# A Generalization of the Graph Packing Theorems of Sauer-Spencer and Brandt 

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#### Abstract

We prove a common generalization of the celebrated Sauer-Spencer packing theorem and a theorem of Brandt concerning finding a copy of a tree inside a graph. This proof leads to the characterization of the extremal graphs in the case of Brandt's theorem: If $G$ is a graph and $F$ is a forest, both on $n$ vertices, and $3 \Delta(G)+\ell^{*}(F) \leq n$, then $G$ and $F$ pack unless $n$ is even, $G=\frac{n}{2} K_{2}$ and $F=K_{1, n-1}$; where $\ell^{*}(F)$ is the difference between the number of leaves and twice the number of nontrivial components of $F$.


## 1 Introduction

Given two graphs $G$ and $H$ both on $n$ vertices, we say that $G$ and $H$ pack if there is a bijection $f: V(G) \rightarrow V(H)$ such that for every $u v \in E(G), f(u) f(v) \notin E(H)$; in other words, edge-disjoint copies of $G$ and $H$ can be found in $K_{n}$, or equivalently, $G$ is isomorphic to a subgraph of the complement of $H$. This concept leads to a natural generalization of a number of problems in extremal graph theory, such as existence of a fixed subgraph, equitable colorings, and Turan-type problems. The study of packing of graphs was started in the 1970s by Bollobás and Eldridge [3], Sauer and Spencer [16], and Catlin [6]. See the surveys by Kierstead et al. [15], Wozniak [18], and Yap [19] for later developments in this field. In the following, we will use $\Delta(G)(\delta(G))$ to denote the maximum (resp., minimum) degree of a graph $G$.

The major conjecture in graph packing is that of Bollobás and Eldridge 3, and independently by Catlin [7], from 1978, that $(\Delta(G)+1)(\Delta(H)+1) \leq n+1$ is sufficient for $G$ and $H$ to pack. Some partial results are known, e.g. [11, 13, 1, 2, 8, 8, 17].

In 1978, Sauer and Spencer proved the following celebrated result.
Theorem 1 (Sauer, Spencer [16). Let $G, H$ be graphs on $n$ vertices such that $2 \Delta(G) \Delta(H)<$ n. Then $G$ and $H$ pack.

[^0]Kaul and Kostochka [12] strengthened the result by characterizing the extremal graphs: if $2 \Delta(G) \Delta(H)=n$ and $G$ and $H$ fail to pack, then $n$ is even, one of the graphs is $\frac{n}{2} K_{2}$, and the other is either $K_{n / 2, n / 2}$ (with $n / 2$ odd) or contains $K_{n / 2+1}$.

Let $\ell(F)$ denote the number of leaves in a forest $F$. In 1994, Brandt [5] proved that if $G$ is a graph and $T$ is a tree, both on $n$ vertices, and $\ell(T) \leq 3 \delta(G)-2 n+4$, then $G$ contains a copy of $T$. This can be rephrased in terms of packing.

Theorem 2 (Brandt [5). If $G$ is a graph and $T$ is a tree, both on $n$ vertices, and

$$
3 \Delta(G)+\ell(T)-2<n,
$$

## then $G$ and $T$ pack.

We need a generalization of this theorem to a forest $F$, which is straightforward and motivates the following definition.

Definition. The excess leaves of a forest $F$, denoted $\ell^{*}(F)$, is $\sum_{v \in V(F)} \max \{d(v)-2,0\}$.
Note that linear forests are precisely the forests with zero excess leaves. We also have that $\ell^{*}(F)$ equals the number of leaves of $F$ minus twice the number of nontrivial components of $F$ (those having at least two vertices), and that for a tree $\left.T, \ell^{*}(T)=\ell(T)-2.\right]$

Corollary 3. If $G$ is a graph and $F$ is a forest, both on $n$ vertices, and $3 \Delta(G)+\ell^{*}(T)<n$, then $G$ and $F$ pack.

Proof. Iteratively add edges joining leaves of distinct nontrivial components of $F$; each such addition does not change $\ell^{*}$. When there is only one nontrivial component left, iteratively add edges from any leaf to the remaining (isolated) vertices; again $\ell^{*}$ is preserved. Now we have a tree, for which $\ell^{*}=\ell-2$. Brandt's theorem now applies, so that $G$ and the new tree pack, and deleting the added edges gives a packing of $G$ with $F$.

Corollary 3 is sharp when $n$ is even, with $G=\frac{n}{2} K_{2}$ and $F=K_{1, n-1}$. We will prove that this is the only pair of extremal graphs, strengthening Brandt's result as follows.

Theorem 4. If $G$ is a graph and $F$ is a forest, both on $n$ vertices, and

$$
3 \Delta(G)+\ell^{*}(F) \leq n,
$$

then $G$ and $F$ pack unless $n$ is even, $G=\frac{n}{2} K_{2}$, and $F=K_{1, n-1}$.
To accomplish this, we will first prove the following theorem, which generalizes both the Sauer-Spencer and Brandt packing theorems.

[^1]Theorem 5. Let $G$ be a graph and $H$ a c-degenerate graph, both on $n$ vertices. Let $d_{1}^{(G)} \geq$ $d_{2}^{(G)} \geq \cdots \geq d_{n}^{(G)}$ be the degree sequence of $G$, and similarly for $H$. If

$$
\sum_{i=1}^{\Delta(G)} d_{i}^{(H)}+\sum_{j=1}^{c} d_{j}^{(G)}<n
$$

then $G$ and $H$ pack.
This strengthens Sauer-Spencer, since $c \leq \Delta(H)$.
This also strengthens Brandt's theorem: if $H$ is a tree, then $c=1$, so the second summation is just $\Delta(G)$. For the first summation,

$$
\sum_{i=1}^{\Delta(G)} d_{i}^{(H)}=2 \Delta(G)+\sum_{i=1}^{\Delta(G)}\left(d_{i}^{(H)}-2\right) \leq 2 \Delta(G)+\ell(H)-2
$$

It is easy to construct examples of graphs $G$ and $H$ for which conditions in the SauerSpencer theorem, Brandt's theorem, or even the Bollobás, Eldridge, and Catlin conjecture are not true, but Theorem 5 does apply.

The proof of Theorem 5 generally follows that of Sauer-Spencer. In the special setting of Brandt's theorem, the proof can be analyzed more closely to show that the only sharpness example is the one mentioned above.

Theorem 5 is sharp itself, with several sharpness examples. It retains all the Sauer-Spencer sharpness examples (with $n$ even) mentioned earlier:

- $H=\frac{n}{2} K_{2}$ and $G \supseteq K_{n / 2+1}$
- $H=\frac{n}{2} K_{2}$ and $G=K_{n / 2, n / 2}$, with $n / 2$ odd
- $H \supseteq K_{n / 2+1}$ and $G=\frac{n}{2} K_{2}$
- $H=K_{n / 2, n / 2}$ and $G=\frac{n}{2} K_{2}$, with $n / 2$ odd

And it has an additional family of sharpness examples:

- $H=K_{s, n-s}$ and $G=\frac{n}{2} K_{2}$, with $n$ even and $s$ odd
(in particular, $H=K_{1, n-1}^{2}$ and $G=\frac{n}{2} K_{2}$ )
We do not know whether these are all the sharpness examples, even if we restrict to the case that $H$ is a forest.

Question 6. What are the extremal graphs for Theorem [5? Do the above listed families of graphs include all the extremal graphs for Theorem 5 when $H$ is a forest?

Note that Theorem 4 shows that the only extremal graphs for that theorem have $n$ even and $\Delta(G)=1$. So, it is natural to ask:

Question 7. By Theorem 目, $3 \Delta(G)+\ell^{*}(F)<n+1$ is a sufficient condition for packing of a graph $G$ and a forest $F$ on $n$ vertices when $n$ is odd or $\Delta(G) \geq 2$. Is this statement sharp? If yes, what are all its sharpness examples?

Degeneracy versions of the Sauer-Spencer packing theorem have been studied before, in 4] and [14. If we think of the condition in Sauer-Spencer as the sum of two terms: $\Delta(G) \Delta(H)+$ $\Delta(H) \Delta(G)<n$, then Theorem 5 can be thought of as replacing $\Delta(H)$ by the degeneracy $c(H)$ in one the terms (in addition to other degree sequence related improvements). The result in [4] replaces $\Delta(G)$ by $c(G)$ in one term and $\Delta(H)$ by $\log \Delta(H)$ in the other. In [14], $\Delta(G)$ is replaced by $(\operatorname{gcol}(G)-1)$ in one term and $\Delta(H)$ by $(\operatorname{gcol}(H)-1)$ in the other, where gcol denotes the game coloring number and $\operatorname{gcol}(G)-1$ lies in between the degeneracy and the maximum degree (see [14] for precise definition and details). It is natural to ask for improvements or extensions of Theorem 5 by considering degree-sum conditions that interplay between maximum degree, degeneracy, and game coloring number. For example, does $\sum_{i=1}^{g c o l(G)-1} d_{i}^{(H)}+\sum_{j=1}^{g c o l(H)-1} d_{j}^{(G)}<n$ suffice for a packing of $G$ and $H$ under the setup of Theorem [5? Or, does $c_{1} \sum_{i=1}^{\lceil\log \Delta(G)\rceil} d_{i}^{(H)}<n$ and $c_{2} \sum_{j=1}^{c(H)} d_{j}^{(G)}<n$ for some fixed constants $c_{1}$ and $c_{2}$ suffice?

## 2 Proofs

Throughout, we think of a bijective mapping $f: V(G) \rightarrow V(H)$ as the multigraph with vertices $V(G)$ and edges labelled by " $G$ " or " $H$ ". We speak of $H$ - and $G$-edges, $H$ - and $G$-neighbors of vertices, and $H$-cliques, $H$-independent sets, etc. A link is a copy of $P_{3}$ with one $H$-edge and one $G$-edge, and a $u v$-link is a link with endpoints $u$ and $v$; a $G H$-link from $u$ to $v$ is a link with endpoints $u, v$ whose edge incident to $u$ is from $G$; similarly we have $H G$ links. From a given mapping $f$, a uv-swap results in a new mapping $f^{\prime}$ with $f^{\prime}(u)=f(v)$, $f^{\prime}(v)=f(u)$, and $f^{\prime}=f$ otherwise. A quasipacking of $G$ with $H$ is a mapping $f$ whose multigraph is simple except for a single pair of vertices joined by both an $H$-edge and a $G$-edge; this pair is called the conflicting edge of the quasipacking.

Consider a pair of graphs $(G, H)$, with $H$ being $c$-degenerate, each on $n$ vertices, that do not pack; furthermore assume that $H$ is edge-minimal with this property. Thus for any edge $e$ in $H, G$ and $H-e$ pack, and so there is a quasipacking of $H$ and $G$ with conflicting edge $e$.

Let $u^{\prime}$ be a vertex of minimum positive degree in $H$, let $x^{\prime} \in N_{H}(u)$, and consider a quasipacking $f$ of $G$ with $H$ with conflicting edge $u^{\prime} x^{\prime}$. Let $u=f^{-1}\left(u^{\prime}\right)$ and $x=f^{-1}\left(x^{\prime}\right)$. We will now consider the set of links from $u$ to each vertex.

Consider a $y \in V(G) \backslash\{u, x\}$. Perform a $u y$-swap: since $G$ and $H$ do not pack, there must be some conflicting edge, and such a conflict must involve an $H$-edge incident to either $u$ or $y$; together with the conflicting $G$-edge, we have a $u y$-link in the original quasipacking. There are two links from $u$ to itself, using the parallel edges $u x$ in each order. Thus there are at least $n$ links from $u$ in the original quasipacking $f$.

The number of $G H$-links from $u$ is at most $\sum_{y \in N_{G}(u)} \operatorname{deg}_{H}(f(y))$. The number of $H G$ links from $u$ is at most $\sum_{z^{\prime} \in N_{H}\left(u^{\prime}\right)} \operatorname{deg}_{G}\left(f^{-1}\left(z^{\prime}\right)\right)$. Hence we have

$$
\begin{equation*}
n \leq \# \text { links from } u \leq \sum_{y \in N_{G}(u)} \operatorname{deg}_{H}(f(y))+\sum_{z^{\prime} \in N_{H}\left(u^{\prime}\right)} \operatorname{deg}_{G}\left(f^{-1}\left(z^{\prime}\right)\right) \leq \sum_{i=1}^{\Delta(G)} d_{i}^{(H)}+\sum_{j=1}^{c} d_{j}^{(G)} . \tag{1}
\end{equation*}
$$

This establishes Theorem 5 .
To prove Theorem 4, suppose additionally that $H$ is a forest, henceforth called $F$, and that $3 \Delta(G)+\ell^{*}(F)=n$. (So, we still assume that $G$ and $F$ do not pack, and that $F$ is edge-minimal with this property.)

If $\Delta(G)=1$, then it is easy to show that $n$ is even, $G=\frac{n}{2} K_{2}$, and $F=K_{1, n-1}$. (In fact, such a $G$ will pack with any bipartite graph that is not complete bipartite.) So we henceforth assume that $\Delta(G)>1$, and seek a contradiction.

Lemma 8. For any leaf $u^{\prime}$ of $F$ and $x^{\prime}$ its neighbor, and a quasipacking $f$ of $G$ with $F$ with $f(u)=u^{\prime}$ and $f(x)=x^{\prime}$ and conflicting edge $u x$, we have the following.

1. For every $y \in V(G) \backslash\{u, x\}$, there is a unique link from $u$ to $y$; there is no link from $u$ to $x$; and there are two links from $u$ to itself.
2. $\operatorname{deg}_{G}(x)=\operatorname{deg}_{G}(u)=\Delta(G)$.
3. For every $w \in N_{G}(u), \operatorname{deg}_{F}(f(w)) \geq 2$.
4. For every $w \notin N_{G}(u), \operatorname{deg}_{F}(f(w)) \leq 2$.

Proof. Note that we now have $\operatorname{deg}_{F}\left(u^{\prime}\right)=1$, so $\sum_{z^{\prime} \in N_{F}\left(u^{\prime}\right)} \operatorname{deg}_{G}\left(f^{-1}\left(z^{\prime}\right)\right)=\operatorname{deg}_{G}(x)$. In this case we can expand on (1):

$$
\begin{align*}
n \leq \# \text { links from } u & \leq \sum_{y \in N_{G}(u)} \operatorname{deg}_{F}(f(y))+\operatorname{deg}_{G}(x)  \tag{2}\\
& \leq \sum_{y \in N_{G}(u)}\left(\operatorname{deg}_{F}(f(y))-2\right)+2 \Delta(G)+\Delta(G)  \tag{3}\\
& \leq \sum_{y \in N_{G}(u)} \max \left\{\operatorname{deg}_{F}(f(y))-2,0\right\}+3 \Delta(G)  \tag{4}\\
& \leq \sum_{i=1}^{n} \max \left\{d_{i}^{(F)}-2,0\right\}+3 \Delta(G)=\ell^{*}(F)+3 \Delta(G)=n, \tag{5}
\end{align*}
$$

so we have equality throughout. Conclusion $i$ follows from having equality in line $(i+1)$ above, for $i \in[4]$.

For a vertex $v$ in a graph $H$, we write $N_{H}[v]$ for the closed neighborhood, i.e. $N_{H}(v) \cup\{v\}$. For a set $S$ of vertices, $N_{H}(S)=\bigcup_{v \in S} N_{H}(v)-S$.

Lemma 9. For any leaf $u^{\prime}$ of $F$ and $x^{\prime}$ its neighbor, and a quasipacking $f$ of $G$ with $F$ with $f(u)=u^{\prime}$ and $f(x)=x^{\prime}$ and conflicting edge $u x$, we have the following.

1. $N_{G}[u]=N_{G}[x]$.
2. Let $Q=N_{G}[u]$. Then $G[Q]$ is a clique component.

Proof. Proof of part 1 .
Let $A=N_{G}(u)-N_{G}[x], B=N_{G}(u) \cap N_{G}(x), C=N_{G}(x)-N_{G}[u]$. Also, let $N_{A}=$ $N_{F}(f(A)), N_{B}=N_{F}(f(B)), N_{C}=N_{F}(f(C))$, and $N_{x}=N_{F}\left(x^{\prime}\right)$.

We will show that $A=N_{A}=N_{C}=C=\varnothing$.


Figure 1: Left: the quasipacking $f$, with $G$-edges dashed and $F$-edges solid. Right: the result after the $u x$-swap.

Note that $B \cup C \cup\{u\}$ is precisely the set of vertices with an $F G$-link from $u$. By Lemma 8(1), there are no $F$-edges from $A$ to $x$, else $x$ would have a $G F$-link; and there are no $F$-edges from $A$ to $B \cup C \cup N_{x}$, else such an endpoint in $B \cup C \cup N_{x}$ would have two links. So for each vertex of $A$ to have exactly one link, $F[f(A)]$ must be a perfect matching. Furthermore, the $F$-edges incident to $A$ only have endpoints in $A \cup N_{A}$. Each vertex of $N_{A}$ must have exactly one $F$-edge from $A$ (to have one link). And by Lemma 8(3), each vertex of $A$ has at least one $F$-neighbor in $N_{A}$. Note that we thus have $\left|N_{A}\right| \geq|A|$. The vertices of $N_{A}, N_{B}, N_{x}$ all have $G F$-links by definition, and to have exactly one, these sets must be disjoint. Thus we have that $\{u, x\}, A, B, C, N_{A}, N_{B}, N_{x}$ is a partition of $V(G)$. See the left side of Figure 1 .

Now perform a $u x$-swap. In Figure 1, we visualize with $V(G)$ fixed, so just the $F$-edges adjacent to $u^{\prime}$ and $x^{\prime}$ move; roughly speaking, we just interchange the roles of $u$ and $x$ and those of $A$ and $C$. The result is again a quasipacking with $u x$ the only conflicting edge. The $F$-neighbors of $u$ are precisely $N_{x}$. Repeating the arguments of the last paragraph, for each vertex of $N_{C}$ to have exactly one link, we must have $\left|N_{C}\right| \geq|C|$. Suppose that $A \neq \varnothing$. Then since $\left|N_{A}\right| \geq|A|, N_{A} \neq \varnothing$ as well. Now, the only possible links (from $x$ ) to vertices in $N_{A}$ are $G F$-links through $C$; hence $N_{C}=N_{A}$, and the $F$-edges incident to $C$ have endpoints in $C \cup N_{A}$. Furthermore, since $N_{C}=N_{A} \neq \varnothing, C \neq \varnothing$ as well; so each vertex of $N_{A}$ has
$F$-degree at least 2 (one edge from $A$ and one from $C$ ). But also, by Lemma 8(3) applied to the original and also this new quasipacking, every vertex of $A$ and $C$ has $F$-degree at least 2, with $F$-edges entirely in $A \cup C \cup N_{A}$. So $F\left[f\left(A \cup C \cup N_{A}\right)\right]$ has minimum degree at least 2 , contradicting that it is a forest, unless $A=C=N_{A}=\varnothing$. Note that this implies that $N_{G}[u]=N_{G}[x]=\{u, x\} \cup B$.

Proof of part 2 .
This time perform a $u y$-swap for some vertex $y \in Q \backslash\{u, x\}$ to get $\tilde{f}$. The result is again a quasipacking with $y x$ the only conflicting edge, with $\tilde{f}(y)=u^{\prime}$. By part $\mathbb{1}, N_{G}[y]=$ $N_{G}[x]=Q$. Since this holds for every $y \in Q \backslash\{u, x\}$, we have that $Q$ is a clique; and since $\operatorname{deg}_{G}(x)=\Delta(G)$ by Lemma $8(2), G[Q]$ is a clique component of $G$.

Let $u^{\prime}$ be a leaf in $F$, and let $x^{\prime}$ be its neighbor. Consider a quasipacking $f$ of $G$ with $F$ with $f(u)=u^{\prime}$ and $f(x)=x^{\prime}$ and conflicting edge $u x$. (Such exists by the extremal choice of $F$, as in the proof of Theorem 5.

Let $G[Q]$ be the clique component of $G$ given in Lemma (9(2). Let $z$ be a vertex of $Q$ with smallest $F$-degree larger than 1 (such a choice is possible, as $\operatorname{deg}_{F}\left(x^{\prime}\right) \geq 2$ by Lemma 8(3)), and let $z^{\prime}=f(z)$. Let $z_{1}, z_{2} \in V(G)$ be two $F$-neighbors of $z$.

In $f, z_{1}$ and $z_{2}$ each have exactly one $F$-edge into $Q$ and at most one other $F$-edge, by Lemma 8(114). So $z_{1}, z_{2}$ have no $F$-neighbors inside $Q$ except $z$. From this and that $Q$ is a $G$-clique in the quasipacking, the set $Q \cup\left\{z_{1}, z_{2}\right\} \backslash\{z\}$ is $F$-independent (whether $z=x$ or not) except perhaps the conflicting edge $u x$. So $Q \cup\left\{z_{1}, z_{2}\right\} \backslash\{u, z\}$ is $F$-independent. Let $X=f\left(Q \cup\left\{z_{1}, z_{2}\right\} \backslash\{u, z\}\right)$.

Let $g: V(G) \rightarrow V(F)$ be a bijection such that $g(Q)=X$. Since $G[Q]$ is a clique component and $X$ is independent, $g$ is a packing if and only if $\left.g\right|_{G-Q}$ is a packing of $G-Q$ with $F-X$.
Claim: $\operatorname{deg}_{F}\left(z^{\prime}\right) \geq 4$.
Suppose to the contrary that $\operatorname{deg}_{F}\left(z^{\prime}\right) \leq 3$. We have taken two of the neighbors of $z$ into $X$, so $\operatorname{deg}_{F-X}\left(z^{\prime}\right) \leq 1$. And $z^{\prime}$ is the only vertex of $F-X$ that may have degree larger than 2 , by Lemma 8(4). That is, $F-X$ is a linear forest. We have that $\Delta(G-Q) \leq \Delta(G) \leq \frac{n}{3}$, so

$$
\begin{aligned}
\delta(\overline{G-Q}) & =|V(G-Q)|-1-\Delta(G-Q) \\
& =n-(\Delta(G)+1)-1-\Delta(G-Q) \\
& \geq n-\frac{3}{2} \Delta(G)-\frac{1}{2} \Delta(G)-2 \\
& \geq \frac{1}{2} n-\frac{1}{2} \Delta(G)-2 \\
& =\frac{1}{2}|V(G-Q)|,
\end{aligned}
$$

and so Dirac's condition for Hamiltonicity applies (10). Since $\overline{G-Q}$ contains a Hamiltonian cycle, it also contains the linear forest $F-X$, i.e. $F-X$ and $G-Q$ pack, a contradiction. This completes the proof of the Claim.

This Claim, together with having $z^{\prime} \notin X$ but its two neighbors $z_{1}, z_{2} \in X$, gives us the inequality

$$
\begin{aligned}
\ell^{*}(F-X) & =\sum_{v \in V(F-X)} \max \left\{\operatorname{deg}_{F-X}(v)-2,0\right\} \\
& \leq-2+\sum_{v \in V(F-X)} \max \left\{\operatorname{deg}_{F}(v)-2,0\right\} \\
& =-2+\sum_{v \in V(F)} \max \left\{\operatorname{deg}_{F}(v)-2,0\right\}-\sum_{v \in X} \max \left\{\operatorname{deg}_{F}(v)-2,0\right\} \\
& =-2+\ell^{*}(F)-\sum_{v \in X} \max \left\{\operatorname{deg}_{F}(v)-2,0\right\}
\end{aligned}
$$

From Lemma $8(3)$, every vertex of $f(Q-u)$ has $F$-degree at least two; and since $z^{\prime}$ was chosen to have smallest $F$-degree among the non-leaves of $f(Q)$, the Claim gives that they must in fact have degree at least four. All these vertices except $z^{\prime}$ are in $X$, so we have at least $\Delta(G)-1$ vertices of $X$ with degree at least 4. Hence $2+\sum_{v \in X} \max \left\{\operatorname{deg}_{F}(v)-2,0\right\} \geq$ $2 \Delta(G)>\Delta(G)+1$, so

$$
\begin{aligned}
3 \Delta(G-Q)+\ell^{*}(F-X) & \leq 3 \Delta(G)+\ell^{*}(F)-2-\sum_{v \in X} \max \left\{\operatorname{deg}_{F}(v)-2,0\right\} \\
& =n-2-\sum_{v \in X} \max \left\{\operatorname{deg}_{F}(v)-2,0\right\} \\
& <n-\Delta(G)-1 \\
& =|V(G-Q)|
\end{aligned}
$$

Thus, by Theorem [5, $G-Q$ and $F-X$ pack, a contradiction. This completes the proof of Theorem 4

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[^1]:    ${ }^{1}$ For these, consider the sum $\sum_{i \geq 0}(i-2) n_{i}$, where $n_{i}$ is the number of vertices with degree $i$.

