Stronger counterexamples to the topological Tverberg conjecture

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Abstract

Denote by Δ_M the M-dimensional simplex. A map $f: \Delta_M \to \mathbb{R}^d$ is an almost r-embedding if $f(\sigma_1) \cap \ldots \cap f(\sigma_r) = \emptyset$ whenever $\sigma_1, \ldots, \sigma_r$ are pairwise disjoint faces. A counterexample to the topological Tverberg conjecture asserts that if r is not a prime power and $d \geq 2r+1$, then there is an almost r-embedding $\Delta_{(d+1)(r-1)} \to \mathbb{R}^d$. This was improved by Blagojević–Frick–Ziegler using a simple construction of higher-dimensional counterexamples by taking k-fold join power of lower-dimensional ones. We improve this further (for d large compared to r): If r is not a prime power and $N = (d+1)r - r \left\lceil \frac{d+2}{r+1} \right\rceil - 2$, then there is an almost r-embedding $\Delta_N \to \mathbb{R}^d$. The improvement follows from our stronger counterexamples to the r-fold van Kampen–Flores conjecture. Our proof is based on generalizations of the Mabillard–Wagner theorem on construction of almost r-embeddings from equivariant maps, and of the Özaydin theorem on existence of equivariant maps.

MSC 2010: 52C35, 55S91, 57S17.

Keywords: The topological Tverberg conjecture, multiple points of maps, equivariant maps, deleted product obstruction.

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1 Introduction and statement of results

Denote by Δ_M the M-dimensional simplex. We omit 'continuous' for maps. A map $f \colon K \to \mathbb{R}^d$ of a union K of closed faces of Δ_M is an **almost** r-embedding if $f(\sigma_1) \cap \ldots \cap f(\sigma_r) = \emptyset$ whenever $\sigma_1, \ldots, \sigma_r$ are pairwise disjoint faces of K. We omit 'for any integers d, r > 0 and $k \geq 0$ ' at the beginnings of statements.

Theorem 1.1'. If r is not a prime power and $d \geq 3r + 1$, then there is an almost r-embedding $\Delta_{(d+1)(r-1)} \to \mathbb{R}^d$.

This is a counterexample to the celebrated topological Tverberg conjecture. Theorem 1.1' follows from Theorem 1.4' of Özaydin and Mabillard–Wagner, together with Lemma 2.1' of

^{*}University of Copenhagen. Email: savvakumov@gmail.com. Supported by the Austrian Science Fund (FWF), Project P31312-N35 and the European Research Council under the European Union's Seventh Framework Programme ERC Grant agreement ERC StG 716424 – CASe.

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Gromov-Blagojević-Frick-Ziegler. For the history, see the surveys [BBZ, Sk16, BZ16, BS17, Sh18] and the references therein. In [AMS+] Theorem 1.1' was improved to $d \ge 2r + 1$.

The following result gives stronger counterexamples to the topological Tverberg conjecture.

Theorem 1.1. If r is not a prime power and

$$N = N(d,r) := (d+1)r - r\left[\frac{d+2}{r+1}\right] - 2,$$

then there is an almost r-embedding $\Delta_N \to \mathbb{R}^d$.

Theorem 1.1 follows from Theorem 1.4 and Lemma 2.1, see the details in §2.

Remark 1.2 (motivation and related work). (a) There naturally appears more general problem: For which a, d there is an almost r-embedding $\Delta_a \to \mathbb{R}^d$?

This problem was considered in [BFZ, §5], where higher-dimensional counterexamples were constructed from lower-dimensional ones: If there is an almost r-embedding $\Delta_a \to \mathbb{R}^d$, then for each k there is an almost r-embedding $\Delta_{k(a+1)-1} \to \mathbb{R}^{k(d+1)-1}$ [BFZ, Lemma 5.2]. The proof (exposed a bit simpler [Sk16, Remark 1.5.c]) is by taking k-fold join power as follows. For two maps $f: \Delta_a \to \mathbb{R}^p$ and $g: \Delta_b \to \mathbb{R}^q$ define the join $f*g: \Delta_{a+b+1} = \Delta_a*\Delta_b \to \mathbb{R}^p*\mathbb{R}^q \subset \mathbb{R}^{p+q+1}$ by the formula

$$(f * g)(\lambda x \oplus (1 - \lambda)y) := \lambda f(x) \oplus (1 - \lambda)f(y), \text{ where } \lambda \in [0, 1].$$

A join of almost r-embeddings is an almost r-embedding. Hence the k-fold join power of an almost r-embedding $\Delta_a \to \mathbb{R}^d$ is an almost r-embedding $\Delta_{k(a+1)-1} \to \mathbb{R}^{k(d+1)-1}$.

According to a private communication by F. Frick this procedure [BFZ, Theorem 5.4] together with the counterexample in [AMS+, Theorem 1.1] gives an almost r-embedding $\Delta_F \to \mathbb{R}^d$ for r not a prime power, d sufficiently large, and F some integer close to $(d+1)r - \frac{r+\frac{1}{2}}{r+1}(d+1)$. Presumably F - (d+1)(r-1) can be arbitrarily large.

Theorem 1.1 provides even stronger counterexamples to the topological Tverberg conjecture: for d large compared to r we have N > (d+1)(r-1), and even N > F. Theorem 1.1 is a partial result on [BFZ, Conjecture 5.5] stating that for r < d not a prime power there is an almost r-embedding $\Delta_{(d+1)r-2} \to \mathbb{R}^d$ and there are no almost r-embeddings $\Delta_{(d+1)r-1} \to \mathbb{R}^d$. (The case $r \geq d$ of the conjecture is trivially covered by known results.) Observe that $N \leq dr - 2$ for r < d. The second part of the conjecture is addressed in [FS20].

- (b) We think counterexamples of Theorem 1.1 are mostly interesting because their proof requires non-trivial ideas, see Theorems 1.4, 2.2, and Lemma 3.2 below. Thus we do not spell out even stronger counterexamples which presumably could be obtained by combining Theorem 1.1 with the procedure of [BFZ, §5] described in (a). Our proof of Theorem 1.1 is independent of Theorem 1.1, of [AMS+], and of the iterated join construction described in (a).
- (c) Let us illustrate Theorem 1.1 by numerical examples. Earlier results gave almost 6-embeddings $\Delta_{280} \to \mathbb{R}^{55}$ and $\Delta_{275} \to \mathbb{R}^{54}$, and, more generally, almost r-embeddings $\Delta_{(d+1)(r-1)} \to \mathbb{R}^d$ for $d \geq 2r+1$, $\Delta_{d(r-1)} \to \mathbb{R}^{d-1}$ for $d \geq 2r+2$, and, even more generally, almost r-embeddings $\Delta_{(d+1-s)(r-1)} \to \mathbb{R}^{d-s}$ for $d \geq 2r+s+1$. Corollary 1.3 below gives an almost 6-embedding $\Delta_{280} \to \mathbb{R}^{54}$, and, more generally, almost r-embeddings $\Delta_{(d+1)(r-1)} \to \mathbb{R}^{d-s}$ for certain r, d, s.

Corollary 1.3. Assume that r is not a prime power.

- (a) For $q \ge r+2$ and d = (r+1)q-1 there is an almost r-embedding $\Delta_{(d+1)(r-1)} \to \mathbb{R}^{d-1}$.
- (b) If $d \ge (s+2)r^2$ for some integer s, then there is an almost r-embedding $\Delta_{(d+1)(r-1)} \to \mathbb{R}^{d-s}$.

Proof. Part (a) follows by Theorem 1.1 because $q \ge r+2$, so $((r+1)q-1)r-rq-2 \ge (r+1)q(r-1)$. Part (b) follows by Theorem 1.1 because $d \ge (s+2)r^2 \ge (s+1)r^2 + r - 1$, hence

$$(d+1)(r-1) \le (d-s+1)r - r\frac{d-s+2+r}{r+1} - 2 \le (d-s+1)r - r\left\lceil \frac{d-s+2}{r+1} \right\rceil - 2.$$

A **complex** is a collection of closed faces (=simplices) of some simplex. A k-complex is a complex containing at most k-dimensional simplices. The body (or geometric realization) |K| of a complex K is the union of simplices of K. Thus continuous or piecewise-linear (PL) maps $|K| \to \mathbb{R}^d$ and continuous maps $|K| \to S^m$ are defined. We abbreviate |K| to K; no confusion should arise.

By general position, any k-complex admits an almost r-embedding in $\mathbb{R}^{k+\lceil \frac{k+1}{r-1} \rceil}$. The following counterexample to the r-fold van Kampen–Flores conjecture allows to sometimes decrease $\lceil \frac{k+1}{r-1} \rceil$ to $\frac{k}{r-1}$ (indeed, take k = sr).

Theorem 1.4'. (Özaydin and Mabillard-Wagner) If $s \geq 3$ and r is not a prime power, then any s(r-1)-complex admits an almost r-embedding in \mathbb{R}^{sr} .

This result follows from the Özaydin and the Mabillard–Wagner Theorems 2.2' and 2.4'. See [MW14, §1, Motivation & Future Work, 2nd paragraph] or the survey [Sk16, Theorems 1.7, 3.1 and 3.2, and Remark 1.9.b].

The following result gives stronger counterexamples to the r-fold van Kampen-Flores conjecture.

Theorem 1.4. If r is not a prime power, then any k-complex admits an almost r-embedding in $\mathbb{R}^{k+\lceil \frac{k+3}{r} \rceil}$.

Theorem 1.4 is easily deduced below from Theorems 2.2 and 2.4. The main new ingredient in the proof of Theorems 1.1 and 1.4 is the following Theorem 2.2.

Acknowledgments. We are grateful to M. Berezovik, F. Frick, A. Magazinov, and the anonymous referees for helpful suggestions.

2 Deduction of Theorems 1.1 and 1.4 from Theorem 2.2

Lemma 2.1'. (Constraint) For M = (sr+2)(r-1) if there is an almost r-embedding of the union of s(r-1)-faces of Δ_M in \mathbb{R}^{sr} , then there is an almost r-embedding $\Delta_M \to \mathbb{R}^{sr+1}$.

Lemma 2.1' is due to Gromov [Gr10, 2.9.c] and Blagojević–Frick–Ziegler [BFZ14, Lemma 4.1.iii and 4.2], [Fr15', proof of Theorem 4]. Lemma 2.1' has a simple proof (see e.g. the survey [Sk16, Lemma 1.8]). The proof shows that

- \mathbb{R}^{sr} and \mathbb{R}^{sr+1} can be replaced by \mathbb{R}^{d-1} and \mathbb{R}^d , respectively.
- s(r-1) and (sr+2)(r-1) can be replaced by k and (k+2)r-2, respectively.

Lemma 2.1 (Constraint). For M = (k+2)r - 2 if there is an almost r-embedding of the union of k-faces of Δ_M in \mathbb{R}^{d-1} , then there is an almost r-embedding $\Delta_M \to \mathbb{R}^d$.

Proof of Theorem 1.1 modulo Theorem 1.4. Theorem 1.1 holds for d=1 because then N=r-2, so Δ_N does not have r non-empty pairwise disjoint faces. Assume further that $d\geq 2$. Denote $k:=d-1-\left\lceil\frac{d+2}{r+1}\right\rceil$. Then N=(k+2)r-2. Since $d\geq 2$ and $r\geq 6$, we have $k\geq 0$. We have

$$\frac{d+2}{r+1} = \frac{d+2 - \frac{d+2}{r+1}}{r} \ge \frac{k+3}{r} \quad \Rightarrow \quad d-1 = k + \left\lceil \frac{d+2}{r+1} \right\rceil \ge k + \left\lceil \frac{k+3}{r} \right\rceil.$$

Since r is not a prime power, by Theorem 1.4 there is an almost r-embedding of the union of k-faces of Δ_N to \mathbb{R}^{d-1} . Then by the Constraint Lemma 2.1 there is an almost r-embedding $\Delta_N \to \mathbb{R}^d$.

Denote by Σ_r the permutation group of r elements. Let $\mathbb{R}^{d \times r} := (\mathbb{R}^d)^r$ be the set of real $d \times r$ -matrices. The group Σ_r acts on $\mathbb{R}^{d \times r}$ by permuting the columns. Denote

$$\delta_r = \delta_{r,d} := \{(x, x, \dots, x) \in \mathbb{R}^{d \times r} \mid x \in \mathbb{R}^d\}.$$

Theorem 2.2'. (Özaydin) If r is not a prime power and $\dim X = d(r-1)$, then there is a Σ_r -equivariant map $X \to \mathbb{R}^{d \times r} - \delta_r$.

This is proved in [Oz], see R. Karasev's short proof in the survey [Sk16, §3.2]. The following result improves the Özaydin Theorem 2.2'.

Theorem 2.2. If r is not a prime power and X is a complex with a free action of Σ_r , then there is a Σ_r -equivariant map $X \to \mathbb{R}^{2 \times r} - \delta_r$.

Remark 2.3 (Relations to other papers). Let X be a complex with a free action of Σ_r , Observe that if dim X < d(r-1), then the existence of an equivariant map $X \to \mathbb{R}^{d \times r} - \delta_r$ follows by general position. The statements [AK19, Theorem 5.1], [AKu19, Theorem 1.1] of other improvements of Theorem 2.2' are obtained from Theorem 2.2 replacing 2 by 1 and imposing stronger restrictions on r.¹

Our proof of Theorem 2.2 is analogous to the argument in [AK19, AKu19]: *Theorem 2.2 follows from the known Lemma 3.1 and the new Lemma 3.2 below* (see also the paragraph after Lemma 3.2). This is different from the Özaydin idea [Oz] and from the short proof in [Sk16, §3.2]. So our argument gives a simple proof of the Özaydin Theorem 2.2'.

For a complex K let $K_{\Lambda}^{\times r}$ be the associated r-fold deleted product:

$$K_{\Delta}^{\times r}:=\bigcup\{\sigma_1\times\cdots\times\sigma_r\ : \sigma_i \text{ a simplex of } K,\ \sigma_i\cap\sigma_j=\emptyset \text{ for every } i\neq j\}.$$

The group Σ_r has a natural action on the set $K_{\Delta}^{\times r}$, permuting the points in an r-tuple (p_1, \ldots, p_r) . This action is evidently free and PL, i.e. compatible with some structure of a complex on $K_{\Delta}^{\times r}$.

Theorem 2.4'. (Mabillard-Wagner) Assume that K is a s(r-1)-complex and $s \geq 3$. There exist an almost r-embedding $K \to \mathbb{R}^{sr}$ if and if there is a Σ_r -equivariant map $K_{\Delta}^{\times r} \to \mathbb{R}^{sr \times r} - \delta_r$.

Theorem 2.4 (Mabillard-Wagner). Assume that K is a k-complex and $rd \geq (r+1)k+3$. There exists an almost r-embedding $f: K \to \mathbb{R}^d$ if and only if there exists a Σ_r -equivariant map $K_{\Delta}^{\times r} \to \mathbb{R}^{d \times r} - \delta_r$.

See the proofs in [MW15], [Sk16, §3], and in [MW16, MW16', Sk17], respectively.²

Proof of Theorem 1.4 modulo Theorems 2.2 and 2.4. Let K be any k-complex and $d:=k+\left\lceil\frac{k+3}{r}\right\rceil$. If d=1, then k=0, so Theorem 1.4 is obvious. Now assume that $d\geq 2$. Since r is not a prime power, by Theorem 2.2 there is a Σ_r -equivariant map $K_{\Delta}^{\times r} \to \mathbb{R}^{2\times r} - \delta_r$. The composition of this map with the r-th power of the inclusion $\mathbb{R}^2 \to \mathbb{R}^d$ gives a Σ_r -equivariant map $K_{\Delta}^{\times r} \to \mathbb{R}^{d\times r} - \delta_r$. We have $rd \geq (r+1)k+3$. Hence by Theorem 2.4 there is an almost r-embedding $K \to \mathbb{R}^d$.

¹For a finite cyclic or dihedral group G, and a certain representation space V of G, G-equivariant maps from the classifying space EG to V-0 were constructed in [BG17]. Theorem 2.2 should also be compared to [Ba93, Theorem 3.6 and the paragraph afterwards]. That reference takes a group G from a certain class and proves that there exists some representation W of G, for which there exist G-equivariant maps $X \to S(W)$ for certain G-spaces X. However, $G = \Sigma_r$ does not belong to that class, and the Σ_r -space S(W) described in [Ba93, Theorem 3.6 and the paragraph afterwards] need not coincide with the Σ_r -space $\mathbb{R}^{2\times r} - \delta_r$ given by Theorem 2.2.

²For a criticism of the proof of Theorem 2.4 in [MW16, MW16'] see [Sk17, §5]; this footnote is not present in the published version of this paper.

3 Proof of Theorem 2.2

Lemma 3.1. Let G be a finite group acting on S^n . If there exists a degree zero G-equivariant self-map of S^n , then any complex X with a free action of G has a G-equivariant map $X \to S^n$.

See the historical remarks and a proof in [AK19, §5]. In particular, this lemma follows from [Ba93, Lemma 3.9]; see [AK19, §5] for a simpler direct proof.³

Denote by $S_{\Sigma_r}^{d(r-1)-1} \subset \mathbb{R}^{d \times r} - \delta_r$ the set formed by all $d \times r$ -matrices in which the sum of the elements in each row is zero, and the sum of the squares of all the matrix elements is 1. This set is invariant under the action of Σ_r . This set is homeomorphic to $S^{d(r-1)-1}$.

Lemma 3.2. If r is not a prime power, then there is a degree zero Σ_r -equivariant self-map of $S := S_{\Sigma_r}^{2r-3} = S_{\Sigma_r}^{2(r-1)-1}$.

Lemma 3.2 is analogous to [AK19, Theorem 4.2] and [AKu19, Theorem 1.4.c,d]. Those theorems are stated in a different language, but can be obtained from Lemma 3.2 by replacing 2r-3 by r-2, and adding stronger restrictions on r. The proofs follow the same plan via Proposition 3.3 (although this proposition is not explicitly stated in [AK19, AKu19]). The binomial coefficients appear in the same way. However, the procedure of obtaining the prescribed sign in front of the binomial coefficient requires additional work. The procedure is easier in [AK19], is intermediate here, and is more complicated in [AKu19] (the proof of [AKu19] also uses additional ideas).

Proof of Lemma 3.2. Since r is not a prime power, the greatest common divisor of the binomial coefficients $\binom{r}{k}$, $k=1,\ldots,r-1$ is 1 [Lu78]. Hence -1 is an integer linear combination of the binomial coefficients. Denote by $C \subset S$ the set of $(2 \times r)$ -matrices whose second row is zero, and the entries of the first row involve only two numbers. A special map is a Σ_r -equivariant self-map f of S which is a local homeomorphism in some neighborhood of C. The identity map of S is a special map of degree 1. Thus the lemma is implied by the following assertion

For any r, any k = 1, ..., r - 1, and any special map f there are special maps f_+, f_- such that deg $f_{\pm} = \deg f \pm \binom{r}{k}$.

This is implied by the following proposition.

Proposition 3.3. For any r, any k = 1, ..., r - 1, and any special map f there are a point $c \in C$ and Σ_r -equivariant homotopies $h_+, h_- : S \times I \to \mathbb{R}^{2r-2}_{\Sigma_r}$ such that

- $(1_{\pm}) \ h_{\pm,0} = f, \ and \ h_{\pm,1} : S \to S \ is \ special,$
- (2_{\pm}) h is a local homeomorphism over a neighborhood of 0, and

$$\deg h_{\pm,1} - \deg f = \deg_0 h = \pm \binom{r}{k} \operatorname{sign}_c f.$$

Here $\deg_0 h$ is the degree of h over 0, and $\operatorname{sign}_c f \in \{+1, -1\}$ is the sign of the preimage c of f(c) under the map f (since $c \in C$ and f is special, f is a homeomorphism in a neighborhood of c; we have $\operatorname{sign}_c f = \operatorname{sign} \det df(c)$ if f is smooth and $\det df(c) \neq 0$).

Informally, we construct h_{\pm} by 'pushing' a certain point $c \in C$ and its orbit towards the origin in $\mathbb{R}^{2\times r}$. See [AK19, Figures 1 and 2]. For h_{+} such a pushing is 'twisted along the reflection with respect to a certain hyperplane'.

³Note that to read the direct proof in [AK19, §5] is simpler than to find the notation required for the statement [Ba93, Lemma 3.9] and deduce Lemma 3.1 from that statement; this footnote is not present in the published version of this paper.

Proof of Proposition 3.3. Definitions of c, G, U and ρ . The objects we construct depend on r, k but we suppress r, k from their notation. Define the vector

$$M := (\underbrace{k-r, \dots, k-r}_{k}, \underbrace{k, \dots, k}_{r-k}).$$

Define the $(2 \times r)$ -matrix $c := \binom{M/|M|}{0} \in C$. The orbit $\Sigma_r c$ of c contains $\binom{r}{k}$ points. The stabilizer of c is $G := \Sigma_k \times \Sigma_{r-k} \subset \Sigma_r$

The standard metric on the sphere is Σ_r -invariant. Hence there is a small ball U centered at c such that $U \cap \sigma U = \emptyset$ for any $\sigma \in \Sigma_r - G$, and $\sigma U = U$ for any $\sigma \in G$. Take a smooth function $\rho': S \to [0,1]$ which is zero outside U, and is one in a neighborhood of c. Define a smooth function $\rho: S \to [0,1]$ by $\rho(x) := \frac{1}{r!} \sum_{\sigma \in \Sigma_r} \rho'(\sigma x)$. Then ρ is zero outside $\Sigma_r U$, is one in a neighborhood of $\Sigma_r c$, and is invariant with respect to the Σ_r -action.

Construction of h_- . For $t \in [0, 1/2]$ define

$$h_{-}(x,t) = h_{-,t}(x) := \begin{cases} f(x) & x \notin \Sigma_r U \\ f(x) - 4t\rho(x)f(\sigma c) & x \in \sigma U, \ \sigma \in \Sigma_r. \end{cases}$$

Clearly, h_{-} is well-defined, is continuous, and is Σ_{r} -equivariant

In this paragraph we prove that

$$h_{-}^{-1}(0) = (\Sigma_r c) \times 1/4$$

for the constructed homotopy $h_-: S \times [0,1/2] \to \mathbb{R}^{2r-2}_{\Sigma_r}$. If $t \in [0,1/2]$ and $h_{-,t}(x) = 0$, then $x \in \Sigma_r U$. Since f is a local homeomorphism and |f(x)| = 1, we have $4t\rho(x) = 1$ and $x = \sigma c$ for some $\sigma \in \Sigma_r$. Then t = 1/4.

Since $h_{-}^{-1}(0) = (\Sigma_r c) \times 1/4$, we have $0 \notin h_{-,1/2}(S)$. Hence there is a homotopy⁴ h_{-} : $S \times [1/2,1] \to \mathbb{R}^{2r-2}_{\Sigma_r} - \{0\}$ between $h_{-,1/2}$ and a map $h_{-,1}$ defined by $h_{-,1}(x) := \frac{h_{-,1/2}(x)}{|h_{-,1/2}(x)|}$. Then $h_{-}^{-1}(0) = (\Sigma_r c) \times 1/4$ for the constructed homotopy $h_{-}: S \times [0,1] \to \mathbb{R}^{2r-2}_{\Sigma_r}$.

Proof of (1_{-}) . Clearly, $h_{-,0} = f$. Since f is a local homeomorphism in some neighborhood of C, the map h_{-1} is such in a neighborhood of $C - \Sigma_r c$. In a neighborhood of σc the map h_{-1} is a shift by $-2f(\sigma c)$ composed with the central projection back to the sphere. This is clearly a homeomorphism in a neighborhood of σc .

Proof of (2_{-}) . Take a sufficiently small neighborhood W of c such that $\rho(W) = 1$ and f(W)is contained in the hemisphere centered at f(c). Then $h_{-,t}(x) = f(x) - 4tf(c)$ for any $x \in W$ and $t \in [0,1/2]$. Let $f(c)^{\perp}$ be the hyperplane tangent to S at f(c). Let $\pi: \mathbb{R}^{2r-2}_{\Sigma_r} \to f(c)^{\perp}$ be the orthogonal projection. Let $\langle f(c) \rangle$ be the line passing through f(c) and the origin (and so orthogonal to $f(c)^{\perp}$). Then $N:=\pi^{-1}(\pi(f(W)))$ is a neighborhood of this line. Observe that $\pi|_{f(W)}$ is a homeomorphism. Denote by π^{-1} its inverse. Define a map

$$\tau:N\to N\quad\text{by}\quad \tau(y):=y+\pi(y)-\pi^{-1}(\pi(y)).$$

Then τ shifts every line in N parallel to $\langle f(c) \rangle$ by the vector $\pi(y) - \pi^{-1}(\pi(y)) \in \langle f(c) \rangle$. Hence τ is a self-homeomorphism of N preserving the orientation. The origin is fixed under τ because $\tau(0) = 0 + \pi(0) - \pi^{-1}(\pi(0)) = c - c = 0$. For $t \in [0, 1/2]$ we have

$$\tau(h_{-,t}(x)) = \tau(f(x) - 4tf(c)) = f(x) - 4tf(c) + \pi(f(x)) - \pi^{-1}(\pi(f(x))) = \pi(f(x)) - 4tf(c).$$

Hence for the decomposition $\mathbb{R}^{2r-2}_{\Sigma_r} = \langle f(c) \rangle \times f(c)^{\perp}$ the map $\tau \circ h_{-|W \times [0,1/2]}$ is the Cartesian product of the maps

$$\pi \circ f: W \to f(c)^{\perp}$$
 and $a: [0, 1/2] \to \langle f(c) \rangle$, where $a(t) := -4t f(c)$.

⁴E.g. take $h_{-,t}(x) = \frac{h_{-,1/2}(x)}{2-2t+(2t-1)|h_{-,1/2}(x)|}$.

Both $\pi \circ f$ and a are embeddings. Hence $\tau \circ h_-|_{W \times [0,1/2]}$ is a homeomorphisms in a neighborhood of (c,1/4). Hence $h_-|_{W \times [0,1/2]}$ and $h_-|_{W \times [0,1]}$ are also such. Since h_- is Σ_r -equivariant and $h_-^{-1}(0) = (\Sigma_r c) \times 1/4$, we see that h_- is a local homeomorphism over a neighborhood of 0.

Denote $D := \partial(S \times [0,1]) = S \times \{0,1\}$. Then

$$\deg h_{-,1} - \deg f = \deg h_{-,1} - \deg h_{-,0} = \deg(h_-|_D : D \to S) = \deg_0 h_-.$$

We also have

$$\frac{\deg_0 h_-}{\binom{r}{k}} = \operatorname{sign}_{(c,1/4)} h_- = \operatorname{sign}_0 \tau^{-1} \cdot \operatorname{sign}_{(c,1/4)} (\tau \circ h_-) = \operatorname{sign}_{(c,1/4)} (\tau \circ h_-) = \\ = \operatorname{sign}_{1/4} a \cdot \operatorname{sign}_c (\pi \circ f) = -\operatorname{sign}_{f(c)} \pi \cdot \operatorname{sign}_c f = -\operatorname{sign}_c f.$$

Construction of h_+ . Define the $(2 \times r)$ -matrix $c_1 := \begin{pmatrix} 0 \\ M/|M| \end{pmatrix} \in S$. Take the hyperplane $c_1^{\perp} \subset \mathbb{R}^{2r-2}_{\Sigma_r}$ orthogonal to c_1 and passing through the origin. Then $c \in c_1^{\perp}$. We may assume that $V := U \cap \rho^{-1}[1/3, 1]$ is a ball by assuming that ρ is radially symmetric in U. Let $q : V \to V[g]$ be the restriction to V of the reflection with respect to the hyperplane c_1^{\perp} . Then q is G-equivariant, $\operatorname{sign}_c q = -1$ and $q^{-1}(c) = c$.

Define a G-equivariant map $\phi': U \times \{0\} \cup (\partial U \cup V) \times [0,1] \to U$

- on $U \times \{0\}$ as the natural homeomorphism;
- on $\partial U \times [0,1]$ as the composition of the projection and the inclusion $\partial U \to U$;
- on $V \times [1/3, 1]$ as the composition of the projection, q, and the inclusion $V \to U$;
- on $V \times [0, 1/3]$ as a G-equivariant homotopy between the identity map ϕ'_0 and $q = \phi'_{1/3}$.

By the Borsuk homotopy extension theorem [FF89, §5.5] ϕ' extends to a homotopy $\psi: U \times [0,1] \to U$. Define a homotopy $\phi: U \times [0,1] \to U$ by considering the average of ψ with respect to G:

$$\phi(x,t) = \frac{1}{|G|} \sum_{g \in G} g\psi(g^{-1}x,t) \in U.$$

We have $\phi = \phi' = \psi$ on $U \times \{0\} \cup (\partial U \cup V) \times [0,1]$ because ϕ' is G-equivariant. The homotopy ϕ is G-equivariant, since for any $m \in G$ from the linearity of the action of G on U one obtains

$$m\phi(x,t) = \frac{1}{|G|} \sum_{g \in G} mg\psi(g^{-1}x,t) = \frac{1}{|G|} \sum_{k=mg \in G} k\psi(k^{-1}mx,t) = \phi(mx,t).$$

Extend ϕ to $\Sigma_r U \times [0,1]$ in a Σ_r -equivariant way. Define for $t \in [0,1/2]$

$$h_{+}(x,t) = h_{+,t}(x) := \begin{cases} f(x) & x \notin \Sigma_{r}U \\ f(\phi(x,2t)) - 4t\rho(x)f(\sigma c) & x \in \sigma U, \ \sigma \in \Sigma_{r}. \end{cases}$$

Clearly, h_+ is well-defined (since ϕ is G-equivariant), is continuous, and is Σ_r -equivariant. In this paragraph we prove that

$$h_{+}^{-1}(0) = (\Sigma_r c) \times 1/4$$

for the constructed homotopy $h_+: S \times [0,1/2] \to \mathbb{R}^{2r-2}_{\Sigma_r}$. If $h_t(x) = 0$ and $t \in [0,1/2]$, then $x \in \Sigma_r U$. Since f is a local homeomorphism and |f(x)| = 1, we have $4t\rho(x) = 1$ and $\phi(x,2t) = \sigma c$ for some $\sigma \in \Sigma_r$. Therefore $\rho(x) \geq \frac{1}{4}$, $t \geq 1/4$, and $x \in \sigma V$. Hence $\phi(x,2t) = \sigma q(\sigma^{-1}x)$. Since $q^{-1}(c) = c$, we have $x = \sigma c$, $\rho(x) = 1$, and t = 1/4.

Then analogously to the construction of h_- we extend h_+ to $S \times [0,1]$, and check the properties (1_+) and (2_+) . In the proof of (2_+) we have $\deg_0 h_+ = -\binom{r}{k} \operatorname{sign}_c q \operatorname{sign}_c f = \binom{r}{k} \operatorname{sign}_c f$. Here $\operatorname{sign}_c q$ appears because for $t \geq 1/6$ and $x \in V$ we have $\phi(x, 2t) = q(x)$.

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