

Balanced subdivisions of cliques in graphs

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Abstract

Given a graph H , a balanced subdivision of H is a graph obtained from H by subdividing every edge the same number of times. In 1984, Thomassen conjectured that for each integer $k \geq 1$, high average degree is sufficient to guarantee a balanced subdivision of K_k . Recently, Liu and Montgomery resolved this conjecture. We give an optimal estimate up to an absolute constant factor by showing that there exists $c > 0$ such that for sufficiently large d , every graph with average degree at least d contains a balanced subdivision of a clique with at least $cd^{1/2}$ vertices. It also confirms a conjecture from Verstraëte: every graph of average degree cd^2 , for some absolute constant $c > 0$, contains a pair of disjoint isomorphic subdivisions of the complete graph K_d . We also prove that there exists some absolute $c > 0$ such that for sufficiently large d , every C_4 -free graph with average degree at least d contains a balanced subdivision of the complete graph K_{cd} , which extends a result of Balogh, Liu and Sharifzadeh.

1 Introduction

Given a graph H , a *subdivision* of H , denote by $\mathbb{T}H$, is a graph obtained from H by subdividing some or all of its edges by drawing new vertices on those edges. In other words, some edges of H are replaced by internally vertex-disjoint paths. The original vertices of H are the *branch vertices* of the $\mathbb{T}H$, and its new vertices are called *subdividing vertices*. Subdivision plays a central role in topological graph theory since Kuratowski [15] showed that a graph is planar if and only if it does not contain a subdivision of a complete graph on five vertices or a subdivision of a complete bipartite graph with three vertices in each partition.

In 1967, Mader [18] proved that for each integer $k \geq 1$, there exists $c > 0$ such that every graph with average degree at least c contains a subdivision of the complete graph K_k . For each integer $k \geq 1$, let $d(k)$ be the smallest number such that each graph with average degree at least $d(k)$ contains a subdivision of K_k . Mader [18], and independently Erdős and Hajnal [5] conjectured that $d(k) = O(k^2)$. The extremal example, the disjoint union of complete regular bipartite subgraphs, first observed by Jung [8], gives a lower bound of $d(k)$, which matches the conjecture. Later, Bollobás and Thomason [3], independently Komlós and Szemerédi [13, 14] confirmed the conjecture in 1990s.

The extremal example suggests that this bound can be improved when some small subgraphs are forbidden. Mader [20] conjectured that every C_4 -free graph contains a subdivision of a clique with order linear in its average degree. Kühn and Osthus [9, 11] proved that every graph with sufficiently large girth contains a subdivision of a clique with order linear in its minimum degree. They [10] also showed that each C_4 -free graph G with average degree k contains a $\mathbb{T}K_{k/\log^{12} k}$.

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Balogh, Liu and Sharifzadeh [2] proved when $k \geq 3$, each C_{2k} -free graph contains a subdivision of a clique with order linear in its average degree. In 2016, Liu and Montgomery [16] solved this conjecture.

A natural question of subdivision is whether we can control the length of paths which replace the edges. Given an integer z , denoted by $\mathsf{T}H^{(z)}$ the subdivision obtained from H by replacing all edges of H with internally vertex-disjoint paths of length z , and we also call it a *balanced H -subdivision*. An old question of Erdős [4] is, for each $\varepsilon > 0$, whether one can find a $\delta > 0$ such that every graph with n vertices and at least εn^2 edges contains a $\mathsf{T}K_{\delta\sqrt{n}}^{(2)}$. Alon, Krivelevich and Sudakov [1] showed the existence of δ , and this result was improved to $\delta = \varepsilon$ by Fox and Sudakov [7]. Erdős' question only considers dense graphs. Generally, Thomassen [21, 22, 23] gave the following conjecture.

Conjecture 1.1 ([23]). *For every $k \geq 1$, there exists a function $f(k)$, if $\delta(G) \geq f(k)$, then G contains a balanced subdivision of K_k .*

Recently, Liu and Montgomery [17] confirmed Conjecture 1.1. Later, a result of Wang [25] implies balanced clique subdivision of order d^c for any $c < 1/2$. We give the following theorem, improving Wang's result.

Theorem 1.2. *There exists an absolute constant $c > 0$ such that every graph with average degree at least d contains a $\mathsf{T}K_{c\sqrt{d}}^{(z)}$ for sufficiently large d and some $z \in \mathbb{N}$.*

Note that the bound above is asymptotically optimal up to a constant factor by considering a disjoint union of complete bipartite graphs $K_{d,d}$. This theorem also resolves the following conjecture from Verstraëte [24]: every graph of average degree cd^2 , for some absolute constant $c > 0$, contains a pair of disjoint isomorphic subdivisions of the complete graph K_d . Indeed, we can divide a balanced subdivision of K_d , which can be found by Theorem 1.2, into two balanced subdivisions of $K_{d/2}$, which are isomorphic.

Mader's conjecture states that every C_4 -free graph has a subdivision of a clique with order linear in its average degree. Balogh, Liu and Sharifzadeh [2] proved that every n -vertex C_4 -free graph with $cn^{3/2}$ edges contains a $\mathsf{T}K_{c'\sqrt{n}}^{(4)}$ for some $c' > 0$. We extend their result by giving the following theorem. The bound is also optimal up to a constant factor.

Theorem 1.3. *There exists an absolute constant $c > 0$ such that every C_4 -free graph with average degree at least d contains a $\mathsf{T}K_{cd}^{(z)}$ for sufficiently large d and some $z \in \mathbb{N}$.*

We shall give a unified approach for Theorem 1.2 and Theorem 1.3. The rest of the paper will be organized as follows. In Section 2, we introduce some necessary notions and the main result (Lemma 2.7) in our proofs. Section 3 is devoted to the proof of Lemma 2.7 and in Section 4 we discuss three crucial ingredients used for embedding balanced subdivisions.

2 Preliminaries

2.1 Notation

Given a graph $G = (V, E)$, we write $|G| = |V(G)|$ for the order of the graph G . Let $\delta(G)$ and $d(G)$ be the minimum and average degree of G respectively. For a set of vertices $W \subseteq V(G)$, denote its external neighbourhood by $N(W) = (\cup_{v \in W} N(v)) \setminus W$. Denote by $G[W]$ the induced subgraph of G on W , and we write $G - W$ for the induced subgraph $G[V(G) \setminus W]$. Given graphs G and H , the graph $G \cup H$ has vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. For a collection \mathcal{P} of graphs, denote by $|\mathcal{P}|$ the number of graphs in \mathcal{P} , and we write $V(\mathcal{P}) = \cup_{G \in \mathcal{P}} V(G)$.

For a path P , the *length* of P , denoted by $l(P)$, is the number of edges in P . Given two vertices x, y , an x, y -path is a path with endvertices x and y . When we say P is a path from

a vertex set A to a disjoint vertex set B , we mean that P has one endvertex in A and another one in B , and has no internal vertices in $A \cup B$.

Let $[n] := \{1, 2, \dots, n\}$. When it is not essential, we omit the floors and ceilings. All logarithms are natural.

2.2 Komlós-Szemerédi graph expander

Komlós and Szemerédi [13, 14] introduced the following sublinear expander, which forms the foundation of our proof.

Definition 2.1 (Sublinear expander). For each $\varepsilon_1 > 0$ and $k > 0$, a graph G is an (ε_1, k) -expander if

$$|N(X)| \geq \varepsilon(|X|, \varepsilon_1, k) \cdot |X|$$

for all $X \subseteq V(G)$ of size $k/2 \leq |X| \leq |V(G)|/2$, where

$$\varepsilon(x, \varepsilon_1, k) := \begin{cases} 0 & \text{if } x < k/5, \\ \varepsilon_1 / \log^2(15x/k) & \text{if } x \geq k/5. \end{cases}$$

Whenever the choices of ε_1, k are clear, we omit them and write $\varepsilon(x)$ for $\varepsilon(x, \varepsilon_1, k)$. Note that $\varepsilon(x, \varepsilon_1, k)$ decreases as x increases when $x \geq k/2$. Komlós and Szemerédi [14] showed that every graph G contains a sublinear expander as dense as G .

Theorem 2.2 ([14]). *There exists $\varepsilon_1 > 0$ such that the following holds for every $k > 0$. Every graph G has an (ε_1, k) -expander H with $d(H) \geq d(G)/2$ and $\delta(H) \geq d(H)/2$.*

Note that, in Theorem 2.2, the sublinear expander H can be much smaller than the original graph G . Indeed, G could be the disjoint union of many copies of such a graph H .

The following lemma is the key property of sublinear expanders that we will use. It roughly says that in a sublinear expander, we can connect two sets of vertices using a short path while avoiding a smaller vertex set.

Lemma 2.3 ([14]). *Let $\varepsilon_1, k > 0$. If G is an n -vertex (ε_1, k) -expander, then any two vertex sets, each of size at least $x \geq k$, are of distance at most $\frac{2}{\varepsilon_1} \log^3(\frac{15n}{k})$ apart. This remains true even after deleting $x \cdot \varepsilon(x)/4$ arbitrary vertices from G .*

It is convenient to work on a bipartite graph, so we use the following well known result.

Proposition 2.4. *Every graph G contains a bipartite subgraph H with $d(H) \geq d(G)/2$.*

Combining this proposition with Theorem 2.2, we get the following corollary immediately.

Corollary 2.5. *There exists $\varepsilon_1 > 0$ such that the following holds for every $k > 0$ and $d \in \mathbb{N}$. Every graph G with $d(G) \geq 8d$ has a bipartite (ε_1, k) -expander H with $\delta(H) \geq d$.*

The following proposition shows that every $K_{s,t}$ -free $(\varepsilon_1, \varepsilon_2 d^{s/(s-1)})$ -expander is an $(\varepsilon_1, \varepsilon_2 d)$ -expander.

Proposition 2.6 (Proposition 5.2 in [16]). *Let $0 < \varepsilon_1 < 1$, $0 < \varepsilon_2 < 1/10^{5t}$, and let $t \geq s \geq 2$ be integers. If G is a $K_{s,t}$ -free, $(\varepsilon_1, \varepsilon_2 d^{s/(s-1)})$ -expander with $\delta(G) \geq d/16$, then G is also an $(\varepsilon_1, \varepsilon_2 d)$ -expander.*

2.3 Main results

The following is a rough proof strategy for Theorem 1.2 and Theorem 1.3. By Corollary 2.5, G contains a bipartite subgraph which has some expansion properties. Then depending on whether the subgraph is dense or not, we divide the proof into two cases. The dense case is handled in Lemma 2.7, and the sparse case is covered in Lemma 2.8 [25].

Lemma 2.7. *There exists $\varepsilon_1 > 0$ such that, for every $0 < \varepsilon_2 < 1/5$ and $s \geq 240$, there exist d_0 and some constant $c > 0$ such that the following holds for each $n \geq d \geq d_0$ and $d \geq \log^s n$.*

(i) *If G is a bipartite n -vertex $(\varepsilon_1, \varepsilon_2 d)$ -expander with $\delta(G) \geq d$, then G contains a $\text{TK}_{c\sqrt{d}}^{(z)}$ for some $z \in \mathbb{N}$;*

(ii) *If G is a C_4 -free bipartite n -vertex $(\varepsilon_1, \varepsilon_2 d^2)$ -expander with $\delta(G) \geq d$, then G contains a $\text{TK}_{cd}^{(z)}$ for some $z \in \mathbb{N}$.*

Lemma 2.8 (Lemma 1.3 in [25]). *There exists $\varepsilon_1 > 0$ such that for any $0 < \varepsilon_2 < 1/5$ and $s \geq 20$, there exist d_0 and some constant $c > 0$ such that the following holds for each $n \geq d \geq d_0$ and $d < \log^s n$. Suppose that G is a $\text{TK}_{d/2}^{(2)}$ -free bipartite n -vertex $(\varepsilon_1, \varepsilon_2 d)$ -expander with $\delta(G) \geq d$. Then G contains a $\text{TK}_{cd}^{(z)}$ for some $z \in \mathbb{N}$.*

We will show that Theorem 1.2 and Theorem 1.3 follow from Lemma 2.7 and Lemma 2.8.

Proof of Theorem 1.2. Given $\varepsilon_2 = 1/10$ and $s = 240$, we have constants ε_1 and d_0 such that the properties in Corollary 2.5, Lemma 2.7 and Lemma 2.8 hold. Let G be a graph with average degree $d(G) = d$ for some $d \geq d_0$. Write $d_1 = d/8$. By Corollary 2.5 with $d = d_1$, G has a bipartite $(\varepsilon_1, \varepsilon_2 d_1)$ -expander H with $\delta(H) \geq d_1$. Let $n = |H|$. If $d_1 \geq \log^s n$, then by Lemma 2.7, G contains a $\text{TK}_{c_1\sqrt{d_1}}^{(z)}$ for some constant $c_1 > 0$ and some $z \in \mathbb{N}$. Otherwise, by Lemma 2.8, G contains either a $\text{TK}_{d_1/2}^{(2)}$ or a $\text{TK}_{c_2 d_1}^{(z)}$ for some constant $c_2 > 0$ and some $z \in \mathbb{N}$. This finishes the proof by taking $c = \min\{1/16, \sqrt{2}c_1/4, c_2/8\}$. \square

Proof of Theorem 1.3. Fix $\varepsilon_2 = 1/10^6$ and $s = 240$, there exist constants ε_1 and d_0 such that the conclusions of Corollary 2.5, Lemma 2.7 and Lemma 2.8 hold. Let G be a C_4 -free graph with average degree at least d for some $d \geq d_0$. Write $d_1 = d/8$. By Corollary 2.5 with $d = d_1$, G has a bipartite $(\varepsilon_1, \varepsilon_2 d_1^2)$ -expander H with $\delta(H) \geq d_1$. Let $n = |H|$. If $d_1 \geq \log^s n$, then by Lemma 2.7, G contains a $\text{TK}_{c_1 d_1}^{(z)}$ for some constant $c_1 > 0$ and some $z \in \mathbb{N}$. Otherwise, by Proposition 2.6 with $s = t = 2$, H is also an $(\varepsilon_1, \varepsilon_2 d_1)$ -expander. Then by Lemma 2.8, G contains either a $\text{TK}_{d_1/2}^{(2)}$ or a $\text{TK}_{c_2 d_1}^{(z)}$ for some constant $c_2 > 0$ and some $z \in \mathbb{N}$. This finishes the proof by taking $c = \min\{1/16, c_1/8, c_2/8\}$. \square

3 Proof of Lemma 2.7

In this section we prove Lemma 2.7. To achieve this, we first introduce some structures from Liu and Montgomery [16, 17].

3.1 Gadgets

Definition 3.1 (Hub [16]). Given integers $h_1, h_2 > 0$, an (h_1, h_2) -hub is a graph consisting of a center vertex u , a set $S_1(u) \subseteq N(u)$ of size h_1 , and pairwise disjoint sets $S_1(z) \subseteq N(z) \setminus \{u\}$ of size h_2 for each $z \in S_1(u)$. Denote by $H(u)$ a hub with center vertex u and write $B_1(u) = \{u\} \cup S_1(u)$ and $S_2(u) = \bigcup_{z \in S_1(u)} S_1(z)$. For any $z \in S_1(u)$, write $B_1(z) = \{z\} \cup S_1(z)$.

Definition 3.2 (Unit [16]). Given integers $h_0, h_1, h_2, h_3 > 0$, an (h_0, h_1, h_2, h_3) -unit F is a graph consisting of a core vertex v , h_0 vertex-disjoint (h_1, h_2) -hubs $H(u_1), \dots, H(u_{h_0})$ and pairwise

disjoint v, u_j -paths of length at most h_3 . By the *exterior* of the unit, denoted by $\text{Ext}(F)$, we mean $\bigcup_{j=1}^{h_0} S_2(u_j)$. Denote by $\text{Int}(F) := V(F) \setminus \text{Ext}(F)$ the *interior* of the unit.

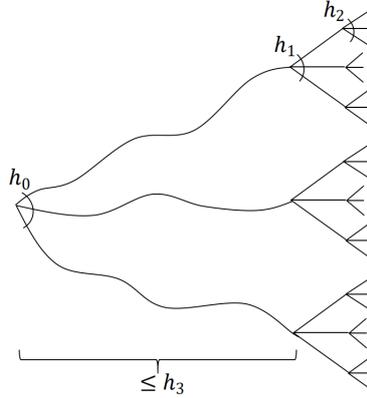


Figure 1: (h_0, h_1, h_2, h_3) -unit

Definition 3.3 (Expansion [17]). Given a vertex v in a graph F , F is a (D, m) -*expansion* of v if $|F| = D$ and v is at distance at most m in F from any other vertex of F .

By the definition of expansion, we have the following property.

Proposition 3.4 ([17]). Let $D, m \in \mathbb{N}$ and $1 \leq D' \leq D$. Then, any graph F which is a (D, m) -*expansion* of v contains a subgraph which is a (D', m) -*expansion* of v .

Liu and Montgomery [17] introduced a structure called *adjuster* which contains a collection of paths whose lengths form a long arithmetic progression of difference 2. We shall use this structure to adjust a path to a desired length.

Definition 3.5 (Adjuster [17]). A (D, m, k) -*adjuster* $\mathcal{A} = (v_1, F_1, v_2, F_2, A)$ in a graph G consists of *core vertices* $v_1, v_2 \in V(G)$, graphs $F_1, F_2 \subseteq G$ and a *center vertex set* $A \subseteq V(G)$ such that the following hold for some $l \in \mathbb{N}$.

- A1** $A, V(F_1)$ and $V(F_2)$ are pairwise disjoint.
- A2** For each $i \in [2]$, F_i is a (D, m) -*expansion* of v_i .
- A3** $|A| \leq 10mk$.
- A4** For each $i \in \{0, 1, \dots, k\}$, there is a v_1, v_2 -path in $G[A \cup \{v_1, v_2\}]$ of length $l + 2i$.

We refer to the subgraphs F_1 and F_2 as the *ends* of the adjuster, and note that $V(\mathcal{A}) = V(F_1) \cup V(F_2) \cup A$. We denote by $l(\mathcal{A})$ the smallest such l for which **A4** hold. Then it immediately follows that $l(\mathcal{A}) \leq |A| + 1 \leq 10mk + 1$. We call a $(D, m, 1)$ -adjuster a *simple adjuster*.

3.2 Proof of Lemma 2.7

Throughout the paper, we always choose m to be the smallest even integer which is larger than $80 \log^4 \frac{n}{d}$ (or $80 \log^4 \frac{n}{d^2}$ for the C_4 -free case) in which $\delta(G) \geq d$. We first introduce the following two lemmas. The first one is from Fox and Sudakov [7], which states the existence of a balanced clique subdivision in a dense graph.

Lemma 3.6 (Theorem 8.1 in [7]). For every $\varepsilon > 0$, if G is a graph with n vertices and εn^2 edges, then G contains a $\text{TK}_{\varepsilon\sqrt{n}}^{(2)}$.

The second lemma is from Balogh, Liu and Sharifzadeh [2] on large balanced clique subdivisions in dense C_4 -free graphs.

Lemma 3.7 (Theorem 1.4 in [2]). *For every $c > 0$ there is a $c' > 0$ such that the following holds. If G is a C_4 -free graph with n vertices and $cn^{3/2}$ edges, then G contains a $\text{TK}_{c'\sqrt{n}}^{(4)}$.*

Then based on the two lemmas above, it remains to consider the case when d is much less than n . An outline of the proof of Lemma 2.7 is as follows. We construct a collection of units whose interiors are pairwise disjoint (see Lemma 3.8), and the core vertices of those units would serve as the branch vertices of a balanced clique subdivision. In order to connect a pair of core vertices of two units, v_1, v_2 say, by a path of fixed length z , we first construct an adjuster. Adapting an approach in Liu and Montgomery [17], we also need to robustly build a desired adjuster by linking simple adjusters. Unlike in [17] and [25] where the simple adjuster is relatively small, we need a large-sized one for our purpose (see Lemma 3.9). To be more precise, we construct an adjuster of order roughly $dm^{O(1)}$. To achieve this, we shall introduce a specific structure (see Definition 4.8). Then we find two paths connecting the exteriors of two units to two expansions of the adjuster. By extending those two paths to v_1, v_2 within two units respectively, we can get two paths P and Q such that the sum of their length is close to z . Finally, by the property of the adjuster, we obtain an intermediate path R of length $z - l(P) - l(Q)$, and thus $P \cup R \cup Q$ is a path as desired.

The following lemma enables us to build a unit whilst avoiding any medium-sized vertex set. This helps us construct many units whose interiors are pairwise disjoint.

Lemma 3.8. *For any $0 < \varepsilon_1, \varepsilon_2 < 1$, $s \geq 240$ and let $c = 1/200$, there exists K such that the following holds for sufficiently large n and d with $d \geq \log^s n$.*

- (i) *Let G be a $\text{TK}_{\sqrt{d}}^{(2)}$ -free bipartite n -vertex $(\varepsilon_1, \varepsilon_2 d)$ -expander with $\delta(G) \geq d$ and $n \geq Kd$. Given any set $W \subseteq V(G)$ with size at most $2cdm^4$, we have that $G - W$ contains a $(c\sqrt{d}, m^4, c\sqrt{d}, 2m)$ -unit.*
- (ii) *Let G be a C_4 -free bipartite n -vertex $(\varepsilon_1, \varepsilon_2 d^2)$ -expander with $\delta(G) \geq d$ and $n \geq Kd^2$. Given any set $W \subseteq V(G)$ with size at most $2cd^2m^4$, we have that $G - W$ contains a $(cd, m^4, cd, 2m)$ -unit.*

We shall make use of the following result to build a desired adjuster robustly.

Lemma 3.9. *There exists some $\varepsilon_1 > 0$ such that, for every $0 < \varepsilon_2 < 1/5$ and integer $s \geq 240$, there exists d_0 and K such that the following holds for each $n \geq d \geq d_0$ and $d \geq \log^s n$.*

- (i) *Let G be a $\text{TK}_{\sqrt{d}}^{(2)}$ -free n -vertex $(\varepsilon_1, \varepsilon_2 d)$ -expander with $\delta(G) \geq d$ and $n \geq Kd$. Let $D = dm^4/10^7$. Let $W \subseteq G$ satisfy $|W| \leq D/\log^3 \frac{n}{d}$. Then $G - W$ contains a (D, m, r) -adjuster for any $r \leq \frac{1}{10}dm^2$.*
- (ii) *Let G be a C_4 -free n -vertex $(\varepsilon_1, \varepsilon_2 d^2)$ -expander with $\delta(G) \geq d$ and $n \geq Kd^2$. Let $D = d^2m^4/10^7$. Let $W \subseteq G$ satisfy $|W| \leq D/\log^3 \frac{n}{d^2}$. Then $G - W$ contains a (D, m, r) -adjuster for any $r \leq \frac{1}{10}d^2m^2$.*

The following lemma helps us find two paths respectively connecting two pairs of vertex sets whilst avoiding a smaller vertex set.

Lemma 3.10. *For any $0 < \varepsilon_1, \varepsilon_2 < 1$, there exists d_0 and K such that the following holds for each $n \geq d \geq d_0$ and $d \geq \log^s n$.*

- (i) *Let G be a $\text{TK}_{\sqrt{d}}^{(2)}$ -free n -vertex $(\varepsilon_1, \varepsilon_2 d)$ -expander with $\delta(G) \geq d$ and $n \geq Kd$. Let $D = dm^4/10^7$, $l \leq dm^3$. Let $W \subseteq V(G)$ satisfy $|W| \leq D/\log^3 \frac{n}{d}$, or*
 - (ii) *Let G be a C_4 -free n -vertex $(\varepsilon_1, \varepsilon_2 d^2)$ -expander with $\delta(G) \geq d$ and $n \geq Kd^2$. Let $D = d^2m^4/10^7$, $l \leq d^2m^3$. Let $W \subseteq V(G)$ satisfy $|W| \leq D/\log^3 \frac{n}{d^2}$.*
- Let $U_i \subseteq V(G) - W$ be disjoint vertex sets of size at least D , $i \in \{1, 2\}$, and $F_j \subseteq G - W - U_1 - U_2$*

be vertex-disjoint (D, m) -expansion of v_j , $j \in \{3, 4\}$. Then, $G - W$ contains vertex-disjoint paths P and Q with $l \leq l(P) + l(Q) \leq l + 13m$ such that both P and Q connect $\{v_1, v_2\}$ to $\{v_3, v_4\}$ for some $v_i \in U_i$ with $i \in \{1, 2\}$.

Throughout the rest of this paper, we always choose

$$\kappa = \sqrt{d} \text{ (or } d \text{ for } C_4\text{-free case)} \quad (1)$$

and recall that m is the smallest even integer which is larger than $80 \log^4 \frac{n}{\kappa^2}$. As $n/\kappa^2 \geq K$, when K is sufficiently large, we obtain that n/κ^2 and also m are sufficiently large, and

$$n/\kappa^2 \geq m^a \quad (2)$$

for any given constant a . Recall that $\varepsilon(x)$ is decreasing, and since $n/\kappa^2 \geq K$ is sufficiently large,

$$\varepsilon(n) = \frac{\varepsilon_1}{\log^2(15n/\varepsilon_2\kappa^2)} \geq \frac{10^5}{\log^3(n/\kappa^2)} \geq \frac{10^5}{m}. \quad (3)$$

Now we are ready to prove Lemma 2.7.

Proof of Lemma 2.7. The proof ideas of (i) and (ii) are similar, and we choose κ as in (1). Note that we shall assume that G is $\text{TK}_\kappa^{(2)}$ -free for (i), otherwise the proof is done. Let $c' = 1/200$ and $z = m^3$. We choose K and d_0 to be sufficiently large.

For (i), if $n < K\kappa^2$, then by Lemma 3.6 with $\varepsilon = 1/K$, G contains a balanced subdivision of a clique of size $\varepsilon\sqrt{n}$. For (ii), if $n < K\kappa^2$, then by Lemma 3.7 with $c_1 = 1/K$, there exists $c'_1 > 0$ such that G contains a balanced subdivision of a clique of size $c'_1\sqrt{n}$. Now it remains to consider the case $n/\kappa^2 \geq K$. Since $n/\kappa^2 \geq K$, we have $m \geq 80 \log^4 K$. We claim that we can greedily pick in G a collection $\{M_1, \dots, M_{c'\kappa}\}$ of $(c'\kappa, m^4, c'\kappa, 2m)$ -units with disjoint interiors. Indeed, this is possible by applying Lemma 3.8 on G with W being the interiors of $(c'\kappa, m^4, c'\kappa, 2m)$ -units we have constructed in each stage of the process, and by the fact that the size of interiors of $c'\kappa$ such units is at most $c'\kappa(2m + 1 + m^4) \cdot c'\kappa \leq 2c'^2\kappa^2m^4$. By the pigeonhole principle, we can find $c'\kappa/2$ such units among them such that their core vertices are in the same part of the bipartition for G . Without loss of generality, these units are $M_1, \dots, M_{c'\kappa/2}$ with core vertices $w_1, \dots, w_{c'\kappa/2}$ and denote by $u_{i,j}$ the center of the j -th hub in M_i , where $1 \leq i \leq c'\kappa/2$ and $1 \leq j \leq c'\kappa$. Let W be the union of the vertices in the $w_i, u_{i,j}$ -paths in all the units, including their endvertices. Then $|W| \leq c'\kappa \cdot (2m + 1) \cdot c'\kappa/2 \leq 2c'^2\kappa^2m$. We will construct a $\text{TK}_{c'\kappa/4}^{(z)}$ as follows. Let \mathcal{P} be a maximum collection of paths under the following rules.

B1 Each path connects one pair of center vertices of hubs from different units, such that it can be extended to a path of length z connecting the core vertices of those two units.

B2 All paths in \mathcal{P} are pairwise disjoint, and the internal vertices of those paths are disjoint from W .

B3 For each pair of units, there is at most one path in \mathcal{P} between their respective hubs.

Let W_1 be the set of vertices of \mathcal{P} . Then by **B1** and **B3**, $|W_1| \leq (m^3 + 1) \binom{c'\kappa/2}{2} \leq c'^2\kappa^2m^3$. Let us call a unit *bad* if more than κm^3 vertices in the interior of this unit have been used in \mathcal{P} , and *good* otherwise. Thus, as the interior of the units are disjoint, there are at most $\frac{c'^2\kappa^2m^3}{\kappa m^3} \leq c'\kappa/4$ bad units.

Claim 3.11. For every pair of good units, there is a path in \mathcal{P} between two of their respective hubs.

Proof. Without loss of generality, we may assume for contradiction that M_1 and M_2 are a pair of good units for which there is no desired path in \mathcal{P} . Let B be the union of the interiors of M_1 and M_2 . Then we have $|B| \leq 2 \cdot 2c'\kappa m^4$. Set $W' = W \cup W_1 \cup B$. Note that $|W'| \leq 2c'^2\kappa^2 m + c'^2\kappa^2 m^3 + 4c'\kappa m^4 \leq 3c'^2\kappa^2 m^3$. By **B3**, M_1 (or M_2) has at least $c'\kappa/2$ hubs whose center vertices are not used in \mathcal{P} . Denote these hubs by $H(u_{1,k})$, $k \in I$ for some index set I with $|I| \geq c'\kappa/2$. Let A_1 be the set of vertices in $\cup_{k \in I} S_1(u_{1,k})$ not used in \mathcal{P} . Similarly, we define A_2 for M_2 . As M_1 is good, we have

$$|A_1| \geq \cup_{k \in I} S_1(u_{1,k}) - \kappa m^3 \geq \frac{c'\kappa}{2} m^4 - \kappa m^3 \geq \frac{c'\kappa m^4}{4}.$$

The last inequality holds as $m \geq 4/c'$ for sufficiently large K . As the hubs in M_1 are disjoint, we have

$$|N_{M_1}(A_1) \setminus W'| \geq \frac{c'\kappa m^4}{4} c'\kappa - 3c'^2\kappa^2 m^3 \geq \frac{c'^2\kappa^2 m^4}{8}.$$

Similarly, $|N_{M_2}(A_2) \setminus W'| \geq c'^2\kappa^2 m^4/8$.

Set $D = \kappa^2 m^4/10^7$. Recall that $n \geq K\kappa^2$ and m is the smallest even integer which is larger than $80 \log^4 \frac{n}{\kappa}$. As $d \geq \log^s n \geq m^{60}$, for sufficiently large K , we have $|W'| \leq 3c'^2\kappa^2 m^3 \leq D/2 \log^3 \frac{n}{\kappa}$. By Lemma 3.9 with $W = W'$, there is a $(D, m, 21m)$ -adjuster in $G - W'$, which is denoted by $\mathcal{A} = (v_1, F_1, v_2, F_2, A)$. By the definitions of expansion and adjuster, we have $|A| \leq 210m^2$, $l(\mathcal{A}) \leq |A| + 1 \leq 220m^2$ and $|V(F_1)| = |V(F_2)| = D$. Let $l' = z - 21m - l(\mathcal{A})$. We have $0 \leq l' \leq \kappa^2 m^3$. Let $U'_1 = N_{M_1}(A_1) \setminus (W' \cup V(F_1) \cup V(F_2))$ and $U'_2 = N_{M_2}(A_2) \setminus (W' \cup V(F_1) \cup V(F_2))$. As the size of U'_1 (or U'_2) is at least $c'^2\kappa^2 m^4/8 - 2D \geq 2D$, there are disjoint vertex sets $U_1 \subseteq U'_1$ and $U_2 \subseteq U'_2$ such that $|U_i| \geq D$, $i \in \{1, 2\}$. As $d \geq \log^s n \geq m^{60}$, $|A \cup W'| \leq 210m^2 + D/2 \log^3 \frac{n}{\kappa} \leq D/\log^3 \frac{n}{\kappa}$. By Lemma 3.10, there are vertex-disjoint paths P_1 and Q_1 in $G - A - W'$ connecting $\{u_1, u_2\}$ to $\{v_1, v_2\}$ for some $u_i \in U_i$, $i \in \{1, 2\}$, and $l' \leq l(P_1) + l(Q_1) \leq l' + 13m$. Without loss of generality, we can assume that P_1 is a u_1, v_1 -path and Q_1 is a u_2, v_2 -path.

As U_1 is a subset of the exterior of M_1 , there exists a path P_2 from w_1 to u_1 of length at most $2m + 2 \leq 4m$ in M_1 . Similarly, there exists a path Q_2 from w_2 to u_2 of length at most $2m + 2 \leq 4m$ in M_2 . Let $P = P_1 \cup P_2$ and $Q = Q_1 \cup Q_2$. Thus, P is a w_1, v_1 -path and Q is a w_2, v_2 -path, with $l' \leq l(P) + l(Q) \leq l' + 21m$.

Now, $l(\mathcal{A}) \leq z - l(P) - l(Q) \leq l(\mathcal{A}) + 21m$. As G is a bipartite graph and w_1, w_2 are in the same part, $l(\mathcal{A})$ and $z - l(P) - l(Q)$ have the same parity. Thus there is a v_1, v_2 -path in $G[A \cup \{v_1, v_2\}]$ of length $z - l(P) - l(Q)$, *R* say, such that $P \cup R \cup Q$ is a w_1, w_2 -path of length z which satisfies **B1-B3**, contradicting the maximality of \mathcal{P} . \square

Finally, for every pair of good units M_i, M_j , there exists a w_i, w_j -path of length z , and by **B2** these paths are disjoint outside of their endvertices. As there are at least $c'\kappa/4$ units are good, we have a $\text{TK}_{c'\kappa/4}^{(z)}$. This finishes the proof by taking $c = \min\{c'/4, \varepsilon, c'_1\}$. \square

4 Proof of main tools

4.1 Constructing units

In this section, we find a collection of units whose interiors are pairwise disjoint, and we prove this by iteratively constructing a unit whilst avoiding the interiors of previous units. To achieve this, we first prove that every $\text{TK}_{\sqrt{d}}^{(2)}$ -free (or C_4 -free) graph maintains its average degree while deleting a vertex set in Lemma 4.1 (or in Corollary 4.3). Then by applying Lemma 4.4 (or Lemma 4.5), we can greedily pick a collection of vertex-disjoint hubs of certain types, and connect them with internally vertex-disjoint short paths such that one of the hubs would be linked to many others (see Claim 4.6), forming the desired unit.

We first show that a $\text{TK}_{\sqrt{d}}^{(2)}$ -free graph keeps its average degree while deleting a medium-sized vertex set.

Lemma 4.1. *Let $s \geq 8x > 0$. There exists $K = K(s, x)$ such that the following holds for each n and d satisfying $n \geq Kd$ and $d \geq \log^s n$. If G is a $\text{TK}_{\sqrt{d}}^{(2)}$ -free n -vertex graph with $\delta(G) \geq d$, then for any vertex set $W \subseteq V(G)$ of size at most dm^x , we have $d(G - W) \geq \frac{d}{2}$.*

Proof. It is easy to see this lemma holds when $|W| \leq \frac{d}{2}$. Suppose to the contrary that there exists some W such that $d(G - W) < \frac{d}{2}$ when $|W| > \frac{d}{2}$. By (2), we have $|W| \leq dm^x \leq n/2$, then

$$e(V(G - W), W) = \sum_{v \in V(G - W)} d_G(v) - 2e(G - W) \geq \frac{d}{2}|G - W|.$$

By Lemma A.1 with $(V_1, V_2, a, c, r) = (W, G - W, \sqrt{d}, d, 2)$ and

$$\alpha = \frac{e(V(G - W), W)}{|G - W||W|} \geq \frac{\frac{d}{2}|G - W|}{|G - W| \cdot dm^x} = \frac{1}{2m^x},$$

if there exists some t such that

$$\left(\frac{1}{2m^x}\right)^t \cdot \frac{d}{2} - \binom{dm^x}{2} \left(\frac{d}{n/2}\right)^t \geq \sqrt{d},$$

then G contains a $\text{TK}_{\sqrt{d}}^{(2)}$, which is a contradiction. We take $t = \log_{8m^x} \sqrt{d}$. Note that $t \geq 1$ as $s \geq 8x$. Then we have

$$\left(\frac{1}{2m^x}\right)^t \cdot \frac{d}{2} \geq \frac{4}{(8m^x)^t} \cdot \frac{d}{2} = 2\sqrt{d}, \quad (4)$$

and

$$\binom{dm^x}{2} \left(\frac{d}{n/2}\right)^t \leq \frac{d^2 m^{2x}}{\left(\frac{n}{2d}\right)^t} \leq \frac{d^2 m^{2x}}{(8m^x)^{8t}} \leq \frac{\log^{8x} n}{\log^{2s} n} \leq 1. \quad (5)$$

The second inequality in (5) holds by (2). By (4) and (5), we have

$$\left(\frac{1}{2m^x}\right)^t \cdot \frac{d}{2} - \binom{dm^x}{2} \left(\frac{d}{n/2}\right)^t > \sqrt{d}.$$

The proof is complete. \square

Then we show that a C_4 -free graph maintains its average degree while deleting a vertex set of moderate size through the following result of K3v3r3i, S3s and Tur3n [12].

Lemma 4.2 ([12]). *Let $G = (A, B)$ be a bipartite graph that does not contain a copy of $K_{s,t}$ with t vertices in A and s vertices in B . Then*

$$|A| \binom{\bar{d}(A)}{s} \leq t \binom{|B|}{s},$$

where $\bar{d}(A) = \sum_{v \in A} \frac{d(v)}{|A|}$ is the average degree in G of the vertices in A .

Thus we can get the following corollary.

Corollary 4.3. *Let $s > 0$ and $x > 0$. There exists K such that the following holds for each n and d satisfying $n \geq Kd^2$ and $d \geq \log^s n$. If G is a C_4 -free n -vertex graph with $\delta(G) \geq d$, then for any vertex set $W \subseteq V(G)$ of size at most $d^2 m^x$, we have $d(G - W) \geq \frac{d}{2}$.*

Proof. Let $H = (V(G - W), W, E)$ be a bipartite subgraph of G , and E be the set of all edges between $V(G - W)$ and W in G . By Lemma 4.2 with $(G, A, B, s, t) = (H, V(G - W), W, 2, 2)$, we have

$$|G - W| \binom{\bar{d}(V(G - W))}{2} \leq 2 \binom{|W|}{2}.$$

Therefore

$$\bar{d}(V(G - W)) \leq \frac{\sqrt{2}|W|}{\sqrt{|G - W|}} + 1 \leq \frac{2d^2m^x}{\sqrt{n}} \leq \frac{d}{2},$$

where the third inequality follows from (2) and the choice of K . Thus we have $d(G - W) \geq \frac{d}{2}$. \square

In order to construct units, we will find many vertex-disjoint hubs, and we use the following lemma to show the existence of these hubs in a $\text{TK}_{\sqrt{d}}^{(2)}$ -free graph.

Lemma 4.4. *Let $s \geq 8x > 0$. There exists K such that the following holds for each $n \geq Kd$, $d \geq \log^s n$ and any $h_1, h_2 \leq \sqrt{d}/100$. If G is a $\text{TK}_{\sqrt{d}}^{(2)}$ -free n -vertex graph with $\delta(G) \geq d$, then for any set $W \subseteq V(G)$ with size at most dm^x , we have that $G - W$ contains an (h_1, h_2) -hub.*

Proof. Let K be sufficiently large. By Lemma 4.1, we have $d(G - W) \geq d/2$, so there exists a subgraph $H \subseteq G - W$ with $\delta(H) \geq d/4$. We choose an arbitrary vertex v in H . As the size of the vertex set of (h_1, h_2) -hub is $1 + h_1 + h_1 \cdot h_2 \leq 2h_1 \cdot h_2 \leq d/4$, we can greedily find a hub with center vertex v as desired. \square

We can also robustly find hubs in a C_4 -free graph.

Lemma 4.5. *Let $s > 0$ and $x > 0$. There exists K such that the following holds for each $n \geq Kd^2$, $d \geq \log^s n$ and any $h_1, h_2 \leq d/100$. If G is a C_4 -free n -vertex graph with $\delta(G) \geq d$, then for any set $W \subseteq V(G)$ with size at most d^2m^x , we have that $G - W$ contains an (h_1, h_2) -hub.*

Proof. Let K be sufficiently large. By Corollary 4.3, we have $d(G - W) \geq d/2$, so there exists a subgraph $H \subseteq G - W$ with $\delta(H) \geq d/4$. We arbitrarily pick a vertex v in H . As H is C_4 -free, for any vertices $x, y \in N_H(v)$, we have $N_H(x) \cap N_H(y) = \{v\}$. Therefore we can in a greedy way find a hub with center vertex v as desired. \square

We now expand hubs to get a unit.

Proof of Lemma 3.8. The proof ideas of (i) and (ii) are similar, and we choose κ as in (1). We choose K to be sufficiently large. Recall that $s \geq 240$, $c = 1/200$ and m is the smallest even integer which is larger than $80 \log^4 \frac{n}{\kappa^2}$, so $\kappa^2 \geq \log^{48} n \geq m^{12}$. Since $n/\kappa^2 \geq K$, we have $m \geq 80 \log^4 K$. For sufficiently large K , we have

$$c\kappa \geq cm^6 \geq 8m^5. \quad (6)$$

We claim that we can find in $G - W$ vertex-disjoint hubs $H(w_1), \dots, H(w_{m^6})$ and $H(u_1), \dots, H(u_{\kappa m^6})$ such that each $H(w_i), 1 \leq i \leq m^6$, is a $(2c\kappa, 2c\kappa)$ -hub and each $H(u_j), 1 \leq j \leq \kappa m^6$, is a $(2m^4, 2c\kappa)$ -hub. Note that the total number of vertices in W and all these hubs we desired is at most

$$2c\kappa^2m^4 + 2(2c\kappa)^2m^6 + 2(2c\kappa)(2m^4)\kappa m^6 \leq 10c\kappa^2m^{10}.$$

Note that we can find a copy of $(2m^4, 2c\kappa)$ -hub in a $(2c\kappa, 2c\kappa)$ -hub. It suffices to show that we can find a $(2c\kappa, 2c\kappa)$ -hub in G avoiding any vertex set of size at most $10c\kappa^2m^{10}$. Indeed, this is possible by applying Lemma 4.4 for (i) or Lemma 4.5 for (ii) with $x = 10$.

Recall that for a hub with a center vertex v , $S_1(v)$ is the vertex set of the neighbours of v in the hub, and $B_1(v) = \{v\} \cup S_1(v)$. We will construct a unit using some vertex w_i as the core vertex. Let \mathcal{P} be a maximum collection of paths connecting different pairs of center vertices $\{w_i, u_j\}$ under the following rules.

C1 Each path has length at most $2m$, and all the paths are pairwise internally vertex-disjoint. Moreover all the vertices of those paths are disjoint from W .

C2 Each path avoids using any vertices in set $B_1(w_{i'})$ or $B_1(u_{j'})$ for $1 \leq i' \leq m^6$ and $1 \leq j' \leq \kappa m^6$, except for at most two vertices each in $B_1(w_i)$ and $B_1(u_j)$ when $\{w_i, u_j\}$ is the pair of vertices being connected.

Claim 4.6. There exists a vertex w_i connected to at least $c\kappa$ vertices u_j via the paths in \mathcal{P} .

Proof. Suppose to the contrary that each vertex w_i is connected to fewer than $c\kappa$ vertices u_j by the paths in \mathcal{P} . Let U be the set of interior vertices in all the paths in \mathcal{P} . Then by **C1**,

$$|U| \leq 2m \cdot m^6 \cdot c\kappa = 2c\kappa m^7. \quad (7)$$

Let W_1 be the vertex set containing the vertices in U and all the vertices in each set $B_1(u_j)$ if u_j has been connected to at least one of the vertices w_i . As there are at most $m^6 \cdot c\kappa$ such vertices u_j , using (6) we have

$$|W_1| \leq |U| + m^6 c\kappa \cdot (2m^4 + 1) \stackrel{(7)}{\leq} 2c\kappa m^7 + 4c\kappa m^{10} \stackrel{(6)}{\leq} c^2 \kappa^2 m^5. \quad (8)$$

For each $1 \leq i \leq m^6$, let $T_i = B_1(w_i) \setminus W_1$. Then by **C2** and the assumption, we have $|T_i| \geq c\kappa$. As the graphs $H(w_i)$ are vertex-disjoint $(2c\kappa, 2c\kappa)$ -hubs, we have $|\cup_{i=1}^{m^6} N_{H(w_i)}(T_i)| \geq 2c\kappa \cdot c\kappa \cdot m^6$, and hence we have

$$|\cup_{i=1}^{m^6} N_{H(w_i)}(T_i) \setminus W_1| \geq 2c\kappa \cdot c\kappa \cdot m^6 - |W_1| \stackrel{(8)}{\geq} c^2 \kappa^2 m^6.$$

At least $\kappa m^6 - m^6 \cdot c\kappa \geq \kappa m^6 / 2$ vertices u_j have not been connected by a path in \mathcal{P} . Without loss of generality, we write these vertices u_1, \dots, u_p , where $p \geq \kappa m^6 / 2$. By **C2**, the set W_1 is disjoint from $\cup_{j=1}^p B_1(u_j)$, we have

$$|\cup_{j=1}^p H(u_j) - W_1| \geq 2m^4 \cdot 2c\kappa \cdot \frac{\kappa m^6}{2} - c^2 \kappa^2 m^5 \geq c^2 \kappa^2 m^6.$$

We will apply Lemma 2.3 to connect $\cup_{i=1}^{m^6} N_{H(w_i)}(T_i) \setminus W_1$ and $\cup_{j=1}^p V(H(u_j)) \setminus W_1$, while avoiding the vertices in $\cup_{i=1}^{m^6} T_i$, W and W_1 . Since $d \geq \log^s n \geq m^{60}$ and $n/\kappa^2 \geq K$ is sufficiently large, we have

$$|\cup_{i=1}^{m^6} T_i| + |W| + |W_1| \leq (2c\kappa + 1)m^6 + 2c\kappa^2 m^4 + c^2 \kappa^2 m^5 \leq 2c^2 \kappa^2 m^5. \quad (9)$$

Hence, setting $y := c^2 \kappa^2 m^6$, we have

$$\frac{1}{4} \cdot \varepsilon(y) \cdot y \geq \frac{1}{4} \cdot \varepsilon(n) \cdot y \stackrel{(3)}{\geq} \frac{1}{4} \cdot \frac{10^5}{m} \cdot y \geq 2c^2 \kappa^2 m^5.$$

Thus, by (9) and Lemma 2.3, there is a shortest path of length at most

$$\frac{2}{\varepsilon_1} \log^3 \frac{15n}{\varepsilon_2 \kappa^2} + 1 \leq \log^4 \frac{n}{\kappa^2} \leq m$$

from some T_i to some $V(H(u_j))$ avoiding W and W_1 . Taking such a path, we extend it to a w_i, u_j -path of length at most $1 + m + 2 \leq 2m$ in $G - (W \cup W_1)$, which together with all the paths in \mathcal{P} satisfies **C1** and **C2**, a contradiction. \square

By Claim 4.6, we have a vertex w_i , which is connected to $c\kappa$ vertices u_j . Without loss of generality, we may take $u_1, u_2, \dots, u_{c\kappa}$ for instance so as to ease the notation. If we can find in every $(2m^4, 2c\kappa)$ -hub $H(u_j)$ an $(m^4, c\kappa)$ -hub which is disjoint from all the internal vertices of the w_i, u_j -paths in \mathcal{P} , $1 \leq j \leq c\kappa$, then all such $(m^4, c\kappa)$ -hubs together with all the w_i, u_j -paths form a $(c\kappa, m^4, c\kappa, 2m)$ -unit. It suffices to find for every $j \in [c\kappa]$ a $(m^4, c\kappa)$ -hub centered at u_j as required above.

By **C2**, for every $j \in [c\kappa]$, at most one vertex in $S_1(u_j)$ is used in the w_i, u_j -paths. Denote by W_2 the vertices in all the w_i, u_j -paths, $1 \leq j \leq c\kappa$. By **C1**, we have that $|W_2| \leq 2c\kappa m$. Therefore, for every u_j , there are at most $2m$ vertices $v \in S_1(u_j)$ such that $|S_1(v) \cap W_2| \geq c\kappa$ vertices from $S_1(v)$ in W_2 . Then we can take a set of m^4 vertices v in $S_1(u_j)$ along with $c\kappa$ vertices in every $S_1(v)$ avoiding W_2 , forming the desired $(m^4, c\kappa)$ -hub. \square

4.2 Constructing an adjuster

In this section, we start by finding a simple adjuster in an expander despite the removal of any medium-sized vertex set by Lemma 4.7. Then, we link simple adjusters together to obtain a desired adjuster. First we state Lemma 4.7, a key ingredient of our proof. It finds a simple adjuster robustly in an expander G , that is, given any subset $W \subseteq V(G)$ with a moderate size, we can construct an adjuster in $G - W$.

Lemma 4.7. *There exists some $\varepsilon_1 > 0$ such that, for every $0 < \varepsilon_2 < 1$ and integer $s \geq 240$, there exists d_0 and K such that the following holds for each $n \geq d \geq d_0$ and $d \geq \log^s n$.*

(i) *If G is a $\text{TK}_{\sqrt{d}}^{(2)}$ -free n -vertex $(\varepsilon_1, \varepsilon_2 d)$ -expander with $\delta(G) \geq d$ and $n \geq Kd$, then for any vertex set $W \subseteq G$ satisfying $|W| \leq 10D$ with $D = dm^4/10^7$, we have that $G - W$ contains a $(D, m/4, 1)$ -adjuster.*

(ii) *If G is a C_4 -free n -vertex $(\varepsilon_1, \varepsilon_2 d^2)$ -expander with $\delta(G) \geq d$ and $n \geq Kd^2$, then for any vertex set $W \subseteq G$ satisfying $|W| \leq 10D$ with $D = d^2m^4/10^7$, we have that $G - W$ contains a $(D, m/4, 1)$ -adjuster.*

Using Lemma 4.7, we can find many vertex-disjoint simple adjusters. Then we can connect them together into a large adjuster.

Proof of Lemma 3.9. The proof ideas of (i) and (ii) are similar, and we choose κ as in (1). Thus $D = \kappa^2 m^4 / 10^7$. Let $\varepsilon_1 > 0$, and K, d_0 be sufficiently large. We prove the lemma by induction on r . For $r = 1$, as $|W| \leq D / \log^3 \frac{n}{\kappa^2}$, by Lemma 4.7, $G - W$ contains a $(D, m/4, 1)$ -adjuster, which is also a $(D, m, 1)$ -adjuster.

Suppose then, for some r with $1 \leq r < \frac{1}{10} \kappa^2 m^2$, $G - W$ contains a (D, m, r) -adjuster, denoted by $\mathcal{A}_1 = (v_1, F_1, v_2, F_2, A_1)$. Let $W_1 = W \cup A_1 \cup V(F_1) \cup V(F_2)$, we have $|W_1| \leq 4D$. Then Lemma 4.7 shows that $G - W_1$ contains a $(D, m/4, 1)$ -adjuster $\mathcal{A}_2 = (v_3, F_3, v_4, F_4, A_2)$. Note that $|F_1 \cup F_2| = |F_3 \cup F_4| = 2D =: x$, and $|W \cup A_1 \cup A_2| \leq D / \log^3 \frac{n}{\kappa^2} + 20rm \leq D / \log^3 \frac{n}{\kappa^2} + 2\kappa^2 m^3 \leq 2D / \log^3 \frac{n}{\kappa^2}$. By (3), we have $1/4 \cdot \varepsilon(x)x \geq 1/4 \cdot \varepsilon(n)x \geq 2D / \log^3 \frac{n}{\kappa^2} \geq |W \cup A_1 \cup A_2|$, then by Lemma 2.3, there is a shortest path P of length at most m , from $V(F_1) \cup V(F_2)$ to $V(F_3) \cup V(F_4)$ avoiding $W \cup A_1 \cup A_2$.

We can assume, without loss of generality, that P is a path from $V(F_1)$ to $V(F_3)$. Then by extending P , using that F_1 and F_3 are (D, m) -expansions of v_1 and v_3 respectively, we can get a v_1, v_3 -path $Q \subseteq F_1 \cup P \cup F_3$ of length at most $3m$. Now we claim that $(v_2, F_2, v_4, F_4, A_1 \cup A_2 \cup V(Q))$ is a $(D, m, r+1)$ -adjuster. Indeed, we easily have that **A1** and **A2** hold, and **A3** holds as $|A_1 \cup A_2 \cup V(Q)| \leq 10mr + 10 \cdot (m/4) + 3m \leq 10(r+1)m$. Finally, let $l = l(\mathcal{A}_1) + l(\mathcal{A}_2) + l(Q)$. For every $i \in \{0, 1, \dots, r+1\}$, there are some $i_1 \in \{0, 1, \dots, r\}$ and $i_2 \in \{0, 1\}$ such that $i = i_1 + i_2$. Let P_1 be a v_1, v_2 -path in $G[A_1 \cup \{v_1, v_2\}]$ of length $l(\mathcal{A}_1) + 2i_1$ and let P_2 be a v_3, v_4 -path in $G[A_2 \cup \{v_3, v_4\}]$ of length $l(\mathcal{A}_2) + 2i_2$. Thus, $P_1 \cup Q \cup P_2$ is a v_2, v_4 -path in $G[A_1 \cup A_2 \cup V(Q)]$ of length $l + 2i$, and therefore **A4** holds. \square

Now we start to prove Lemma 4.7. We first introduce the concept of octopus used to find a simple adjuster.

Definition 4.8 (Octopus). Given integers $r_1, r_2, r_3, r_4 > 0$, an (r_1, r_2, r_3, r_4) -octopus $\mathcal{B} = (A, R, \mathcal{D}, \mathcal{P})$ is a graph consisting of a *core* $(r_1, r_2, 1)$ -adjuster A , one of the ends of A , called R , and

- a family \mathcal{D} of r_3 vertex-disjoint $(r_1, r_2, 1)$ -adjusters, which are disjoint from A , and
- a minimal family \mathcal{P} of internally vertex-disjoint paths of length at most r_4 , such that each adjuster in \mathcal{D} has at least one end which is connected to R by a subpath from a path in \mathcal{P} , and all the paths are disjoint from all center sets of the adjusters in $\mathcal{D} \cup \{A\}$. It is easy to observe that $|\mathcal{P}| \leq |\mathcal{D}|$.

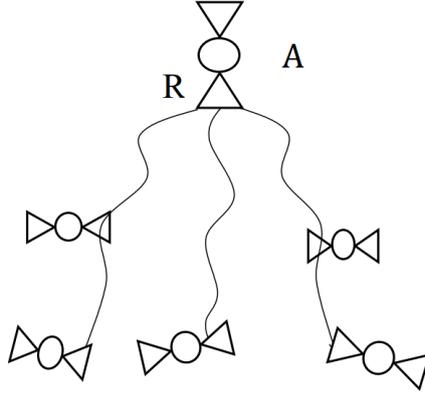


Figure 2: octopus

An outline of the proof of Lemma 4.7 is as follows. We can find an expander $H \subseteq G - W$, but note that H may be much smaller than G , so we may not find a simple adjuster of desired size directly. To overcome this, we first find many vertex-disjoint small simple adjusters. Then we connect them such that by averaging one end of a small simple adjuster would be linked to many others by internally vertex-disjoint paths, forming an octopus (see Claim 4.9). This process can be seen as that we expand one end of the small simple adjuster to a desired size. Finally, using the similar idea, we can expand another end to get the desired simple adjuster.

Proof of Lemma 4.7. The proof ideas of (i) and (ii) are similar, and we choose κ as in (1). Recall m is the smallest even integer that larger than $80 \log^4 \frac{n}{\kappa^2}$. Let K be sufficiently large. We first prove that there are m^{30} pairwise disjoint $(\kappa^2/800, m/400, 1)$ -adjusters in $G - W$. Note that such m^{30} adjusters have total size at most $(2 \cdot \kappa^2/800 + 10 \cdot m/400) \cdot m^{30} \leq \kappa^2 m^{30}/20$. It suffices to show that for any set $W' \subseteq V(G)$ with size $\kappa^2 m^{30}/10$, there is a $(\kappa^2/800, m/400, 1)$ -adjuster in $G - W'$.

For (i), by Lemma 4.1 with $(W, x) = (W', 30)$, we have $d(G - W') \geq d/2$, and by Corollary 2.5 with $G = G - W'$, there exists a bipartite $(\varepsilon_1, \varepsilon_2 d)$ -expander $H \subseteq G - W'$ with $\delta(H) \geq d/16$. Then by Lemma 2.3, there exists a shortest cycle C in H of length at most $m/16$ and whose length is denoted by $2r$. Thereafter, we arbitrarily pick two vertices $v_1, v_2 \in V(C)$ of distance $r - 1$ apart on C , together with $d/800$ distinct vertices in $N_{H-C}(v_1), N_{H-C}(v_2)$ respectively, then we get a $(d/800, m/400, 1)$ -adjuster as desired. Note that every vertex in the ends of the adjuster has no neighbour in the same end, except for v_1 and v_2 .

For (ii), by Corollary 4.3 with $(W, x) = (W', 30)$, we have $d(G - W') \geq d/2$, and by Corollary 2.5 with $G = G - W'$, there exists a bipartite $(\varepsilon_1, \varepsilon_2 d^2)$ -expander $H \subseteq G - W'$ with $\delta(H) \geq d/16$. Then by Lemma 2.3, there exists a shortest cycle C in H of length at most $m/16$ and whose length is denoted by $2r$. We can arbitrarily pick one vertex $v_1 \in V(C)$, which has at least $d/20$

neighbours in $H - C$. As G is C_4 -free, every pair of vertices in $N_{H-C}(v_1)$ has no other common neighbour than v_1 . Thus $N_{H-C}(N_{H-C}(v_1))$ has size at least $(d/16 - 2r)d/20 \geq d^2/400$. Choose another vertex $v_2 \in V(C)$ which is at distance $r - 1$ from v_1 in C . Now we find two vertex-disjoint $(d^2/800, 2)$ -expansions of v_1 and v_2 respectively, F_1, F_2 say, such that every vertex in $N_{F_1}(v_1)$ and $N_{F_2}(v_2)$ has at most d neighbours in the same expansion. Thus F_1 and F_2 together with C form the $(d^2/800, m/400, 1)$ -adjuster as desired.

We will sometimes call an adjuster is *touched* by a path if they intersect in at least one vertex, and *untouched* otherwise. Now we give a claim, which helps us find internally vertex-disjoint short paths connecting a vertex set to many ends of different adjusters.

Claim 4.9. Given G, m, s and κ as above. Let $x \geq 5$ and $X \subseteq V(G)$ be an arbitrary vertex set of size at most $\kappa^2 m^{x-1}/2$. Let $B \subseteq G - X$ be a graph with order at least $\kappa^2 m^x/800$ and \mathcal{U} be a subfamily of vertex-disjoint $(\kappa^2/800, m/400, 1)$ -adjusters in $G - (X \cup V(B))$ with $|\mathcal{U}| \geq m^{2x}$. Let \mathcal{P}_B be a maximum collection of internally vertex-disjoint paths of length at most $m/8$ in $G - X$, each connecting $V(B)$ to one end from different adjusters in \mathcal{U} . Then $V(B)$ can be connected to $1600m^{x+4}$ ends from different adjusters in \mathcal{U} via a subpath from a path in \mathcal{P}_B .

Proof. Suppose to the contrary that there are less than $1600m^{x+4}$ such ends from different adjusters, and denote by P the set of internal vertices of those paths. Then we have $|P| \leq 1600m^{x+4} \cdot m/8 = 200m^{x+5}$, and there are at least $m^{2x} - 1600m^{x+4}$ adjusters in \mathcal{U} untouched by the paths in \mathcal{P}_B . Arbitrarily pick m^x adjusters among those adjusters, and let B' be the union of their ends. Then we have $|B'| = m^x \cdot 2\kappa^2/800 = \kappa^2 m^x/400$. As $s \geq 240$, and thus $d \geq m^{60}$, we have

$$|X \cup P| \leq \frac{\kappa^2 m^{x-1}}{2} + 200m^{x+5} \leq \kappa^2 m^{x-1}.$$

Let $y := \kappa^2 m^x/800$. By (3), we have $1/4 \cdot \varepsilon(y)y \geq 1/4 \cdot \varepsilon(n)y \geq \kappa^2 m^{x-1} \geq |X \cup P|$. Then by Lemma 2.3, there is a path of length at most $m/8$ from $V(B)$ to $V(B')$, avoiding $X \cup P$, that is, this path connects $V(B)$ to an end of one more adjuster in \mathcal{U} , contrary to the maximality of \mathcal{P}_B . \square

In order to build a $(D, m/4, 1)$ -adjuster, we shall first construct many octopuses using those $(\kappa^2/800, m/400, 1)$ -adjusters we found above. Let Z be the union of the center sets and core vertices of all those adjusters.

Claim 4.10. There are m^5 $(\kappa^2/800, m/400, 800m^4, m/8)$ -octopuses $\mathcal{B}_j = (A_j, R_j, \mathcal{D}_j, \mathcal{P}_j)$, $1 \leq j \leq m^5$ in $G - W$ such that the following rules hold.

D1 A_j are pairwise disjoint adjusters, $1 \leq j \leq m^5$.

D2 $A_i \notin \mathcal{D}_j$, $1 \leq i, j \leq m^5$.

D3 \mathcal{D}_j contains every adjuster other than A_j which intersects at least one path in \mathcal{P}_j , $1 \leq j \leq m^5$.

D4 Paths in \mathcal{P}_i are vertex-disjoint from Z and A_j , except for $i = j$, $1 \leq i, j \leq m^5$.

D5 Every two paths from distinct $\mathcal{P}_i, \mathcal{P}_j$ are mutually vertex-disjoint, $1 \leq i < j \leq m^5$.

Proof. We will construct the octopuses iteratively. Suppose that we have constructed q octopuses so far. Let $W_1 = W \cup Z$. Let U be the union of the vertex sets of the ends of the core adjusters of octopuses we have constructed. Then we have $|U| \leq m^5 \cdot 2\kappa^2/800 = \kappa^2 m^5/400$. Let us call an adjuster *used* if it is used to construct an octopus, and *unused* otherwise. There are at most $m^5 \cdot (800m^4 + 1)$ adjusters are used until now, so there are more than m^{20} unused adjusters.

Let $P = \cup_{j=1}^q V(\mathcal{P}_j)$. Then $|P| \leq m/8 \cdot 800m^4 \cdot m^5 \leq m^{11}$. Arbitrarily pick a subfamily \mathcal{B} of m^6 unused adjusters, and let B be the union of their ends. Then $|B| = m^6 \cdot 2\kappa^2/800 = \kappa^2 m^6/400$.

Let \mathcal{U} be the family of unused adjusters, except for the adjusters we picked above. Then we have $|\mathcal{U}| \geq m^{12}$. Note that $|W_1 \cup U \cup P| \leq (10D + m^{30} \cdot m/16) + \kappa^2 m^5/400 + m^{11} \leq \kappa^2 m^5/2$ as $d \geq \log^s n \geq m^{60}$. Then by Claim 4.9 with $(B, \mathcal{U}, x, X) = (B, \mathcal{U}, 6, W_1 \cup U \cup P)$, $V(B)$ can be connected to $1600m^{10}$ ends from different adjusters in \mathcal{U} via some internally vertex-disjoint paths of length at most $m/8$ in $G - W_1 - U - P$.

By the pigeonhole principle, there exists an adjuster in \mathcal{B} , say A_{q+1} , such that A_{q+1} has an end R_{q+1} connected to at least $800m^4$ adjusters, say \mathcal{D}'_{q+1} , via a subfamily of internally vertex-disjoint paths, denoted by \mathcal{P}'_{q+1} , and then observe that we can choose $\mathcal{D}_{q+1} \subseteq \mathcal{D}'_{q+1}$ of exactly $800m^4$ adjusters and a system of paths \mathcal{P}_{q+1} such that every path is an initial segment of one in \mathcal{P}'_{q+1} and thus **D3** holds. Denote by L_{q+1} the other end of A_{q+1} . As we only use adjusters outside the octopuses which have been constructed during the process, **D1** and **D2** hold. As we find paths in \mathcal{P}_{q+1} avoiding $W_1 \cup U \cup P$, **D4** and **D5** hold. Thus, $A_{q+1}, R_{q+1}, \mathcal{D}_{q+1}$ and \mathcal{P}_{q+1} form a $(\kappa^2/800, m/400, 800m^4, m/8)$ -octopus. \square

Now we have m^5 octopuses $\mathcal{B}_j = (A_j, R_j, \mathcal{D}_j, \mathcal{P}_j)$, $1 \leq j \leq m^5$. Let B_1 be the union of L_j , $1 \leq j \leq m^5$. Then we have $|B_1| = m^5 \cdot \kappa^2/800$. There are at most $m^5 \cdot (800m^4 + 1)$ adjusters used, and thus at least m^{20} adjusters unused. Let \mathcal{U}' be the family of unused adjusters. Reset $P = \bigcup_{j=1}^{m^5} V(\mathcal{P}_j)$, then $|P| \leq m^{11}$. By definition, inside each $\mathcal{B}_j = (A_j, R_j, \mathcal{D}_j, \mathcal{P}_j)$, $j \in [m^5]$, every adjuster $A \in \mathcal{D}_j$ intersects $V(\mathcal{P}_j)$ and thus there exists a shortest path in A of length at most $m/400$ connecting a core vertex of A to $V(\mathcal{P}_j)$, and denote by \mathcal{Q}_j the disjoint union of such paths taken over all adjusters in \mathcal{D}_j . Let $Q = \bigcup_{j=1}^{m^5} V(\mathcal{Q}_j)$. Then $|Q| \leq m^5 \cdot 800m^4 \cdot (m/400 + 1) \leq m^{11}$. Recall $s \geq 240$ and $d \geq \log^s n \geq m^{60}$, we have $|W_1 \cup P \cup Q| \leq (10D + m^{30} \cdot m/16) + m^{11} + m^{11} \leq \kappa^2 m^4/2$. Then by Claim 4.9 with $(B, \mathcal{U}, x, X) = (B_1, \mathcal{U}', 5, W_1 \cup P \cup Q)$, $V(B_1)$ can be connected to $800m^9$ ends from different adjusters in \mathcal{U}' via some internally vertex-disjoint paths of length at most $m/8$ in $G - W_1 - P - Q$.

By the pigeonhole principle, there exists a core adjuster A_k such that L_k is connected to a family \mathcal{D}'_k of at least $800m^4$ adjusters, via a subfamily of internally vertex-disjoint paths, denote by \mathcal{P}'_k . Then A_k, L_k, \mathcal{D}'_k and \mathcal{P}'_k form a $(\kappa^2/800, m/400, 800m^4, m/8)$ -octopus. Note that $(A_k, R_k, \mathcal{D}_k, \mathcal{P}_k)$ is also a $(\kappa^2/800, m/400, 800m^4, m/8)$ -octopus.

For the adjuster A_k , denote by C_k the center vertex set of A_k , and note that L_k, R_k are $(\kappa^2/800, m/400)$ -expansions of vertices v_1, v_2 respectively. Let $F'_1 := G[V(L_k) \cup V(\mathcal{P}'_k) \cup V(\mathcal{D}'_k)]$, and F'_2 be the component of $G[V(R_k) \cup V(\mathcal{P}_k) \cup V(\mathcal{D}_k)] - V(\mathcal{P}'_k)$ containing v_2 . Indeed, paths in \mathcal{P}_k and \mathcal{P}'_k are disjoint from Z , and $V(\mathcal{P}_k)$ and $V(\mathcal{P}'_k)$ are disjoint. Recall that for every adjuster in \mathcal{D}_k , every vertex in the ends of the adjuster has at most κ neighbours in the same end, except for its core vertices. As $d \geq \log^s n \geq m^{60}$, and $V(\mathcal{P}'_k)$ is disjoint from Z and Q , F'_2 has size at least

$$|V(\mathcal{D}_k)| - \kappa|V(\mathcal{P}'_k)| \geq 800m^4 \cdot \frac{\kappa^2}{800} \cdot 2 - \kappa \cdot \frac{m}{8} \cdot 800m^4 \geq \kappa^2 m^4,$$

and the distance between v_2 and each $v \in V(F'_2)$ is at most $m/400 + m/8 + m/400 + m/32 + m/400 \leq m/4$. Then by Proposition 3.4, there exists a subgraph of F'_2 , denoted by F_2 , which is a $(\kappa^2 m^4, m/4)$ -expansion of v_2 . Similarly, we can find F_1 , which is a $(\kappa^2 m^4, m/4)$ -expansion of v_1 . Recall that $C_k \cup \{v_1, v_2\}$ is an even cycle of length $2r' \leq m/16$, and the distance between v_1 and v_2 on $C_k \cup \{v_1, v_2\}$ is $r' - 1$. Thus, $(v_1, F_1, v_2, F_2, C_k)$ is a $(\kappa^2 m^4, m/4, 1)$ -adjuster, and by Proposition 3.4, there exists a $(D, m/4, 1)$ -adjuster in $G - W$. \square

4.3 Connecting vertices with paths of desired length

Our goal now is connecting two vertex sets to two expansions with two paths, whose combined length is in the desired range. We first connect one vertex set to an expansion by the following lemma.

Lemma 4.11. *For any $0 < \varepsilon_1, \varepsilon_2 < 1$, there exists d_0 and K such that the following holds for $n \geq d \geq d_0$ and $d \geq \log^s n$.*

(i) Let G be a $\text{TK}_{\sqrt{d}}^{(2)}$ -free n -vertex $(\varepsilon_1, \varepsilon_2 d)$ -expander with $\delta(G) \geq d$ and $n \geq Kd$. Let $D = dm^4/10^7$. Suppose $U \subseteq V(G)$ is a vertex set with $|U| \geq D$, F is a (D, m) -expansion of v in $G - U$, and $W \subseteq V(G) \setminus (U \cup V(F))$ satisfies $|W| \leq 2D/\log^3 \frac{n}{d}$. Then for any $l \leq dm^3$, there is a v, v' -path in $G - W$ for some $v' \in U$ of length between l and $l + 11m$.

(ii) Let G be a C_4 -free n -vertex $(\varepsilon_1, \varepsilon_2 d^2)$ -expander with $\delta(G) \geq d$ and $n \geq Kd^2$. Let $D = d^2 m^4/10^7$. Suppose $U \subseteq V(G)$ is a vertex set with $|U| \geq D$, F is a (D, m) -expansion of v in $G - U$, and $W \subseteq V(G) \setminus (U \cup V(F))$ satisfies $|W| \leq 2D/\log^3 \frac{n}{d^2}$. Then for any $l \leq d^2 m^3$, there is a v, v' -path in $G - W$ for some $v' \in U$ of length between l and $l + 11m$.

Proof. The proof ideas of (i) and (ii) are similar, and we choose κ as in (1). Thus $D = \kappa^2 m^4/10^7$ and $l \leq \kappa^2 m^3$. Let K be sufficiently large. Let (P, v_1, F_1) be such that $l(P)$ is maximised subject to the following properties.

E1 P is a v, v_1 -path in $G - W$.

E2 $l(P) \leq l + 7m$.

E3 F_1 is a $(D, 3m)$ -expansion of v_1 in $G - W$ with $V(F_1) \cap V(P) = \{v_1\}$.

Note that $P = G[v_1]$, $v_1 = v$, and $F_1 = F$ satisfy **E1-E3**, therefore such a tuple (P, v_1, F_1) exists.

We claim that $l(P) \geq l$. Suppose to the contrary that $l(P) < l$. Note that for sufficiently large K , we have $|W \cup V(P \cup F_1)| \leq 2D/\log^3 \frac{n}{\kappa^2} + D + l \leq 2D$. By Lemma 3.8, $G - W - V(P \cup F_1)$ contains a $(c\kappa, m^4, c\kappa, 2m)$ -unit U' with core vertex v_2 . Note that $|W \cup V(P)| \leq 3D/\log^3 \frac{n}{\kappa^2}$. By (3), we have $1/4 \cdot \varepsilon(D)D \geq 1/4 \cdot \varepsilon(n)D \geq 3D/\log^3 \frac{n}{\kappa^2} \geq |W \cup V(P)|$. Then by Lemma 2.3, there is a path Q' from $V(U')$ to $V(F_1)$ of length at most m , avoiding $W \cup V(P) \setminus \{v_1\}$. Without loss of generality we can assume that Q' has endvertices $v'_1 \in F_1$ and $v'_2 \in U'$. By **E3** and the fact that U' is a $(c\kappa, m^4, c\kappa, 2m)$ -unit, we can extend Q' to a v_1, v_2 -path Q of length at most $2m + 2 + m + 3m \leq 7m$ which is vertex-disjoint from $P - v_1$. Using the definition of unit, we can get $F_2 \subseteq U' \setminus V(Q) \cup \{v_2\}$ which is a $(D, 3m)$ -expansion of v_2 . Let $P_1 = P \cup Q$. As $l(Q) \leq 7m$, P_1 is a v, v_2 -path of length at least $l(P) + 1$ and at most $l(P) + 7m$. Then, (P_1, v_2, F_2) satisfies **E1-E3** with $l(P_1) > l(P)$, a contradiction. Therefore, $l(P) \geq l$.

Now, note that $|W \cup V(P_1)| \leq 3D/\log^3 \frac{n}{\kappa^2}$. By (3), we have $\varepsilon(D)D/4 \geq \varepsilon(n)D/4 \geq 3D/\log^3 \frac{n}{\kappa^2} \geq |W \cup V(P_1)|$. Then by Lemma 2.3, there is a path R of length at most m , from some $r_1 \in U$ to some $r_2 \in V(F_1)$ avoiding $W \cup V(P) \setminus \{v_1\}$. Let Q_1 be a path from v_1 to r_2 in F_1 of length at most $3m$. Then, $P \cup Q_1 \cup R$ is a v, r_1 -path in $G - W$ of length at least $l(P) \geq l$ and at most $l + 7m + 3m + m = l + 11m$ by **E2**. \square

Combining Lemma 4.11 with Lemma 2.3, we can prove Lemma 3.10 as follows.

Proof of Lemma 3.10. The proof ideas of (i) and (ii) are similar, and we choose κ as in (1). Thus $D = \kappa^2 m^4/10^7$ and $l \leq \kappa^2 m^3$. Given G, W, U_1 and U_2 as stated, we choose K to be sufficiently large. Note that $|U_1 \cup U_2| \geq 2D$ and $|V(F_3) \cup V(F_4)| = 2D$. By (3), we have $1/4 \cdot \varepsilon(2D) \cdot 2D \geq 1/4 \cdot \varepsilon(n) \cdot 2D \geq D/\log^3 \frac{n}{\kappa^2} \geq |W|$. Then by Lemma 2.3, there is a shortest path $P' \subseteq G - W$ from $U_1 \cup U_2$ to $V(F_3) \cup V(F_4)$ of length at most m . Without loss of generality, we can assume that P' goes from some $v_1 \in U_1$ to $v'_3 \in V(F_3)$. As F_3 is a (D, m) -expansion of v_3 , there is a v_1, v_3 -path P of length at most $2m$ in $U_1 \cup P' \cup F_3$.

Let $W' = W \cup V(P)$. Then $|W'| \leq D/\log^3 \frac{n}{\kappa^2} + 2m + 2 \leq 2D/\log^3 \frac{n}{\kappa^2}$. By Lemma 4.11 with $(U, F, D, m, W, l) = (U_2, F_4, D, m, W', l)$, there is a path Q in $G - W'$ from some $v_2 \in U_2$ to v_4 of length between l and $l + 11m$. As $l \leq l(P) + l(Q) \leq l + 13m$, the paths P and Q are desired. \square

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We want to mention that Fernández, Hyde, Liu, Pikhurko and Wu [6] obtained a similar result regarding Theorem 1.2. The results in two papers are finished independently.

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A Dependent random choice

The following lemma can be regarded as a bipartite version of dependent random choice, of which the proof follows from that of Fox and Sudakov [7].

Lemma A.1. *Given integers a, t, n_1, n_2, c, r and a constant $\alpha > 0$, let $G = (V_1, V_2, E)$ be a bipartite graph such that $|V_1| = n_1$, $|V_2| = n_2$ and $|E| \geq \alpha n_1 n_2$. If it holds that*

$$\alpha^t n_1 - \binom{n_1}{r} \left(\frac{c}{n_2}\right)^t \geq a,$$

then there exists a set $A_0 \subseteq V_1$ of size at least a such that every r -subset of A_0 has at least c common neighbours in V_2 .

Proof. Pick a set of t vertices of V_2 uniformly at random with repetition, say b_1, b_2, \dots, b_t . Let A denote the set of common neighbours for all vertices b_i and $X = |A|$. By linearity of expectation,

$$\mathbb{E}(X) = \sum_{v \in V_1} \mathbb{P}(b_1, \dots, b_t \in N(v)) = \sum_{v \in V_1} \left(\frac{d(v)}{n_2} \right)^t.$$

By the convexity of the function $f(x) = x^t$, we have

$$\mathbb{E}(X) \geq \frac{n_1 (\sum_{v \in V_1} d(v) / n_1)^t}{n_2^t} \geq \frac{n_1 (\alpha n_2)^t}{n_2^t} \geq \alpha^t n_1.$$

Let Y denote the random variable counting the number of r -subsets in A with fewer than c common neighbours in V_2 . Therefore, the probability that a randomly chosen b_i is one of the common neighbours of such an r -set is at most $\frac{c}{n_2}$. Hence, since we made random choices of b_i uniformly and independently, the probability that such an r -tuple be contained in A is at most $\left(\frac{c}{n_2}\right)^t$. As there are at most $\binom{|V_1|}{r}$ subsets of size r , it follows that

$$\mathbb{E}(Y) \leq \binom{|V_1|}{r} \left(\frac{c}{n_2}\right)^t.$$

Again, by linearity of expectation, it holds that

$$\mathbb{E}(X - Y) \geq \alpha^t n_1 - \binom{n_1}{r} \left(\frac{c}{n_2}\right)^t \geq a.$$

Hence there exists a choice of A for which $X - Y \geq a$. Delete one vertex from each subset r -subset of A with fewer than c common neighbours and let A_0 be the remaining subset of A . Thus, $|A_0| \geq a$ and every r -subset of A_0 has at least c common neighbours. \square