- Suppose  $f_1$  and  $f_0$  both depend on  $x_i$ . If  $\deg(f_0) \neq \deg(f_1)$  then  $\deg_i(f) = 1 + \max(\deg_i(f_0), \deg_i(f_1))$  and so  $w_i(f) = \frac{1}{2} \min(w_i(f_0), w_i(f_1))$ . If  $\deg(f_0) = \deg(f_1)$  then every monomial of  $f_0$  appears in  $f = x_j(f_1 - f_0) + f_0$  with the same coefficient and therefore  $w_i(f) \leq w_i(f_0) = \frac{1}{2}(w_i(f_0) + w_i(f_1))$ .

In every case, we have  $w_i(f) \leq (w_i(f_0) + w_i(f_1))/2$ , as required.

For the induction step, assume  $|J| \geq 2$ . By the case  $|J| = 1$ , we have  $w_i(f) \leq \frac{1}{2}(w_i(f_0) + w_i(f_1))$ . Apply the induction hypothesis separately to  $f_0$  and  $f_1$  with the set of variables  $J - \{j\}$ :

$$\begin{aligned}
w_i(f) &\leq \frac{1}{2}(w_i(f_0) + w_i(f_1)) \\
&\leq \frac{1}{2} \left( \sum_{\beta \in PA(J-\{j\})} w_i(f_{0,\beta}) + \sum_{\beta \in PA(J-\{j\})} w_i(f_{1,\beta}) \right) \\
&\leq \sum_{\alpha \in PA(J)} w_i(f_\alpha).
\end{aligned}$$

□

To complete the proof of [Lemma 1.2](#), apply [Claim 2.2](#) with  $J$  being the minimum size hitting set  $H$ , and sum over all  $i \in [n] - H$  to get:

$$\sum_{i \in [n]-H} w_i(f) \leq 2^{-|H|} \sum_{i \in [n]-H} \sum_{\alpha \in PA(J)} w_i(f_\alpha) \leq 2^{-|H|} \sum_{\alpha \in PA(H)} w(f_\alpha) \leq W_{d-1},$$

since  $\deg(f_\alpha) \leq d - 1$  for all  $\alpha \in PA(H)$ .

Combining with [\(Eq. \(1\)\)](#) gives  $W_d \leq W_{d-1} + |H| \cdot 2^{-d}$  as required to prove [Lemma 1.2](#).

### 3 Bounds on $C^*$

[Lemma 1.2](#) implies that  $C_d \leq \sum_{i=1}^d 2^{-i} h_i$ . Combined with [Lemma 1.3](#) gives  $C_d \leq \sum_{i=1}^d i^3 2^{-i}$  and the limiting value  $C^* \leq \sum_{i=1}^{\infty} i^3 2^{-i}$ . This sum is equal to 26 (which can be shown, for example, by using  $\sum_{i \geq 1} \binom{i}{j} 2^{-i} = 2$  for all  $j$ , and  $i^3 = 6 \binom{i}{3} + 6 \binom{i}{2} + 1$ ) and thus  $C^* \leq 26$ . As noted in the introduction,  $R_d \geq 2^d - 1$  and so  $C^* \geq 1$ .

The best upper and lower bounds we know on  $C^*$  are:

**Theorem 3.1.**  $3/2 \leq C^* \leq 6.614$ .

For the upper bound, [Lemma 1.2](#) implies that for any positive integer  $d$ ,

$$C^* \leq C_d + \sum_{i=d+1}^{\infty} 2^{-i} h_i.$$

Since  $C_d \leq d/2$  by the result of Nisan and Szegedy mentioned in the introduction, we have

$$C^* \leq \min_d \left( \frac{d}{2} + \sum_{i=d+1}^{\infty} i^3 2^{-i} \right).$$

The minimum occurs at the largest  $d$  for which the summand  $d^3 2^{-d} > 1/2$  which is 12. Evaluating the right hand side for  $d = 12$  gives  $C^* \leq 6.614$ .

We lower bound  $C^*$  by exhibiting, for each  $d$  a function  $\Xi_d$  of degree  $d$  with  $l(d) = \frac{3}{2} 2^d - 2$  relevant variables. (A similar construction was found independently by Shinkar and Tal [[ST17](#)]. It is more convenient to switch our Boolean set to be  $\{-1, 1\}$ ).

We define  $\Xi_d : \{-1, 1\}^{l(d)} \rightarrow \{-1, 1\}$  as follows.  $\Xi_1 : \{-1, 1\} \rightarrow \{-1, 1\}$  is the identity function and for all  $d > 1$ ,  $\Xi_d$  on  $l(d) = 2l(d-1) + 2$  variables is defined in terms of  $\Xi_{d-1}$  as follows:

$$\Xi_d(s, t, \vec{x}, \vec{y}) = \frac{s+t}{2}\Xi_{d-1}(\vec{x}) + \frac{s-t}{2}\Xi_{d-1}(\vec{y})$$

for all  $s, t \in \{-1, 1\}$  and  $\vec{x}, \vec{y} \in \{-1, 1\}^{l(d-1)}$ . It is evident from the definition that  $\deg(\Xi_d) = 1 + \deg(\Xi_{d-1})$  which is  $d$  by induction (as for the base case  $d = 1$ ,  $\Xi_1$  is linear). It is easily checked that  $\Xi_d$  depends on all of its variables and that  $\Xi_d(s, t, \vec{x}, \vec{y})$  equals  $s * \Xi_{d-1}(\vec{x})$  if  $s = t$  and equals  $s * \Xi_{d-1}(\vec{y})$  if  $s \neq t$ , and is therefore Boolean.

## 4 Proof of Lemma 1.3

In this section, we will show that for any Boolean function  $f$ ,  $h(f) \leq d(f)^3$ .

In an unpublished argument, Nisan and Smolensky (see Lemma 6 of [BDW02]) proved  $h_i \leq d(f)bs(f)$ , where  $bs(f)$  is the block sensitivity of  $f$ . A result from [NS94] (see Theorem 4 of [BDW02]) says  $bs(f) \leq 2d(f)^2$ , which implies that  $h(f) \leq 2d(f)^3$ . We now show how to eliminate the factor 2 in the upper bound.

Recall that for Boolean functions  $f : \{0, 1\}^n \rightarrow \{0, 1\}$  and  $g : \{0, 1\}^m \rightarrow \{0, 1\}$ , their *composition*

$$f \circ g = f(g(t_{1,1}, t_{1,2}, \dots), g(t_{2,1}, t_{2,2}, \dots), \dots)$$

is a Boolean function in  $mn$  variables with variable set  $\{t_{i,j} : i \in [n], j \in [m]\}$ . We begin by showing that degree and maxonomial hitting set size are multiplicative, i.e.,  $d(f \circ g) = d(f)d(g)$  and  $h(f \circ g) = h(f)h(g)$ . The former property is well known: the set of monomials of  $f \circ g$  is the set of all monomials of the form  $c_M \prod_{x_i \in M} m_i$ , where  $M = c_M \prod_{x_i \in M} x_i$  is a monomial of  $f(x_1, x_2, \dots)$  and, for all relevant  $i$ ,  $m_i$  is a monomial of  $g(t_{i,1}, t_{i,2}, \dots)$ . The degree of such a monomial is maximized when  $M$  and all corresponding  $m_i$ 's are maxonomials, in which case its degree is  $\sum_{x_i \in M} d(g) = d(f)d(g)$ .

We now show that  $h(f \circ g) = h(f)h(g)$ . It is easy to check that  $S_0 = \{(i, j) : i \in S_1, j \in S_2\}$  is a maxonomial hitting set of  $f \circ g$ , where  $S_1$  is any maxonomial hitting set of  $f(x_1, x_2, \dots)$  and  $S_2$  is any maxonomial hitting set of  $g(t_{1,1}, t_{1,2}, \dots)$ ; therefore,  $h(f \circ g) \leq h(f)h(g)$ .

We now show that  $h(f \circ g) \geq h(f)h(g)$ . Let  $S \subseteq \{(i, j) : i \in [n], j \in [m]\}$  be a maxonomial hitting set of  $f \circ g$ . Let  $S_i$  be the set of pairs in  $S$  with first coordinate  $i$ , and let  $S'$  be the set of all  $i \in [n]$  such that  $S_i$  is a maxonomial hitting set of  $g(t_{i,1}, t_{i,2}, \dots)$ . We claim that  $S'$  is a maxonomial hitting set of  $f(x_1, x_2, \dots)$ . (Suppose not. Then there is a maxonomial  $M_f$  that  $S'$  does not cover. For each  $i$  such that  $x_i \in M_f$ , there is a maxonomial  $M_i$  of  $g(t_{i,1}, t_{i,2}, \dots)$  that is not hit by  $S_i$ . Then,  $\prod_{i: x_i \in M_f} M_i$  is a maxonomial of  $f \circ g$  that is not hit by  $S$ .) This implies  $|S'| \geq h(f)$ . Since  $i \in S'$  implies  $S_i \geq h(g)$ ,  $|S| \geq h(f)h(g)$ . Therefore  $h(f \circ g) \geq h(f)h(g)$ , and so  $h(f \circ g) = h(f)h(g)$ .

Returning to the proof of the main result, assume for the sake of contradiction, that there exists a degree  $d$  Boolean  $f$  with  $h(f) \geq d^3 + 1$ . We have  $d > 1$ , since the only functions with degree  $d \leq 1$ , the constant and univariate functions, have maxonomial hitting set size  $d$ . Consider the function  $F := \circ^{d^3} f$ , the composition of  $f$  with itself  $d^3$  times. By the multiplicative property of Boolean degree and hitting set size, we have  $d(F) = d^{d^3}$  and  $h(F) = (d^3 + 1)^{d^3} > (d^3)^{d^3} (1 + \frac{1}{d^3})^{d^3} > 2(d^3)^{d^3} = 2d(F)^3$ , contradicting  $h(F) \leq 2d(F)^3$ . Therefore no such  $f$  exists and  $h(f) \leq d(f)^3$  for all Boolean  $f$ .

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## References

- [BDW02] Harry Buhrman and Ronald De Wolf, *Complexity measures and decision tree complexity: a survey*, Theoretical Computer Science **288** (2002), no. 1, 21–43. [1](#), [5](#)
- [FI18] Yuval Filmus and Ferdinand Ihringer, *Boolean constant degree functions on the slice are juntas*, arXiv preprint arXiv:1801.06338 (2018). [2](#)
- [HKP11] Pooya Hatami, Raghav Kulkarni, and Denis Pankratov, *Variations on the sensitivity conjecture*, Theory of Computing Library, Graduate Surveys **4** (2011), 1–27. [1](#)
- [MP88] M. Minsky and S. Papert, *Perceptrons*. [1](#)
- [NS94] Noam Nisan and Mario Szegedy, *On the degree of boolean functions as real polynomials*, Computational complexity **4** (1994), no. 4, 301–313. [1](#), [5](#)
- [O’D14] Ryan O’Donnell, *Analysis of boolean functions*, Cambridge University Press, 2014. [1](#)
- [ST17] Igor Shinkar and Avishay Tal, 2017, Private communication. [4](#)