Γ-graphic delta-matroids and their applications

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Abstract

For an abelian group Γ, a Γ-labelled graph is a graph whose vertices are labelled by elements of Γ. We prove that a certain collection of edge sets of a Γ-labelled graph forms a delta-matroid, which we call a Γ-*graphic* delta-matroid, and provide a polynomial-time algorithm to solve the separation problem, which allows us to apply the symmetric greedy algorithm of Bouchet to find a maximum weight feasible set in such a delta-matroid. We present two algorithmic applications on graphs; Maximum Weight Packing of Trees of Order Not Divisible by k and Maximum Weight S-Tree Packing. We also discuss various properties of Γ-graphic delta-matroids.

1 Introduction

We introduce the class of Γ-graphic delta-matroids arising from graphs whose vertices are labelled by elements of an abelian group Γ. This allows us to show that the following problems are solvable in polynomial time by using the symmetric greedy algorithm [1].

Maximum Weight Packing of Trees of Order Not Divisible by k
Input: An integer $k \geq 2$, a graph G, and a weight $w : E(G) \to \mathbb{Q}$.
Problem: Find vertex-disjoint trees $T_1, T_2, \ldots, T_m$ for some m such that $|V(T_i)| \not\equiv 0 \pmod{k}$ for each $i \in \{1, \ldots, m\}$ and $\sum_{i=1}^{m} \sum_{e \in E(T_i)} w(e)$ is maximized.

For a vertex set S of a graph G, a subgraph of G is an S-*tree* if it is a tree intersecting S.

Maximum Weight S-Tree Packing
Input: A graph G, a nonempty subset S of $V(G)$, and a weight $w : E(G) \to \mathbb{Q}$.
Problem: Find vertex-disjoint S-trees $T_1, T_2, \ldots, T_m$ for some m such that $\bigcup_{i=1}^{m} V(T_i) = V(G)$ and $\sum_{i=1}^{m} \sum_{e \in E(T_i)} w(e)$ is maximized.

Let Γ be an abelian group. We assume that Γ is an additive group. A Γ-*labelled graph* is a pair (G, γ) of a graph G and a map $\gamma : V(G) \to \Gamma$. A subgraph H of G is γ-*nonzero* if, for each component C of H,

(G1) $\sum_{v \in V(C)} \gamma(v) \neq 0$ or $\gamma|_{V(C)} \equiv 0$, and

(G2) if $\gamma|_{V(C)} \equiv 0$, then $G[V(C)]$ is a component of G.

A subset F of $E(G)$ is γ-*nonzero* in G if a subgraph $(V(G), F)$ is γ-nonzero. A subset F of $E(G)$ is *acyclic* in G if a subgraph $(V(G), F)$ has no cycle.

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Bouchet [1] introduced delta-matroids which are set systems $(E, \mathcal{F})$ satisfying certain axioms. Our first theorem proves that the set of acyclic γ-nonzero sets in a Γ-labelled graph (G, γ) forms a delta-matroid, which we call a Γ-*graphic* delta-matroid. For sets X and Y, let $X \triangle Y = (X - Y) \cup (Y - X)$.

Theorem 1.1. *Let Γ be an abelian group and (G, γ) be a Γ-labelled graph. If $\mathcal{F}$ is the set of acyclic γ-nonzero sets in G, then the following hold.*

(1) $\mathcal{F} \neq \emptyset$.

(2) *For $X, Y \in \mathcal{F}$ and $e \in X \triangle Y$, there exists $f \in X \triangle Y$ such that $X \triangle \{e, f\} \in \mathcal{F}$.*

Bouchet [1] proved that the symmetric greedy algorithm finds a maximum weight set in $\mathcal{F}$ for a delta-matroid $(E, \mathcal{F})$. But it requires the *separation oracle*, which determines, for two disjoint subsets X and Y of E, whether there exists a set $F \in \mathcal{F}$ such that $X \subseteq F$ and $F \cap Y = \emptyset$. We provide the separation oracle that runs in polynomial time for Γ-graphic delta-matroids given by Γ-labelled graphs. As a consequence, we prove the following theorem.

> Maximum Weight Acyclic γ-nonzero Set
> **Input:** A Γ-labelled graph (G, γ) and a weight $w : E(G) \to \mathbb{Q}$.
> **Problem:** Find an acyclic γ-nonzero set F in G maximizing $\sum_{e \in F} w(e)$.

Theorem 1.2. Maximum Weight Acyclic γ-nonzero Set *is solvable in polynomial time.*

From Theorem 1.2, we can easily deduce that both Maximum Weight Packing of Trees of Order Not Divisible by k and Maximum Weight S-Tree Packing are solvable in polynomial time.

Corollary 1.3. Maximum Weight Packing of Trees of Order Not Divisible by k *is solvable in polynomial time.*

Proof. Let $\Gamma = \mathbb{Z}_k$ and $\gamma : V(G) \to \mathbb{Z}_k$ be a map such that $\gamma(v) = 1$ for each $v \in V(G)$. Then, an edge set F is an acyclic γ-nonzero set in (G, γ) if and only if there exist vertex-disjoint trees $T_1, \ldots, T_m$ for some m such that $\bigcup_{i=1}^{m} E(T_i) = F$ and $|V(T_i)| \not\equiv 0 \pmod{k}$ for each $i \in \{1, \ldots, m\}$. □

Corollary 1.4. Maximum Weight S-Tree Packing *is solvable in polynomial time.*

Proof. We may assume that every component of G has a vertex in S. Let $\Gamma = \mathbb{Z}$ and $\gamma : V(G) \to \mathbb{Z}$ be a map such that

$$\gamma(v) = \begin{cases} 1 & \text{if } v \in S, \\ 0 & \text{otherwise.} \end{cases}$$

Then, an edge set F is an acyclic γ-nonzero set in (G, γ) if and only if there exist vertex-disjoint S-trees $T_1, \ldots, T_m$ for some m such that $\bigcup_{i=1}^{m} V(T_i) = V(G)$ and $\bigcup_{i=1}^{m} E(T_i) = F$. □

One of the major motivations to introduce Γ-graphic delta-matroids is to generalize the concept of graphic delta-matroids introduced by Oum [8], which are precisely $\mathbb{Z}_2$-graphic delta-matroids. Oum [8] proved that every minor of graphic delta-matroids is graphic. We will prove that every minor of a Γ-graphic delta-matroid is Γ-graphic.

A delta-matroid $(E, \mathcal{F})$ is *even* if $|X \triangle Y|$ is even for all $X, Y \in \mathcal{F}$. Oum [8] proved that every graphic delta-matroid is even. We characterize even Γ-graphic delta-matroids as follows.

Theorem 1.5. *Let Γ be an abelian group. Then a Γ-graphic delta-matroid is even if and only if it is graphic.*

Bouchet [2] proved that for a symmetric or skew-symmetric matrix A over a field $\mathbb{F}$, the set of index sets of nonsingular principal submatrices of A forms a delta-matroid, which we call a delta-matroid *representable over* $\mathbb{F}$. Oum [8] proved that every graphic delta-matroid is representable over GF(2). Our next theorem partially characterizes a pair of an abelian group Γ and a field $\mathbb{F}$ such that every Γ-graphic delta-matroid is representable over $\mathbb{F}$.

If $\mathbb{F}_1$ is a subfield of a field $\mathbb{F}_2$, then $\mathbb{F}_2$ is an *extension field* of $\mathbb{F}_1$, denoted by $\mathbb{F}_2/\mathbb{F}_1$. The *degree* of a field extension $\mathbb{F}_2/\mathbb{F}_1$, denoted by $[\mathbb{F}_2 : \mathbb{F}_1]$, is the dimension of $\mathbb{F}_2$ as a vector space over $\mathbb{F}_1$.

Theorem 1.6. *Let p be a prime, k be a positive integer, and $\mathbb{F}$ be a field of characteristic p. If $[\mathbb{F} : \mathrm{GF}(p)] \geq k$, then every $\mathbb{Z}_p^k$-graphic delta-matroid is representable over $\mathbb{F}$.*

For a prime p, an abelian group is an *elementary abelian p-group* if every nonzero element has order p.

Theorem 1.7. *Let $\mathbb{F}$ be a finite field of characteristic p and Γ be an abelian group. If every Γ-graphic delta-matroid is representable over $\mathbb{F}$, then Γ is an elementary abelian p-group.*

Theorems 1.6 and 1.7 allow us to partially characterize pairs of a finite field $\mathbb{F}$ and an abelian group Γ for which every Γ-graphic delta-matroid is representable over $\mathbb{F}$ as follows. We omit its easy proof.

Corollary 1.8. *Let Γ be a finite abelian group of order at least 2 and $\mathbb{F}$ be a finite field.*

(i) *For every prime p and integers $1 \leq k \leq \ell$, every $\mathbb{Z}_p^k$-graphic delta-matroid is representable over $\mathrm{GF}(p^\ell)$.*

(ii) *If every Γ-graphic delta-matroid is representable over $\mathbb{F}$, then Γ is isomorphic to $\mathbb{Z}_p^k$ and $\mathbb{F}$ is isomorphic to $\mathrm{GF}(p^\ell)$ for a prime p and positive integers k and ℓ.*

We suspect that the following could be the complete characterization.

Conjecture 1.9. *Let Γ be a finite abelian group of order at least 2 and $\mathbb{F}$ be a finite field. Then every Γ-graphic delta-matroid is representable over $\mathbb{F}$ if and only if $(\Gamma, \mathbb{F}) = (\mathbb{Z}_p^k, \mathrm{GF}(p^\ell))$ for some prime p and positive integers $k \leq \ell$.*

This paper is organized as follows. In Section 2, we review some terminologies and results on delta-matroids and graphic delta-matroids. In Section 3, we introduce Γ-graphic delta-matroids. We show that the class of Γ-graphic delta-matroids is closed under taking minors in Section 4. In Section 5, we present a polynomial-time algorithm to solve Maximum Weight Acyclic γ-nonzero Set, proving Theorem 1.2. We characterize even Γ-graphic delta-matroids in Section 6. In Section 7, we prove Theorems 1.6 and 1.7.

2 Preliminaries

In this paper, all graphs are finite and may have parallel edges and loops. A graph is *simple* if it has neither loops nor parallel edges. For a graph G, *contracting* an edge e is an operation to obtain a new graph G/e from G by deleting e and identifying ends of e. For a set X and a positive integer s, let $\binom{X}{s}$ be the set of s-element subsets of X. For two sets A and B, let $A \triangle B = (A - B) \cup (B - A)$. For a function $f : X \to Y$ and a subset $A \subseteq X$, we write $f|_A$ to denote the restriction of f on A.

Delta-matroids. Bouchet [1] introduced delta-matroids. A *delta-matroid* is a pair $M = (E, \mathcal{F})$ of a finite set E and a nonempty set $\mathcal{F}$ of subsets of E such that if $X, Y \in \mathcal{F}$ and $x \in X \triangle Y$, then there is $y \in X \triangle Y$ such that $X \triangle \{x, y\} \in \mathcal{F}$. We write $E(M) = E$ to denote the *ground set* of M. An element of $\mathcal{F}$ is called a *feasible* set. An element of E is a *loop* of M if it is not contained in any feasible set of M. An element of E is a *coloop* of M if it is contained in every feasible set of M.

Minors. For a delta-matroid $M = (E, \mathcal{F})$ and a subset X of E, we can obtain a new delta-matroid $M \triangle X = (E, \mathcal{F} \triangle X)$ from M where $\mathcal{F} \triangle X = \{F \triangle X : F \in \mathcal{F}\}$. This operation is called *twisting* a set X in M. A delta-matroid N is *equivalent* to M if $N = M \triangle X$ for some set X.

If there is a feasible subset of $E - X$, then $M \setminus X = (E - X, \mathcal{F} \setminus X)$ is a delta-matroid where $\mathcal{F} \setminus X = \{F \in \mathcal{F} : F \cap X = \emptyset\}$. This operation of obtaining $M \setminus X$ is called the *deletion* of X in M. A delta-matroid N is a *minor* of a delta-matroid M if $N = M \triangle X \setminus Y$ for some subsets X, Y of E.

A delta-matroid is *normal* if $\emptyset$ is feasible. A delta-matroid is *even* if $|X \triangle Y|$ is even for all feasible sets X and Y. It is easy to see that all minors of even delta-matroids are even.

The following theorem gives the minimal obstruction for even delta-matroids, which is implied by Bouchet [3, Lemma 5.4].

Theorem 2.1 (Bouchet [3]). *A delta-matroid is even if and only if it does not have a minor isomorphic to* $(\{e\}, \{\emptyset, \{e\}\})$.

Lemma 2.2. *Let N be a minor of a delta-matroid M such that $|E(M)| > |E(N)|$. Then there exists an element $e \in E(M) - E(N)$ such that N is a minor of $M \setminus e$ or a minor of $M\triangle\{e\} \setminus e$.*

Proof. Since N is a minor of M and $|E(M)| > |E(N)|$, there exist $X, Y \subseteq E$ such that $N = M\triangle X \setminus Y$ and $|Y| \geq 1$. So there exists $e \in Y = E(M) - E(N)$. If $e \notin X$, then $N = (M \setminus e)\triangle X \setminus (Y - \{e\})$ and so N is a minor of $M \setminus e$. So we may assume that $e \in X$. Then $N = (M\triangle\{e\} \setminus e)\triangle(X \setminus \{e\}) \setminus (Y - \{e\})$ and so N is a minor of $M\triangle\{e\} \setminus e$. □

Representable delta-matroids. For an $R \times C$ matrix A and subsets X of R and Y of C, we write $A[X, Y]$ to denote the $X \times Y$ submatrix of A. For an $E \times E$ square matrix A and a subset X of E, we write $A[X]$ to denote $A[X, X]$, which is called an $X \times X$ *principal* submatrix of A.

For an $E \times E$ square matrix A, let $\mathcal{F}(A) = \{X \subseteq E : A[X] \text{ is nonsingular}\}$. We assume that $A[\emptyset]$ is nonsingular and so $\emptyset \in \mathcal{F}(A)$. Bouchet [2] proved that, $(E, \mathcal{F}(A))$ is a delta-matroid if A is an $E \times E$ symmetric or skew-symmetric matrix. A delta-matroid $M = (E, \mathcal{F})$ is *representable over* a field $\mathbb{F}$ if $\mathcal{F} = \mathcal{F}(A)\triangle X$ for a symmetric or skew-symmetric matrix A over $\mathbb{F}$ and a subset X of E. Since $\emptyset \in \mathcal{F}(A)$, it is natural to define representable delta-matroids with twisting so that the empty set is not necessarily feasible in representable delta-matroids.

A delta-matroid is *binary* if it is representable over GF(2). Note that all diagonal entries of a skew-symmetric matrix are zero, even if the characteristic of a field is 2.

Proposition 2.3 (Bouchet [2]). *Let $M = (E, \mathcal{F})$ be a delta-matroid. Then M is normal and representable over a field $\mathbb{F}$ if and only if there is an $E \times E$ symmetric or skew-symmetric matrix A over $\mathbb{F}$ such that $\mathcal{F} = \mathcal{F}(A)$.*

Lemma 2.4 (Geelen [5, page 27]). *Let M be a delta-matroid representable over a field $\mathbb{F}$. Then M is even if and only if M is representable by a skew-symmetric matrix over $\mathbb{F}$.*

Pivoting. For a finite set E and a symmetric or skew-symmetric $E \times E$ matrix A, if A is represented by

$$A = \begin{array}{c c} & \begin{array}{cc} X & Y \end{array} \\ \begin{array}{c} X \\ Y \end{array} & \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \end{array}$$

after selecting a linear ordering of E and $A[X] = \alpha$ is nonsingular, then let

$$A * X = \begin{array}{c c} & \begin{array}{cc} X & Y \end{array} \\ \begin{array}{c} X \\ Y \end{array} & \begin{pmatrix} \alpha^{-1} & \alpha^{-1}\beta \\ -\gamma\alpha^{-1} & \delta - \gamma\alpha^{-1}\beta \end{pmatrix} \end{array}$$

This operation is called *pivoting*. Tucker [11] proved that when $A[X]$ is nonsingular, $A * X[Y]$ is nonsingular if and only if $A[X\triangle Y]$ is nonsingular for each subset Y of E. Hence, if X is a feasible set of a delta-matroid $M = (E, \mathcal{F}(A))$, then $M\triangle X = (E, \mathcal{F}(A * X))$. It implies that all minors of delta-matroids representable over a field $\mathbb{F}$ are representable over $\mathbb{F}$ [4].

Greedy algorithm. Let $M = (E, \mathcal{F})$ be a set system such that E is finite and $\mathcal{F} \neq \emptyset$. A pair (X, Y) of disjoint subsets X and Y of E is *separable* in M if there exists a set $F \in \mathcal{F}$ such that $X \subseteq F$ and $Y \cap F = \emptyset$. The following theorem characterizes delta-matroids in terms of a greedy algorithm. Note that this greedy algorithm requires an oracle which answers whether a pair (X, Y) of disjoint subsets X and Y of E is separable in M.

Theorem 2.5 (Bouchet [1]; see Moffatt [7]). *Let $M = (E, \mathcal{F})$ be a set system such that E is finite and $\mathcal{F} \neq \emptyset$. Then M is a delta-matroid if and only if the symmetric greedy algorithm in Algorithm 1 gives a set $F \in \mathcal{F}$ maximizing $\sum_{e\in F} w(e)$ for each $w : E \to \mathbb{R}$.*

Algorithm 1 Symmetric greedy algorithm

function SYMMETRIC GREEDY ALGORITHM(M, w) $\triangleright$ $M = (E, \mathcal{F})$ and $w : E \to \mathbb{R}$
Enumerate $E = \{e_1, e_2, \dots, e_n\}$ such that $|w(e_1)| \geq |w(e_2)| \geq \cdots \geq |w(e_n)|$
$X \leftarrow \emptyset$ and $Y \leftarrow \emptyset$
for $i \leftarrow 1$ **to** n **do**
if $w(e_i) \geq 0$ **then**
if $(X \cup \{e_i\}, Y)$ is separable **then**
$X \leftarrow X \cup \{e_i\}$
else
$Y \leftarrow Y \cup \{e_i\}$
end if
else
if $(X, Y \cup \{e_i\})$ is separable **then**
$Y \leftarrow Y \cup \{e_i\}$
else
$X \leftarrow X \cup \{e_i\}$
end if
end if
end for
end function
return X $\triangleright$ $X \in \mathcal{F}$

Graphic delta-matroids. Oum [8] introduced graphic delta-matroid. A *graft* is a pair (G, T) of a graph G and a subset T of $V(G)$. A subgraph H of G is T*-spanning* in G if $V(H) = V(G)$, for each component C of H, either

(i) $|V(C) \cap T|$ is odd, or

(ii) $V(C) \cap T = \emptyset$ and $G[V(C)]$ is a component of G.

An edge set F of G is T*-spanning* in G if a subgraph $(V(G), F)$ is T-spanning in G. For a graft (G, T), let $\mathcal{G}(G, T) = (E(G), \mathcal{F})$ where $\mathcal{F}$ is the set of acyclic T-spanning sets in G. Oum [8] proved that $\mathcal{G}(G, T)$ is an even binary delta-matroid. A delta-matroid is *graphic* if it is equivalent to $\mathcal{G}(G, T)$ for a graft (G, T).

3 Delta-matroids from group-labelled graphs

Let Γ be an abelian group. A Γ*-labelled graph* (G, γ) is a pair of a graph G and a map $\gamma : V(G) \to \Gamma$. We say $\gamma \equiv 0$ if $\gamma(v) = 0$ for all $v \in V(G)$. A Γ-labelled graph (G, γ) and a Γ'-labelled graph (G', γ') are *isomorphic* if there are a graph isomorphism f from G to G' and a group isomorphism $\phi : \Gamma \to \Gamma'$ such that $\phi(\gamma(v)) = \gamma'(f(v))$ for each $v \in V(G)$.

A subgraph H of G is γ*-nonzero* if, for each component C of H,

(G1) $\sum_{v \in V(C)} \gamma(v) \neq 0$ or $\gamma|_{V(C)} \equiv 0$, and

(G2) if $\gamma|_{V(C)} \equiv 0$, then $G[V(C)]$ is a component of G.

An edge set F of $E(G)$ is γ*-nonzero* in G if a subgraph $(V(G), F)$ is γ-nonzero. An edge set F of $E(G)$ is *acyclic* in G if a subgraph $(V(G), F)$ has no cycle.

For an abelian group Γ and a Γ-labelled graph (G, γ), let $\mathcal{F}$ be the set of acyclic γ-nonzero sets in G. Now we are ready to show Theorem 1.1, which proves that $(E(G), \mathcal{F})$ is a delta-matroid. We denote $(E(G), \mathcal{F})$ by $\mathcal{G}(G, \gamma)$. A delta-matroid M is Γ*-graphic* if there exist a Γ-labelled graph (G, γ) and $X \subseteq E(G)$ such that $M = \mathcal{G}(G, \gamma) \triangle X$.

Theorem 1.1. *Let Γ be an abelian group and (G, γ) be a Γ-labelled graph. If $\mathcal{F}$ is the set of acyclic γ-nonzero sets in G, then the following hold.*

(1) $\mathcal{F} \neq \emptyset$.

(2) *For* $X, Y \in \mathcal{F}$ *and* $e \in X \triangle Y$*, there exists* $f \in X \triangle Y$ *such that* $X \triangle \{e, f\} \in \mathcal{F}$.

Proof. By considering each component, we may assume that G is connected. If $\gamma \equiv 0$, then we choose a vertex v of G and a map $\gamma' : V(G) \to \Gamma$ such that $\gamma'(u) \neq 0$ if and only if $u = v$. Then the set of acyclic γ-nonzero sets in G is equal to the set of acyclic γ'-nonzero sets in G. Hence, we can assume that γ is not identically zero. Therefore, a subgraph H of G is γ-nonzero if and only if $\sum_{u \in V(C)} \gamma(u) \neq 0$ for each component C of H.

Let us first prove (1), stating that $\mathcal{F} \neq \emptyset$. Let $S = \{v \in V(G) : \gamma(v) \neq 0\}$ and T be a spanning tree of G. Then by the assumption, we have $S \neq \emptyset$. We may assume that $\sum_{u \in V(G)} \gamma(u) = 0$ because otherwise $E(T)$ is acyclic γ-nonzero in G. Let e be an edge of T such that one of two components C_1 and C_2 of $T \setminus e$ has exactly one vertex in S. Then $\sum_{u \in V(C_1)} \gamma(u) = -\sum_{u \in V(C_2)} \gamma(u) \neq 0$. So $E(T) - \{e\}$ is acyclic γ-nonzero in G, and (1) holds.

Now let us prove (2). We proceed by induction on $|E(G)|$. It is obvious if $|E(G)| = 0$. If there is an edge $g = vw$ in $X \cap Y$, then let $\gamma' : V(G/g) \to \Gamma$ such that, for each vertex x of G/g,

$$\gamma'(x) = \begin{cases} \gamma(v) + \gamma(w) & \text{if } x \text{ is the vertex of } G/g \text{ corresponding to } g, \\ \gamma(x) & \text{otherwise.} \end{cases}$$

Then both $X - \{g\}$ and $Y - \{g\}$ are acyclic γ'-nonzero sets in G/g. Let $e \in (X - \{g\}) \triangle (Y - \{g\}) = X \triangle Y$. By the induction hypothesis, there exists $f \in X \triangle Y$ such that $(X - \{g\}) \triangle \{e, f\}$ is an acyclic γ'-nonzero set in G/g.

We now claim that $X \triangle \{e, f\}$ is an acyclic γ-nonzero set in G. It is obvious that $X \triangle \{e, f\}$ is acyclic in G. If $\gamma' \equiv 0$, then $\gamma(v) = -\gamma(w) \neq 0$ and $\gamma(u) = 0$ for every u in $V(G) - \{v, w\}$. Then X is not γ-nonzero, contradicting our assumption. Hence, $\gamma' \not\equiv 0$ and let C be a component of $(V(G), X \triangle \{e, f\})$. If C contains g, then $\sum_{u \in V(C)} \gamma(u) = \sum_{u \in V(C/g)} \gamma'(u) \neq 0$. If C does not contain g, then $\sum_{u \in V(C)} \gamma(u) = \sum_{u \in V(C)} \gamma'(u) \neq 0$. It implies that $X \triangle \{e, f\}$ is γ-nonzero in G, so the claim is verified.

Therefore we may assume that $X \cap Y = \emptyset$. Let $H_1 = (V(G), X)$ and $H_2 = (V(G), Y)$.

Case 1. $e \in X$.

Let C be the component of H_1 containing e and C_1, C_2 be two components of $C \setminus e$. If both $\sum_{u \in V(C_1)} \gamma(u)$ and $\sum_{u \in V(C_2)} \gamma(u)$ are nonzero, then $X \triangle \{e\}$ is acyclic γ-nonzero and so we can choose $f = e$. So we may assume that $\sum_{u \in V(C_1)} \gamma(u) = 0$ and therefore

$$\sum_{u \in V(C_2)} \gamma(u) = \sum_{u \in V(C)} \gamma(u) - \sum_{u \in V(C_1)} \gamma(u) \neq 0.$$

If there exists $f \in Y$ joining a vertex in $V(C_1)$ to a vertex in $V(G) - V(C_1)$, then $X \triangle \{e, f\}$ is acyclic γ-nonzero. Therefore, we may assume that there is a component D_1 of H_2 such that $V(D_1) \subseteq V(C_1)$. Since $\sum_{u \in V(D_1)} \gamma(u) \neq 0$, there is a vertex x of D_1 such that $\gamma(x) \neq 0$. So $\gamma|_{V(C_1)} \not\equiv 0$ and there is an edge f of C_1 such that one of the components of $C_1 \setminus f$, say U, has exactly one vertex v with $\gamma(v) \neq 0$. If U' is the component of $C_1 \setminus f$ other than U, then $\sum_{u \in V(U')} \gamma(u) = -\sum_{u \in V(U)} \gamma(u) \neq 0$. So $X \triangle \{e, f\}$ is acyclic γ-nonzero.

Case 2. $e \in Y$.

Let $\tilde{H} = (V(G), X \cup \{e\})$. If $\tilde{H}$ contains a cycle D, then, since X and Y are acyclic, D is a unique cycle of $\tilde{H}$ and there is an edge $f \in E(D) - Y$. Then $X \triangle \{e, f\}$ is acyclic γ-nonzero. Therefore, we can assume that e joins two distinct components C', C'' of H_1.

Since $\sum_{u \in V(C')} \gamma(u) \neq 0$, there is an edge f of C' such that one of the components of $C' \setminus f$, say U, has exactly one vertex v with $\gamma(v) \neq 0$. If U' is the component of $C' \setminus f$ other than U, then $\sum_{u \in V(U')} \gamma(u) = -\sum_{u \in V(U)} \gamma(u) \neq 0$. So $X \triangle \{e, f\}$ is acyclic γ-nonzero. □

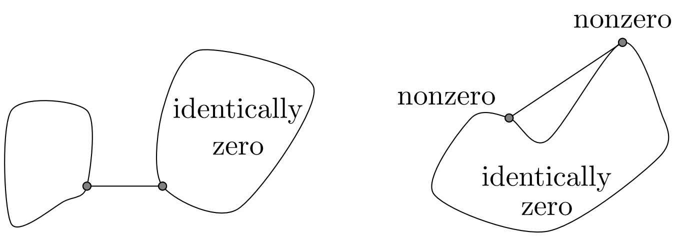

Figure 1: A γ-bridge and a γ-tunnel.

4 Minors of group-labelled graphs

Let Γ be an abelian group. Now we define minors of Γ-labelled graphs as follows. Let (G,γ) be a Γ-labelled graph and $e = uv$ be an edge of G. Then $(G,\gamma)\setminus e = (G\setminus e, \gamma)$ is the Γ-labelled graph obtained by *deleting* the edge e from (G,γ). For an isolated vertex v of G, $(G,\gamma)\setminus v = (G\setminus v, \gamma|_{V(G)-\{v\}})$ is the Γ-labelled graph obtained by *deleting* the vertex v from (G,γ). If e is not a loop, then let $(G,\gamma)/e = (G/e, \gamma')$ such that, for each $x \in V(G/e)$,

$$\gamma'(x) = \begin{cases} \gamma(u)+\gamma(v) & \text{if } x \text{ is the vertex of } G/e \text{ corresponding to } e, \\ \gamma(x) & \text{otherwise.} \end{cases}$$

If e is a loop, then let $(G,\gamma)/e = (G,\gamma)\setminus e$. *Contracting* the edge e is an operation obtaining $(G,\gamma)/e$ from (G,γ). For an edge set $X = \{e_1,\ldots,e_t\}$, let $(G,\gamma)/X = (G,\gamma)/e_1/\ldots/e_t$ and $(G,\gamma)\setminus X = (G\setminus X,\gamma)$. A Γ-labelled graph (G',γ') is a *minor* of (G,γ) if (G',γ') is obtained from (G,γ) by deleting some edges, contracting some edges, and deleting some isolated vertices. Let $\kappa(G,\gamma)$ be the number of components C of G such that $\gamma(x) = 0$ for all $x \in V(C)$. An edge e of G is a *γ-bridge* if $\kappa((G,\gamma)\setminus e) > \kappa(G,\gamma)$. A non-loop edge $e = uv$ of G is a *γ-tunnel* if, for the component C of G containing e, the following hold:

(i) For each $x \in V(C)$, $\gamma(x) \neq 0$ if and only if $x \in \{u,v\}$.

(ii) $\gamma(u)+\gamma(v) = 0$.

From the definition of a γ-tunnel, it is easy to see that an edge e is a γ-tunnel in G if and only if $\kappa((G,\gamma)/e) > \kappa(G,\gamma)$.

The following lemmas are analogous to properties of graphic delta-matroids in Oum [8, Propositions 8, 9, 10, and 11].

Lemma 4.1. *Let (G,γ) be a Γ-labelled graph and e be an edge of G. The following are equivalent.*

(i) *Every acyclic γ-nonzero set in G contains e.*

(ii) *The edge e is a γ-bridge in G.*

(iii) *Every γ-nonzero set in G contains e.*

Proof. We may assume that G is connected. If $\gamma \equiv 0$, then we choose a vertex v of G and take a map $\gamma' : V(G) \to \Gamma$ such that $\gamma'(v) \neq 0$ and $\gamma'(u) = 0$ for all $u \neq v$. Then an edge set of G is γ-nonzero if and only if it is γ'-nonzero, and e is a γ-bridge if and only if it is a γ'-bridge. So we can assume that $\gamma \not\equiv 0$. Therefore, an edge set F of G is γ-nonzero in G if and only if $\sum_{u\in V(C)} \gamma(u) \neq 0$ for each component C of a subgraph $(V(G),F)$. It is obvious that (iii) implies (i).

We first prove that (i) implies (ii). Suppose that e is not a γ-bridge. By (1) of Theorem 1.1, $G\setminus e$ has an acyclic γ-nonzero set F. If $G\setminus e$ is connected, then F is acyclic γ-nonzero in G. So we may assume that $G\setminus e$ has exactly two components C_1 and C_2. Since e is not a γ-bridge in G, we have $\gamma|_{V(C_1)}, \gamma|_{V(C_2)} \not\equiv 0$. Hence, $\sum_{u\in V(D)} \gamma(u) \neq 0$ for every component D of $(V(G),F)$ by (G1). Therefore, F is acyclic γ-nonzero in G not containing e.

Now let us prove that (ii) implies (iii). Let $e = uv$ be a γ-bridge. Then $G \setminus e$ contains a component C such that $\gamma|_{V(C)} \equiv 0$. We may assume that $u \in V(C)$ and $v \notin V(C)$. Suppose that G has a γ-nonzero set F which does not contain e. Let D be a component of $(V(G), F)$ containing u. Then $V(D) \subseteq V(C)$ and so $\sum_{u \in V(D)} \gamma(u) = 0$, contradicting that F is γ-nonzero in G. Therefore, every γ-nonzero set in G contains the edge e. $\square$

Lemma 4.2. *Let (G, γ) be a Γ-labelled graph. Then, for an edge e of G,*

$$\mathcal{G}((G,\gamma) \setminus e) = \begin{cases} \mathcal{G}(G,\gamma) \setminus e & \text{if } e \text{ is not a } \gamma\text{-bridge,} \\ \mathcal{G}(G,\gamma) \triangle \{e\} \setminus e & \text{otherwise.} \end{cases}$$

Proof. We may assume that G is connected. If $\gamma \equiv 0$, then we choose a vertex v of G and take a map $\gamma' : V(G) \to \Gamma$ such that $\gamma'(v) \neq 0$ and $\gamma'(u) = 0$ for all $u \neq v$. Then an edge set of G is γ-nonzero if and only if it is γ'-nonzero, and e is a γ-bridge if and only if it is a γ'-bridge. So we can assume that $\gamma \not\equiv 0$. Therefore, an edge set F of G is γ-nonzero in G if and only if $\sum_{u \in V(C)} \gamma(u) \neq 0$ for each component C of a subgraph $(V(G), F)$.

We first consider the case that e is not a γ-bridge. Then $\gamma|_{V(C)} \not\equiv 0$ for each component C of $G \setminus e$. So an edge set F of $G \setminus e$ is γ-nonzero in $G \setminus e$ if and only if $\sum_{u \in V(D)} \gamma(u) \neq 0$ for each component D of a subgraph $(V(G), F)$. Therefore, $\mathcal{G}((G,\gamma) \setminus e) = \mathcal{G}(G,\gamma) \setminus e$.

So it is enough to consider the case that e is a γ-bridge. Since $\gamma \not\equiv 0$ and e is a γ-bridge, $G \setminus e$ consists of two components C_1 and C_2 such that $\gamma|_{V(C_1)} \equiv 0$ and $\gamma|_{V(C_2)} \not\equiv 0$. Let v_1 and v_2 be the ends of e such that $v_i \in V(C_i)$ for $i \in \{1, 2\}$.

Let F be a feasible set of $\mathcal{G}(G,\gamma)$, which means that F is acyclic γ-nonzero in G. By Lemma 4.1, F contains e. Let D be the component of $(V(G), F)$ containing e and D_i be the component of $D \setminus e$ containing v_i for $i \in \{1, 2\}$. Since $V(D_1) \subseteq V(C_1)$ and $\gamma|_{V(C_1)} \equiv 0$, we have $V(D_1) = V(C_1)$ and $\sum_{u \in V(D_2)} \gamma(u) = \sum_{u \in V(D)} \gamma(u) - \sum_{u \in V(D_1)} \gamma(u) \neq 0$. Therefore, $F - \{e\}$ is an acyclic γ-nonzero set in $G \setminus e$, which implies that $F - \{e\} = F \triangle \{e\}$ is a feasible set of $\mathcal{G}((G,\gamma) \setminus e)$.

Conversely, let F be a feasible set of $\mathcal{G}((G,\gamma) \setminus e)$. Then F is an acyclic γ-nonzero set in $G \setminus e$. We claim that $F \cup \{e\}$ is an acyclic γ-nonzero set in G. Let D be the component of $(V(G), F \cup \{e\})$ containing e and D_i be the component of $D \setminus e$ containing v_i for $i \in \{1, 2\}$. Then $V(D_i) \subseteq V(C_i)$. So $\gamma|_{V(D_1)} \equiv 0$ and, since F is acyclic γ-nonzero in $G \setminus e$, we have $\sum_{u \in V(D_2)} \gamma(u) \neq 0$. Hence $\sum_{u \in V(D)} \gamma(u) = \sum_{u \in V(D_1)} \gamma(u) + \sum_{u \in V(D_2)} \gamma(u) \neq 0$. Therefore, $F \cup \{e\} = F \triangle \{e\}$ is an acyclic γ-nonzero set in G, which implies that F is a feasible set of $\mathcal{G}(G,\gamma) \triangle \{e\} \setminus e$. $\square$

Lemma 4.3. *Let (G, γ) be a Γ-labelled graph and e be a non-loop edge of G. Then the following are equivalent.*

(i) *No acyclic γ-nonzero set in G contains e.*

(ii) *The edge e is a γ-tunnel in G.*

(iii) *No γ-nonzero set in G contains e.*

Proof. It is obvious that (iii) implies (i). We first show that (i) implies (ii). We may assume that G is connected. Let H be a spanning tree containing e. Then we may assume that $\gamma \not\equiv 0$ and $\sum_{u \in V(H)} \gamma(u) = 0$ because otherwise $E(H)$ is acyclic γ-nonzero in G. Let $S = \{v \in V(G) : \gamma(v) \neq 0\}$. Then $|S| \geq 2$. If S contains a vertex not in $\{x, y\}$, then let x' be a vertex in $S - \{x, y\}$ maximizing $d_H(x, x')$. Let f be an edge incident with x' on the path from x to x'. Then $H \setminus f$ has components D_1, D_2 such that $x' \in V(D_1)$ and $x \in V(D_2)$. Then $\sum_{u \in V(D_1)} \gamma(u) = \gamma(x') \neq 0$ by the choice of x' and $\sum_{u \in V(D_2)} \gamma(u) = \sum_{u \in V(H)} \gamma(u) - \sum_{u \in V(D_1)} \gamma(u) = -\gamma(x') \neq 0$. Hence $E(H) - \{f\}$ is an acyclic γ-nonzero set in G containing e. Therefore, $S = \{x, y\}$ and e is a γ-tunnel in G.

Now let us prove that (ii) implies (iii). Let $e = xy$ be a γ-tunnel of G and C be a component of G containing e. Suppose that G has a γ-nonzero set F containing e and let D be the component of $(V(G), F)$ containing e. Since $V(D) \subseteq V(C)$ and e is a γ-tunnel, we have that $\gamma|_{V(D)} \not\equiv 0$ and $\sum_{u \in V(D)} \gamma(u) = \gamma(x) + \gamma(y) = 0$, contradicting (G1). Hence G has no γ-nonzero set containing e. $\square$

Lemma 4.4. *Let (G,γ) be a Γ-labelled graph. Then, for an edge e of G,*

$$\mathcal{G}((G,\gamma)/e) = \begin{cases} \mathcal{G}(G,\gamma)\triangle\{e\} \setminus e & \text{if } e \text{ is neither a loop nor a } \gamma\text{-tunnel,} \\ \mathcal{G}(G,\gamma) \setminus e & \text{otherwise.} \end{cases}$$

Proof. Let e^* be the vertex of G/e corresponding to e, and let $\gamma^* : V(G/e) \to \Gamma$ be a map such that $(G/e, \gamma^*) = (G,\gamma)/e$. We first consider the case that e is neither a γ-tunnel nor a loop. Let F be a feasible set of $\mathcal{G}((G,\gamma)/e)$, which implies that F is acyclic γ^*-nonzero in G/e. We aim to prove that $F \cup \{e\}$ is acyclic γ-nonzero in G.

Let C be the component of $(V(G), F \cup \{e\})$ containing e and $C^* = C/e$. Then C^* is a component of $(V(G/e), F)$. If $\sum_{u\in V(C^*)} \gamma^*(u) \neq 0$, then $\sum_{u\in V(C)} \gamma(u) = \sum_{u\in V(C^*)} \gamma^*(u) \neq 0$ and therefore $F \cup \{e\}$ is acyclic γ-nonzero in G because e is not a loop. So we may assume that $\gamma^*|_{V(C^*)} \equiv 0$ and $(G/e)[V(C^*)]$ is a component of G/e. Then $G[V(C)]$ is a component of G and $\gamma|_{V(C)} \equiv 0$ because e is not a γ-tunnel. Since e is not a loop, $F \cup \{e\}$ is acyclic γ-nonzero in G, which means that $F \cup \{e\} = F\triangle\{e\}$ is a feasible set of $\mathcal{G}(G,\gamma)$. Hence F is a feasible set of $\mathcal{G}(G,\gamma)\triangle\{e\} \setminus e$.

Conversely, let F be a feasible set of $\mathcal{G}(G,\gamma)\triangle\{e\} \setminus e$, meaning that $F \cup \{e\}$ is acyclic γ-nonzero in G. Trivially, F is acyclic in G/e. We claim that F is γ^*-nonzero in G/e. Let C^* be the component of $(V(G/e), F)$ containing e^* and let C be the component of $(V(G), F \cup \{e\})$ containing e. Then $C^* = C/e$. If $\sum_{u\in V(C)} \gamma(u) \neq 0$, then $\sum_{u\in V(C^*)} \gamma^*(u) = \sum_{u\in V(C)} \gamma(u) \neq 0$ and therefore F is γ^*-nonzero in G/e. So we may assume that $\gamma|_{V(C)} \equiv 0$ and $G[V(C)]$ is a component of G. Then we know that $\gamma^*|_{V(C^*)} \equiv 0$ and $(G/e)[V(C^*)]$ is a component of G/e. Hence F is γ^*-nonzero in G/e. This proves that $\mathcal{G}((G,\gamma)/e) = \mathcal{G}(G,\gamma)\triangle\{e\} \setminus e$ if e is neither a loop nor a γ-tunnel.

If e is a loop, then by Lemma 4.2, $\mathcal{G}(G,\gamma) \setminus e = \mathcal{G}((G,\gamma) \setminus e) = \mathcal{G}((G,\gamma)/e)$.

Now we may assume that $e = xy$ is a γ-tunnel in G. First, let us show that every feasible set F of $\mathcal{G}((G,\gamma)/e)$ is feasible in $\mathcal{G}(G,\gamma) \setminus e$. Let C^* be the component of $(V(G/e), F)$ containing e^*. Since e is a γ-tunnel in G, we have $\gamma^*|_{V(C^*)} \equiv 0$. Hence, $(G/e)[V(C^*)]$ is a component of G/e because F is acyclic γ^*-nonzero in G/e. Let C be the component of $(V(G), F \cup \{e\})$ containing e. Then $C/e = C^*$ and $C \setminus e$ has two components C_1, C_2 such that $x \in V(C_1)$ and $y \in V(C_2)$. Observe that $\sum_{u\in V(C_1)} \gamma(u) = \gamma(x) \neq 0$ and $\sum_{u\in V(C_2)} \gamma(u) = \gamma(y) \neq 0$. Hence F is acyclic γ-nonzero in G and F is a feasible set of $\mathcal{G}(G,\gamma) \setminus e$.

Conversely, we want to show that every feasible set F of $\mathcal{G}(G,\gamma) \setminus e$ is feasible in $\mathcal{G}((G,\gamma)/e)$. Observe that F is acyclic γ-nonzero in G not containing e. Let C_1 and C_2 be components of a subgraph $(V(G), F)$ containing x and y, respectively. Since e is a γ-tunnel, we know that $\sum_{u\in V(C_1)} \gamma(u) = \gamma(x) \neq 0$, $\sum_{u\in V(C_2)} \gamma(u) = \gamma(y) \neq 0$, and $G[V(C_1) \cup V(C_2)]$ is a component of G. So let C^* be a component of $(V(G/e), F)$ containing e^*. Since $\gamma^*(e^*) = \gamma(x) + \gamma(y) = 0$ and $G[V(C_1) \cup V(C_2)]$ is a component of G, we have $\gamma^*|_{V(C^*)} \equiv 0$ and $(G/e)[V(C^*)]$ is a component of G/e. So F is acyclic γ^*-nonzero in G/e and therefore F is a feasible set of $\mathcal{G}((G,\gamma)/e)$. □

We omit the proof of the following lemma.

Lemma 4.5. *Let (G,γ) be a Γ-labelled graph and v be an isolated vertex of G. Then $\mathcal{G}((G,\gamma) \setminus v) = \mathcal{G}(G \setminus v, \gamma|_{V(G)-\{v\}})$.*

Proposition 4.6. *Let (G,γ) be a Γ-labelled graph and $M = \mathcal{G}(G,\gamma)\triangle X$ for some $X \subseteq E(G)$.*

(i) *If (G',γ') is a minor of (G,γ), then $\mathcal{G}(G',\gamma')$ is a minor of M.*

(ii) *If M' is a minor of M, then there exists a minor (G',γ') of (G,γ) such that $M' = \mathcal{G}(G',\gamma')\triangle X'$ for some $X' \subseteq E(G')$.*

Proof. We may assume that $X = \emptyset$. Lemmas 4.2, 4.4, and 4.5 imply (i) and Lemmas 2.2, 4.2, 4.4, and 4.5 imply (ii). □

5 Maximum weight acyclic γ-nonzero set

In this section, we prove that one can find a maximum weight acyclic γ-nonzero set in a Γ-labelled graph (G, γ) in polynomial time by applying the symmetric greedy algorithm on Γ-graphic delta-matroids. Let us first state the problem.

> Maximum Weight Acyclic γ-nonzero Set
> **Input:** A Γ-labelled graph (G, γ) and a weight $w : E(G) \to \mathbb{Q}$.
> **Problem:** Find an acyclic γ-nonzero set F in G maximizing $\sum_{e\in F} w(e)$.

Recall that Theorem 2.5 allows us to find a maximum weight feasible set in a delta-matroid by using the symmetric greedy algorithm in Algorithm 1. As we proved that the set of acyclic γ-nonzero sets in a Γ-labelled graph (G, γ) forms a Γ-graphic delta-matroid in Section 3, we can apply Theorem 2.5 to solve Maximum Weight Acyclic γ-nonzero Set, but it requires a subroutine that decides in polynomial time whether a pair of two disjoint sets X and Y of $E(G)$ is separable in $\mathcal{G}(G, \gamma)$. In the remainder of this section, we focus on developing this subroutine.

We assume that the addition of two elements of Γ and testing whether an element of Γ is zero can be done in time polynomial in the length of the input.

Theorem 5.1. *Given a Γ-labelled graph (G, γ) and disjoint subsets X, Y of $E(G)$, one can decide in polynomial time whether G has an acyclic γ-nonzero set F such that $X \subseteq F$ and $Y \cap F = \emptyset$.*

To prove Theorem 5.1, we will characterize separable pairs (X, Y) in $\mathcal{G}(G, \gamma)$. Recall that, for a Γ-labelled graph (G, γ), $\kappa(G, \gamma)$ is the number of components C of G such that $\gamma|_{V(C)} \equiv 0$.

Lemma 5.2. *Let Γ be an abelian group and (G, γ) be a Γ-labelled graph. Then $\kappa((G, \gamma) \setminus e) \geq \kappa(G, \gamma)$ and $\kappa((G, \gamma)/e) \geq \kappa(G, \gamma)$ for every edge e of G.*

Proof. We may assume that G is connected and $\kappa(G, \gamma) = 1$. Then $\gamma \equiv 0$ and therefore $\kappa((G, \gamma)\setminus e) \geq 1$ and $\kappa((G, \gamma)/e) = 1$. $\square$

Lemma 5.3. *Let Γ be an abelian group, (G, γ) be a Γ-labelled graph, and X be an acyclic set of edges of G. Let $\gamma' : V(G/X) \to \Gamma$ be a map such that $(G/X, \gamma') = (G, \gamma)/X$. Then the following hold.*

(1) *If $\kappa((G, \gamma)/X) = \kappa(G, \gamma)$ and F is an acyclic γ'-nonzero set in G/X, then $F \cup X$ is an acyclic γ-nonzero set in G.*

(2) *If $\kappa((G, \gamma)/X) > \kappa(G, \gamma)$, then G has no acyclic γ-nonzero set containing X.*

Proof. Let us first prove (1). By considering each component, we may assume that G is connected. Since X is acyclic, $F \cup X$ is acyclic in G.

If $\kappa((G, \gamma)/X) = \kappa(G, \gamma) = 1$, then $\gamma \equiv 0$ and F is the edge set of a spanning tree of G/X by (G2). Hence $F \cup X$ is the edge set of a spanning tree of G, which implies that $F \cup X$ is acyclic γ-nonzero in G.

If $\kappa((G, \gamma)/X) = \kappa(G, \gamma) = 0$, then let $H' = (V(G/X), F)$ be a subgraph of G/X and $H = (V(G), F \cup X)$ be a subgraph of G. Then, for each component C of H, there exists a component C' of H' such that $C' = C/(E(C)\cap X)$. Then $\sum_{u\in V(C)} \gamma(u) = \sum_{u\in V(C')} \gamma'(u) \neq 0$ by (G1). Hence $F \cup X$ is an acyclic γ-nonzero set in G and (1) holds.

Now let us prove (2). We proceed by induction on $|X|$.

If $|X| = 1$, then $e \in X$ is a γ-tunnel and by Lemma 4.3, there is no acyclic γ-nonzero set containing X. So we may assume that $|X| > 1$. Let $e \in X$ and $X' = X - \{e\}$.

By the induction hypothesis, we may assume that $\kappa((G, \gamma)/X') = \kappa(G, \gamma)$. Let $\gamma'' : V(G/X') \to \Gamma$ be a map such that $(G/X', \gamma'') = (G, \gamma)/X'$. Since $\kappa((G, \gamma)/X) = \kappa((G, \gamma)/X'/e) > \kappa((G, \gamma)/X')$, by the induction hypothesis, G/X' has no acyclic γ''-nonzero set containing e. Therefore, G has no acyclic γ-nonzero set containing X. $\square$

Lemma 5.4. *Let Γ be an abelian group, (G, γ) be a Γ-labelled graph, and Y be a set of edges of G. Then the following hold.*

(1) *If* $\kappa((G,\gamma)\setminus Y)=\kappa(G,\gamma)$ *and* F *is an acyclic* γ*-nonzero set in* $G\setminus Y$*, then* F *is an acyclic* γ*-nonzero set in* G*.*

(2) *If* $\kappa((G,\gamma)\setminus Y)>\kappa(G,\gamma)$*, then* G *has no acyclic* γ*-nonzero set* F *such that* $Y\cap F=\emptyset$*.*

Proof. Let us first prove (1). By considering each component, we may assume that G is connected.

If $\kappa((G,\gamma)\setminus Y)=\kappa(G,\gamma)=1$, then $\gamma\equiv 0$ and the set F is the edge set of a spanning tree of $G\setminus Y$ by (G2). Then F is an acyclic γ-nonzero set in G.

If $\kappa((G,\gamma)\setminus Y)=\kappa(G,\gamma)=0$, then for each component C of $G\setminus Y$, we have $\gamma|_{V(C)}\not\equiv 0$. Then, $\sum_{v\in V(C)}\gamma(v)\neq 0$ for each component C of $(V(G),F)$. So F is an acyclic γ-nonzero set in G.

Let us show (2). We proceed by induction on $|Y|$. If $|Y|=1$, then $e\in Y$ is a γ-bridge so it is done by Lemma 4.1. Now we assume $|Y|\geq 2$. Let $e\in Y$ and $Y'=Y-\{e\}$. By the induction hypothesis, we may assume that $\kappa(G\setminus Y',\gamma)=\kappa(G,\gamma)$. Since $\kappa(G\setminus Y,\gamma)=\kappa(G\setminus Y'\setminus e,\gamma)>\kappa(G\setminus Y',\gamma)$, by the induction hypothesis, every acyclic γ-nonzero set in $G\setminus Y'$ contains e. Since every acyclic γ-nonzero set F in G not intersecting Y' is an acyclic γ-nonzero set in $G\setminus Y'$, every acyclic γ-nonzero set in G intersects Y. ☐

Proposition 5.5. *Let* Γ *be an abelian group and* (G,γ) *be a* Γ*-labelled graph. Let* X *and* Y *be disjoint subsets of* $E(G)$ *such that* X *is acyclic in* G*. Then* $\kappa((G,\gamma)/X\setminus Y)=\kappa(G,\gamma)$ *if and only if* G *has an acyclic* γ*-nonzero set* F *such that* $X\subseteq F$ *and* $Y\cap F=\emptyset$*.*

Proof. Let us prove the forward direction. By Lemma 5.2, $\kappa((G,\gamma)/X\setminus Y)=\kappa((G,\gamma)/X)=\kappa(G,\gamma)$. Let $\gamma':V(G/X\setminus Y)\to\Gamma$ be a map such that $(G/X\setminus Y,\gamma')=(G,\gamma)/X\setminus Y$. By (1) of Theorem 1.1, there exists an acyclic γ'-nonzero set F' in $G/X\setminus Y$. Since $\kappa((G,\gamma)/X\setminus Y)=\kappa((G,\gamma)/X)$, F' is acyclic γ'-nonzero in G/X by (1) of Lemma 5.4. Since $\kappa((G,\gamma)/X)=\kappa(G,\gamma)$, $F:=F'\cup X$ is acyclic γ-nonzero in G by (1) of Lemma 5.3. Therefore, F is an acyclic γ-nonzero set in G such that $X\subseteq F$ and $Y\cap F=\emptyset$.

Now let us prove the backward direction. Let F be an acyclic γ-nonzero set in G such that $X\subseteq F$ and $Y\cap F=\emptyset$. Let $\gamma':V(G/X)\to\Gamma$ be a map such that $(G/X,\gamma')=(G,\gamma)/X$. Then $F-X$ is an acyclic γ'-nonzero set in G/X not intersecting Y, so we have $\kappa((G,\gamma)/X\setminus Y)=\kappa((G,\gamma)/X)$ by (2) of Lemma 5.4. Since F is an acyclic γ-nonzero set containing X in G, we have $\kappa((G,\gamma)/X)=\kappa(G,\gamma)$ by (2) of Lemma 5.3. ☐

Proof of Theorem 5.1. Given a Γ-labelled graph (G,γ) and disjoint subsets X, Y of $E(G)$, we can compute $\kappa((G,\gamma)/X\setminus Y)$ in polynomial time and therefore, by Proposition 5.5, we can decide whether there exists an acyclic γ-nonzero set F in G such that $X\subseteq F$ and $Y\cap F=\emptyset$. ☐

Now we are ready to show Theorem 1.2

Theorem 1.2. Maximum Weight Acyclic γ-nonzero Set *is solvable in polynomial time.*

Proof. Let $M=\mathcal{G}(G,\gamma)$ be a Γ-graphic delta-matroid. The set of acyclic γ-nonzero sets in G is equal to the set of feasible sets of M. By Theorem 5.1, we can decide in polynomial time whether a pair (X,Y) of disjoint subsets X and Y of $E(G)$ is separable in M. It implies that the symmetric greedy algorithm in Algorithm 1 for M and w runs in polynomial time. By Theorem 2.5, we can obtain an acyclic γ-nonzero set F in G maximizing $\sum_{e\in F}w(e)$. ☐

6 Even Γ-graphic delta-matroids

In this section, we show that every even Γ-graphic delta-matroid is graphic.

Lemma 6.1. *Let* (G,γ) *be a* Γ*-labelled graph, and* $\eta:V(G)\to\mathbb{Z}_2$ *such that* $\eta(v)=0$ *if and only if* $\gamma(v)=0$ *for each* $v\in V(G)$*. If* $\mathcal{G}(G,\gamma)$ *is even, then, for each connected subgraph* H *of* G*,* $\sum_{u\in V(H)}\eta(u)=0$ *if and only if* $\sum_{u\in V(H)}\gamma(u)=0$*.*

Proof. We proceed by induction on $|V(H)|$. We may assume that $|V(H)| \geq 2$. Then there is a vertex $v \in V(H)$ such that $H \setminus v$ is connected. Let $H' = H \setminus v$. By the induction hypothesis, $\sum_{u \in V(H')} \eta(u) = 0$ if and only if $\sum_{u \in V(H')} \gamma(u) = 0$.

Let (G', γ') be a Γ-labelled graph obtained from (G, γ) by deleting all edges in $E(G) - E(H)$, contracting edges in $E(H \setminus v)$, and deleting loops and parallel edges. Let $e = vw$ be a unique edge of G'. Observe that $\gamma'(v) = \gamma(v)$ and $\gamma'(w) = \sum_{u \in V(H')} \gamma(u)$. Let $\eta' : V(G') \to \mathbb{Z}_2$ be a map such that $\eta'(v) = \eta(v)$ and $\eta'(w) = \sum_{u \in V(H')} \eta(u)$.

Let us first check the backward direction. If $\sum_{u \in V(H)} \gamma(u) = \gamma'(v) + \gamma'(w) = 0$, then $\eta'(v) = \eta'(w)$ and so $\sum_{u \in V(H)} \eta(u) = \eta'(v) + \eta'(w) = 0$.

Now let us prove the forward direction. Suppose that $\sum_{u \in V(H)} \eta(u) = 0$ and $\sum_{u \in V(H)} \gamma(u) \neq 0$. Then $\eta'(v) + \eta'(w) = 0$ and $\gamma'(v) + \gamma'(w) \neq 0$ and therefore all of $\gamma'(v)$, $\gamma'(w)$, and $\gamma'(v) + \gamma'(w)$ are nonzero. Hence $\mathcal{G}(G', \gamma') = (\{e\}, \{\emptyset, \{e\}\})$. By Theorem 2.1 and (i) of Proposition 4.6, $\mathcal{G}(G, \gamma)$ is not even, which is a contradiction. □

Proposition 6.2. *Let (G, γ) be a Γ-labelled graph. If $\mathcal{G}(G, \gamma)$ is even, then there is a map $\eta : V(G) \to \mathbb{Z}_2$ such that $\mathcal{G}(G, \gamma) = \mathcal{G}(G, \eta)$.*

Proof. Let $\eta : V(G) \to \mathbb{Z}_2$ is a map such that, for every $u \in V(G)$, $\eta(u) = 0$ if and only if $\gamma(u) = 0$. Let F be a set of edges of G. Then, for each component C of $(V(G), F)$, $\gamma|_{V(C)} \equiv 0$ if and only if $\eta|_{V(C)} \equiv 0$ and, by Lemma 6.1, $\sum_{u \in V(C)} \gamma(u) \neq 0$ if and only if $\sum_{u \in V(C)} \eta(u) \neq 0$. Therefore, F is acyclic γ-nonzero in G if and only if it is acyclic η-nonzero in G. □

We are ready to prove Theorem 1.5.

Theorem 1.5. *Let Γ be an abelian group. Then a Γ-graphic delta-matroid is even if and only if it is graphic.*

Proof of Theorem 1.5. Let M be an even Γ-graphic delta-matroid. By twisting, we may assume that $M = \mathcal{G}(G, \gamma)$ for a Γ-labelled graph (G, γ). By Proposition 6.2, M is $\mathbb{Z}_2$-graphic. Conversely, Oum [8, Theorem 5] proved that every graphic delta-matroid is even. □

7 Representations of Γ-graphic delta-matroids

We aim to study the condition on an abelian group Γ and a field $\mathbb{F}$ such that every Γ-graphic delta-matroid is representable over $\mathbb{F}$. Recall that a delta-matroid $M = (E, \mathcal{F})$ is representable over $\mathbb{F}$ if there is an $E \times E$ symmetric or skew-symmetric A over $\mathbb{F}$ such that $\mathcal{F} = \{F \subseteq E : A[X] \text{ is nonsingular}\} \triangle X$ for some $X \subseteq E$. If every Γ-graphic delta-matroid is representable over $\mathbb{F}$, then to prove this, we will construct symmetric matrices over $\mathbb{F}$ representing Γ-graphic delta-matroids.

For a graph $G = (V, E)$, let $\vec{G}$ be an orientation obtained from G by arbitrarily assigning a direction to each edge. Let $I_{\vec{G}} = (a_{ve})_{v \in V,\, e \in E}$ be a $V \times E$ matrix over $\mathbb{F}$ such that, for a vertex $v \in V$ and an edge $e \in E$,

$$a_{ve} = \begin{cases} 1 & \text{if } v \text{ is the head of a non-loop edge } e \text{ in } \vec{G}, \\ -1 & \text{if } v \text{ is the tail of a non-loop edge } e \text{ in } \vec{G}, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 7.1. *Let $G = (V, E)$ be a graph and $\vec{G}_1$, $\vec{G}_2$ be orientations of G. If $W \subseteq V$, $F \subseteq E$, and $|W| = |F|$, then $\det(I_{\vec{G}_1}[W, F]) = \pm \det(I_{\vec{G}_2}[W, F])$.*

Proof. The matrix $I_{\vec{G}_1}$ can be obtained from $I_{\vec{G}_2}$ by multiplying -1 to some columns. □

By slightly abusing the notation, we simply write I_G to denote $I_{\vec{G}}$ for some orientation $\vec{G}$ of G. The following two lemmas are easy exercises.

Lemma 7.2 (see Oxley [9, Lemma 5.1.3])**.** *Let G be a graph and F be an edge set of G. Then F is acyclic if and only if column vectors of I_G indexed by the elements of F are linearly independent.*

Lemma 7.3 (see Matoušek and Nešetřil [6, Lemma 8.5.3])**.** *Let $G = (V, E)$ be a tree. Then* $\det(I_G[V - \{v\}, E]) = \pm 1$ *for any vertex $v \in V$.*

Lemma 7.4. *Let Γ be an abelian group with at least one nonzero element, and (G, γ) be a Γ-labelled graph. Then there is a Γ-labelled graph (H, η) such that*

(i) *$\eta(v) \neq 0$ for each vertex $v \in V(H)$ and*

(ii) *(G, γ) is a minor of (H, η).*

Proof. Let $Z(G, \gamma)$ be the set of vertices $v \in V(G)$ such that $\gamma(v) = 0$. We proceed by induction on $|Z(G, \gamma)|$.

We may assume that $v \in Z(G, \gamma)$. Choose a nonzero element $g \in \Gamma$. Let G' be a graph obtained from G by adding a new vertex w adjacent only to v, and γ' be a map from $V(G')$ to Γ such that

$$\gamma'(u) = \begin{cases} g & \text{if } u = v, \\ -g & \text{if } u = w, \\ \gamma(u) & \text{otherwise.} \end{cases}$$

Then the Γ-labelled graph (G, γ) is isomorphic to $(G', \gamma')/vw$. We know $|Z(G', \gamma')| = |Z(G, \gamma)| - 1$. By the induction hypothesis, there is a Γ-labelled graph (H, η) such that $Z(H, \eta) = \emptyset$ and (G', γ') is a minor of (H, η). The latter implies that (G, γ) is a minor of (H, η). $\square$

Theorem 7.5 (Binet-Cauchy theorem)**.** *Let X and Y be finite sets. Let M be an $X \times Y$ matrix and N be a $Y \times X$ matrix with $|Y| \geq |X| = s$. Then*

$$\det(MN) = \sum_{S \in \binom{Y}{s}} \det(M[X, S]) \cdot \det(N[S, X]).$$

It is straightforward to prove the following lemma from the Binet-Cauchy theorem.

Corollary 7.6. *Let X, Y, Z be finite sets. Let L, M, N be $X \times Y$, $Y \times Z$, $Z \times X$ matrices, respectively, with $|Y|, |Z| \geq |X| = s$. Then*

$$\det(LMN) = \sum_{S \in \binom{Y}{s},\, T \in \binom{Z}{s}} \det(L[X, S]) \cdot \det(M[S, T]) \cdot \det(N[T, X]).$$

Theorem 1.6. *Let p be a prime, k be a positive integer, and $\mathbb{F}$ be a field of characteristic p. If $[\mathbb{F} : \mathrm{GF}(p)] \geq k$, then every $\mathbb{Z}_p^k$-graphic delta-matroid is representable over $\mathbb{F}$.*

Proof. Let M be a $\mathbb{Z}_p^k$-graphic delta-matroid. By twisting, we may assume $M = \mathcal{G}(G, \gamma)$ for a $\mathbb{Z}_p^k$-labelled graph (G, γ). Let $V = V(G)$ and $E = E(G)$. We may assume that $\gamma(v) \neq 0$ for each vertex v of V because otherwise, by applying Lemma 7.4, we can replace (G, γ) by a $\mathbb{Z}_p^k$-labelled graph (H, η) such that $\eta(u) \neq 0$ for each $u \in V(H)$ and $\mathcal{G}(G, \gamma)$ is a minor of $\mathcal{G}(H, \eta)$.

Since $[\mathbb{F} : \mathrm{GF}(p)] \geq k$, there exists a set $\{\alpha_1, \ldots, \alpha_k\}$ of linearly independent vectors of $\mathbb{F}$. So we can take an injective group homomorphism ϕ from $\mathbb{Z}_p^k$ to $\mathbb{F}$ such that

$$\phi(c_1, \ldots, c_k) = \sum_{i=1}^{k} c_i \alpha_i.$$

Let $B = (b_{vw})$ be a $V \times V$ diagonal matrix over $\mathbb{F}$ such that $b_{vv} = 1/\phi(\gamma(v))$, and let $A := I_G^T B I_G$. Then A is an $E \times E$ symmetric matrix over $\mathbb{F}$. Observe that $\det(B) = 1/\prod_{v \in V} \phi(\gamma(v)) \neq 0$ and $\det(B[V - \{v\}]) = \det(B)\phi(\gamma(v))$ for each $v \in V$.

Let F be a set of edges of G.

Claim 1. *If F is not acyclic in G, then* $\det(A[F]) = 0$.

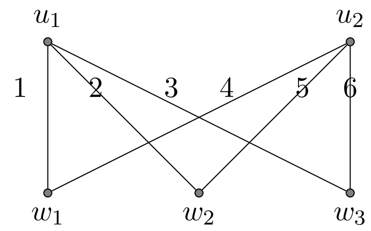

Figure 2: A graph G in the proof of Lemma 7.9.

Proof. We have $\mathrm{rank}(A[F]) \leq \mathrm{rank}(I_G[V,F])$. By Lemma 7.2, $\mathrm{rank}(I_G[V,F]) < |F|$, which implies that $\det(A[F]) = 0$. $\square$

Claim 2. *For a connected subgraph H of G, $A[E(H)]$ is nonsingular if and only if H is a tree and $\sum_{v \in V(H)} \gamma(v) \neq 0$.*

Proof. Let $V_H = V(H)$ and $E_H = E(H)$. Let us first show that if H is a tree, then $\det(A[E_H]) = \det(B[V_H])\phi\left(\sum_{v \in V_H} \gamma(v)\right)$. For $X, Y \subseteq V$ with $|X| = |Y|$, we have $\det(B[X,Y]) \neq 0$ if and only if $X = Y$ because B is a diagonal matrix. Then by Lemma 7.3 and Corollary 7.6, we know that

$$\begin{aligned}
\det(A[E_H]) &= \sum_{v,w \in V_H} \det(I_G^T[E_H, V_H - \{v\}]) \det(B[V_H - \{v\}, V_H - \{w\}]) \det(I_G[V_H - \{w\}, E_H]) \\
&= \sum_{v \in V_H} \det(I_G[V_H - \{v\}, E_H])^2 \det(B[V_H - \{v\}]) \\
&= \det(B[V_H]) \sum_{v \in V_H} \phi(\gamma(v)) = \det(B[V_H])\phi\left(\sum_{v \in V_H} \gamma(v)\right)
\end{aligned}$$

If $A[E_H]$ is nonsingular, then H is a tree by Claim 1 and therefore $\sum_{v \in V_H} \gamma(v) \neq 0$. Conversely, if H is a tree and $\sum_{v \in V_H} \gamma(v) \neq 0$, then $\det(A[E_H]) \neq 0$ because ϕ is injective and $\det(B[V_H]) \neq 0$. $\square$

By Claim 2, $A[F]$ is nonsingular if and only if F is acyclic γ-nonzero in G, which implies that $\mathcal{G}(G, \gamma)$ is representable over $\mathbb{F}$. $\square$

Now we show that for some pairs of an abelian group Γ and a finite field $\mathbb{F}$, not every Γ-graphic delta-matroid is representable over $\mathbb{F}$. Let $R(n;m)$ be the Ramsey number that is the minimum integer t such that any coloring of edges of K_t into m colors induces a monochromatic copy of K_n.

Theorem 7.7 (Ramsey [10])**.** *For positive integers m and n, $R(n;m)$ is finite.*

Corollary 7.8. *Let k be a positive integer and $\mathbb{F}$ be a finite field of order m. If $N \geq R(k;m)$, then each $N \times N$ symmetric matrix A over $\mathbb{F}$ has a $k \times k$ principal submatrix A' such that all non-diagonal entries are equal.*

Lemma 7.9. *Let $\mathbb{F}$ be a field. If every $\mathbb{Z}_2$-graphic delta-matroid is representable over $\mathbb{F}$, then the characteristic of $\mathbb{F}$ is 2.*

Proof. Let $G = (V, E)$ be a graph such that $V = \{u_1, u_2, w_1, w_2, w_3\}$ and $E = \{u_i w_j : i \in \{1,2\},\ j \in \{1,2,3\}\}$. We label edges u_1w_1, u_1w_2, u_1w_3 by 1, 2, 3, respectively and label edges u_2w_1, u_2w_2, u_2w_3 by 4, 5, 6, respectively. See Figure 7.

Let $\gamma : V \to \mathbb{Z}_2$ be a map such that $\gamma(x) = 1$ for each $x \in V$ and $M = (E, \mathcal{F})$ be a $\mathbb{Z}_2$-graphic delta-matroid $\mathcal{G}(G, \gamma)$. Then M is even by Oum [8]. Since $\gamma(x) \neq 0$ for each $x \in V$, we have $\emptyset \in \mathcal{F}$. By Lemmas 2.3 and 2.4, there exists an $E \times E$ skew-symmetric matrix A such that $\mathcal{F} = \mathcal{F}(A)$. Since the set of 2-element acyclic γ-nonzero sets is $\{\{1,2\},\{1,3\},\{2,3\},\{1,4\},\{2,5\},\{3,6\},\{4,5\},\{4,6\},\{5,6\}\}$, by scaling rows and columns simultaneously, we may assume that

$$A = \begin{array}{c c} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} & \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ -1 & 0 & t_1 & 0 & 1 & 0 \\ -1 & -t_1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & t_2 & t_3 \\ 0 & -1 & 0 & -t_2 & 0 & t_4 \\ 0 & 0 & -1 & -t_3 & -t_4 & 0 \end{pmatrix} \end{array}$$

such that $t_i \neq 0$ for $1 \leq i \leq 4$. We know that $\{1,2,4,5\}$, $\{1,3,4,6\}$, $\{2,3,5,6\}$ and $\{1,2,3,4,5,6\}$ are not acyclic γ-nonzero sets in G. Hence, $\det(A[\{1,2,4,5\}]) = (t_2-1)^2 = 0$, $\det(A[\{1,3,4,6\}]) = (t_3-1)^2 = 0$, $\det(A[\{2,3,5,6\}]) = (t_1t_4-1)^2 = 0$ and $\det(A[\{1,2,3,4,5,6\}]) = (t_1t_4+t_2+t_3-1)^2 = 0$. Then, we have $1 = t_1t_4 = -t_2 - t_3 + 1 = -1$. Therefore, the characteristic of $\mathbb{F}$ is 2. $\square$

Theorem 1.7. *Let $\mathbb{F}$ be a finite field of characteristic p, and Γ be an abelian group. If every Γ-graphic delta-matroid is representable over $\mathbb{F}$, then Γ is an elementary abelian p-group.*

Proof. If $\Gamma = \mathbb{Z}_2$, then the conclusion follows by Lemma 7.9. So we may assume that $|\Gamma| \geq 3$.

Suppose that Γ is not an elementary abelian p-group. Then Γ has a nonzero element g whose order is not equal to p. There exists a nonzero $h \in \Gamma$ such that $h \neq g$ since $|\Gamma| \geq 3$. Let $n := R(p+1; |\mathbb{F}|)$ and $G = (V, E)$ be a graph isomorphic to $K_{1,n+1}$. Let u be a leaf of G and e be the edge incident with u. Let $\gamma : V \to \Gamma$ be a map such that, for each $v \in V$,

$$\gamma(v) = \begin{cases} h & \text{if } v = u, \\ g & \text{if } v \neq u \text{ and } v \text{ is a leaf,} \\ -g & \text{otherwise.} \end{cases}$$

Let $M = \mathcal{G}(G, \gamma)$ be a Γ-graphic delta-matroid. Then M is normal because $\gamma(v) \neq 0$ for each vertex v of G. By Proposition 2.3, we may assume that M is equal to $(E, \mathcal{F}(A))$ for an $E \times E$ symmetric or skew-symmetric matrix A over $\mathbb{F}$. Since $\emptyset$ and $\{e\}$ are acyclic γ-nonzero in G, the delta-matroid M is not even. So A is not skew-symmetric by Lemma 2.4.

Since every 1-element subset of $E - \{e\}$ is not γ-nonzero in G, all diagonal entries of $A[E - \{e\}]$ are zero. By Corollary 7.8, there exists $F \subseteq E - \{e\}$ such that $|F| = p + 1$ and all non-diagonal entries of $A[F]$ are identical. Since the characteristic of the field $\mathbb{F}$ is p, the submatrix $A[F]$ is singular. Therefore, F is not acyclic γ-nonzero in G.

For each component C of a subgraph $(V(G), F)$, we have $\sum_{v \in V(C)} \gamma(v) \in \{h, g, pg\}$. This implies that F is acyclic γ-nonzero in G, which is a contradiction. $\square$

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