A Generalization of the Graph Packing Theorems of Sauer-Spencer and Brandt

Hemanshu Kaul

Benjamin Reiniger

Abstract

We prove a common generalization of the celebrated Sauer-Spencer packing theorem and a theorem of Brandt concerning finding a copy of a tree inside a graph. This proof leads to the characterization of the extremal graphs in the case of Brandt's theorem: If G is a graph and F is a forest, both on n vertices, and $3\Delta(G) + \ell^*(F) \leq n$, then G and Fpack unless n is even, $G = \frac{n}{2}K_2$ and $F = K_{1,n-1}$; where $\ell^*(F)$ is the difference between the number of leaves and twice the number of nontrivial components of F.

1 Introduction

Given two graphs G and H both on n vertices, we say that G and H pack if there is a bijection $f: V(G) \to V(H)$ such that for every $uv \in E(G)$, $f(u)f(v) \notin E(H)$; in other words, edge-disjoint copies of G and H can be found in K_n , or equivalently, G is isomorphic to a subgraph of the complement of H. This concept leads to a natural generalization of a number of problems in extremal graph theory, such as existence of a fixed subgraph, equitable colorings, and Turan-type problems. The study of packing of graphs was started in the 1970s by Bollobás and Eldridge [3], Sauer and Spencer [16], and Catlin [6]. See the surveys by Kierstead et al. [15], Wozniak [18], and Yap [19] for later developments in this field. In the following, we will use $\Delta(G)$ ($\delta(G)$) to denote the maximum (resp., minimum) degree of a graph G.

The major conjecture in graph packing is that of Bollobás and Eldridge [3], and independently by Catlin [7], from 1978, that $(\Delta(G) + 1)(\Delta(H) + 1) \leq n + 1$ is sufficient for G and H to pack. Some partial results are known, e.g. [11, 13, 1, 2, 8, 9, 17].

In 1978, Sauer and Spencer proved the following celebrated result.

Theorem 1 (Sauer, Spencer [16]). Let G, H be graphs on n vertices such that $2\Delta(G)\Delta(H) < n$. Then G and H pack.

Department of Applied Mathematics, Illinois Institute of Technology, Chicago, IL 60616. Email: kaul@iit.edu, ben.reiniger-math@yahoo.com.

Kaul and Kostochka [12] strengthened the result by characterizing the extremal graphs: if $2\Delta(G)\Delta(H) = n$ and G and H fail to pack, then n is even, one of the graphs is $\frac{n}{2}K_2$, and the other is either $K_{n/2,n/2}$ (with n/2 odd) or contains $K_{n/2+1}$.

Let $\ell(F)$ denote the number of leaves in a forest F. In 1994, Brandt [5] proved that if G is a graph and T is a tree, both on n vertices, and $\ell(T) \leq 3\delta(G) - 2n + 4$, then G contains a copy of T. This can be rephrased in terms of packing.

Theorem 2 (Brandt [5]). If G is a graph and T is a tree, both on n vertices, and

$$3\Delta(G) + \ell(T) - 2 < n,$$

then G and T pack.

We need a generalization of this theorem to a forest F, which is straightforward and motivates the following definition.

Definition. The excess leaves of a forest F, denoted $\ell^*(F)$, is $\sum_{v \in V(F)} \max\{d(v) - 2, 0\}$.

Note that linear forests are precisely the forests with zero excess leaves. We also have that $\ell^*(F)$ equals the number of leaves of F minus twice the number of nontrivial components of F (those having at least two vertices), and that for a tree T, $\ell^*(T) = \ell(T) - 2$.¹

Corollary 3. If G is a graph and F is a forest, both on n vertices, and $3\Delta(G) + \ell^*(T) < n$, then G and F pack.

Proof. Iteratively add edges joining leaves of distinct nontrivial components of F; each such addition does not change ℓ^* . When there is only one nontrivial component left, iteratively add edges from any leaf to the remaining (isolated) vertices; again ℓ^* is preserved. Now we have a tree, for which $\ell^* = \ell - 2$. Brandt's theorem now applies, so that G and the new tree pack, and deleting the added edges gives a packing of G with F.

Corollary 3 is sharp when n is even, with $G = \frac{n}{2}K_2$ and $F = K_{1,n-1}$. We will prove that this is the only pair of extremal graphs, strengthening Brandt's result as follows.

Theorem 4. If G is a graph and F is a forest, both on n vertices, and

$$3\Delta(G) + \ell^*(F) \le n,$$

then G and F pack unless n is even, $G = \frac{n}{2}K_2$, and $F = K_{1,n-1}$.

To accomplish this, we will first prove the following theorem, which generalizes both the Sauer-Spencer and Brandt packing theorems.

¹For these, consider the sum $\sum_{i>0} (i-2)n_i$, where n_i is the number of vertices with degree *i*.

Theorem 5. Let G be a graph and H a c-degenerate graph, both on n vertices. Let $d_1^{(G)} \ge d_2^{(G)} \ge \cdots \ge d_n^{(G)}$ be the degree sequence of G, and similarly for H. If

$$\sum_{i=1}^{\Delta(G)} d_i^{(H)} + \sum_{j=1}^c d_j^{(G)} < n,$$

then G and H pack.

This strengthens Sauer-Spencer, since $c \leq \Delta(H)$.

This also strengthens Brandt's theorem: if H is a tree, then c = 1, so the second summation is just $\Delta(G)$. For the first summation,

$$\sum_{i=1}^{\Delta(G)} d_i^{(H)} = 2\Delta(G) + \sum_{i=1}^{\Delta(G)} \left(d_i^{(H)} - 2 \right) \le 2\Delta(G) + \ell(H) - 2.$$

It is easy to construct examples of graphs G and H for which conditions in the Sauer-Spencer theorem, Brandt's theorem, or even the Bollobás, Eldridge, and Catlin conjecture are not true, but Theorem 5 does apply.

The proof of Theorem 5 generally follows that of Sauer-Spencer. In the special setting of Brandt's theorem, the proof can be analyzed more closely to show that the only sharpness example is the one mentioned above.

Theorem 5 is sharp itself, with several sharpness examples. It retains all the Sauer-Spencer sharpness examples (with n even) mentioned earlier:

- $H = \frac{n}{2}K_2$ and $G \supseteq K_{n/2+1}$
- $H = \frac{n}{2}K_2$ and $G = K_{n/2,n/2}$, with n/2 odd
- $H \supseteq K_{n/2+1}$ and $G = \frac{n}{2}K_2$
- $H = K_{n/2,n/2}$ and $G = \frac{n}{2}K_2$, with n/2 odd

And it has an additional family of sharpness examples:

• $H = K_{s,n-s}$ and $G = \frac{n}{2}K_2$, with *n* even and *s* odd (in particular, $H = K_{1,n-1}$ and $G = \frac{n}{2}K_2$)

We do not know whether these are all the sharpness examples, even if we restrict to the case that H is a forest.

Question 6. What are the extremal graphs for Theorem 5? Do the above listed families of graphs include all the extremal graphs for Theorem 5 when H is a forest?

Note that Theorem 4 shows that the only extremal graphs for that theorem have n even and $\Delta(G) = 1$. So, it is natural to ask: **Question 7.** By Theorem 4, $3\Delta(G) + \ell^*(F) < n+1$ is a sufficient condition for packing of a graph G and a forest F on n vertices when n is odd or $\Delta(G) \ge 2$. Is this statement sharp? If yes, what are all its sharpness examples?

Degeneracy versions of the Sauer-Spencer packing theorem have been studied before, in [4] and [14]. If we think of the condition in Sauer-Spencer as the sum of two terms: $\Delta(G)\Delta(H) + \Delta(H)\Delta(G) < n$, then Theorem 5 can be thought of as replacing $\Delta(H)$ by the degeneracy c(H) in one the terms (in addition to other degree sequence related improvements). The result in [4] replaces $\Delta(G)$ by c(G) in one term and $\Delta(H)$ by $\log \Delta(H)$ in the other. In [14], $\Delta(G)$ is replaced by $(\operatorname{gcol}(G) - 1)$ in one term and $\Delta(H)$ by $(\operatorname{gcol}(H) - 1)$ in the other, where gool denotes the game coloring number and $\operatorname{gcol}(G) - 1$ lies in between the degeneracy and the maximum degree (see [14] for precise definition and details). It is natural to ask for improvements or extensions of Theorem 5 by considering degree-sum conditions that interplay between maximum degree, degeneracy, and game coloring number. For example, does $\sum_{i=1}^{gcol(G)-1} d_i^{(H)} + \sum_{j=1}^{gcol(H)-1} d_j^{(G)} < n$ suffice for a packing of G and H under the set-up of Theorem 5? Or, does $c_1 \sum_{i=1}^{\lceil \log \Delta(G) \rceil} d_i^{(H)} < n$ and $c_2 \sum_{j=1}^{c(H)} d_j^{(G)} < n$ for some fixed constants c_1 and c_2 suffice?

2 Proofs

Throughout, we think of a bijective mapping $f: V(G) \to V(H)$ as the multigraph with vertices V(G) and edges labelled by "G" or "H". We speak of H- and G-edges, H- and G-neighbors of vertices, and H-cliques, H-independent sets, etc. A link is a copy of P_3 with one H-edge and one G-edge, and a uv-link is a link with endpoints u and v; a GH-link from u to v is a link with endpoints u, v whose edge incident to u is from G; similarly we have HGlinks. From a given mapping f, a uv-swap results in a new mapping f' with f'(u) = f(v), f'(v) = f(u), and f' = f otherwise. A quasipacking of G with H is a mapping f whose multigraph is simple except for a single pair of vertices joined by both an H-edge and a G-edge; this pair is called the conflicting edge of the quasipacking.

Consider a pair of graphs (G, H), with H being c-degenerate, each on n vertices, that do not pack; furthermore assume that H is edge-minimal with this property. Thus for any edge e in H, G and H - e pack, and so there is a quasipacking of H and G with conflicting edge e.

Let u' be a vertex of minimum positive degree in H, let $x' \in N_H(u)$, and consider a quasipacking f of G with H with conflicting edge u'x'. Let $u = f^{-1}(u')$ and $x = f^{-1}(x')$. We will now consider the set of links from u to each vertex.

Consider a $y \in V(G) \setminus \{u, x\}$. Perform a *uy*-swap: since G and H do not pack, there must be some conflicting edge, and such a conflict must involve an H-edge incident to either u or y; together with the conflicting G-edge, we have a *uy*-link in the original quasipacking. There are two links from u to itself, using the parallel edges ux in each order. Thus there are at least n links from u in the original quasipacking f.

The number of GH-links from u is at most $\sum_{y \in N_G(u)} \deg_H(f(y))$. The number of HG-links from u is at most $\sum_{z' \in N_H(u')} \deg_G(f^{-1}(z'))$. Hence we have

$$n \le \# \text{ links from } u \le \sum_{y \in N_G(u)} \deg_H(f(y)) + \sum_{z' \in N_H(u')} \deg_G(f^{-1}(z')) \le \sum_{i=1}^{\Delta(G)} d_i^{(H)} + \sum_{j=1}^c d_j^{(G)}.$$
(1)

This establishes Theorem 5.

To prove Theorem 4, suppose additionally that H is a forest, henceforth called F, and that $3\Delta(G) + \ell^*(F) = n$. (So, we still assume that G and F do not pack, and that F is edge-minimal with this property.)

If $\Delta(G) = 1$, then it is easy to show that *n* is even, $G = \frac{n}{2}K_2$, and $F = K_{1,n-1}$. (In fact, such a *G* will pack with any bipartite graph that is not complete bipartite.) So we henceforth assume that $\Delta(G) > 1$, and seek a contradiction.

Lemma 8. For any leaf u' of F and x' its neighbor, and a quasipacking f of G with F with f(u) = u' and f(x) = x' and conflicting edge ux, we have the following.

- 1. For every $y \in V(G) \setminus \{u, x\}$, there is a unique link from u to y; there is no link from u to x; and there are two links from u to itself.
- 2. $\deg_G(x) = \deg_G(u) = \Delta(G).$
- 3. For every $w \in N_G(u)$, $\deg_F(f(w)) \ge 2$.
- 4. For every $w \notin N_G(u)$, $\deg_F(f(w)) \leq 2$.

Proof. Note that we now have $\deg_F(u') = 1$, so $\sum_{z' \in N_F(u')} \deg_G(f^{-1}(z')) = \deg_G(x)$. In this case we can expand on (1):

$$n \le \#$$
 links from $u \le \sum_{y \in N_G(u)} \deg_F(f(y)) + \deg_G(x)$ (2)

$$\leq \sum_{y \in N_G(u)} \left(\deg_F(f(y)) - 2 \right) + 2\Delta(G) + \Delta(G) \tag{3}$$

$$\leq \sum_{y \in N_G(u)} \max\{ \deg_F(f(y)) - 2, 0\} + 3\Delta(G)$$
(4)

$$\leq \sum_{i=1}^{n} \max\{d_i^{(F)} - 2, 0\} + 3\Delta(G) = \ell^*(F) + 3\Delta(G) = n,$$
 (5)

so we have equality throughout. Conclusion i follows from having equality in line (i + 1) above, for $i \in [4]$.

For a vertex v in a graph H, we write $N_H[v]$ for the closed neighborhood, i.e. $N_H(v) \cup \{v\}$. For a set S of vertices, $N_H(S) = \bigcup_{v \in S} N_H(v) - S$.

Lemma 9. For any leaf u' of F and x' its neighbor, and a quasipacking f of G with F with f(u) = u' and f(x) = x' and conflicting edge ux, we have the following.

- 1. $N_G[u] = N_G[x]$.
- 2. Let $Q = N_G[u]$. Then G[Q] is a clique component.

Proof. Proof of part 1.

Let $A = N_G(u) - N_G[x]$, $B = N_G(u) \cap N_G(x)$, $C = N_G(x) - N_G[u]$. Also, let $N_A = N_F(f(A))$, $N_B = N_F(f(B))$, $N_C = N_F(f(C))$, and $N_x = N_F(x')$.

We will show that $A = N_A = N_C = C = \emptyset$.

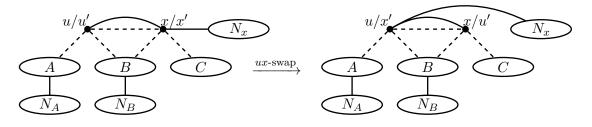


Figure 1: Left: the quasipacking f, with G-edges dashed and F-edges solid. Right: the result after the ux-swap.

Note that $B \cup C \cup \{u\}$ is precisely the set of vertices with an FG-link from u. By Lemma 8(1), there are no F-edges from A to x, else x would have a GF-link; and there are no F-edges from A to $B \cup C \cup N_x$, else such an endpoint in $B \cup C \cup N_x$ would have two links. So for each vertex of A to have exactly one link, F[f(A)] must be a perfect matching. Furthermore, the F-edges incident to A only have endpoints in $A \cup N_A$. Each vertex of N_A must have exactly one F-edge from A (to have one link). And by Lemma 8(3), each vertex of A has at least one F-neighbor in N_A . Note that we thus have $|N_A| \ge |A|$. The vertices of N_A, N_B, N_x all have GF-links by definition, and to have exactly one, these sets must be disjoint. Thus we have that $\{u, x\}, A, B, C, N_A, N_B, N_x$ is a partition of V(G). See the left side of Figure 1.

Now perform a ux-swap. In Figure 1, we visualize with V(G) fixed, so just the *F*-edges adjacent to u' and x' move; roughly speaking, we just interchange the roles of u and x and those of A and C. The result is again a quasipacking with ux the only conflicting edge. The *F*-neighbors of u are precisely N_x . Repeating the arguments of the last paragraph, for each vertex of N_C to have exactly one link, we must have $|N_C| \ge |C|$. Suppose that $A \ne \emptyset$. Then since $|N_A| \ge |A|, N_A \ne \emptyset$ as well. Now, the only possible links (from x) to vertices in N_A are *GF*-links through *C*; hence $N_C = N_A$, and the *F*-edges incident to *C* have endpoints in $C \cup N_A$. Furthermore, since $N_C = N_A \ne \emptyset$, $C \ne \emptyset$ as well; so each vertex of N_A has *F*-degree at least 2 (one edge from *A* and one from *C*). But also, by Lemma 8(3) applied to the original and also this new quasipacking, every vertex of *A* and *C* has *F*-degree at least 2, with *F*-edges entirely in $A \cup C \cup N_A$. So $F[f(A \cup C \cup N_A)]$ has minimum degree at least 2, contradicting that it is a forest, unless $A = C = N_A = \emptyset$. Note that this implies that $N_G[u] = N_G[x] = \{u, x\} \cup B$.

Proof of part 2.

This time perform a *uy*-swap for some vertex $y \in Q \setminus \{u, x\}$ to get \tilde{f} . The result is again a quasipacking with yx the only conflicting edge, with $\tilde{f}(y) = u'$. By part 1, $N_G[y] = N_G[x] = Q$. Since this holds for every $y \in Q \setminus \{u, x\}$, we have that Q is a clique; and since $\deg_G(x) = \Delta(G)$ by Lemma 8(2), G[Q] is a clique component of G.

Let u' be a leaf in F, and let x' be its neighbor. Consider a quasipacking f of G with F with f(u) = u' and f(x) = x' and conflicting edge ux. (Such exists by the extremal choice of F, as in the proof of Theorem 5.)

Let G[Q] be the clique component of G given in Lemma 9(2). Let z be a vertex of Q with smallest F-degree larger than 1 (such a choice is possible, as deg_F(x') \geq 2 by Lemma 8(3)), and let z' = f(z). Let $z_1, z_2 \in V(G)$ be two F-neighbors of z.

In f, z_1 and z_2 each have exactly one F-edge into Q and at most one other F-edge, by Lemma 8(1,4). So z_1, z_2 have no F-neighbors inside Q except z. From this and that Q is a G-clique in the quasipacking, the set $Q \cup \{z_1, z_2\} \setminus \{z\}$ is F-independent (whether z = x or not) except perhaps the conflicting edge ux. So $Q \cup \{z_1, z_2\} \setminus \{u, z\}$ is F-independent. Let $X = f(Q \cup \{z_1, z_2\} \setminus \{u, z\})$.

Let $g: V(G) \to V(F)$ be a bijection such that g(Q) = X. Since G[Q] is a clique component and X is independent, g is a packing if and only if $g|_{G-Q}$ is a packing of G-Q with F-X.

Claim: $\deg_F(z') \ge 4$.

Suppose to the contrary that $\deg_F(z') \leq 3$. We have taken two of the neighbors of z into X, so $\deg_{F-X}(z') \leq 1$. And z' is the only vertex of F - X that may have degree larger than 2, by Lemma 8(4). That is, F - X is a linear forest. We have that $\Delta(G - Q) \leq \Delta(G) \leq \frac{n}{3}$, so

$$\begin{split} \delta(G-Q) &= |V(G-Q)| - 1 - \Delta(G-Q) \\ &= n - (\Delta(G) + 1) - 1 - \Delta(G-Q) \\ &\geq n - \frac{3}{2}\Delta(G) - \frac{1}{2}\Delta(G) - 2 \\ &\geq \frac{1}{2}n - \frac{1}{2}\Delta(G) - 2 \\ &= \frac{1}{2}|V(G-Q)|, \end{split}$$

and so Dirac's condition for Hamiltonicity applies ([10]). Since $\overline{G-Q}$ contains a Hamiltonian cycle, it also contains the linear forest F - X, i.e. F - X and G - Q pack, a contradiction. This completes the proof of the Claim.

This Claim, together with having $z' \notin X$ but its two neighbors $z_1, z_2 \in X$, gives us the inequality

$$\begin{split} \ell^*(F-X) &= \sum_{v \in V(F-X)} \max\{ \deg_{F-X}(v) - 2, 0 \} \\ &\leq -2 + \sum_{v \in V(F-X)} \max\{ \deg_F(v) - 2, 0 \} \\ &= -2 + \sum_{v \in V(F)} \max\{ \deg_F(v) - 2, 0 \} - \sum_{v \in X} \max\{ \deg_F(v) - 2, 0 \} \\ &= -2 + \ell^*(F) - \sum_{v \in X} \max\{ \deg_F(v) - 2, 0 \}. \end{split}$$

From Lemma 8(3), every vertex of f(Q - u) has *F*-degree at least two; and since z' was chosen to have smallest *F*-degree among the non-leaves of f(Q), the Claim gives that they must in fact have degree at least four. All these vertices except z' are in *X*, so we have at least $\Delta(G) - 1$ vertices of *X* with degree at least 4. Hence $2 + \sum_{v \in X} \max\{\deg_F(v) - 2, 0\} \ge 0$

$$2\Delta(G) > \Delta(G) + 1$$
, so

$$\begin{aligned} 3\Delta(G-Q) + \ell^*(F-X) &\leq 3\Delta(G) + \ell^*(F) - 2 - \sum_{v \in X} \max\{\deg_F(v) - 2, 0\} \\ &= n - 2 - \sum_{v \in X} \max\{\deg_F(v) - 2, 0\} \\ &< n - \Delta(G) - 1 \\ &= |V(G-Q)|. \end{aligned}$$

Thus, by Theorem 5, G - Q and F - X pack, a contradiction. This completes the proof of Theorem 4.

Acknowledgment. The authors thank the anonymous referees for their helpful suggestions for improving the exposition.

References

- M. Aigner and S. Brandt. Embedding arbitrary graphs of maximum degree two. J. London Math. Soc. (2), 48(1):39–51, 1993.
- [2] N. Alon and E. Fischer. 2-factors in dense graphs. Discrete Math., 152(1-3):13–23, 1996.
- B. Bollobás and S. E. Eldridge. Packings of graphs and applications to computational complexity. J. Combin. Theory Ser. B, 25(2):105–124, 1978.

- [4] B. Bollobás, A. Kostochka, and K. Nakprasit. Packing d-degenerate graphs. J. Combin. Theory Ser. B, 98(1):85–94, 2008.
- [5] S. Brandt. Subtrees and subforests of graphs. J. Combin. Theory Ser. B, 61(1):63–70, 1994.
- [6] P. A. Catlin. Subgraphs of graphs. I. Discrete Math., 10:225–233, 1974.
- [7] P. A. Catlin. Embedding subgraphs and coloring graphs under extremal degree conditions. Pro-Quest LLC, Ann Arbor, MI, 1976. Thesis (Ph.D.)–The Ohio State University.
- [8] B. Csaba. On the Bollobás–Eldridge conjecture for bipartite graphs. Combinatorics, Probability and Computing, 16(5):661–691, 2007.
- [9] B. Csaba, A. Shokoufandeh, and E. Szemerédi. Proof of a conjecture of Bollobás and Eldridge for graphs of maximum degree three. *Combinatorica*, 23(1):35–72, 2003. Paul Erdős and his mathematics (Budapest, 1999).
- [10] G. A. Dirac. Some theorems on abstract graphs. Proceedings of the London Mathematical Society, s3-2(1):69–81, 1952.
- [11] N. Eaton. A near packing of two graphs. J. Combin. Theory Ser. B, 80(1):98–103, 2000.
- [12] H. Kaul and A. Kostochka. Extremal graphs for a graph packing theorem of Sauer and Spencer. Combin. Probab. Comput., 16(3):409–416, 2007.
- [13] H. Kaul, A. Kostochka, and G. Yu. On a graph packing conjecture by Bollobás, Eldridge and Catlin. Combinatorica, 28(4):469–485, 2008.
- [14] H. A. Kierstead and A. V. Kostochka. Efficient graph packing via game colouring. Combin. Probab. Comput., 18(5):765–774, 2009.
- [15] H. A. Kierstead, A. V. Kostochka, and G. Yu. Extremal graph packing problems: Ore-type versus Dirac-type. In *Surveys in combinatorics 2009*, volume 365 of *London Math. Soc. Lecture Note Ser.*, pages 113–135. Cambridge Univ. Press, Cambridge, 2009.
- [16] N. Sauer and J. Spencer. Edge disjoint placement of graphs. J. Combin. Theory Ser. B, 25(3):295– 302, 1978.
- [17] W. C. van Batenburg and R. J. Kang. Packing graphs of bounded codegree. Combin. Probab. Comput., 27(5):725–740, 2018.
- [18] M. Woźniak. Packing of graphs. Dissertationes Math. (Rozprawy Mat.), 362:78, 1997.
- [19] H. P. Yap. Packing of graphs—a survey. In Proceedings of the First Japan Conference on Graph Theory and Applications (Hakone, 1986), volume 72, pages 395–404, 1988.