Technische Universität Ilmenau
Institut für Mathematik

**Preprint No.** M **10/15**

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November 2010

**Impressum:**
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# Zero dynamics and funnel control of linear differential-algebraic systems

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submitted to: *Mathematics of Control, Signals and Systems*

*November 11, 2010*

**Abstract**

We study the class of linear differential-algebraic $m$-input $m$-output systems which have a transfer function with proper inverse. A sufficient condition for the transfer function to have proper inverse it that the system has 'strict and non-positive relative degree'. We present two main results: First, a so called 'zero dynamics form' is derived: this form is – within the class of system equivalence – a simple ("almost normal") form of the DAE; it is a counterpart to the well-known Byrnes-Isidori form for ODE systems with strictly proper transfer function. The 'zero dynamics form' is exploited to characterize structural properties such as asymptotically stable zero dynamics, minimum phase, and high-gain stabilizability. The zero dynamics are characterized by $(A, E, B)$-invariant subspaces. Secondly, it is shown that the 'funnel controller' (that is a static nonlinear output error feedback) achieves, for all DAE systems with asymptotically stable zero dynamics and transfer function with proper inverse, tracking of a reference signal by the output signal within a pre-specified funnel. This funnel determines the transient behaviour.

**Keywords:** Differential-algebraic systems, strict relative degree, zero dynamics, minimum phase, stabilization, high-gain output feedback, funnel control

**Nomenclature**

| | | |
|---|---|---|
| $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$ | | set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, set of all integers, resp. |
| $\mathbb{R}_{\geq 0}$ | $=$ | $[0, \infty)$ |
| $\mathbb{C}_+$, $\mathbb{C}_-$ | | the open set of complex numbers with positive, negative real part, resp. |
| $\mathbf{Gl}_n(\mathbb{R})$ | | the set of invertible real $n \times n$ matrices |
| $\mathbb{R}[s]$ | | the ring of polynomials with coefficients in $\mathbb{R}$ |
| $\mathbb{R}(s)$ | | the quotient field of $\mathbb{R}[s]$ |
| $R^{n,m}$ | | the set of $n \times m$ matrices with entries in a ring $R$ |
| $\|x\|$ | $=$ | $\sqrt{x^\top x}$, the Euclidean norm of $x \in \mathbb{R}^n$ |
| $\|M\|$ | $=$ | $\max\left\{\|M\,x\| \,\middle\vert\, x \in \mathbb{R}^m,\ \|x\| = 1\right\}$, induced matrix norm of $M \in \mathbb{R}^{n,m}$ |
| $M^{-1}\mathcal{Y}$ | $=$ | $\{\ x \in \mathbb{R}^m \ \mid Mx \in \mathcal{Y}\ \}$, the pre-image of the set $\mathcal{Y} \subseteq \mathbb{R}^n$ under $M \in \mathbb{R}^{n,m}$ |
| $L^\infty(\mathcal{T}; \mathbb{R}^n)$ | | the set of essentially bounded functions $f : \mathcal{T} \to \mathbb{R}^n$ |
| $\mathcal{C}^\ell(\mathcal{T}; \mathbb{R}^n)$ | | the set of $\ell$-times continuously differentiable functions $f : \mathcal{T} \to \mathbb{R}^n$ |
| $\mathcal{B}^\ell(\mathcal{T}; \mathbb{R}^n)$ | $=$ | $\{f \in \mathcal{C}^\ell(\mathcal{T}; \mathbb{R}^n) \,\vert\, \frac{\mathrm{d}^i}{\mathrm{d}t^i} f \in L^\infty(\mathcal{T}; \mathbb{R}^n) \text{ for } i = 0, \ldots, \ell\}$ |

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# 1 Introduction

We consider linear differential-algebraic systems of the form

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t)\,, \end{aligned} \tag{1.1}$$

where $E, A \in \mathbb{R}^{n,n}$, $B \in \mathbb{R}^{n,m}$, $C \in \mathbb{R}^{p,n}$ are such that the pencil $sE - A \in \mathbb{R}[s]^{n,n}$ is *regular*, i.e. $\det(sE - A) \in \mathbb{R}[s] \setminus \{0\}$; the set of these systems is denoted by $\Sigma_{n,m,p}$ and we write $[E, A, B, C] \in \Sigma_{n,m,p}$.
The functions $u : \mathbb{R} \to \mathbb{R}^m$ and $y : \mathbb{R} \to \mathbb{R}^p$ are called *input* and *output* of the system, resp. A trajectory $(x, u, y) : \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p$ is said to be a *solution* of (1.1) if, and only if, it belongs to the *behaviour* of (1.1):

$$\mathfrak{B}_{(1.1)} := \left\{ \, (x, u, y) \in \mathcal{C}^1(\mathbb{R}; \mathbb{R}^n) \times \mathcal{C}(\mathbb{R}; \mathbb{R}^m) \times \mathcal{C}(\mathbb{R}; \mathbb{R}^m) \;\middle|\; (x, u, y) \text{ solves (1.1) for all } t \in \mathbb{R} \, \right\}.$$

More smoothness for $u$ and $y$ is required for some results.
The *transfer function* of $[E, A, B, C] \in \Sigma_{n,m,p}$ is defined by

$$G(s) = C(sE - A)^{-1}B \in \mathbb{R}(s)^{p,m}$$

and throughout the paper we will assume that $G(s)$ has proper inverse over $\mathbb{R}(s)$ or stronger, has a strict relative degree. Both notions are defined as follows.

**Definition 1.1.** A rational matrix function $G(s) \in \mathbb{R}(s)^{p,m}$ is called *proper* if, and only if, $\lim_{s\to\infty} G(s) = D$ for some $D \in \mathbb{R}^{p,m}$; and *strictly proper* if, and only if, $\lim_{s\to\infty} G(s) = 0$. We say that the square matrix function $G(s) \in \mathbb{R}(s)^{m,m}$ has *strict relative degree* $\rho \in \mathbb{Z}$ if, and only if,

$$\rho \;=\; \operatorname{sr\,deg} G(s) \;:=\; \sup\left\{ k \in \mathbb{Z} \;\middle|\; \lim_{s\to\infty} s^k\, G(s) \in \mathbf{Gl}_m(\mathbb{R}) \right\} \in \mathbb{Z}\,.$$

$\diamond$

The notion of strict relative degree generalizes (see [8] and the references therein) what is known for transfer function of ODE systems $[I_n, A, B, C] \in \Sigma_{n,m,m}$:

$$G(s) = C(sI_n - A)^{-1}B = CBs^{-1} + CABs^{-2} + CA^2Bs^{-3} + \ldots\,,$$

is strictly proper and has strict relative degree $\rho \in \mathbb{N}$ if, and only if,

$$CA^iB = 0 \quad \text{for} \quad i = 0, \ldots, \rho - 2 \quad \text{and} \quad CA^{\rho-1}B \in \mathbf{Gl}_m(\mathbb{R})\,.$$

In the single-input, single-output case, i.e.

$$G(s) = \frac{p(s)}{q(s)} \qquad \text{for } p(s), q(s) \in \mathbb{R}[s],\ q(s) \neq 0,$$

it is clear that

$$\operatorname{sr\,deg} G(s) = \deg q(s) - \deg p(s) \leq 0 \quad \iff \quad G(s) \text{ has proper inverse.}$$

This equivalence does, in general, not hold in the multi-input, multi output case:

**Proposition 1.2** (Strict relative degree implies proper inverse)**.**
*For $G(s) \in \mathbb{R}(s)^{m,m}$ we have*

$$\operatorname{sr\,deg} G(s) \leq 0 \quad \underset{\text{i.g.}}{\overset{\Longrightarrow}{\not\Longleftarrow}} \quad G(s) \textit{ has proper inverse.}$$

The proof is in Appendix 8.2.

We will show that the assumption '$G(s)$ has proper inverse' suffices to derive a so called 'zero dynamics form' of $[E, A, B, C] \in \Sigma_{n,m,m}$ within the equivalence class defined by:

**Definition 1.3** (System equivalence)**.**
Two systems $[E_i, A_i, B_i, C_i] \in \Sigma_{n,m,p}$, $i = 1, 2$, are called *system equivalent* if, and only if,

$$\exists\, W, T \in \mathbf{Gl}_n(\mathbb{R}) \;:\; \begin{bmatrix} sE_1 - A_1 & B_1 \\ C_1 & 0 \end{bmatrix} = \begin{bmatrix} W & 0 \\ 0 & I_p \end{bmatrix} \begin{bmatrix} sE_2 - A_2 & B_2 \\ C_2 & 0 \end{bmatrix} \begin{bmatrix} T & 0 \\ 0 & I_m \end{bmatrix};$$

we write

$$[\, E_1\,,\, A_1\,,\, B_1\,,\, C_1\,] \overset{W,T}{\sim} [\, E_2\,,\, A_2\,,\, B_2\,,\, C_2\,]\,.$$

◇

It is easy to see that system equivalence is an equivalence relation on $\Sigma_{n,m,p}$.

Weierstraß proved the following for regular pencils $sE - A$.

**Proposition 1.4** (Weierstraß form [5, Th. XII.3])**.**
*For any regular matrix pencil $sE - A \in \mathbb{R}[s]^{n,n}$, there exist $W, T \in \mathbf{Gl}_n(\mathbb{R})$ such that*

$$sE - A \;=\; W \begin{bmatrix} sI_{n_s} - A_s & 0 \\ 0 & sN - I_{n_f} \end{bmatrix} T, \tag{1.2}$$

*for some $A_s \in \mathbb{R}^{n_s,n_s}$ and nilpotent $N \in \mathbb{R}^{n_f,n_f}$.* ◇

The *index of nilpotency* of a nilpotent matrix $N \in \mathbb{R}^{k,k}$ is defined to be the smallest $\nu \in \mathbb{N}$ such that $N^\nu = 0$. It can be shown (see e.g. [10, Lem. 2.10]) that the index of nilpotency $\nu$ of $N$ in (1.2) is uniquely defined by the regular pencil $sE - A$; $\nu$ is therefore called the *index* of the pencil $sE - A$ if the nilpotent block is present and $\nu = 0$ if it is absent ($n_f = 0$).

The following is immediate from Proposition 1.4.

**Corollary 1.5** (Decoupled DAE)**.**
*Let $[E, A, B, C] \in \Sigma_{n,m,p}$. Then there exist $W, T \in \mathbf{Gl}_n(\mathbb{R})$ such that*

$$[E, A, B, C] \overset{W,T}{\sim} \left[ \begin{bmatrix} I_{n_s} & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} A_s & 0 \\ 0 & I_{n_f} \end{bmatrix}, \begin{bmatrix} B_s \\ B_f \end{bmatrix}, \begin{bmatrix} C_s & C_f \end{bmatrix} \right], \tag{1.3}$$

*for some $B_s \in \mathbb{R}^{n_f,m}$, $B_f \in \mathbb{R}^{n_f,m}$, $C_s \in \mathbb{R}^{p,n_s}$, $C_f \in \mathbb{R}^{p,n_s}$, $A_s \in \mathbb{R}^{n_s,n_s}$ and nilpotent $N \in \mathbb{R}^{n_f,n_f}$. This is interpreted, in terms of the DAE (1.1), as follows: $(x, u, y) \in \mathfrak{B}_{(1.1)}$ if, and only if,*

$$\begin{pmatrix} x_s(\cdot) \\ x_f(\cdot) \end{pmatrix} := Tx(\cdot)$$

*solves the decoupled DAEs*

$$\begin{aligned} \dot{x}_s(t) &= A_s\, x_s(t) + B_s\, u(t) \\ y_s(t) &= C_s\, x_s(t), \end{aligned} \tag{1.4a}$$

$$\begin{aligned} N\dot{x}_f(t) &= x_f(t) + B_f\, u(t) \\ y_f(t) &= C_f\, x_f(t), \end{aligned} \tag{1.4b}$$

$$y(t) = y_s(t) + y_f(t). \tag{1.4c}$$

◇

If $(x, u, y) \in \mathfrak{B}_{(1.1)}$ and in addition $u \in \mathcal{C}^{\nu-1}(\mathbb{R};\mathbb{R}^m)$, then by repeated multiplication of (1.4b) by $N$ from the left, differentiation, and using the identity

$$(sN - I_{n_f})^{-1} = -I_{n_f} - sN - s^2N^2 - \ldots - s^{\nu-1}N^{\nu-1}, \quad \text{if } \nu \text{ is the index of nilpotency of } N, \tag{1.5}$$

it is easy to see that the solution of (1.4b) satisfies

$$x_f(\cdot) = -\sum_{k=0}^{\nu-1} N^k B_f u^{(k)}(\cdot)\,. \tag{1.6}$$

We are now in a position to interpret the relative degree of the transfer function of $[E, A, B, C] \in \Sigma_{n,m,p}$. Since the transfer function is invariant under system equivalence, we have

$$\begin{aligned} G(s) = C(sE - A)^{-1}B &\overset{(1.3)}{=} C_f(sN - I_{n_f})^{-1}B_f + C_s(sI_{n_s} - A_s)^{-1}B_s \\ &\overset{(1.5)}{=} \underbrace{-\sum_{i=0}^{\nu-1} C_f N^i B_f\, s^i}_{=:P(s)} + \underbrace{\sum_{i\geq 1} C_s A_s^{i-1} B_s\, s^{-i}}_{=:G_{\rm sp}(s)} \end{aligned} \tag{1.7}$$

where $G_{\rm sp}(s)$ is strictly proper; if $G(s)$ has strict relative degree $\rho \leq 0$, then

$$\operatorname{sr\,deg} G(s) = -\deg P(s) = -\max\left\{\ i \in \{0, \ldots, \nu-1\}\ \middle|\ C_f N^i B_f \in \mathbf{Gl}_m(\mathbb{R})\ \right\}.$$

This means that system $y(s) = G(s)\,u(s)$ can be represented as a parallel decomposition of a strictly proper system and a system of differentiators of the input, see Fig. 1.

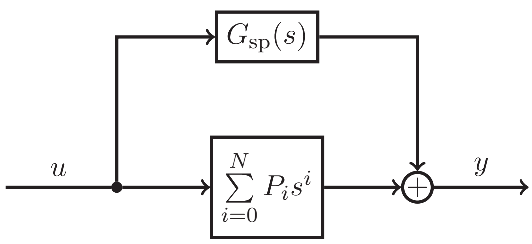



Figure 1: Parallel decomposition of a strictly proper system and differentiators

Finally, we recall different concepts of controllability and observability for DAEs (1.1). For brevity, we do not define the concepts in system theoretic terms but only give the algebraic characterizations in Proposition 1.6; the latter will be used in our proofs. Different notions of controllability and observability at infinity are used in the literature: [3] compares the algebraically formulated controllability/observability concepts of [14] and [16]; we go along with those in [14]. For system theoretic notions of the concepts see [4, Secs. 2 & 3].

**Proposition 1.6** (Controllability and observability).
*A system (1.1) is*

(i) *R-controllable* $\iff$ $\operatorname{rk}[sE - A, B] = n$ *for all* $s \in \mathbb{C}$

(ii) *stabilizable* $\iff$ $\operatorname{rk}[sE - A, B] = n$ *for all* $s \in \overline{\mathbb{C}}_+$

(iii) *controllable at infinity* $\iff$ $\operatorname{rk}[E,\, B] = n$

| | | | |
|---|---|---|---|
| (iv) | *controllable* | $\iff$ | *it is R-controllable and controllable at infinity* |
| (v) | *R-observable* | $\iff$ | $\operatorname{rk}[sE^\top - A^\top, C^\top] = n$ *for all* $s \in \mathbb{C}$ |
| (vi) | *detectable* | $\iff$ | $\operatorname{rk}[sE^\top - A^\top, C^\top] = n$ *for all* $s \in \overline{\mathbb{C}}_+$ |
| (vii) | *observable at infinity* | $\iff$ | $\operatorname{rk}[E^\top, C^\top] = n$ |
| (viii) | *observable* | $\iff$ | *it is R-observable and observable at infinity.* |

*The properties* (i)-(viii) *are invariant under system equivalence.*

The present note is organized as follows: Throughout we study those systems $[E, A, B, C] \in \Sigma_{n,m,m}$ where the transfer function $C(sE - A)^{-1}B$ has proper inverse; hence we do not consider systems with strictly proper transfer function, i.e. ODEs without feedthrough. In Section 2, we derive the 'zero dynamics form'; this is a counterpart to the Byrnes-Isidori form for strictly proper systems $[I_n, A, B, C] \in \Sigma_{n,m,m}$. In Section 3, we investigate zero dynamics and give, exploiting the zero dynamics form, a simple representation for the zero dynamics. The zero dynamics may be further characterized by $(A, E, B)$-invariant subspaces in Section 4. In Section 5, the concept of asymptotically stable zero dynamics is defined and characterized. In Section 6, the above findings will be exploited to show that the so called 'funnel controller' (that is an output feedback controller such that the error between a given reference signal and the output evolves within a pre-specified funnel) is applicable to any system $[E, A, B, C] \in \Sigma_{n,m,m}$ with asymptotically stable zero dynamics. Finally, in Section 7 our findings are illustrated and simulated by examples for $[E, A, B, C] \in \Sigma_{n,m,m}$: the first is a mechanical system with springs, masses and dampers and the second is a 2-input 2-output system. We have delegated two sections into an appendix, Section 8: relevant facts on rational matrix functions are collected in Section 8.1, and all proofs of the results in the preceding sections are delegated to Section 8.2.

## 2 Zero dynamics form

The first main result of this paper is to show that any system $[E, A, B, C] \in \Sigma_{n,m,m}$, such that $C(sE - A)^{-1}B$ has proper inverse, is system equivalent to a system in so called zero dynamics form. The latter is the counterpart to the Byrnes-Isidori form for strictly proper systems, see [9, Sec. 5.1] and [8]. Although the notion of zero dynamics is not necessary to derive this form, and in fact will be introduced in the following Section 3, the form allows to read off the zero dynamics as will be shown in Section 3. It gives more structural insight into the strict relative degree (see also Remark 2.7) and is the main mathematical tool to prove tracking control results in Section 6.

**Definition 2.1** (Zero dynamics form)**.**
System $[E, A, B, C] \in \Sigma_{n,m,m}$ is said to be in *zero dynamics form* if, and only if,

$$[E, A, B, C] = \left[ \begin{bmatrix} 0 & 0 & 0 & E_{14} \\ 0 & I_{n_2} & 0 & 0 \\ E_{31} & 0 & N_{33} & E_{34} \\ 0 & 0 & 0 & N_{44} \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12} & 0 & 0 \\ A_{21} & Q & 0 & 0 \\ 0 & 0 & I_{n_3} & 0 \\ 0 & 0 & 0 & I_{n_4} \end{bmatrix}, \begin{bmatrix} I_m \\ 0_{n_2,m} \\ 0_{n_3,m} \\ 0_{n_4,m} \end{bmatrix}, [I_m,\ 0_{m,n_2},\ 0_{m,n_3},\ 0_{m,n_4}] \right] \tag{2.1}$$

for some $n_2, n_3, n_4 \in \mathbb{N}_0$, $E_{14} \in \mathbb{R}^{m,n_4}, A_{11} \in \mathbb{R}^{m,m}, A_{12} \in \mathbb{R}^{m,n_2}, A_{21} \in \mathbb{R}^{n_2,m}, Q \in \mathbb{R}^{n_2,n_2}, E_{31} \in \mathbb{R}^{n_3,m}, N_{33} \in \mathbb{R}^{n_3,n_3}, E_{34} \in \mathbb{R}^{n_3,n_4}, N_{44} \in \mathbb{R}^{n_4,n_4}$ such that $N_{33}$ and $N_{44}$ are nilpotent and $\operatorname{rk}[E_{31}, N_{33}] = n_3$. ◇

**Remark 2.2** (Controllability and observability at infinity).
Note that condition $\operatorname{rk}[\,E_{31}\,,\ N_{33}\,] = n_3$ on the zero dynamics form is equivalent to that the subsystem

$$\left[\begin{bmatrix} 0 & 0 & 0 \\ 0 & I_{n_2} & 0 \\ E_{31} & 0 & N_{33} \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{21} & Q & 0 \\ 0 & 0 & I_{n_3} \end{bmatrix}, \begin{bmatrix} I_m \\ 0_{n_2,m} \\ 0_{n_3,m} \end{bmatrix}, [I_m,\ 0_{m,n_2},\ 0_{m,n_3}]\right] \tag{2.2}$$

of (2.1) is controllable at infinity. As a consequence, $n_4 = 0$ if, and only if, the system (2.1) is controllable at infinity. If (2.1) is observable at infinity, then $n_3 = 0$. ◇

We are now in a position to state the main result of the present note.

**Theorem 2.3** (Zero dynamics form).
*Suppose $[E, A, B, C] \in \Sigma_{n,m,m}$ is such that $C(sE - A)^{-1}B$ has proper inverse. Then there exist $W, T \in \mathbf{Gl}_n(\mathbb{R})$ such that $[E, A, B, C] \overset{W,T}{\sim} [\hat{E}, \hat{A}, \hat{B}, \hat{C}]$ where the latter is in zero dynamics form (2.1). Furthermore, the following holds:*

(i) *$N_{33}^{\nu} = 0$ and $N_{44}^{\nu} = 0$, where $\nu$ denotes the index of the pencil $sE - A$.*

(ii) *the transfer function satisfies*

$$C(sE - A)^{-1}B \;=\; -\left(A_{11} + A_{12}(sI_{n_2} - Q)^{-1}A_{21}\right)^{-1}.$$

(iii) $\operatorname{sr\,deg}\left(C(sE - A)^{-1}B\right) = 0 \quad \Longleftrightarrow \quad A_{11} \in \mathbf{Gl}_m(\mathbb{R})$.

(iv) $\operatorname{sr\,deg}\left(C(sE - A)^{-1}B\right) = \rho < 0 \quad \Longleftrightarrow \quad \left[\ A_{11} = 0\ \wedge\ \operatorname{sr\,deg}\left(A_{12}(sI_{n_2} - Q)^{-1}A_{21}\right) = -\rho\ \right]$.

*For uniqueness we have:*

(v) *If $[E, A, B, C]$, $[\hat{E}, \hat{A}, \hat{B}, \hat{C}] \in \Sigma_{n,m,m}$ are in zero dynamics form (2.1) and*

$$[\hat{E}, \hat{A}, \hat{B}, \hat{C}] \overset{W,T}{\sim} [E, A, B, C] \qquad \text{for some} \quad W, T \in \mathbf{Gl}_n(\mathbb{R}), \tag{2.3}$$

*then*

$$W = T^{-1} = \begin{bmatrix} I_m & 0 & 0 & 0 \\ 0 & W_{22} & 0 & 0 \\ 0 & 0 & W_{33} & W_{34} \\ 0 & 0 & 0 & W_{44} \end{bmatrix}, \quad \text{where } W_{ii} \in \mathbf{Gl}_{n_i}(\mathbb{R}),\ i = 2, 3, 4,\ W_{34} \in \mathbb{R}^{n_3, n_4}. \tag{2.4}$$

(vi) *The dimensions $n_2, n_3, n_4$ are unique; the matrices $N_{33}$, $N_{44}$, $Q$ are unique up to similarity, so in particular the spectrum of $Q$ is unique; and $A_{11} = -\lim_{s\to\infty}\left(C(sE - A)^{-1}B\right)^{-1}$.*

The proof is in Appendix 8.2.

**Remark 2.4** (How close is the zero dynamics form to a normal form?).
Equation (8.17b) in Step 7 of the proof of Theorem 2.3 shows that $E_{31}$, $E_{14}$, $A_{21}$, $A_{12}$ could be normalized by multiplication with invertible matrices from the left or right; and (8.17c) shows that the only "critical entry" is $E_{34}$. It is easy to present an example such that (8.17c) is satisfied for $E_{34} = 0$ and $\hat{E}_{34} \neq 0$. Therefore, the zero dynamics form is not a normal form, but "almost": Transform the uncontrollable subsystem $[N_{\bar{c}}, I_{n_{f,\bar{c}}}, 0, C_{f,\bar{c}}]$, obtained in Step 1 of the proof of Theorem 2.3, into observability form (following [4, Sec. 2-5.]) such that the observable variables are separated from the unobservable ones and carry on with the proof as in Theorem 2.3. This should result in a normal form with $5 \times 5$ block structure. ◇

**Remark 2.5** (Properties of the zero dynamics form).
Let $[E, A, B, C] \in \Sigma_{n,m,m}$ such that $C(sE - A)^{-1}B$ has proper inverse. Then, by Theorem (2.3), $[E, A, B, C]$ is system equivalent to a system in zero dynamics form (2.1). Since regularity of the pencil $sE - A$ is invariant under system equivalence, the pencil

$$\begin{bmatrix} -A_{11} & -A_{12} \\ -A_{21} & sI - Q \end{bmatrix} \quad \text{is regular and} \quad \operatorname{rk}[A_{11}, A_{12}] = \operatorname{rk} \begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix} = m\,.$$

$\diamond$

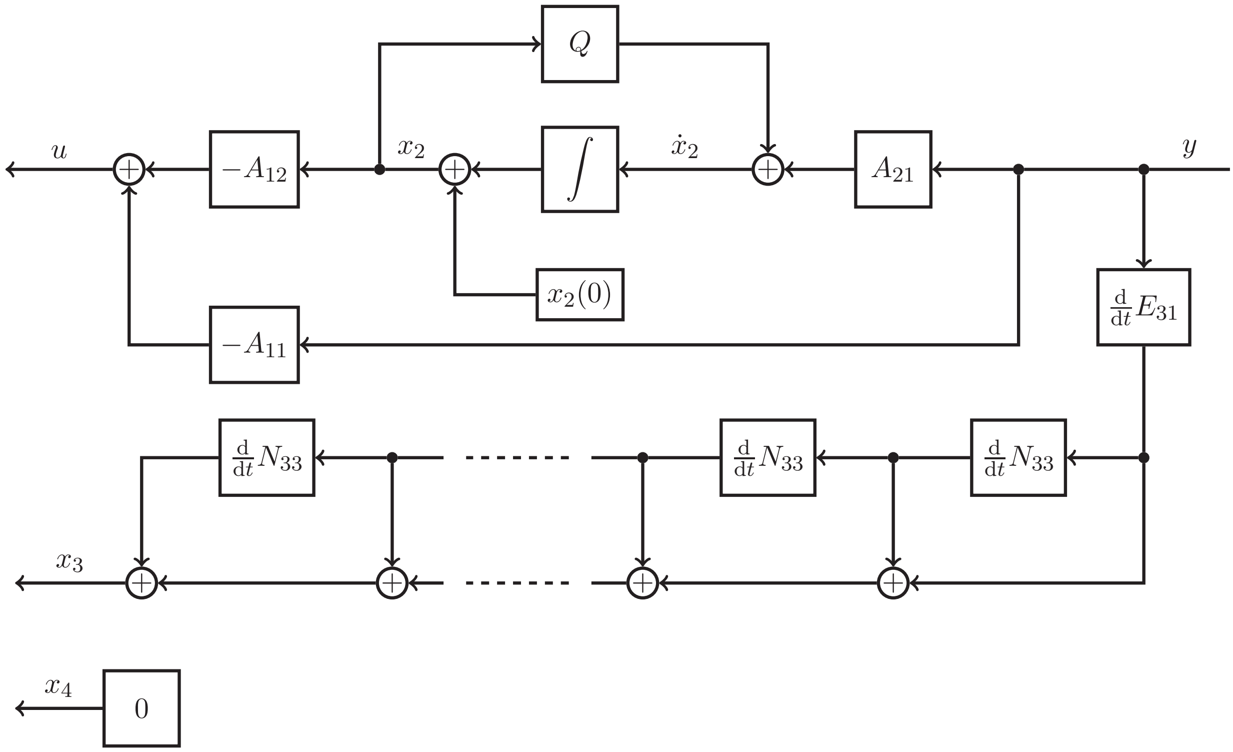



Figure 2: System $[E, A, B, C] \in \Sigma_{n,m,m}$ in zero dynamics form

A simple consequence of the zero dynamics form is that the DAE (1.1) can be written in the following "decoupled form", see also Fig. 2:

**Corollary 2.6** (DAE of zero dynamics form).
*Suppose $[E, A, B, C] \in \Sigma_{n,m,m}$ is such that $C(sE - A)^{-1}B$ has proper inverse and let $\nu$ be the index of $sE - A$. The behaviour of the DAE* (1.1) *may be interpreted, in terms of the zero dynamics form* (2.1) *in Theorem 2.3, as follows: $(x, u, y) \in \mathfrak{B}_{(1.1)} \cap \big(\mathcal{C}^1(\mathbb{R}; \mathbb{R}^n) \times \mathcal{C}^1(\mathbb{R}; \mathbb{R}^m) \times \mathcal{C}^\nu(\mathbb{R}; \mathbb{R}^m)\big)$ if, and only if, $(Tx, u, y)$ solves*

$$\boxed{\begin{array}{rcl} 0 &=& A_{11}\, y(t) + A_{12}\, x_2(t) + u(t) \\ \dot{x}_2(t) &=& Q\, x_2(t) + A_{21}\, y(t) \\ x_3(t) &=& \sum_{i=0}^{\nu-1} N_{33}^i E_{31}\, y^{(i+1)}(t) \\ x_4(t) &=& 0, \end{array}} \tag{2.5}$$

*where $Tx = \big(y,\, x_2^\top,\, x_3^\top,\, x_4^\top\big)^\top \in \mathcal{C}^1(\mathbb{R}; \mathbb{R}^{m+n_2+n_3+n_4})$.*

**Proof:** Let $Tx = \left(x_1^\top, x_2^\top, x_3^\top, x_4^\top\right)^\top \in \mathcal{C}^1(\mathbb{R};\mathbb{R}^{m+n_2+n_3+n_4})$ for $T$ as in Theorem 2.3. Then the DAE associated with (1.1) is system equivalent to

$$\left.\begin{array}{rcl} E_{14}\,\dot{x}_4(t) &=& A_{11}\,x_1(t) + A_{12}\,x_2(t) + u(t) \\ \dot{x}_2(t) &=& A_{21}\,x_1(t) + Q\,x_2(t) \\ E_{31}\,\dot{x}_1(t) + N_{33}\,\dot{x}_3(t) + E_{34}\,\dot{x}_4(t) &=& x_3(t) \\ N_{44}\,\dot{x}_4(t) &=& x_4(t) \\ y(t) &=& x_1(t)\,. \end{array}\right\} \tag{2.6}$$

Since $N_{44}^\nu = 0$, a similar argument as for (1.6) applied to $N_{44}\dot{x}_4(t) = x_4(t)$ gives $x_4 = 0$. Therefore (2.6) yields the first equation in (2.5). Since $N_{33}^\nu = 0$ and $y \in \mathcal{C}^\nu(\mathbb{R};\mathbb{R}^m)$, repeating the argument used for (1.6) yields that the third equation in (2.6) and in (2.5) are equivalent. This completes the proof. $\square$

The zero dynamics form (2.5) allows the following interpretation.

**Remark 2.7.**
Consider $[E,A,B,C] \in \Sigma_{n,m,m}$ such that its transfer function $C(sE-A)^{-1}B$ has proper inverse.

(i) The DAE in zero dynamics form (2.5) provides a realization for the inverse system:

$$\begin{array}{rcrl} \dot{x}_2(t) &=& Q\,x_2(t) & +\ A_{21}\,y(t) \\ u(t) &=& -A_{12}\,x_2(t) & -\ A_{11}\,y(t)\,. \end{array}$$

This is a time-domain counterpart of the representation (2.3) of the transfer function. In terms of Corollary 2.6: if $(x,u,y) \in \mathfrak{B}_{(1.1)}$, then the input $u(\cdot)$ is uniquely determined by $x(\cdot)$:

$$u(t) \overset{(2.5)}{=} -A_{11}C\,x(t) - A_{12}\,[0_{n_2,m}, I_{n_2}, 0_{n_2,n_3}, 0_{n_2,n_4}]\,T\,x(t)\,. \tag{2.7}$$

Note that the assumption $y \in \mathcal{C}^\nu(\mathbb{R};\mathbb{R}^m)$ is not required for this observation since the third equation in (2.5) is irrelevant.

(ii) Theorem 2.3 (iii) and (iv) allows to interpret the non-positive strict relative degree $\rho$ of the transfer function of $[E,A,B,C] \in \Sigma_{n,m,m}$: First, suppose that $\rho < 0$. Then $A_{11} = 0$, and $A_{12}(sI-Q)^{-1}A_{21}$ has strict relative degree $-\rho \geq 1$, i.e.

$$A_{12}Q^iA_{21} = 0 \quad \text{for } i = 0,\ldots,-\rho-2 \qquad \text{and} \qquad A_{12}Q^{-\rho-1}A_{21} \in \mathbf{Gl}_m(\mathbb{R})\,, \tag{2.8}$$

and (2.5) yields, for any $(x,u,y) \in \mathfrak{B}_{(1.1)}$ and $Tx = (y^\top, x_2^\top, x_3^\top, x_4^\top)^\top$,

$$\begin{array}{rcrl} \dot{x}_2(t) &=& Q\,x_2(t) & +\ A_{21}y(t) \\ u(t) &=& -A_{12}\,x_2(t)\,. & \end{array}$$

We see that $-\rho$ is the least number of times one has to differentiate the input $u(\cdot)$ so that the output $y(\cdot)$ occurs explicitly in the equation for $u^{(-\rho)}(\cdot)$.
If $\rho = 0$, then $A_{11} \in \mathbf{Gl}_m(\mathbb{R})$ and (2.5) yields $u(t) = -A_{11}y(t) - A_{12}x_2(t)$; thus $y(\cdot)$ occurs explicitly in the equation for $u(\cdot)$.
An algebraic characterization of non-positive strict relative degree in terms of the zero dynamics form is the following: Let $[E,A,B,C]$ be in zero dynamics form (2.1). Then, invoking $CAB = A_{11}$ and (2.8), an easy calculation yields that $C(sE-A)^{-1}B$ has strict relative degree $\rho \leq 0$ if, and only if,

$$CA^iB = 0 \quad \text{for } i = 1,\ldots,-\rho \qquad \text{and} \qquad CA^{-\rho+1}B \in \mathbf{Gl}_m(\mathbb{R})\,.$$

(iii) If the transfer function of $[E,A,B,C]$ has strict relative degree zero, then it has a realization as an ODE system with feedthrough; the latter is, in the notation of Theorem 2.3, given by

$$\begin{array}{rcrcr}\dot{x}_2(t) & = & (Q - A_{21}A_{11}^{-1}A_{12})\,x_2(t) & - & A_{21}A_{11}^{-1}\,u(t)\\ y(t) & = & -A_{11}^{-1}A_{12}\,x_2(t) & - & A_{11}^{-1}\,u(t)\,.\end{array}$$

◇

# 3 Zero dynamics

In this section we introduce the central concept of zero dynamics for DAE systems (1.1) and give, exploiting the zero dynamics form (2.1), a simple representation of it.

**Definition 3.1** (Zero dynamics)**.**
The *zero dynamics* of system (1.1) is defined as the set of trajectories

$$\mathcal{ZD}_{(1.1)} := \left\{ \, (x,u,y) \in \mathfrak{B}_{(1.1)} \;\middle|\; y = 0 \, \right\} .$$

◇

By linearity of (1.1), the set $\mathcal{ZD}_{(1.1)}$ is a real vector space.
In the following we show that the zero dynamics form (2.5) of the DAE is quite useful for a simple representation of the zero dynamics.

**Remark 3.2** (Representation of zero dynamics)**.**
Suppose $[E,A,B,C] \in \Sigma_{n,m,m}$ so that $C(sE-A)^{-1}B$ has proper inverse, and use the notation of Theorem 2.3. Then Corollary 2.6 yields

$$(x,u,y) \in \mathcal{ZD}_{(1.1)} \quad \Longleftrightarrow \quad y(t) = 0 \;\wedge\; \dot{x}_2(t) = Qx_2(t) \;\wedge\; u(t) = -A_{12}x_2(t) \;\wedge\; x_3(t) = 0.$$

Therefore, the zero dynamics may be written as

$$\mathcal{ZD}_{(1.1)} = \left\{ \left( T^{-1}\begin{pmatrix} 0_m \\ e^{Q\cdot}x_2^0 \\ 0_{n_3} \\ 0_{n_4} \end{pmatrix},\; -A_{12}e^{Q\cdot}x_2^0,\; 0 \right) \;\middle|\; x_2^0 \in \mathbb{R}^{n_2} \right\} . \tag{3.1}$$

◇

Next we show that the zero dynamics is a direct summand of the behaviour of the system.

**Remark 3.3** (Zero dynamics and behaviour)**.**
It may be interesting to see that for any $[E,A,B,C] \in \Sigma_{n,m,m}$ so that $C(sE-A)^{-1}B$ has proper inverse, the behaviour $\mathfrak{B}_{(1.1)}$ can be decomposed, in terms of the transformation matrix $T$ from Theorem 2.3, into a direct sum of the zero dynamics and a summand as

$$\begin{aligned}&\mathfrak{B}_{(1.1)}\\ =&\mathcal{ZD}_{(1.1)} \oplus \left\{ \, (x,u,y) \in \mathcal{C}^1(\mathbb{R};\mathbb{R}^n) \times \mathcal{C}(\mathbb{R};\mathbb{R}^m) \times \mathcal{C}(\mathbb{R};\mathbb{R}^m) \;\middle|\; \begin{array}{l} (x,u,y) \text{ solves (1.1) and} \\ [0_{n_2,m}, I_{n_2}, 0_{n_2,n_3}, 0_{n_2,n_4}]Tx(0) = 0 \end{array} \right\} .\end{aligned}$$

In terms of (3.1), the sum is immediate from

$$\big(x(\cdot),u(\cdot),y(\cdot)\big) = \left( T^{-1}\begin{pmatrix} 0_m \\ e^{Q\cdot}x_2^0 \\ 0_{n_3} \\ 0_{n_4} \end{pmatrix},\; -A_{12}e^{Q\cdot}x_2^0,\; 0 \right) + \left( x(\cdot) - T^{-1}\begin{pmatrix} 0_m \\ e^{Q\cdot}x_2^0 \\ 0_{n_3} \\ 0_{n_4} \end{pmatrix},\; u(\cdot) + A_{12}e^{Q\cdot}x_2^0,\; y(\cdot) \right),$$

for any $(x,u,y) \in \mathfrak{B}_{(1.1)}$, where $x_2^0 = [0, I_{n_2}, 0, 0]Tx(0)$. The direct sum also follows from (3.1). ◇

Finally, we show that the zero dynamics carries in a certain sense the structure of a dynamical system.

**Remark 3.4** (Zero dynamics are a dynamical system)**.**
Suppose $[E, A, B, C] \in \Sigma_{n,m,m}$ and let $\nu$ be the index of the pencil $sE - A$. The *transition map* of system (1.1) is defined, in terms of Proposition 1.4, as

$$\begin{aligned} \varphi : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{C}^{\nu-1}(\mathbb{R};\mathbb{R}^m) \quad &\to \quad \mathbb{R}^n \\ (t, t_0, x^0, u(\cdot)) \quad &\mapsto \quad T^{-1}\begin{bmatrix} e^{A_s(t-t_0)} & 0 \\ 0 & 0 \end{bmatrix} T x^0 + \int_{t_0}^{t} T^{-1}\begin{bmatrix} e^{A_s(t-s)} & 0 \\ 0 & 0 \end{bmatrix} W^{-1} B u(s)\,\mathrm{d}s \\ &\qquad - \sum_{k=0}^{\nu-1} T^{-1}\begin{bmatrix} 0 & 0 \\ 0 & N^k \end{bmatrix} W^{-1} B u^{(k)}(t)\,. \end{aligned}$$

We have shown in [1, Prop. 2.20] that for any $(t_0, x^0, u) \in \mathbb{R} \times \mathbb{R}^n \times \mathcal{C}^{\nu-1}(\mathbb{R};\mathbb{R}^m)$ the map $t \mapsto x(t) := \varphi(t, t_0, x^0, u(\cdot))$ solves the initial value problem

$$E\dot{x}(t) = Ax(t) + Bu(t)\,, \qquad x(t_0) = x^0 \tag{3.2}$$

if, and only if,

$$x^0 \in \mathcal{V}_{t_0,u(\cdot)} := \left\{ x \in \mathbb{R}^n \;\middle|\; x + \sum_{i=0}^{\nu-1} T^{-1}\begin{bmatrix} 0 & 0 \\ 0 & N^i \end{bmatrix} W^{-1} B u^{(i)}(t_0) \in \operatorname{im} T^{-1}\begin{bmatrix} I_{n_s} \\ 0 \end{bmatrix} \right\}.$$

Therefore, consistency of the initial value $x^0$ depends on the initial time $t_0$ and the input $u(\cdot)$. The *output map* of system (1.1) is defined by

$$\eta : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m,\ (t, x, u) \mapsto Cx\,.$$

It is readily verified that the structure $(\mathbb{R}, \mathbb{R}^m, \mathcal{C}^{\nu-1}(\mathbb{R};\mathbb{R}^m), \mathbb{R}^n, \mathbb{R}^m, \varphi, \eta)$, where $\varphi : \mathcal{D}_\varphi \to \mathbb{R}^n$ is the restriction of the transition map (by abuse of notation we write the same symbol) on

$$\mathcal{D}_\varphi := \left\{ (t, t_0, x^0, u) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{C}^{\nu-1}(\mathbb{R};\mathbb{R}^m) \;\middle|\; x^0 \in \mathcal{V}_{t_0,u(\cdot)},\ C\varphi(\cdot\,; t_0, x^0, u(\cdot)) = 0 \right\},$$

is an $\mathbb{R}$-linear time-invariant dynamical system as defined in [7, Defs. 2.1.1, 2.1.24, 2.1.26].
Next let $u \in \mathcal{C}^{\nu-1}(\mathbb{R};\mathbb{R}^m)$. As a consequence of uniqueness and global existence of the solution of the initial value problem (3.2) for $x^0 \in \mathcal{V}_{0,u(\cdot)}$ (see again [1, Prop. 2.20]), the map

$$\Psi : \mathcal{D}_{\varphi,0} \to \mathcal{ZD}_{(1.1)}\,, \quad (0, 0, x^0, u(\cdot)) \mapsto \big(\varphi(\cdot\,; 0, x^0, u(\cdot)), u(\cdot), C\varphi(\cdot\,; 0, x^0, u(\cdot))\big)$$

is well-defined, where

$$\mathcal{D}_{\varphi,0} := \left\{ (0, 0, x^0, u) \in \mathcal{D}_\varphi \right\} \subset \mathcal{D}_\varphi\,.$$

Most importantly, if $C(sE - A)^{-1}B$ has proper inverse, then $\Psi$ is an isomorphism: it is surjective since a pre-image of $(x, u, 0) \in \mathcal{ZD}_{(1.1)}$ is $(0, 0, x(0), u(\cdot)) \in \mathcal{D}_{\varphi,0}$ (note that $u \in \mathcal{C}^{\nu-1}(\mathbb{R};\mathbb{R}^m)$ by Remark 3.2), it is injective by uniqueness of the solution of the initial value problem (3.2). In this sense, we may say that $\mathcal{ZD}_{(1.1)}$ is a dynamical system. $\diamond$

# 4 $(\mathbf{A}, \mathbf{E}, \mathbf{B})$-invariant subspaces

In this section we recall the concept of $(A, E, B)$-invariant subspaces and show that the function vector space of zero dynamics of system (1.1) is, under some conditions, isomorphic to the supremal (in fact maximal) $(A, E, B)$-invariant subspace included in $\ker C$.

**Definition 4.1** ($(A, E, B)$-invariance)**.**
Let $(A, E, B) \in \mathbb{R}^{n,n} \times \mathbb{R}^{n,n} \times \mathbb{R}^{n,m}$. A subspace $\mathcal{V} \subseteq \mathbb{R}^n$ is called $(A, E, B)$-*invariant* if, and only if,

$$A\mathcal{V} \subseteq E\mathcal{V} + \operatorname{im} B.$$

$\diamond$

The concept of $(A, E, B)$-invariance is well-known, see e.g. [11, 12], and used, for example in [13], to derive the reachable and controllable subspaces of (1.1).

The existence of the *supremal* $(A, E, B)$-*invariant subspace* included in $\ker C$

$$\mathcal{V}^*(A, E, B; \ker C) := \sup \{ \ \mathcal{V} \subseteq \mathbb{R}^n \ \mid \mathcal{V} \text{ is an } (A, E, B)\text{-invariant subspace and } \mathcal{V} \subseteq \ker C \ \}$$

follows since the sum of $(A, E, B)$-invariant subspaces included in $\ker C$ is $(A, E, B)$-invariant and included in $\ker C$; actually, the supremum is a maximum.
The space $\mathcal{V}^*(A, E, B; \ker C)$ is the limit of the following sequence of subspaces (see e.g. [13, Lemma 2.1]):

$$\mathcal{V}_0 = \ker C, \quad \mathcal{V}_{i+1} = \ker C \cap A^{-1}(E\mathcal{V}_i + \operatorname{im} B), \quad i \in \mathbb{N}_0$$

actually, it terminates after finitely many steps:

$$\exists\, k^* \in \mathbb{N}\ \forall\, i \in \mathbb{N} \ : \ \mathcal{V}_{k^*+i} = \mathcal{V}_{k^*} = \mathcal{V}^*(A, E, B; \ker C)\,.$$

The following proposition gives a simple and useful representation of $\mathcal{V}^*(A, E, B; \ker C)$ under the assumptions of Theorem 2.3.

**Proposition 4.2** (Representation of $\mathcal{V}^*(A, E, B; \ker C)$)**.**
*Suppose* $[E, A, B, C] \in \Sigma_{n,m,m}$ *such that* $C(sE - A)^{-1}B$ *has proper inverse. Then, in terms of the matrices in Theorem 2.3, we have*

$$\mathcal{V}^*(A, E, B; \ker C) = \operatorname{im}\, T^{-1} \begin{bmatrix} 0_{m,n_2} \\ I_{n_2} \\ 0_{n_3,n_2} \\ 0_{n_4,n_2} \end{bmatrix}.$$

The proof is in Appendix 8.2.

Now, as a consequence of the simple representation of the zero dynamics in (3.1) and the maximal $(A, E, B)$-invariant subspace included in $\ker C$, we are able to show that the state $x(\cdot)$ of $(x, u, y) \in \mathcal{ZD}_{(1.1)}$ evolves in $\mathcal{V}^*(A, E, B; \ker C)$.

**Proposition 4.3** (Characterization of zero dynamics)**.**
*Suppose* $[E, A, B, C] \in \Sigma_{n,m,m}$ *so that* $C(sE - A)^{-1}B$ *has proper inverse. Then*

$$(x, u, y) \in \mathcal{ZD}_{(1.1)} \quad \Longleftrightarrow \quad \Big[\ \forall\, t \in \mathbb{R} : \ x(t) \in \mathcal{V}^*(A, E, B; \ker C)\ \Big]\,.$$

**Proof:** "$\Rightarrow$" follows from Remark 3.2 and Proposition 4.2. To see "$\Leftarrow$", note that by Corollary 2.6 the first $m$ components of $Tx$ coincide with the vector $y$; then Proposition 4.2 and the assumption yield $y = 0$, hence $(x, u, y) \in \mathcal{ZD}_{(1.1)}$. $\square$

The following theorem shows that the zero dynamics of a DAE (with proper inverse transfer function) is isomorphic to the maximal $(A, E, B)$-invariant subspace included in the kernel of $C$.

**Theorem 4.4** (Vector space isomorphism).
*Let* $[E, A, B, C] \in \Sigma_{n,m,m}$ *such that* $C(sE - A)^{-1}B$ *has proper inverse. Then the linear map*

$$\begin{aligned}\varphi : \mathcal{V}^*(A, E, B; \ker C) &\to \mathcal{ZD}_{(1.1)},\\ x^0 &\mapsto \big(x(\cdot), u(\cdot), Cx(\cdot)\big), \text{ where } (x(\cdot), u(\cdot)) \text{ solves the initial value problem}\end{aligned}$$

$$\left.\begin{aligned}\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} \dot{x} \\ \dot{u} \end{pmatrix} &= \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}\begin{pmatrix} x \\ u \end{pmatrix}\\ \begin{pmatrix} x(0) \\ u(0) \end{pmatrix} &= \begin{pmatrix} x^0 \\ -A_{12}[0_{n_2,m}, I_{n_2}, 0_{n_2,n_3}, 0_{n_2,n_4}]Tx^0 \end{pmatrix},\end{aligned}\right\} \tag{4.1}$$

*where we use the notation from Theorem 2.3, is a vector space isomorphism.*

**Proof:** If $x^0 \in \mathcal{V}^*(A, E, B; \ker C)$, then Proposition 4.2 yields, for some $x_2^0 \in \mathbb{R}^{n_2}$, that $Tx^0 = \big((0_m)^\top, (x_2^0)^\top, (0_{n_3})^\top, (0_{n_4})^\top\big)^\top$. Since (4.1) gives $y(\cdot) = 0$, it follows from Corollary 2.6 that DAE (2.5) with initial value $x(0) = Tx^0$ has unique solution

$$y(t) = 0, \quad x_2(t) = e^{Qt}x_2^0, \quad x_3(t) = 0, \quad x_4(t) = 0, \quad u(t) = -A_{12}x_2(t) = -A_{12}e^{Qt}x_2^0\,.$$

Hence, the initial value problem (4.1) has a unique solution for any $x^0 \in \mathcal{V}^*(A, E, B; \ker C)$. Furthermore, $\varphi$ is well-defined and injective; surjectivity is a consequence of Proposition 4.3 and the representation (2.7) of $u(\cdot)$. □

# 5 Stable zero dynamics

The main result of this section is Theorem 5.4 which characterizes stable zero dynamics and, most importantly, shows that any system with asymptotically stable zero dynamics can be stabilized by high-gain output feedback. Note that in the present section and in the following Section 6 we consider the restriction of solutions of $[E, A, B, C]$ to $[0, \infty)$.

**Definition 5.1** (Stability of zero dynamics).
The zero dynamics of system (1.1) is called *asymptotically stable* if, and only if,

$$\forall\, \varepsilon > 0\ \exists\, \delta > 0\ \forall\, (x, u, y) \in \mathcal{ZD}_{(1.1)} \text{ s.t. } \|(x(0), u(0))\| < \delta\ \forall\, t \geq 0\ :\ \|(x(t), u(t))\| < \varepsilon$$

and

$$\forall\, (x, u, y) \in \mathcal{ZD}_{(1.1)} :\ \lim_{t\to\infty} \big(x(t), u(t)\big) = 0.$$

◇

As an immediate consequence of the representation of $\mathcal{ZD}_{(1.1)}$ in Remark 3.2 we have that asymptotic stability of the zero dynamics can be read off as follows:

**Corollary 5.2** (Stable zero dynamics).
*Let* $[E, A, B, C] \in \Sigma_{n,m,m}$ *so that* $C(sE - A)^{-1}B$ *has proper inverse. The zero dynamics are asymptotically stable if, and only if, the matrix* $Q$ *in* (2.1) *satisfies* $\sigma(Q) \subset \mathbb{C}_-$.

To state the main result of this section, we need to recall the definition of transmission zeros and poles of a transfer function.

**Definition 5.3** (Smith-McMillan form).
Let $G(s) \in \mathbb{R}(s)^{p,m}$ with *Smith-McMillan form*

$$U^{-1}(s)G(s)V^{-1}(s) = \operatorname{diag}\left(\frac{\varepsilon_1(s)}{\psi_1(s)}, \ldots, \frac{\varepsilon_r(s)}{\psi_r(s)}, 0, \ldots, 0\right) \in \mathbb{R}(s)^{p,m},$$

where $U(s) \in \mathbb{R}[s]^{p,p}$, $V(s) \in \mathbb{R}[s]^{m,m}$ are unimodular, $\operatorname{rk} G(s) = r$, $\varepsilon_i(s), \psi_i(s) \in \mathbb{R}[s]$ are monic, coprime and satisfy $\varepsilon_i(s) \mid \varepsilon_{i+1}(s)$, $\psi_{i+1}(s) \mid \psi_i(s)$ for $i = 1, \ldots, r-1$. $s_0 \in \mathbb{C}$ is called *transmission zero* of $G(s)$ if $\varepsilon_r(s_0) = 0$ and a *pole* of $G(s)$ if $\psi_1(s_0) = 0$. ◇

We are now in a position to state the main result of this section and show characterizations of the zero dynamics in terms of a determinant, minimum phase and high-gain stabilizability.

**Theorem 5.4** (Stable zero dynamics). *Let $[E, A, B, C] \in \Sigma_{n,m,m}$ such that $C(sE - A)^{-1}B$ has proper inverse. Then the following statements are equivalent:*

(i) *The zero dynamics of system (1.1) are asymptotically stable;*

(ii)
$$\forall\, s \in \overline{\mathbb{C}}_+ \;:\; \det \begin{bmatrix} sE - A & B \\ C & 0 \end{bmatrix} \neq 0\,;$$

(iii) *system (1.1) is* minimum phase, *i.e.*

  (a) *(1.1) is stabilizable,*

  (b) *(1.1) is detectable,*

  (c) *$C(sE - A)^{-1}B$ has no transmission zeros in $\overline{\mathbb{C}}_+$ ;*

(iv) *(1.1) is* high-gain stabilizable, *i.e.*

$$\exists\, k^* \geq 0 \;\forall\, k \in \mathbb{R}, |k| \geq k^* \;\forall\, \textit{sln. } x(\cdot) \textit{ of } \text{‘}u(t) = ky(t) \;\&\; (1.1)\text{’} \;:\; \lim_{t\to\infty} x(t) = 0.$$

The proof is in Appendix 8.2.

**Remark 5.5** (High-gain stabilizability).
Being familiar with high-gain control for minimum phase systems with strictly proper transfer function, it might surprise that high-gain stabilizability in Theorem 5.4 (iv) does neither depend on the relative degree of the system nor on the sign of the high frequency gain. The reason is that, in view of Corollary 2.6, the closed-loop system ‘$u(t) = ky(t)$ & (1.1)’ is equivalent to

$$\begin{aligned} -(A_{11} + kI_m)\, y(t) &= A_{12}x_2(t) \\ \dot{x}_2(t) &= Qx_2(t) + A_{21}y(t) \\ x_3(t) &= \textstyle\sum_{i=0}^{\nu-1} N_{33}^i\, E_{31}\, y^{(i+1)}(t) \end{aligned}$$

and, if $k > \|A_{11}\|$, equivalent to

$$\begin{aligned} y(t) &= -(A_{11} + kI_m)^{-1}\, A_{12}x_2(t) \\ \dot{x}_2(t) &= \left[Q - A_{21}(A_{11} + kI_m)^{-1}A_{12}\right]\, x_2(t) \\ x_3(t) &= \textstyle\sum_{i=0}^{\nu-1} N_{33}^i\, E_{31}\, y^{(i+1)}(t)\,. \end{aligned}$$

Note that $x_2 \in \mathcal{C}^\infty(\mathbb{R};\mathbb{R}^{n-2})$ yields $y \in \mathcal{C}^\infty(\mathbb{R};\mathbb{R}^m)$ and so the algebraic equation for $x_3$ is well defined. By Corollary 5.2 , system (1.1) has asymptotically stable zero dynamics if, and only if, $\sigma(Q) \subset \mathbb{C}_-$. Since

$$\lim_{k\to\pm\infty} \sigma\left(Q - A_{21}(A_{11} + kI_m)^{-1}A_{12}\right) \;=\; \sigma(Q)\,,$$

the assumptions '$|k|$ sufficiently large' and '(1.1) has asymptotically stable zero dynamics' yields exponential decay of $x_2(\cdot)$ , and therefore $x_3(\cdot)$ and $y(\cdot)$ decay exponentially, too. On the other hand, it is immediate that high-gain stabilizability implies $\sigma(Q) \subset \mathbb{C}_-$. $\diamond$

# 6 Funnel control

We have seen in Theorem 5.4 that any system $[E, A, B, C] \in \Sigma_{n,m,m}$, where $C(sE - A)^{-1}B$ has proper inverse and system (1.1) has asymptotically stable zero dynamics, can be stabilized by output feedback $u(t) = k\,y(t)$ for sufficiently large $|k|$. This controller is simple, it does not require any specific system data but only structural assumptions; however, one has to find out what "sufficiently large" means and the aim is that the control law does not explicitly depend on the system data. To resolve this problem, one may consider the adaptive controller

$$\left.\begin{array}{rcl} u(t) & = & -k(t)\,y(t) \\ \dot{k}(t) & = & \|y(t)\|^2\,, \qquad k(0) = k^0 \end{array}\right\} \tag{6.1}$$

for DAE systems $[E, A, B, C]$ with proper inverse transfer function. This result is not proved here; the proof is similar to that of Theorem 6.2. The drawback of the control strategy (6.1) is that, albeit $k(\cdot)$ is bounded, it is monotonically increasing and potentially so large that it is very sensitive to noise corrupting the output measurement. Further drawbacks are that (6.1) does not tolerate mild output perturbations, tracking would require an internal model and, most importantly, transient behaviour is not taken into account. These issues are discussed for ODE systems (with strictly proper transfer function of strict relative degree one and asymptotically stable zero dynamics) in the survey [8].

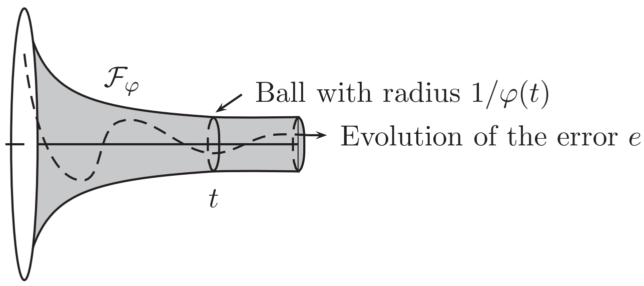



Figure 3: Error evolution in the funnel $\mathcal{F}_\varphi$ with "width $\infty$" at $t = 0$, i.e. $\varphi(0) = 0$

To overcome these drawbacks, the concept of "funnel control" is introduced (see [8] and the references therein): For any function $\varphi$ belonging to

$$\Phi^\mu := \left\{\varphi \in \mathcal{B}^\mu(\mathbb{R}_{\geq 0}, \mathbb{R}) \;\middle|\; \varphi(0) = 0, \quad \varphi(s) > 0 \text{ for all } s > 0 \text{ and } \liminf_{s\to\infty} \varphi(s) > 0\right\},$$

we associate the *performance funnel*

$$\mathcal{F}_\varphi := \left\{(t, e) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^m_{\geq 0} \;\middle|\; \varphi(t)\|e\| < 1\right\}, \tag{6.2}$$

see Fig. 3. We assume sufficient smoothness of $\varphi \in \Phi^\mu$, that is, $\mu = \nu + 1$, where $\nu$ is the index of $sE - A$. The control objective is feedback control so that the tracking error evolves within $\mathcal{F}_\varphi$ and all variables are bounded. More specific, the transient behaviour is supposed to satisfy

$$\|e(t)\| < 1/\varphi(t) \quad \forall\, t > 0,$$

and, moreover, if $\varphi$ is chosen so that $\varphi(t) \geq 1/\lambda$ for all $t$ sufficiently large, then the tracking error remains smaller than $1/\lambda$.

To ensure error evolution within the funnel, we introduce, for $\hat{k} \in \mathbb{R}\backslash\{0\}$, the *funnel controller*:

$$\boxed{\begin{aligned} u(t) &= -k(t)\, e(t), \\ k(t) &= \frac{\hat{k}}{1 - \varphi(t)^2 \|e(t)\|^2} \end{aligned}} \qquad e(t) = y(t) - y_{\text{ref}}(t)\,. \tag{6.3}$$

In view of the high-gain property derived in Theorem 5.4, we see intuitively that, in order to maintain the error evolution within the funnel, high gain values may only be required if the norm of the error, i.e. $\|e(t)\|$, is close to the funnel boundary $\varphi(t)^{-1}$. This intuition underpins the choice of the gain $k(t)$ in (6.3). The control design (6.3) has two advantages: $k(\cdot)$ is non-monotone and (6.3) is a static time-varying proportional output feedback of striking simplicity.

Before we state our main result, some remarks on the consistency of the initial value of the closed-loop system are necessary.

**Remark 6.1** (Consistent initial values)**.**
Suppose that $[E, A, B, C] \in \Sigma_{n,m,m}$ such that the assumptions of Theorem 6.2 hold.

(i) An initial value $x^0 \in \mathbb{R}^n$ is consistent for the closed-loop system (1.1), (6.3) if, and only if, there exists a solution $x : [0, \omega) \to \mathbb{R}^n$ for some $\omega \in (0, \infty]$ of the initial value problem (1.1), (6.3), $x(0) = x^0$.

(ii) If, in terms of Theorem 2.3, $n_3 = 0$, then $x^0 \in \mathbb{R}^n$ is consistent for the closed-loop system (1.1), (6.3) if, and only if,

$$x^0 + T^{-1} \begin{pmatrix} \hat{k}(A_{11} - \hat{k}I)^{-1} y_{\text{ref}}(0) \\ 0 \\ 0 \end{pmatrix} \in \operatorname{im} T^{-1} \begin{bmatrix} -(A_{11} - \hat{k}I)^{-1} A_{12} \\ I_{n_2} \\ 0_{n_4, n_2} \end{bmatrix}.$$

Invoking $n_3 = 0$, "$\Rightarrow$" follows immediately from (8.22); and "$\Leftarrow$" follows from a careful inspection of the proof of Theorem 6.2.

(iii) In practice, consistency of the initial state of the "unknown" system should be satisfied as far as the DAE $[E, A, B, C]$ is the correct model. $\diamond$

We are now in a position to state the main result of this section.

**Theorem 6.2** (Funnel control)**.**
*Suppose that $[E, A, B, C] \in \Sigma_{n,m,m}$ has asymptotically stable zero dynamics and the transfer function $G(s) = C(sE - A)^{-1}B$ has proper inverse. Let $\nu$ be the index of $sE - A$. Let $\varphi \in \Phi^{\nu+1}$ define a performance funnel $\mathcal{F}_\varphi$. Then for any consistent initial value $x^0 \in \mathbb{R}^n$, initial gain $\hat{k} \in \mathbb{R}$ with $|\hat{k}| > \lim_{s\to\infty} \|G^{-1}(s)\|$, and any reference signal $y_{\text{ref}} \in \mathcal{B}^{\nu+1}(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$, the application of the funnel controller* (6.3) *to* (1.1) *yields a closed-loop initial-value problem with the following properties:*

(i) *Precisely one maximal continuously differential solution* $x(\cdot)\colon [0,\omega) \to \mathbb{R}^n$ *exists and this solution is global (i.e.* $\omega = \infty$*).*

(ii) *The global solution* $x(\cdot)$ *is bounded and the tracking error* $e(\cdot) = Cx(\cdot) - y_{\mathrm{ref}}(\cdot)$ *evolves uniformly within the performance funnel* $\mathcal{F}_\varphi$*; more precisely,*

$$\exists\, \varepsilon > 0\ \forall\, t \geq 0 : \ \|e(t)\| \leq \varphi(t)^{-1} - \varepsilon\,. \tag{6.4}$$

(iii) *The gain function* $k$ *is bounded:* $\|k\|_\infty \leq \dfrac{\hat{k}}{1-(1-\varepsilon\|\varphi\|_\infty)^2}$.

The proof is in Appendix 8.2.

**Remark 6.3** (Initial data of the funnel controller)**.**
Theorem 6.2 does not require that $[E, A, B, C]$ has a non-positive strict relative degree. However, if it has one, then Theorem 2.3 (iii) and (iv) yield that

$$k(0) = \hat{k} \quad \text{with } |\hat{k}| > \begin{cases} \|A_{11}\|, & \text{if } \ \operatorname{sr\,deg}\big(C(sE-A)^{-1}B\big) = 0 \\ 0, & \text{if } \ \operatorname{sr\,deg}\big(C(sE-A)^{-1}B\big) < 0\,. \end{cases}$$

Therefore, additional information on the system class, namely $\hat{k} > \|A_{11}\|$, is required for the funnel controller (6.3) only if $[E, A, B, C]$ is an ODE system with feedthrough; see Remark 2.7 (iii). Otherwise, any $\hat{k} > 0$ is good enough. ◇

**Remark 6.4** (Weakening the assumptions of Theorem 6.2)**.**

(i) The result of Theorem 6.2 is valid for a much bigger class of systems. This is revealed by a careful inspection of its proof: In fact, the pencil $sE - A$ must not be regular, it is sufficient to assume that $[E, A, B, C]$ is system equivalent to a system in form (2.1) (where not necessarily $\operatorname{rk}[E_{31}, N_{33}] = n_3$ holds) such that $\sigma(Q) \subseteq \mathbb{C}_-$; then we may choose $\nu$ to be the maximum of the indices of nilpotency of $N_{33}$ and $N_{44}$ and funnel control is feasible. Note that existence and uniqueness of the solution is guaranteed by the control law to that extent that the resulting semi-explicit DAE (8.24) is index 1. This issue is illustrated in the example in Section 7.2.

(ii) The smoothness assumptions on the reference trajectory $y_{\mathrm{ref}}$ and the funnel function $\varphi$ can be further relaxed. A close look at the zero dynamics form (2.1) of $[E, A, B, C]$ reveals: Denoting the index of nilpotency of $N_{33}$ by $\nu_3$ ($\nu_3 = 0$, if $n_3 = 0$), and setting $\mu = \max\{1, \nu_3\}$, it is only required that

$$y_{\mathrm{ref}} \in \mathcal{B}^{\mu+1}(\mathbb{R}_{\geq 0}; \mathbb{R}^m) \qquad \text{and} \qquad \varphi \in \Phi^{\mu+1}.$$

Note that, by Theorem 2.3 (v), the number $\nu_3$ is an invariant of the system $[E, A, B, C]$ and, by construction of the zero dynamics form in the proof of Theorem 2.3, it holds that $\nu_3 \leq \nu$.
If $[E, A, B, C]$ is observable at infinity, then $\nu_3 = 0$ by Remark 2.2, and thus $y_{\mathrm{ref}} \in \mathcal{B}^2(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$ and $\varphi \in \Phi^2$ are sufficient to guarantee the statements (i)–(iii) of Theorem 6.2.

(iii) Remark 6.1 also remains valid under the weaker assumptions of (i) and (ii).

◇

# 7 Examples and simulations

For purposes of illustration, we consider two examples of differential-algebraic systems (1.1) and apply the funnel controller (6.3). The first example in Section 7.1 is a mechanical system with springs, masses and dampers with single-input spatial distance between the two masses and single-output position of one mass; the second example in Section 7.2 is an academic example of a 2-input 2-output system with singular matrix pencil.

As reference signal $y_{\rm ref} : \mathbb{R}_{\geq 0} \to \mathbb{R}$, we take components of the (chaotic) solution of the following initial-value problem for the Lorenz system

$$\begin{array}{rcll} \dot{\xi}_1(t) &=& 10\,(\xi_2(t) - \xi_1(t)), & \xi_1(0) = 5 \\ \dot{\xi}_2(t) &=& 28\,\xi_1(t) - \xi_1(t)\,\xi_3(t) - \xi_2(t), & \xi_2(0) = 5 \\ \dot{\xi}_3(t) &=& \xi_1(t)\,\xi_2(t) - \frac{8}{3}\,t\xi_3(t), & \xi_3(0) = 5\,. \end{array} \tag{7.1}$$

It is well known that the unique global solution of (7.1) is bounded with bounded derivative on the positive real axis, see for example [15, App. C]. The first and second components of the solution of (7.1) are depicted in Fig. 4.

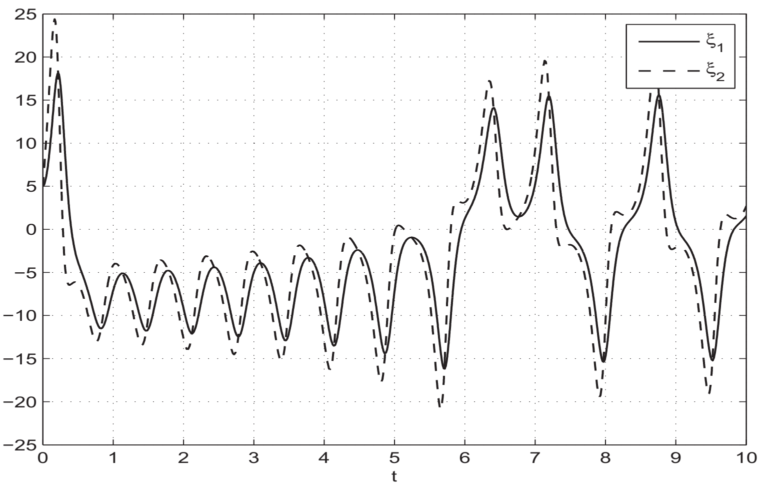



Figure 4: Components $\xi_1(\cdot)$ and $\xi_2(\cdot)$ of the Lorenz system (7.1)

The funnel $\mathcal{F}_\varphi$ is determined by the function

$$\varphi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}, \quad t \mapsto 0.5\ te^{-t} + 2\ \arctan t\,. \tag{7.2}$$

Note that this prescribes an exponentially (exponent 1) decaying funnel in the transient phase $[0, T]$, where $T \approx 3$, and a tracking accuracy quantified by $\lambda = 1/\pi$ thereafter, see Fig. 6d.
All numerical simulations are performed by MATLAB.

## 7.1 Position control of a mechanical system with springs, masses and dampers

We are indebted to our colleague Professor P.C. Müller (BU Wuppertal) for providing the following mechanical system illustrated in Fig. 5 to us.

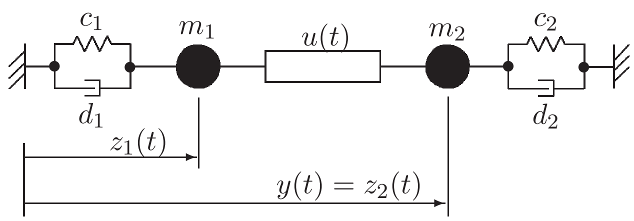



Figure 5: Mass-spring-damper system

The masses $m_1$, $m_2$, damping constants $d_1$, $d_2$ and spring constants $c_1$, $c_2$ are all assumed to be positive. The input $u(t) = z_2(t) - z_1(t)$ is the spatial distance between the masses $m_1$ and $m_2$; as output $y(t) = z_2(t)$ we take the position of the mass $m_2$. Then the mechanical system in Fig. 5 may be modelled by the second-order differential-algebraic equation

$$\begin{aligned} m_1\ddot{z}_1(t) + d_1\dot{z}_1(t) + c_1 z_1(t) - \lambda(t) &= 0 \\ m_2\ddot{z}_2(t) + d_2\dot{z}_2(t) + c_2 z_2(t) + \lambda(t) &= 0 \\ z_2(t) - z_1(t) &= u(t) \\ y(t) &= z_2(t) \end{aligned} \tag{7.3}$$

where $\lambda(\cdot)$ is a constraint force viewed as a variable. Defining $x(t) = (z_1(t), \dot{z}_1(t), z_2(t), \dot{z}_2(t), \lambda(t))^\top$, the model (7.3) may be rewritten as the linear differential-algebraic input-output system (1.1) for

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & m_1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & m_2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -c_1 & -d_1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -c_2 & -d_2 & -1 \\ 1 & 0 & -1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}^\top. \tag{7.4}$$

We may immediately see that the pencil $sE - A$ is regular and has index $\nu = 3$.
The transfer function is

$$G(s) = C(sE - A)^{-1}B = -\frac{m_1 s^2 + d_1 s + c_1}{(m_1 + m_2)s^2 + (d_1 + d_2)s + (c_1 + c_2)},$$

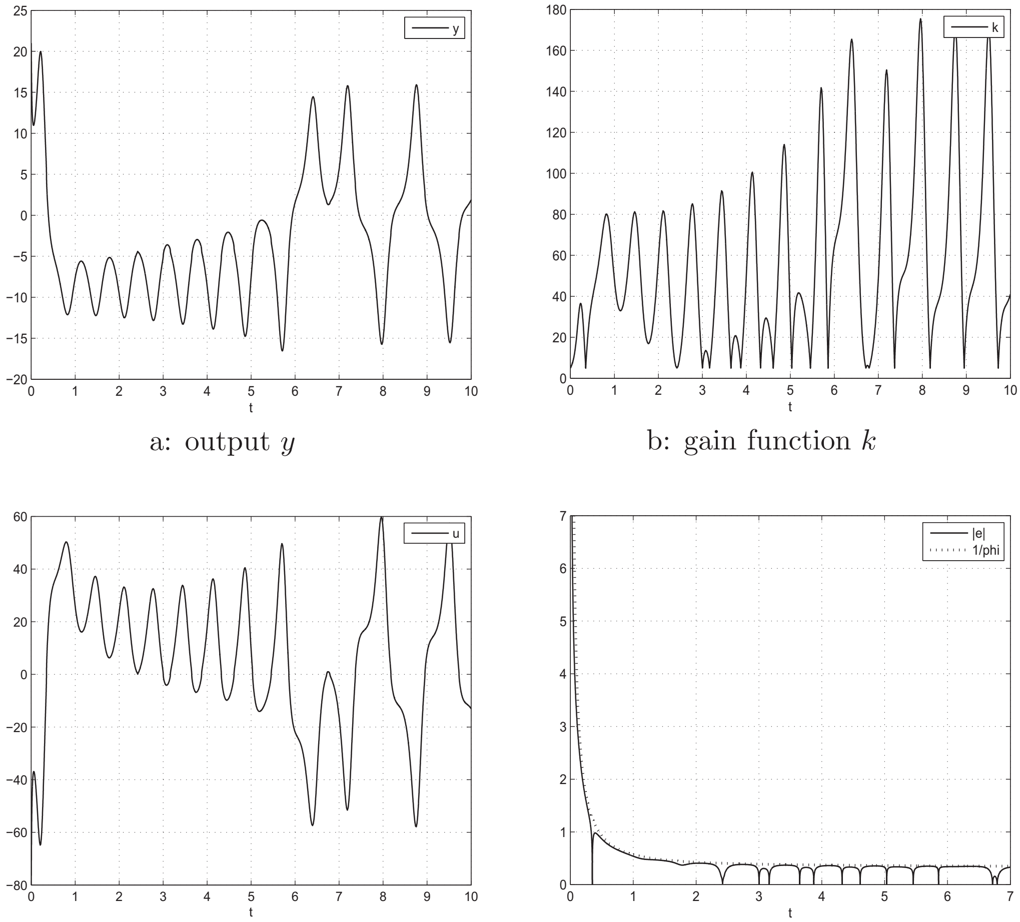
has strict relative degree $\operatorname{sr\,deg} G(s) = 0$ and proper inverse: $\lim_{s\to\infty} G^{-1}(s) = -(m_1 + m_2)/m_1$.
The zero dynamics of (7.4) is asymptotically stable: setting $y(\cdot) = 0$ in (7.3) yields $z_2(\cdot) = 0$, $\lambda(\cdot) = 0$, $z_1(\cdot) = -u(\cdot)$ and $m_1\, z_1(t) + d_1\dot{z}_1(t) + c_1 z_1(t) = 0$ for all $t \geq 0$; positivity of $m_1$, $d_1$ and $c_1$ then gives $\lim_{t\to\infty} \dot{z}_1(t) = \lim_{t\to\infty} z_1(t) = 0$.
The constants in (7.3) and initial position of masses are chosen, for the simulations, as

$$m_1 = 1, \quad m_2 = 3, \quad c_1 = 2, \quad c_2 = 1, \quad d_1 = 3, \quad d_2 = 5, \qquad z_1(0) = -59, \quad z_2(0) = 21. \tag{7.5}$$

In view of Remark 6.3 and Theorem 2.3 (vi), we set

$$\hat{k} = 5 > 4 = -\lim_{s\to\infty} G^{-1}(s).$$

Now some straightforward calculations show that the closed-loop system (6.3), (7.3) has uniquely determined initial velocities $\dot{z}_1(0)$, $\dot{z}_2(0)$ as well as initial constraint force $\lambda(0)$ and that the initialization is consistent.




a: output $y$ b: gain function $k$

c: input $u$ d: norm of error $|e(t)|$ and funnel

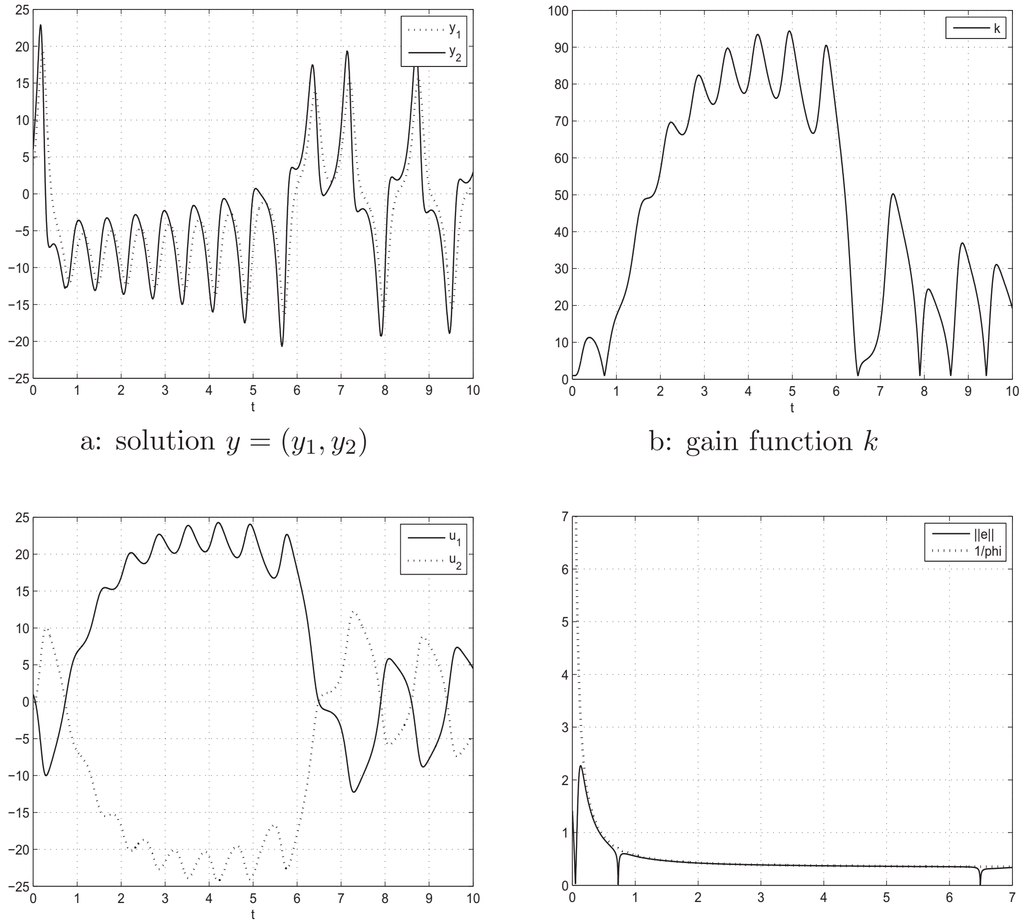
Figure 6: Simulation of the funnel controller (6.3) with funnel boundary specified in (7.2) and reference signal $y_{\text{ref}}(\cdot) = \xi_1(\cdot)$ given in (7.1) applied to the mechanical model (7.3) with data (7.5).

Then all assumptions of Theorem 6.2 are satisfied and we may apply the funnel controller (6.3) with funnel boundary specified in (7.2) and reference signal $y_{\text{ref}}(\cdot) = \xi_1(\cdot)$ given in (7.1). The simulations over the time interval $[0, 10]$ are depicted in Fig. 6: Fig. 6a shows the output $y(\cdot)$ tracking the rather "vivid" reference signal $y_{\text{ref}}(\cdot)$ within the funnel shown in Fig. 6d. Note that the input $u(\cdot)$ in Fig. 6c as well as the gain function $k(\cdot)$ in Fig. 6b have spikes at those times $t$ when the norm of the error $\|e(t)\|$ is "close" to the funnel boundary $\varphi(t)^{-1}$; this is due to rapid change of the reference signal. We stress that the gain function $k(\cdot)$ is non-monotone.

### 7.2 2-input 2-output system with singular matrix pencil

Consider the academic example

$$[E, A, B, C] := \left[ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & 2 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \right]. \tag{7.6}$$




a: solution $y = (y_1, y_2)$

b: gain function $k$

c: input $u = (u_1, u_2)$

d: norm of error $\|e(t)\|$ and funnel

Figure 7: Simulation of the funnel controller (6.3) with funnel boundary specified in (7.2) and reference signal $y_{\rm ref}(\cdot) = (\xi_1(\cdot), \xi_2(\cdot))^\top$ given in (7.1) applied to system (7.6) with initial data (7.7).

It is immediate that the pencil $sE - A$ is singular, i.e. $\det(sE - A) = 0$. However, in view of Remark 6.4 (i), funnel control as suggested in Theorem 6.2 is still feasible if the other assumptions are satisfied: Obviously, $[E, A, B, C]$ is in zero dynamics form (2.1) with $A_{11} = 0_{2,2}$ and asymptotically stable zero dynamics since $\sigma(Q) = \{-1\}$. We may choose the initial data of the closed-loop system (6.3), (7.6) as

$$\hat{k} = 1, \qquad x^0 = (4, 6, -1)^\top \tag{7.7}$$

so that, by Remark 6.1 (ii), they are consistent.
The simulations, over the time interval $[0, 10]$, of the funnel controller (6.3) with funnel boundary specified in (7.2) and reference signal $y_{\rm ref}(\cdot) = (\xi_1(\cdot), \xi_2(\cdot))^\top$ given in (7.1), applied to system (7.6) with initial data (7.7) are depicted in Fig. 7: Similar to the example in Section 7.1, an action of the input components in Fig. 7c and the gain function in Fig. 7b is required only if the error $\|e(t)\|$ is close to the funnel boundary $\varphi(t)^{-1}$. Note that $y(t) \approx y_{\rm ref}(t)$ if, and only if, $k(t) \approx \hat{k} = 1$.

# 8 Appendix

The following Section 8.1 is the fundament for the proofs in Section 8.2.

## 8.1 Elementary properties of rational matrices

In the present section we collect the relevant properties of rational matrices for our results.
If $G(s) \in \mathbb{R}(s)^{m,m}$, then it is well known, see for example [4, Thm. 2-6.2], that

$$\exists\, P(s) \in \mathbb{R}[s]^{m,m}\ \exists \text{ strictly proper } G_{\rm sp}(s) \in \mathbb{R}(s)^{m,m} \ : \ G(s) = P(s) + G_{\rm sp}(s)\,; \tag{8.1}$$

and it is easy to see that

$$\operatorname{sr}\deg G(s) = \rho \le 0 \quad \iff \quad \rho = -\deg P(s) \text{ and the leading coefficient of } P(s) \text{ is invertible}\,. \tag{8.2}$$

In passing, we also note that if $P_N \in \mathbf{Gl}_m(\mathbb{R})$, then $P(s) = \sum_{i=0}^N P_i s^i \in \mathbb{R}[s]^{m,m}$ is invertible over $\mathbb{R}(s)$; however, if $P(s)$ is invertible over $\mathbb{R}(s)$, then $P_N$ is not necessarily invertible, for an example see [2, p. 257]. Note further that, if $G(s) \in \mathbb{R}(s)^{m,m}$ has some non-positive strict relative degree, then (8.2) yields that the leading coefficient of $P(s)$ is invertible and therefore $P(s)$ is invertible; and thus the Sherman-Morrison-Woodbury formula (see [6, p. 50]) gives

$$G^{-1}(s) = P^{-1}(s) - P^{-1}(s)\, G_{\rm sp}(s) \left[I + P^{-1}(s) G_{\rm sp}(s)\right]^{-1} P^{-1}(s), \tag{8.3}$$

whence

$$\operatorname{sr}\deg G(s) = -\operatorname{sr}\deg G^{-1}(s)\,. \tag{8.4}$$

To characterize transfer functions with proper inverse we first have to show the following technical lemma.

**Lemma 8.1.** *Suppose $E \in \mathbb{R}^{n,n}, B, C^\top \in \mathbb{R}^{n,m}$ satisfy*

$$\operatorname{im} E + \operatorname{im} B = \mathbb{R}^n, \qquad \operatorname{im} E^\top + \operatorname{im} C^\top = \mathbb{R}^n, \qquad \operatorname{rk}\, B = \operatorname{rk}\, C = m\,. \tag{8.5}$$

*Let* $m_1 := \dim(\operatorname{im} E \cap \operatorname{im} B)$, $r := n - m + m_1$ *and* $m_2 := m - m_1$. *Then*

(i) $\operatorname{im} E \cap \operatorname{im} B \neq \{0\} \iff \operatorname{im} E^\top \cap \operatorname{im} C^\top \neq \{0\} \iff \operatorname{rk}\, E > n - m\,,$

(ii) $\dim(\operatorname{im} E \cap \operatorname{im} B) = \dim(\operatorname{im} E^\top \cap \operatorname{im} C^\top)$.

(iii) *There exist $W_1, T_1 \in \mathbf{Gl}_n(\mathbb{R})$ and $W_2, T_2 \in \mathbf{Gl}_m(\mathbb{R})$, $B_{11} \in \mathbb{R}^{r,m_1}$, $C_{11} \in \mathbb{R}^{m_1,r}$ such that*

$$W_1 E T_1 = \begin{bmatrix} I_r & 0_{r,m_2} \\ 0_{m_2,r} & 0_{m_2,m_2} \end{bmatrix}, \quad W_1 B T_2 = \begin{bmatrix} B_{11} & 0_{r,m_2} \\ 0_{m_2,m_1} & I_{m_2} \end{bmatrix}, \quad W_2 C T_1 = \begin{bmatrix} C_{11} & 0_{m_1,m_2} \\ 0_{m_2,r} & I_{m_2} \end{bmatrix}. \tag{8.6}$$

(iv) *If $m_1 = 0$, then (8.6) reduces to*

$$W_1 E T_1 = \begin{bmatrix} I_{n-m} & 0_{n-m,m} \\ 0_{m,n-m} & 0_{m,m} \end{bmatrix}, \qquad W_1 B = \begin{bmatrix} 0_{n-m,m} \\ I_m \end{bmatrix}, \qquad C T_1 = \begin{bmatrix} 0_{m,n-m} & I_m \end{bmatrix}. \tag{8.7}$$

**Proof:** First note that (8.5) yields

$$\left.\begin{array}{rcl} n &=& \dim(\operatorname{im} E + \operatorname{im} B) = \operatorname{rk}\, E + \operatorname{rk}\, B - \dim(\operatorname{im} E \cap \operatorname{im} B) \\ &=& \operatorname{rk}\, E + m - \dim(\operatorname{im} E \cap \operatorname{im} B), \\ n &=& \dim(\operatorname{im} E^\top + \operatorname{im} C^\top) = \operatorname{rk}\, E^\top + \operatorname{rk}\, C^\top - \dim(\operatorname{im} E^\top \cap \operatorname{im} C^\top) \\ &=& \operatorname{rk}\, E^\top + m - \dim(\operatorname{im} E^\top \cap \operatorname{im} C^\top)\,, \end{array}\right\} \tag{8.8}$$

and hence assertions (i) and (ii) are immediate. We show (iii):
*Step 1*: We show

$$\exists\, W_1 \in \mathbf{Gl}_n(\mathbb{R})\ \exists\, T_2 \in \mathbf{Gl}_m(\mathbb{R})\ :\ W_1\, E = \begin{bmatrix} E_1 \\ 0_{n-r,n} \end{bmatrix}, \qquad W_1\, B\, T_2 = \begin{bmatrix} B_{11} & 0 \\ 0 & I_{m_2} \end{bmatrix}. \tag{8.9}$$

Note that (8.8) yields $\operatorname{rk} E = n - m + m_1 = r$. So

$$\exists\, W \in \mathbf{Gl}_n(\mathbb{R})\ \exists\, E_1 \in \mathbb{R}^{r,n}\ :\ WE = \begin{bmatrix} E_1 \\ 0 \end{bmatrix} \quad \text{and} \quad \operatorname{rk} E_1 = \operatorname{rk} E = r.$$

Then

$$\operatorname{im} \begin{bmatrix} E_1 \\ 0 \end{bmatrix} + \operatorname{im} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \stackrel{(8.5)}{=} \mathbb{R}^n, \qquad \text{where } \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = WB \ \text{ for } \ B_1 \in \mathbb{R}^{r,m},\ B_2 \in \mathbb{R}^{n-r,m}.$$

and so $\operatorname{im} B_2 = \mathbb{R}^{n-r}$ which gives

$$\exists\, T \in \mathbf{Gl}_m(\mathbb{R})\ \exists\, \tilde{B}_{22} \in \mathbf{Gl}_{n-r}(\mathbb{R})\ :\ WBT = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} T = \begin{bmatrix} \tilde{B}_{11} & \tilde{B}_{12} \\ \tilde{B}_{21} & \tilde{B}_{22} \end{bmatrix}$$

and hence (8.9) follows for

$$W_1 := \begin{bmatrix} I_r & -\tilde{B}_{12}\tilde{B}_{22}^{-1} \\ 0 & I_{n-r} \end{bmatrix} W \quad \text{and} \quad T_2 := T \begin{bmatrix} I_{m_1} & 0 \\ -\tilde{B}_{22}^{-1}\tilde{B}_{21} & \tilde{B}_{22}^{-1} \end{bmatrix}.$$

*Step 2*: We show (8.6). Since

$$\operatorname{im} E_1^\top + \operatorname{im} C^\top = \operatorname{im}[E_1^\top, 0] + \operatorname{im} C^\top = \operatorname{im}[E_1^\top, 0]W_1^\top + \operatorname{im} C^\top = \operatorname{im} E^\top + \operatorname{im} C^\top \stackrel{(8.5)}{=} \mathbb{R}^n,$$

we may apply Step 1 to $E_1^\top$ and $C^\top$ and conclude

$$\exists\, V \in \mathbf{Gl}_n(\mathbb{R})\ \exists\, W_2^\top \in \mathbf{Gl}_m(\mathbb{R})\ \exists\, E_2 \in \mathbf{Gl}_r(\mathbb{R})\ :\ VE_1^\top = \begin{bmatrix} E_2 \\ 0 \end{bmatrix}, \quad VC^\top W_2^\top = \begin{bmatrix} C_{11} & 0 \\ 0 & I_{m_2} \end{bmatrix}.$$

Therefore,

$$W_1 E V^\top = \begin{bmatrix} E_2^\top & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad W_2 C V^\top = \begin{bmatrix} C_{11}^\top & 0 \\ 0 & I_{m_2} \end{bmatrix}$$

and (8.6) follows for

$$T_1 := V^\top \begin{bmatrix} E_2^{-\top} & 0 \\ 0 & 0 \end{bmatrix}.$$

The statement (iv) follows from a careful inspection of the proof of (iii). □

**Proposition 8.2** (Transfer function with proper inverse)**.**
*Let $[E, A, B, C] \in \Sigma_{n,m,m}$ be controllable at infinity and observable at infinity. Then the following statements are equivalent:*

(i) *The transfer function $C(sE - A)^{-1}B$ has proper inverse.*

(ii) $\operatorname{im} E \oplus \operatorname{im} B = \mathbb{R}^n \quad \wedge \quad \operatorname{im} E^\top \oplus \operatorname{im} C^\top = \mathbb{R}^n \quad \wedge \quad \operatorname{rk} B = \operatorname{rk} C = m$ .

**Proof:** Note that

$$G(s) \text{ has inverse over } \mathbb{R}(s) \qquad \Longrightarrow \qquad \text{rk } B = \text{rk } C = m \tag{8.10}$$

and, by assumption,

$$\operatorname{im} E + \operatorname{im} B = \mathbb{R}^n \qquad \wedge \qquad \operatorname{im} E^\top + \operatorname{im} C^\top = \mathbb{R}^n \,. \tag{8.11}$$

"(i)⇒(ii)": By (8.10) and (8.11) and in view of Lemma 8.1(i), it remains to show that

$$\dim\left(\operatorname{im} E \cap \operatorname{im} B\right) =: m_1 = 0.$$

Seeking a contradiction, suppose that $m_1 > 0$. Then by Lemma 8.1(iii) there exist $W_1, T_1 \in \mathbf{Gl}_n(\mathbb{R})$ and $W_2, T_2 \in \mathbf{Gl}_m(\mathbb{R})$, such that, for $m_2 = m - m_1$, $r = n - m + m_1$ and some $B_{11} \in \mathbb{R}^{r,m_1}$, $C_{11} \in \mathbb{R}^{m_1,r}$, equation (8.6) holds. Define

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} := W_1 A T_2 \,, \qquad \text{where } \; A_{11} \in \mathbb{R}^{r,r} \; \text{ and } \; A_{12}, A_{21}, A_{22} \; \text{ accordingly}$$

we conclude

$$W_1\, G(s)\, T_2 = \begin{bmatrix} C_{11} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} sI - A_{11} & -A_{12} \\ -A_{21} & -A_{22} \end{bmatrix}^{-1} \begin{bmatrix} B_{11} & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix}$$

where, by invoking the matrix inversion formula [2, Prop. 2.8.7],

$$\begin{aligned}
G_{11}(s) &= C_{11}(sI_{n_2} - A_{11})^{-1}B_{11} + C_{11}(sI_{n_2} - A_{11})^{-1}A_{12}G_{22}(s)A_{21}(sI_{n_2} - A_{11})^{-1}B_{11} \\
G_{12}(s) &= C_{11}(sI_{n_2} - A_{11})^{-1}A_{12}G_{22}(s) \\
G_{21}(s) &= G_{22}(s)A_{21}(sI_{n_2} - A_{11})^{-1}B_{11} \\
G_{22}(s) &= -\left(A_{22} + A_{21}(sI_r - A_{11})^{-1}A_{12}\right)^{-1}.
\end{aligned}$$

A repeated application of the matrix inversion formula [2, Prop. 2.8.7] yields

$$(W_1\, G(s)\, T_2)^{-1} = \begin{bmatrix} H_{11}(s) & * \\ * & * \end{bmatrix}$$

with improper

$$H_{11}(s) = (G_{11}(s) - G_{12}(s)G_{22}^{-1}(s)G_{21}(s))^{-1} = (C_{11}(sI_{n_2} - A_{11})^{-1}B_{11})^{-1}.$$

Therefore, $(W_1\, G(s)\, T_2)^{-1}$ is improper and so is $G^{-1}(s)$. This contradicts (i).
"(ii)⇒(i)": Since Lemma 8.1(iv) holds, we may set

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} := W_1 A T_2 \,, \quad \text{where } \; A_{11} \in \mathbb{R}^{n-m,n-m} \; \text{ and } \; A_{12}, A_{21}, A_{22} \; \text{ accordingly.}$$

Then an application of the matrix inversion formula [2, Prop. 2.8.7] gives

$$C(sE - A)^{-1}B = \begin{bmatrix} 0 & I_m \end{bmatrix} \begin{bmatrix} sI - A_{11} & -A_{12} \\ -A_{21} & -A_{22} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ I_m \end{bmatrix} = -\left(A_{22} + A_{21}(sI_{n-m} - A_{11})^{-1}A_{12}\right)^{-1},$$

and thus

$$\lim_{s\to\infty} G^{-1}(s) = \lim_{s\to\infty} \left(-A_{22} - A_{21}(sI_{n_2} - A_{11})^{-1}A_{12}\right) = -A_{22}\,. \qquad \square$$

## 8.2 Proofs

This section contains all proofs of the statements in Sections 1–6.

**Proof of Proposition 1.2:**
Suppose (8.1) holds. Then (8.2) follows and hence $P^{-1}(s)$ exists and is proper and we may apply (8.3) to conclude that $G^{-1}(s)$ exists and is proper.
To see that the converse is, in general, false, consider the following counterexample:

$$[E, A, B, C] := \left[ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 2 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \right]$$

has transfer function

$$G(s) = C(sE - A)^{-1}B = C\,\frac{1}{2} \begin{bmatrix} 1-s & s+1 & -2 \\ s-2 & -(s+2) & 2 \\ -1 & 1 & 0 \end{bmatrix} B = \frac{1}{2} \begin{bmatrix} 1-s & s+1 \\ s-2 & -(s+2) \end{bmatrix}$$

and inverse transfer function

$$G^{-1}(s) = -\begin{bmatrix} 1+2/s & 1+1/s \\ 1-2/s & 1-1/s \end{bmatrix} \underset{s\to\infty}{\longrightarrow} \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}.$$

Therefore, $G^{-1}(s)$ exists and is proper; however, $G(s)$ does not have a strict relative degree since

$$s^{-1}G(s) \underset{s\to\infty}{\longrightarrow} \frac{1}{2} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \notin \mathbf{Gl}_2(\mathbb{R}).$$

This completes the proof. $\square$

**Proof of Theorem 2.3:**
We proceed in several steps.
*Step 1*: We show that there exist $W_2, T_3 \in \mathbf{Gl}_n(\mathbb{R})$ such that

$$[\,E\,,\,A\,,\,B\,,\,C\,] \overset{W_3,T_3}{\sim} \left[ \begin{bmatrix} \hat{E}_{11} & 0 & \hat{E}_{13} \\ \hat{E}_{21} & \hat{N}_{22} & \hat{E}_{23} \\ 0 & 0 & \hat{N}_{33} \end{bmatrix}, \begin{bmatrix} \hat{A}_{11} & 0 & 0 \\ 0 & I_{\hat{n}_2} & 0 \\ 0 & 0 & I_{\hat{n}_3} \end{bmatrix}, \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \\ 0 \end{bmatrix}, [\hat{C}_1, \;\; 0, \;\; \hat{C}_3] \right], \tag{8.12a}$$

$$\hat{N}_{22}^{\nu} = 0 \text{ and } \hat{N}_{33}^{\nu} = 0, \tag{8.12b}$$

$$[\hat{E}_{11},\, \hat{A}_{11},\, \hat{B}_1,\, \hat{C}_1] \in \Sigma_{\hat{n}_1,m,m} \text{ is controllable at infinity and observable at infinity.} \tag{8.12c}$$

Corollary 1.5 yields (1.3) for some $W_1, T_1 \in \mathbf{Gl}_n(\mathbb{R})$. It follows from [4, Sec. 2-5.] that the system (1.4b) may be decomposed into controllability and observability form so that, for some $T_2 \in \mathbf{Gl}_{n_f}(\mathbb{R})$,

$$[N, I_{n_f}, B_f, C_f] \overset{T_2^{-1},T_2}{\sim}$$
$$\left[ \left[\begin{array}{cc|c} N_{co} & 0 & N_{13} \\ N_{21} & N_{c\overline{o}} & N_{23} \\ \hline 0 & 0 & N_{\overline{c}} \end{array}\right], \left[\begin{array}{cc|c} I_{n_{f,co}} & 0 & 0 \\ 0 & I_{n_{f,c\overline{o}}} & 0 \\ \hline 0 & 0 & I_{n_{f,\overline{c}}} \end{array}\right], \left[\begin{array}{c} B_{f,co} \\ B_{f,c\overline{o}} \\ \hline 0 \end{array}\right], \left[\begin{array}{cc|c} C_{f,co}, & 0 & C_{f,\overline{c}} \end{array}\right] \right],$$

where $N_{\overline{c}} \in \mathbb{R}^{n_{f,\overline{c}},n_{f,\overline{c}}}$, $C_{f,\overline{c}} \in \mathbb{R}^{m,n_{f,\overline{c}}}$, $N_{c\overline{o}} \in \mathbb{R}^{n_{f,c\overline{o}},n_{f,c\overline{o}}}$, $N_{co} \in \mathbb{R}^{n_{f,co},n_{f,co}}$, $C_{f,co} \in \mathbb{R}^{m,n_{f,co}}$, $B_{f,co} \in \mathbb{R}^{n_{f,co},m}$ and $N_{co}^{\nu} = 0$, $N_{c\overline{o}}^{\nu} = 0$, $N_{\overline{c}}^{\nu} = 0$. The system $[N_{co}, I_{n_{f,co}}, B_{f,co}, C_{f,co}]$ is both controllable and observable; this is equivalent to

$$\operatorname{rk}[\,N_{co},\, B_{f,co}\,] = \operatorname{rk}\left(\begin{bmatrix} N_{co} \\ C_{f,co}\end{bmatrix}\right) = n_{f,co}\,. \tag{8.13}$$

Setting $W_3 := W_1 \begin{bmatrix} I_{n_s} & 0 \\ 0 & T_2^{-1}\end{bmatrix}$, $T_3 := \begin{bmatrix} I_{n_s} & 0 \\ 0 & T_2\end{bmatrix} T_1$, we arrive at

$$\begin{aligned}
&[\,E\,,\,A\,,\,B\,,\,C\,] \overset{W_3,T_3}{\sim} \\
&\left[\left[\begin{array}{cc|c|c} I_{n_s} & 0 & 0 & 0 \\ 0 & N_{co} & 0 & N_{13} \\ \hline 0 & N_{21} & N_{c\overline{o}} & N_{23} \\ \hline 0 & 0 & 0 & N_{\overline{c}} \end{array}\right], \left[\begin{array}{cc|c|c} A_s & 0 & 0 & 0 \\ 0 & I_{n_{f,co}} & 0 & 0 \\ \hline 0 & 0 & I_{n_{f,c\overline{o}}} & 0 \\ \hline 0 & 0 & 0 & I_{n_{f,\overline{c}}} \end{array}\right], \left[\begin{array}{c} B_s \\ B_{f,co} \\ \hline B_{f,c\overline{o}} \\ \hline 0 \end{array}\right], \left[\begin{array}{cc|c|c} C_s, & C_{f,co} & 0 & C_{f,\overline{c}} \end{array}\right]\right] \\
&=: \left[\begin{bmatrix} \hat{E}_{11} & 0 & \hat{E}_{13} \\ \hat{E}_{21} & \hat{N}_{22} & \hat{E}_{23} \\ 0 & 0 & \hat{N}_{33}\end{bmatrix}, \begin{bmatrix} \hat{A}_{11} & 0 & 0 \\ 0 & I_{\hat{n}_2} & 0 \\ 0 & 0 & I_{\hat{n}_3}\end{bmatrix}, \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \\ 0 \end{bmatrix}, [\hat{C}_1, \;\; 0, \;\; \hat{C}_3]\right]
\end{aligned} \tag{8.14}$$

and clearly (8.12b) holds true.

*Step 2*: We show that, for $\hat{n}_1 = n_s + n_{f,co}$,

$$\operatorname{im}\hat{E}_{11} \oplus \operatorname{im}\hat{B}_1 = \mathbb{R}^{\hat{n}_1} \quad \wedge \quad \operatorname{im}\hat{E}_{11}^{\top} \oplus \operatorname{im}\hat{C}_1^{\top} = \mathbb{R}^{\hat{n}_1} \quad \wedge \quad \operatorname{rk}\hat{B}_1 = \operatorname{rk}\hat{C}_1 = m. \tag{8.15}$$

Since the transfer function is invariant under system equivalence, an application of the matrix inversion formula [2, Prop. 2.8.7] to (8.14) yields

$$\begin{aligned}
C(sE-A)^{-1}B &= [\hat{C}_1, 0, \hat{C}_3] \left[\begin{array}{cc|c} s\hat{E}_{11} - \hat{A}_{11} & 0 & s\hat{E}_{13} \\ s\hat{E}_{21} & s\hat{N}_{22} - I_{\hat{n}_2} & s\hat{E}_{23} \\ \hline 0 & 0 & s\hat{N}_{33} - I_{\hat{n}_3} \end{array}\right]^{-1} \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \\ 0 \end{bmatrix} \\
&= [\hat{C}_1, 0, \hat{C}_3] \left[\begin{array}{c|c} \begin{bmatrix} s\hat{E}_{11} - \hat{A}_{11} & 0 \\ s\hat{E}_{21} & s\hat{N}_{22} - I_{\hat{n}_2} \end{bmatrix}^{-1} & \begin{matrix} * \\ * \end{matrix} \\ \hline \begin{matrix} 0 & \quad 0 \end{matrix} & * \end{array}\right] \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \\ 0 \end{bmatrix} \\
&= [\hat{C}_1, 0] \begin{bmatrix} (s\hat{E}_{11} - \hat{A}_{11})^{-1} & 0 \\ * & (s\hat{N}_{22} - I_{\hat{n}_2})^{-1} \end{bmatrix} \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \end{bmatrix} \\
&= \hat{C}_1 (s\hat{E}_{11} - \hat{A}_{11})^{-1} \hat{B}_1.
\end{aligned}$$

Since (8.12c) holds and $C(sE-A)^{-1}B$ has proper inverse by assumption, we may apply Proposition 8.2 to conclude (8.15).

*Step 3*: We show that $[E, A, B, C]$ is system equivalent to an DAE in zero dynamics form (2.1).
Since (8.15) holds and $\operatorname{rk}\hat{B}_1 = \operatorname{rk}\hat{C}_1 = m$, we may apply Lemma 8.1(iv) and multiply the matrices in (8.7) by permutation matrices so that

$$\hat{W}\hat{E}_{11}\hat{T} = \begin{bmatrix} 0_{m,m} & 0_{m,\hat{n}_1-m} \\ 0_{\hat{n}_1-m,m} & I_{\hat{n}_1-m}\end{bmatrix}, \; \hat{W}\hat{B}_1 = \begin{bmatrix} I_m \\ 0_{\hat{n}_1-m,m}\end{bmatrix}, \; \hat{C}_1\hat{T} = \begin{bmatrix} I_m, & 0_{m,\hat{n}_1-m}\end{bmatrix} \text{ for some } \hat{W}, \hat{T} \in \mathbf{Gl}_{\hat{n}_1}(\mathbb{R}).$$

Partitioning the matrices

$$\hat{W}\hat{A}_{11}\hat{T} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{Q} \end{bmatrix}, \quad \hat{W}\hat{E}_{13} = \begin{bmatrix} \tilde{E}_{14} \\ \tilde{E}_{24} \end{bmatrix}, \quad \hat{E}_{23} = \tilde{E}_{34}, \quad \hat{B}_2 = \tilde{B}_3,$$
$$\hat{E}_{21}\hat{T} = \begin{bmatrix} \tilde{E}_{31}, & \tilde{E}_{32} \end{bmatrix}, \qquad \hat{N}_{22} = \tilde{N}_{33}, \qquad \hat{N}_{33} = \tilde{N}_{44}, \quad \hat{C}_3 = \tilde{C}_4 \tag{8.16}$$

accordingly and setting

$$W := W_3 \begin{bmatrix} \hat{W}^{-1} & 0 \\ 0 & I_{\hat{n}_2+\hat{n}_3} \end{bmatrix}, \qquad T := \begin{bmatrix} \hat{T}^{-1} & 0 \\ 0 & I_{\hat{n}_2+\hat{n}_3} \end{bmatrix} T_3 \,,$$

we obtain that

$$[E, A, B, C] \overset{W,T}{\sim} [\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}]$$

for

$$[\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}] := \left[ \begin{bmatrix} 0 & 0 & 0 & \tilde{E}_{14} \\ 0 & I_{\tilde{n}_2} & 0 & \tilde{E}_{24} \\ \tilde{E}_{31} & \tilde{E}_{32} & \tilde{N}_{33} & \tilde{E}_{34} \\ 0 & 0 & 0 & \tilde{N}_{44} \end{bmatrix}, \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} & 0 & 0 \\ \tilde{A}_{21} & \tilde{Q} & 0 & 0 \\ 0 & 0 & I_{\tilde{n}_3} & 0 \\ 0 & 0 & 0 & I_{\tilde{n}_4} \end{bmatrix}, \begin{bmatrix} I_m \\ 0_{\tilde{n}_2,m} \\ \tilde{B}_3 \\ 0_{\tilde{n}_4,m} \end{bmatrix}, \begin{bmatrix} I_m, 0_{m,\tilde{n}_2}, 0_{m,\tilde{n}_3}, \tilde{C}_4 \end{bmatrix} \right]$$

In the following steps $a)$-$d)$, we show successively that an equivalence action $\overset{W_i,T_i}{\sim}$ applied to $[\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}]$ removes $\tilde{E}_{32}$, $\tilde{E}_{24}$, $\tilde{B}_3$ and $\tilde{C}_4$.
*3a)* $[\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}] \overset{W_a,T_a}{\sim} [\tilde{E}_a, \tilde{A}_a, \tilde{B}_a, \tilde{C}_a]$ for

$$W_a = \begin{bmatrix} I_m & 0 & 0 & -\tilde{A}_{11}\tilde{C}_4 \\ 0 & I_{\tilde{n}_2} & 0 & -\tilde{A}_{21}\tilde{C}_4 \\ 0 & 0 & I_{\tilde{n}_3} & 0 \\ 0 & 0 & 0 & I_{\tilde{n}_4} \end{bmatrix}, \qquad T_a = \begin{bmatrix} I_m & 0 & 0 & \tilde{C}_4 \\ 0 & I_{\tilde{n}_2} & 0 & 0 \\ 0 & 0 & I_{\tilde{n}_3} & 0 \\ 0 & 0 & 0 & I_{\tilde{n}_4} \end{bmatrix},$$

and so $[\tilde{E}_a, \tilde{A}_a, \tilde{B}_a, \tilde{C}_a]$ preserves the structure of $[\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}]$ and changes $\tilde{C}_4$ to $0_{m,\tilde{n}_4}$.
*3b)* $[\tilde{E}_a, \tilde{A}_a, \tilde{B}_a, \tilde{C}_a] \overset{W_b,T_b}{\sim} [\tilde{E}_b, \tilde{A}_b, \tilde{B}_b, \tilde{C}_b]$ for

$$W_b = \begin{bmatrix} I_m & 0 & 0 & 0 \\ 0 & I_{\tilde{n}_2} & 0 & 0 \\ \tilde{B}_3 & 0 & I_{\tilde{n}_3} & 0 \\ 0 & 0 & 0 & I_{\tilde{n}_4} \end{bmatrix}, \qquad T_b = \begin{bmatrix} I_m & 0 & 0 & 0 \\ 0 & I_{\tilde{n}_2} & 0 & 0 \\ -\tilde{B}_3\tilde{A}_{11} & -\tilde{B}_3\tilde{A}_{12} & I_{\tilde{n}_3} & 0 \\ 0 & 0 & 0 & I_{\tilde{n}_4} \end{bmatrix},$$

and so $[\tilde{E}_b, \tilde{A}_b, \tilde{B}_b, \tilde{C}_b]$ preserves the structure of $[\tilde{E}_a, \tilde{A}_a, \tilde{B}_a, \tilde{C}_a]$ and changes $\tilde{B}_3$ to $0_{\tilde{n}_3,m}$.
*3c)* $[\tilde{E}_b, \tilde{A}_b, \tilde{B}_b, \tilde{C}_b] \overset{W_c,T_c}{\sim} [\tilde{E}_c, \tilde{A}_c, \tilde{B}_c, \tilde{C}_c]$ for

$$W_c = \begin{bmatrix} I_m & 0 & 0 & 0 \\ 0 & I_{\tilde{n}_2} & 0 & 0 \\ 0 & -L_{32} & I_{\tilde{n}_3} & 0 \\ 0 & 0 & 0 & I_{\tilde{n}_4} \end{bmatrix}, \qquad T_c = \begin{bmatrix} I_m & 0 & 0 & 0 \\ 0 & I_{\tilde{n}_2} & 0 & 0 \\ L_{32}\tilde{A}_{21} & L_{32}\tilde{Q} & I_{\tilde{n}_3} & 0 \\ 0 & 0 & 0 & I_{\tilde{n}_4} \end{bmatrix}, \qquad L_{32} = \sum_{k=0}^{\nu-1} \tilde{N}_{33}^k \tilde{E}_{32} \tilde{Q}^k,$$

and so $[\tilde{E}_c, \tilde{A}_c, \tilde{B}_c, \tilde{C}_c]$ preserves the structure of $[\tilde{E}_b, \tilde{A}_b, \tilde{B}_b, \tilde{C}_b]$ and changes $\tilde{E}_{32}$ to $0_{\tilde{n}_3,\tilde{n}_2}$.
*3d)* $[\tilde{E}_c, \tilde{A}_c, \tilde{B}_c, \tilde{C}_c] \overset{W_d,T_d}{\sim} [\tilde{E}_d, \tilde{A}_d, \tilde{B}_d, \tilde{C}_d]$ for

$$W_d = \begin{bmatrix} I_m & 0 & 0 & \tilde{A}_{12}L_{24} \\ 0 & I_{\tilde{n}_2} & 0 & \tilde{Q}L_{24} \\ 0 & 0 & I_{\tilde{n}_3} & 0 \\ 0 & 0 & 0 & I_{\tilde{n}_4} \end{bmatrix}, \qquad T_d = \begin{bmatrix} I_m & 0 & 0 & 0 \\ 0 & I_{\tilde{n}_2} & 0 & -L_{24} \\ 0 & 0 & I_{\tilde{n}_3} & 0 \\ 0 & 0 & 0 & I_{\tilde{n}_4} \end{bmatrix}, \qquad L_{24} = \sum_{k=0}^{\nu-1} \tilde{Q}^k \tilde{E}_{24} \tilde{N}_{44}^k,$$

and so $[\tilde E_d,\tilde A_d,\tilde B_d,\tilde C_d]$ preserves the structure of $[\tilde E_c,\tilde A_c,\tilde B_c,\tilde C_c]$ and changes $\tilde E_{24}$ to $0_{\tilde n_2,\tilde n_4}$. Therefore,

$$[\tilde E_d,\tilde A_d,\tilde B_d,\tilde C_d]=\left[\begin{bmatrix}0&0&0&\tilde E^d_{14}\\0&I_{\tilde n_2}&0&0\\\tilde E^d_{31}&0&\tilde N^d_{33}&\tilde E^d_{34}\\0&0&0&\tilde N^d_{44}\end{bmatrix},\begin{bmatrix}\tilde A^d_{11}&\tilde A^d_{12}&0&0\\\tilde A^d_{21}&\tilde Q^d&0&0\\0&0&I_{\tilde n_3}&0\\0&0&0&I_{\tilde n_4}\end{bmatrix},\begin{bmatrix}I_m\\0_{\tilde n_2,m}\\0_{\tilde n_3,m}\\0_{\tilde n_4,m}\end{bmatrix},[I_m,0_{m,\tilde n_2},0_{m,\tilde n_3},0_{m,\tilde n_4}]\right]$$

has block structure as in (2.1); however, it remains to show that the rank condition on $[\tilde E^d_{31},\tilde N^d_{33}]$ holds. This is, in general, not the case and a further transformation is required.
By [4, Sec. 2-5.], we may transform $[\tilde N^d_{33},I_{\tilde n_3},\tilde E^d_{31},0]$ into controllability form; that means there exist $T_e\in\mathbf{Gl}_{\tilde n_3}(\mathbb R)$, nilpotent $N_c\in\mathbb R^{n_{3c},n_{3c}}$, $N_{\bar c}\in\mathbb R^{n_{3\bar c},n_{3\bar c}}$, and matrices $N_{12}\in\mathbb R^{n_{3c},n_{3\bar c}}$, $E_{31}\in\mathbb R^{n_{3c},m}$ such that

$$[\tilde N^d_{33},I_{\tilde n_3},\tilde E^d_{31},0]\overset{T_e^{-1},T_e}{\sim}\left[\begin{bmatrix}N_c&N_{12}\\0&N_{\bar c}\end{bmatrix},\begin{bmatrix}I_{n_{3c}}&0\\0&I_{n_{3\bar c}}\end{bmatrix},\begin{bmatrix}E_{31}\\0\end{bmatrix},\begin{bmatrix}0,&0\end{bmatrix}\right],\qquad \operatorname{rk}[\,E_{31}\,,\,N_c\,]=n_{3c}.$$

Finally,

$$[\tilde E_d,\tilde A_d,\tilde B_d,\tilde C_d]\overset{\breve W,\breve T}{\sim}[\breve E,\breve A,\breve B,\breve C],\quad\text{for}\quad \breve W^{-1}=\breve T=\begin{bmatrix}I_m&0&0&0\\0&I_{\tilde n_2}&0&0\\0&0&T_e&0\\0&0&0&I_{\tilde n_4}\end{bmatrix}$$

where $[\breve E,\breve A,\breve B,\breve C]$ is in zero dynamics form (2.1) and satisfies $\breve n_2=\tilde n_2$, $\breve n_3=n_{3c}$, $\breve n_4=n_{3\bar c}+\tilde n_4$, $[\breve E_{31},\breve N_{33}]=[E_{31},N_c]$, $\breve N_{44}=\begin{bmatrix}N_{\bar c}&*\\0&\tilde N_{44}\end{bmatrix}$, and $\operatorname{rk}[\breve E_{31},\breve N_{33}]=\breve n_3$. This proves the claim of Step 3.

*Step 4*: Assertion (i) follows from (8.12b).

*Step 5*: We show assertion (ii). Very similar to Step 2, we apply the matrix inversion formula [2, Prop. 2.8.7] to (2.1) to conclude

$$\begin{aligned}C(sE-A)^{-1}B&=\begin{bmatrix}I,&0,&0,&0\end{bmatrix}\left[\begin{array}{ccc|c}-A_{11}&-A_{12}&0&sE_{14}\\-A_{21}&sI-Q&0&0\\sE_{31}&0&sN_{33}-I&sE_{34}\\\hline 0&0&0&sN_{44}-I\end{array}\right]^{-1}\begin{bmatrix}I\\0\\0\\0\end{bmatrix}\\&=\begin{bmatrix}I,&0,&0,&0\end{bmatrix}\left[\begin{array}{c|c}\begin{bmatrix}-A_{11}&-A_{12}&0\\-A_{21}&sI-Q&0\\sE_{31}&0&sN_{33}-I\end{bmatrix}^{-1}&\begin{matrix}*\\*\\*\end{matrix}\\\hline \begin{matrix}0&0&0\end{matrix}&(sN_{44}-I)^{-1}\end{array}\right]\begin{bmatrix}I\\0\\0\\0\end{bmatrix}\\&=\begin{bmatrix}I,&0\end{bmatrix}\begin{bmatrix}-A_{11}&-A_{12}\\-A_{21}&sI-Q\end{bmatrix}^{-1}\begin{bmatrix}I\\0\end{bmatrix}\\&=-\big(A_{11}+A_{12}(sI_{n_2}-Q)^{-1}A_{21}\big)^{-1}.\end{aligned}$$

*Step 6*: We show assertions (iii) and (iv). To see "$\Rightarrow$", observe that (8.4) yields that $\big(C(sE-A)^{-1}B\big)^{-1}$ has strict relative degree $-\rho\ge 0$. If $\rho=0$, then $A_{11}\in\mathbf{Gl}_m(\mathbb R)$; if $-\rho\ge 1$, then $A_{11}=0$ and $A_{12}(sI-Q)^{-1}A_{21}$ has strict relative degree $-\rho$. "$\Leftarrow$" is straightforward.

*Step 7*: We show assertions (v). Suppose (2.3) holds and the two systems in zero dynamics form (2.1):

$$[E, A, B, C] = \left[ \begin{bmatrix} 0 & 0 & 0 & E_{14} \\ 0 & I_{n_2} & 0 & 0 \\ E_{31} & 0 & N_{33} & E_{34} \\ 0 & 0 & 0 & N_{44} \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12} & 0 & 0 \\ A_{21} & Q & 0 & 0 \\ 0 & 0 & I_{n_3} & 0 \\ 0 & 0 & 0 & I_{n_4} \end{bmatrix}, \begin{bmatrix} I_m \\ 0_{n_2,m} \\ 0_{n_3,m} \\ 0_{n_4,m} \end{bmatrix}, [I_m,\ 0_{m,n_2},\ 0_{m,n_3},\ 0_{m,n_4}] \right]$$

$$[\hat{E}, \hat{A}, \hat{B}, \hat{C}] = \left[ \begin{bmatrix} 0 & 0 & 0 & \hat{E}_{14} \\ 0 & I_{\hat{n}_2} & 0 & 0 \\ \hat{E}_{31} & 0 & \hat{N}_{33} & \hat{E}_{34} \\ 0 & 0 & 0 & \hat{N}_{44} \end{bmatrix}, \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} & 0 & 0 \\ \hat{A}_{21} & \hat{Q} & 0 & 0 \\ 0 & 0 & I_{\hat{n}_3} & 0 \\ 0 & 0 & 0 & I_{\hat{n}_4} \end{bmatrix}, \begin{bmatrix} I_m \\ 0_{\hat{n}_2,m} \\ 0_{\hat{n}_3,m} \\ 0_{\hat{n}_4,m} \end{bmatrix}, [I_m,\ 0_{m,\hat{n}_2},\ 0_{m,\hat{n}_3},\ 0_{m,\hat{n}_4}] \right]$$

where

$$\begin{aligned} &E_{14} \in \mathbb{R}^{m,n_4}, \quad E_{31} \in \mathbb{R}^{n_3,m}, \quad N_{33} \in \mathbb{R}^{n_3,n_3}, \quad E_{34} \in \mathbb{R}^{n_3,n_4}, \quad N_{44} \in \mathbb{R}^{n_4,n_4}, \\ &A_{11} \in \mathbb{R}^{m,m}, \quad Q \in \mathbb{R}^{n_2,n_2}, \quad A_{12} \in \mathbb{R}^{m,n_2}, \quad A_{21} \in \mathbb{R}^{n_2,m}, \\ &\hat{E}_{14} \in \mathbb{R}^{m,\hat{n}_4}, \quad \hat{E}_{31} \in \mathbb{R}^{\hat{n}_3,m}, \quad \hat{N}_{33} \in \mathbb{R}^{\hat{n}_3,\hat{n}_3}, \quad \hat{E}_{34} \in \mathbb{R}^{\hat{n}_3,\hat{n}_4}, \quad \hat{N}_{44} \in \mathbb{R}^{\hat{n}_4,\hat{n}_4}, \\ &\hat{A}_{11} \in \mathbb{R}^{m,m}, \quad \hat{Q} \in \mathbb{R}^{\hat{n}_2,\hat{n}_2}, \quad \hat{A}_{12} \in \mathbb{R}^{m,\hat{n}_2}, \quad \hat{A}_{21} \in \mathbb{R}^{\hat{n}_2,m}, \end{aligned}$$

such that $N_{33}$, $N_{44}$, $\hat{N}_{33}$, $\hat{N}_{44}$ are nilpotent and $\operatorname{rk}[\, E_{31}\,,\, N_{33}\,] = n_3$, $\operatorname{rk}[\, \hat{E}_{31}\,,\, \hat{N}_{33}\,] = \hat{n}_3$.
The equations $WB = \hat{B}$, $CT = \hat{C}$ give

$$W = \begin{bmatrix} I_m & W_{12} & W_{13} & W_{14} \\ 0 & W_{22} & W_{23} & W_{24} \\ 0 & W_{32} & W_{33} & W_{34} \\ 0 & W_{42} & W_{43} & W_{44} \end{bmatrix}, \quad T^{-1} = \begin{bmatrix} I_m & 0 & 0 & 0 \\ T_{21} & T_{22} & T_{23} & T_{24} \\ T_{31} & T_{32} & T_{33} & T_{34} \\ T_{41} & T_{42} & T_{43} & T_{44} \end{bmatrix}$$

for some matrices $W_{ij}$, $T_{ij}$ of dimensions corresponding to the partitioning of $E$ and $\hat{E}$, resp.
In the following, we investigate the $(i, j)$-th block, $i, j \in \{1, 2, 3, 4\}$, in the matrix equations $WE = \hat{E}T^{-1}$ and $WA = \hat{A}T^{-1}$: Block $(2, 2)$ of $WE = \hat{E}T^{-1}$ gives $W_{22} = T_{22}$, and blocks $(3, 3)$, $(3, 4)$, $(4, 3)$ and $(4, 4)$ of $WA = \hat{A}T^{-1}$ yield

$$W_{33} = T_{33}, \quad W_{34} = T_{34}, \quad W_{43} = T_{43}, \quad W_{44} = T_{44}.$$

Block $(2, 3)$ of $WA = \hat{A}T^{-1}$ and $WE = \hat{E}T^{-1}$ yield $W_{23} = \hat{Q}T_{23}$ and $W_{23}N_{33} = T_{23}$. Hence,

$$W_{23} = \hat{Q}W_{23}N_{33} = \ldots = \hat{Q}^{\nu}W_{23}N_{33}^{\nu} = 0 \quad \text{and} \quad T_{23} = W_{23}\hat{N}_{33} = 0.$$

By the same argument, block $(2, 4)$ of $WE = \hat{E}T^{-1}$ gives $W_{24} = T_{24} = 0$,
block $(4, 2)$ of $WE = \hat{E}T^{-1}$ gives $W_{42} = T_{42} = 0$,
block $(3, 2)$ of $WE = \hat{E}T^{-1}$ gives $W_{32} = T_{32} = 0$,
blocks $(1, 3)$ and $(1, 4)$ of $WA = \hat{A}T^{-1}$ give $W_{13} = \hat{A}_{12}T_{23} = 0$ and $W_{14} = \hat{A}_{12}T_{24} = 0$,
blocks $(3, 1)$ and $(4, 1)$ of $WA = \hat{A}T^{-1}$ give $T_{31} = W_{32}A_{21} = 0$ and $T_{41} = W_{42}A_{21} = 0$,
blocks $(2, 1)$ and $(1, 2)$ of $WE = \hat{E}T^{-1}$ give $W_{12} = \hat{E}_{14}T_{42} = 0$ and $T_{21} = W_{23}E_{31} = 0$.
blocks $(4, 1)$ and $(4, 3)$ of $WE = \hat{E}T^{-1}$ give $W_{43}E_{31} = 0$ and $W_{43}N_{33} = \hat{N}_{44}T_{43} = \hat{N}_{44}W_{43}$.
Hence, for $k = 1, \ldots, \nu - 1$, we have $W_{43}N_{33}^{k}E_{31} = \hat{N}_{44}^{k}W_{43}E_{31} = 0$. Therefore,

$$W_{43}\underbrace{[E_{31}\,,\, N_{33}E_{31}\,,\, \ldots\,,\, N_{33}^{\nu-1}E_{31}\,]}_{=:K_\infty} = 0.$$

By [4, Thm. 2-2.1], the matrix $K_\infty$ has full row rank if, and only if, $[\,E_{31}\,,\,N_{33}\,]$ has full row rank; so the assumption $\operatorname{rk}[\,E_{31}\,,\,N_{33}\,]=n_3$ on the zero dynamics form yields to $W_{43}=0$. Therefore, $W$ has the structure as indicated in (2.4) and it remains to show that $n_i=\hat{n}_i$ for $i=2,3,4$. Since $W\in\mathbf{Gl}_n(\mathbb{R})$, we obtain $n_2=\hat{n}_2$ and

$$\begin{bmatrix} W_{33} & W_{34}\\ 0 & W_{44}\end{bmatrix}\in\mathbf{Gl}_{n_3+n_4}(\mathbb{R})\quad\text{for}\quad W_{33}\in\mathbb{R}^{\hat{n}_3,n_3},\ W_{34}\in\mathbb{R}^{\hat{n}_3,n_4},\ W_{44}\in\mathbb{R}^{\hat{n}_4,n_4}.$$

Hence, $W_{33}$ has full column rank and $W_{44}$ has full row rank, whence $n_3\le\hat{n}_3$ and $n_4\ge\hat{n}_4$. Reversing the roles of $[E,A,B,C]$ and $[\hat{E},\hat{A},\hat{B},\hat{C}]$, we obtain $\hat{n}_3\le n_3$ and $\hat{n}_4\ge n_4$ and, thus, $n_3=\hat{n}_3$, $n_4=\hat{n}_4$. This shows $W_{ii}\in\mathbf{Gl}_{n_i}(\mathbb{R})$ for $i=2,3,4$.

*Step 8*: We show assertion (vi). The fact $W_{ii}\in\mathbf{Gl}_{n_i}(\mathbb{R})$ yields $n_i=\hat{n}_i$ for $i=2,3,4$, resp. From (2.3) and (2.4) we see that

$$W_{33}N_{33}=\hat{N}_{33}W_{33},\quad W_{44}N_{44}=\hat{N}_{44}W_{44},\quad W_{22}Q=\hat{Q}W_{22}, \tag{8.17a}$$

$$W_{33}E_{31}=\hat{E}_{31},\quad E_{14}=\hat{E}_{14}W_{44},\quad W_{22}A_{21}=\hat{A}_{21},\quad A_{12}=\hat{A}_{12}W_{22}, \tag{8.17b}$$

$$W_{33}E_{34}+W_{34}N_{44}=\hat{N}_{33}W_{34}+\hat{E}_{34}W_{44}. \tag{8.17c}$$

Now (8.17a) shows that $N_{33}$, $N_{44}$ and $Q$ are unique up to similarity. Finally, the formula for $A_{11}$ in (vi) follows from (ii). This completes the proof. □

**Proof of Proposition 4.2:**
First note that for $[\hat{E},\hat{A},\hat{B},\hat{C}]$ in zero dynamics form (2.1) we have

$$T\,\mathcal{V}^*(A,E,B;\ker C)\;=\;\mathcal{V}^*(\hat{A},\hat{E},\hat{B};\ker\hat{C})$$

and therefore it suffices to show that

$$\mathcal{V}^*(\hat{A},\hat{E},\hat{B};\ker\hat{C})=\operatorname{im}V\qquad\text{for}\quad V:=\begin{bmatrix}0_{m,n_2}\\ I_{n_2}\\ 0_{n_3,n_2}\\ 0_{n_4,n_2}\end{bmatrix}. \tag{8.18}$$

First note that $\operatorname{im}V\subseteq\ker\hat{C}=\ker[I_m,0_{m,n_2},0_{m,n_3},0_{m,n_4}]$ and $(\hat{A},\hat{E},\hat{B})$-invariance of $\operatorname{im}V$ follows from

$$\operatorname{im}\begin{bmatrix}A_{12}I_{n_2}\\ QI_{n_2}\\ 0_{n_3,n_2}\\ 0_{n_4,n_2}\end{bmatrix}\subseteq\operatorname{im}\begin{bmatrix}0_{n_2,n_2}\\ I_{n_2}\\ 0_{n_3,n_2}\\ 0_{n_4,n_2}\end{bmatrix}+\operatorname{im}\begin{bmatrix}I_m\\ 0_{n_2,m}\\ 0_{n_3,m}\\ 0_{n_4,m}\end{bmatrix}. \tag{8.19}$$

We prove next that $\operatorname{im}\tilde{V}\subseteq\operatorname{im}V$ for any $\tilde{V}\in\mathbb{R}^{n,k}$, $k\in\mathbb{N}$, such that $\operatorname{im}\tilde{V}$ is $(\hat{A},\hat{E},\hat{B})$-invariant and $\operatorname{im}\tilde{V}\subseteq\ker\hat{C}$.
Let $[V_1^\top,V_2^\top,V_3^\top,V_4^\top]^\top:=\tilde{V}$ for $V_1\in\mathbb{R}^{m,k},V_2\in\mathbb{R}^{n_2,k},V_3\in\mathbb{R}^{n_3,k},V_4\in\mathbb{R}^{n_4,k}$. $(\hat{A},\hat{E},\hat{B})$-invariance of $\operatorname{im}\tilde{V}$ yields

$$\operatorname{im}\begin{bmatrix}A_{11}V_1+A_{12}V_2\\ A_{21}V_1+QV_2\\ V_3\\ V_4\end{bmatrix}\subseteq\operatorname{im}\begin{bmatrix}E_{14}V_4\\ V_2\\ E_{31}V_1+N_{33}V_3+E_{34}V_4\\ N_{44}V_4\end{bmatrix}+\operatorname{im}\begin{bmatrix}I_m\\ 0_{n_2,m}\\ 0_{n_3,m}\\ 0_{n_4,m}\end{bmatrix}. \tag{8.20}$$

The assumption $\operatorname{im}\tilde{V} \subseteq \ker \hat{C}$ gives $V_1 = 0$. Then $\operatorname{im} V_4 \subseteq \operatorname{im} N_{44}V_4$ and an iterative application of the inclusion leads to

$$\operatorname{im} V_4 \subseteq \operatorname{im} N_{44}V_4 \subseteq \ldots \subseteq \operatorname{im} N_{44}^{\nu}V_4 = \{0\}.$$

Therefore, $V_4 = 0$ and (8.20) now gives $\operatorname{im} V_3 \subseteq N_{33}V_3$. Repeating the argument used for $V_4$ we find $V_3 = 0$, thus

$$\forall\, x \in \mathbb{R}^k : \ \tilde{V}x = \begin{bmatrix} 0 \\ I_{n_2} \\ 0 \\ 0 \end{bmatrix} V_2 x \ \in \operatorname{im} V.$$

This completes the proof. □

For the proof of Theorem 5.4 we first show the following technical lemma.

**Lemma 8.3.** *Consider* $[\,I_n,\, A,\, B,\, C\,] \in \Sigma_{n,m,p}$ *and assume that* $\mu \in \sigma(A)$ *is not a pole of* $C(sI - A)^{-1}B \in \mathbb{R}(s)^{p,m}$*. Then*

$$\operatorname{rk}\,[\mu I - A,\, B] < n \qquad \vee \qquad \operatorname{rk}\,[\mu I - A^\top,\, C^\top] < n\,.$$

**Proof:** Without loss of generality, we assume that $A$ is in Jordan form and $A, B, C$ are partitioned as follows

$$A = \begin{bmatrix} \lambda_1 I_{n_1}{+}N_1 & & \\ & \ddots & \\ & & \lambda_k I_{n_k}{+}N_k \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ \vdots \\ B_k \end{bmatrix}, \quad C = \begin{bmatrix} C_1, & \ldots, & C_k \end{bmatrix},$$

where $\sigma(A) = \{\lambda_1, \ldots, \lambda_k\}$, $\mu = \lambda_1$ and $N_1, \ldots, N_k$ are nilpotent with indices of nilpotency $\nu_1, \ldots, \nu_k$ and appropriate formats. Then

$$C(sI - A)^{-1}B \ = \ \sum_{i=1}^{k} \sum_{j=0}^{\nu_i - 1} \frac{C_i N_i^j B_i}{(s - \lambda_i)^j}$$

and the set of poles of $C(sI - A)^{-1}B$ is given by

$$\left\{ \ \lambda_i \in \sigma(A) \ \middle| \ i \in \{1\ldots,k\} \ \wedge \ \exists\, j \in \{0, \ldots, \nu_i - 1\} \ : \ C_i N_i^j B_i \neq 0_{p,m} \ \right\}.$$

Suppose $\mu$ is not a pole of $C(sI - A)^{-1}B$ and $\operatorname{rk}\,[\mu I - A^\top,\, C^\top] = n$; then $C_1 N_1^{\nu_1 - 1} B_1 = 0$ and $\operatorname{rk}\,[N_1^\top,\, C_1^\top] = n_1$. Since

$$\begin{bmatrix} N_1 \\ C_1 \end{bmatrix} (N_1^{\nu_1 - 1} B_1) = 0\,,$$

we conclude $N_1^{\nu_1 - 1}B_1 = 0$ and so

$$N_1^{\nu_1 - 1} \cdot [\,N_1\,,\, B_1\,] = 0\,.$$

Since $N_1^{\nu_1 - 1} \neq 0$, it follows that $\operatorname{rk}\,[N_1,\, B_1] < n_1$ and thus $\operatorname{rk}\,[\mu I - A,\, B] < n$.
Analogously, one may show that '$\mu$ is not a pole of $C(sI - A)^{-1}B$ and $\operatorname{rk}\,[\mu I - A,\, B] = n$' yields $\operatorname{rk}\,[\mu I - A^\top,\, C^\top] < n$; this is omitted. This completes the proof of the lemma. □

**Proof of Theorem 5.4:**
It is readily verified that without restriction of generality we may assume that system $[E, A, B, C]$ is

in zero dynamics form (2.1).
"(i)⇔(ii)": Since $\det(sN - I_k) = (-1)^k$ for any nilpotent $N \in \mathbb{R}^{k,k}$, we have

$$\det \begin{bmatrix} -A_{11} & -A_{12} & 0 & sE_{14} & I \\ -A_{21} & sI - Q & 0 & 0 & 0 \\ sE_{31} & 0 & sN_{33} - I & sE_{34} & 0 \\ 0 & 0 & 0 & sN_{44} - I & 0 \\ I & 0 & 0 & 0 & 0 \end{bmatrix} = (-1)^m \det \begin{bmatrix} I & -A_{12} & 0 & sE_{14} & -A_{11} \\ 0 & sI - Q & 0 & 0 & -A_{21} \\ 0 & 0 & sN_{33} - I & sE_{34} & sE_{31} \\ 0 & 0 & 0 & sN_{44} - I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix}$$
$$= (-1)^{m+n_3+n_4} \cdot \det(sI_{n_2} - Q)\,.$$

Now the claim is a consequence of Corollary 5.2.
"(i)⇒(iii)(a)": Suppose that, for some $\lambda \in \overline{\mathbb{C}}_+$ and $v_1 \in \mathbb{C}^{1,n_2}$, $v_2 \in \mathbb{C}$, $v_3 \in \mathbb{C}^{1,n_3}$, $v_4 \in \mathbb{C}^{1,n_4}$, we have

$$\begin{bmatrix} v_1, & v_2, & v_3, & v_4 \end{bmatrix} \begin{bmatrix} -A_{11} & -A_{12} & 0 & \lambda E_{14} & I \\ -A_{21} & \lambda I - Q & 0 & 0 & 0 \\ \lambda E_{31} & 0 & \lambda N_{33} - I & \lambda E_{34} & 0 \\ 0 & 0 & 0 & \lambda N_{44} - I & 0 \end{bmatrix} = 0. \tag{8.21}$$

Then $v_1 = 0$, $v_3 = 0$ and thus $v_4 = 0$. Equation $v_2(\lambda I - Q) = 0$ yields, since $\sigma(Q) \subset \mathbb{C}_-$ holds by assumption and Corollary 5.2, that $v_2 = 0$.
"(i)⇒(iii)(b)": Detectability of the system can be shown similarly and is omitted.
"(i)⇒(iii)(c)": First note that if $G(s) \in \mathbf{Gl}_m(\mathbb{R}(s))$, then its Smith-McMillan form has, in terms of Definition 5.3, the form

$$U^{-1}(s)G(s)V^{-1}(s) = \operatorname{diag}\left(\frac{\varepsilon_1(s)}{\psi_1(s)}, \ldots, \frac{\varepsilon_m(s)}{\psi_m(s)}\right),$$

and the Smith-McMillan form of $G^{-1}(s)$ is

$$\Pi\, V(s)G^{-1}(s)U(s)\,\Pi = \operatorname{diag}\left(\frac{\psi_m(s)}{\varepsilon_m(s)}, \ldots, \frac{\psi_1(s)}{\varepsilon_1(s)}\right), \quad \text{where} \quad \Pi = \begin{bmatrix} & & 1 \\ & \diagup & \\ 1 & & \end{bmatrix} \in \mathbb{R}^{m,m}\,.$$

Now formula (2.3) of the transfer function yields that the set of transmission zeros of $C(sE - A)^{-1}B$ coincides with the set of poles of $A_{11} + A_{12}(sI_{n_2} - Q)^{-1}A_{21}$. Since the latter is a subset of $\sigma(Q)$, the claim follows from Corollary 5.2.
"(iii)⇒(i)": By Corollary 5.2, we have to show that every $\lambda \in \sigma(Q)$ satisfies $\lambda \in \mathbb{C}_-$. We distinguish two cases:
*Case 1*: $\lambda$ is a pole of $A_{11} + A_{12}(sI_{n_2} - Q)^{-1}A_{21}$.
By the preliminary thoughts, $\lambda$ is then a transmission zero of $C(sE - A)^{-1}B$ and assumption (iii)(c) implies that $\lambda \in \mathbb{C}_-$.
*Case 2*: $\lambda$ is not a pole of $A_{11} + A_{12}(sI_{n_2} - Q)^{-1}A_{21}$.
In this case, we can apply Lemma 8.3 to $[I_{n_2}, Q, A_{21}, A_{12}] \in \Sigma_{n_2,m,p}$ to see that we are at least in one of the following situations:

$$(\alpha)\ \operatorname{rk}[\lambda I_{n_2} - Q,\, A_{21}] < n_2 \qquad \text{or} \qquad (\beta)\ \operatorname{rk}[\lambda I_{n_2} - Q^\top,\, A_{12}^\top] < n_2\,.$$

If $(\alpha)$ holds, then

$$\exists\, v_2 \in \mathbb{C}^{n_2} \setminus \{0\} \ : \ v_2^\top[\lambda I_{n_2} - Q,\, A_{21}] = 0\,,$$

and for $v_1 := 0_{1,1}$, $v_3 := 0_{1,n_3}$, $v_4 := 0_{1,n_4}$, we obtain that (8.21) holds true. Now (iii)(a) gives $\lambda \in \mathbb{C}_-$. The case $(\beta)$ is treated analogously and omitted.
"(i)⇔(iv)" is a consequence of Remark 5.5. This completes the proof. □

**Proof of Theorem 6.2:** We proceed in several steps:
*Step 1*: In view of Corollary 2.6, the closed-loop system (1.1), (6.3) is given by

$$0 = (A_{11} - k(t)I_m)e(t) + A_{12}x_2(t) + A_{11}y_{\rm ref}(t), \tag{8.22a}$$

$$\dot x_2(t) = Qx_2(t) + A_{21}e(t) + A_{21}y_{\rm ref}(t), \tag{8.22b}$$

$$x_3(t) = \sum_{i=0}^{\nu-1} N_{33}^i\ E_{31}e^{(i+1)}(t) + \sum_{i=0}^{\nu-1} N_{33}^i\ E_{31}y_{\rm ref}^{(i+1)}(t)\,, \tag{8.22c}$$

$$x_4(t) = 0\,, \tag{8.22d}$$

$$k(t) = \tfrac{\hat k}{1-\varphi(t)^2\|e(t)\|^2}. \tag{8.22e}$$

We seek for a local solution $(e(\cdot), x_2(\cdot), x_3(\cdot), x_4(\cdot), k(\cdot))$ of (8.22) so that $e(\cdot)$ evolves within the funnel, that means $(t, e(t))$ belongs to the set $\widetilde{\mathcal{D}} := \{\ (t,e) \in [0,\infty) \times \mathbb{R}^m\ \mid \varphi(t)\,\|e\| < 1\ \}$. Note that solution means in particular that (8.22b) holds and this is implied by $e \in \mathcal{C}^{\nu+1}(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$; this will be shown in Step 6.
Note also that the design of $k(\cdot)$ together with the assumption $|\hat k| > \lim_{s\to\infty}\|G^{-1}(s)\| = \|A_{11}\|$ ensures that $k(t)I_m - A_{11} \in \mathbf{Gl}_m(\mathbb{R})$ for all $t \in [0,\omega)$ as long as there exists a solution $(x,e,k) : [0,\omega) \to \mathbb{R}^{n_2+m+1}$, for $\omega \in (0,\infty]$, to the closed-loop system (1.1), (6.3) such that $(t,e(t)) \in \widetilde{\mathcal{D}}$. As a consequence, (8.22a) is equivalent to

$$e(t)\ =\ (k(t)I_m - A_{11})^{-1}\,(A_{12}x_2(t) + A_{11}y_{\rm ref}(t)). \tag{8.23}$$

Therefore, $e(\cdot)$ evolves within the funnel if, and only if, $(t, x_2(t), k(t))$ belongs to

$$\mathcal{D} := \left\{\ (t,x_2,k) \in [0,\infty)\times\mathbb{R}^{n_2}\times[\hat k,\infty)\ \middle|\ \varphi(t)\,\big\|(kI_m - A_{11})^{-1}(A_{12}x_2 + A_{11}y_{\rm ref}(t))\big\| < 1\ \right\}.$$

*Step 2*: We show that there exists a solution $(x_2, k)$ to (8.22a), (8.22b), (8.22e). For this, it remains to seek for a solution $(x_2, k)$ of the time-varying non-linear semi-explicit DAE

$$\begin{aligned} \dot x_2(t) &= f(t, x_2(t), k(t)) \\ 0 &= g(t, x_2(t), k(t)), \end{aligned} \tag{8.24}$$

where

$$f\colon \mathcal{D} \to \mathbb{R}^{n_2},\quad (t,x_2,k) \mapsto (Q + A_{21}(kI_m - A_{11})^{-1}A_{12})x_2 + A_{21}(kI_m - A_{11})^{-1}A_{11}y_{\rm ref}(t) + A_{21}y_{\rm ref}(t)$$

$$g\colon \mathcal{D} \to \mathbb{R},\quad (t,x_2,k) \mapsto k - \frac{\hat k}{1 - \varphi(t)^2\|(kI_m - A_{11})^{-1}(A_{12}x_2 + A_{11}y_{\rm ref}(t))\|^2}.$$

To rewrite (8.24) as an ODE, we first record two technical facts:

$$\forall\, k \geq \hat k\ \forall\, \ell \in \mathbb{N}\ :\ \frac{\partial^\ell}{\partial k^\ell}(kI_m - A_{11})^{-1} = (-1)^\ell\ \ell!\ (kI_m - A_{11})^{-(1+\ell)} \tag{8.25a}$$

$$\forall\, k \geq \hat k\ \forall\, \eta \in \mathbb{R}^m\ :\ (1 - k^{-1}\|A_{11}\|)\,\|\eta\|^2\ \leq\ \eta^\top(I_m - k^{-1}A_{11})\,\eta\,. \tag{8.25b}$$

We obtain, for $\psi \in \mathbb{R}^m$,

$$\begin{aligned} \frac{\partial}{\partial k}\,\|(kI_m - A_{11})^{-1}\,\psi\|^2 &\overset{(8.25a)}{=} -2\left((kI_m - A_{11})^{-2}\,\psi\right)^\top \left((kI_m - A_{11})^{-1}\,\psi\right) \\ &= -2\left((kI_m - A_{11})^{-2}\,\psi\right)^\top (kI_m - A_{11})\,\left((kI_m - A_{11})^{-2}\,\psi\right) \\ &\overset{(8.25b)}{\leq} -2k\,(1 - k^{-1}\|A_{11}\|)\ \|(kI_m - A_{11})^{-2}\,\psi\|^2\,. \end{aligned} \tag{8.26}$$

Since $\nu \geq 1$, there holds that $y_{\text{ref}} \in \mathcal{C}^2(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$ and $\varphi \in \Phi^2$, whence $\frac{\partial}{\partial t} g(t, x_2, k)$, $\frac{\partial}{\partial x_2} g(t, x_2, k)$, and $\frac{\partial}{\partial k} g(t, x_2, k)$ are well defined and continuously differentiable on $\mathcal{D}$. Now we conclude, for all $(t, x_2, k) \in \mathcal{D}$ and $\psi(x_2, t) := A_{12}x_2 + A_{11}y_{\text{ref}}(t)$, that

$$\begin{aligned}
\frac{\partial}{\partial k} g(t, x_2, k) \quad &= \quad 1 - \frac{\hat{k}\, \varphi(t)^2}{(1 - \varphi(t)^2 \|(kI_m - A_{11})^{-1}\psi(x_2, t)\|^2)^2} \, \frac{\partial}{\partial k} \|(kI_m - A_{11})^{-1}\, \psi(x_2, t)\|^2 \\
&\overset{(8.26)}{\geq} \quad 1 + \frac{2\, k\, \hat{k}\, \varphi(t)^2\, (1 - k^{-1}\|A_{11}\|)}{(1 - \varphi(t)^2 \|(kI_m - A_{11})^{-1}\psi(x_2, t)\|^2)^2} \; \|(kI_m - A_{11})^{-2}\, \psi(x_2, t)\|^2 \\
&\geq \quad 1
\end{aligned} \tag{8.27}$$

and since

$$0 \overset{(8.24)}{=} \frac{\partial g}{\partial t}(t, x_2(t), k(t)) + \frac{\partial g}{\partial x_2}(t, x_2(t), k(t))\; f(t, x_2(t), k(t)) + \frac{\partial g}{\partial k}(t, x_2(t), k(t))\; \dot{k}(t)\,,$$

it follows from (8.27) that

$$h\colon \mathcal{D} \to \mathbb{R}, \quad (t, x_2, k) \mapsto -\frac{\frac{\partial g}{\partial t}(t, x_2, k) + \frac{\partial g}{\partial x_2}(t, x_2, k)\, f(t, x_2, k)}{\frac{\partial g}{\partial k}(t, x_2, k)}$$

is well defined and continuously differentiable. Therefore, the DAE (8.24) is equivalent to the ODE

$$\begin{aligned}
\dot{x}_2(t) &= f(t, x_2(t), k(t)) \\
\dot{k}(t) &= h(t, x_2(t), k(t))\,.
\end{aligned} \tag{8.28}$$

*Step 3*: Both functions $f(\cdot, \cdot, \cdot)$ and $g(\cdot, \cdot, \cdot)$ are continuously differentiable on $\mathcal{D}$, and $\mathcal{D}$ is a relatively open, non-empty set in $[0, \infty) \times \mathbb{R}^{n_2} \times [\hat{k}, \infty)$. This allows to apply [17, §10, Thm. VI] to conclude existence of a unique solution $(x_2, k) : [0, \omega) \to \mathbb{R}^{n_2+1}$ for maximal $\omega \in (0, \infty]$ of the initial value problem (8.28), $(x_2(0), k(0)) = (x_2^0, \hat{k})$.
It follows from Step 1-2, that equivalently there exists a unique and maximal solution $(x, k) : [0, \omega) \to \mathbb{R}^{n+1}$ of the closed-loop system (1.1) for any consistent initial value $x^0 \in \mathbb{R}^n$ such that $Tx^0 = \left((y^0)^\top, (x_2^0)^\top, (x_3^0)^\top, 0\right)^\top$.

*Step 4*: We show $x_2 \in L^\infty([0, \omega), \mathbb{R}^{n_2})$ and $k \in L^\infty([0, \omega), \mathbb{R})$.
Note that $e(\cdot)$ as in (8.23) evolves within the funnel and $y_{\text{ref}}(\cdot)$ is bounded by assumption, i.e. $e, y_{\text{ref}} \in L^\infty([0, \omega), \mathbb{R}^m)$, thus $y \in L^\infty([0, \omega), \mathbb{R}^m)$. Since $\sigma(Q) \subseteq \mathbb{C}_-$ by assumption and Corollary 5.2, equation (8.22b) yields

$$\begin{aligned}
\exists\, \lambda, M > 0\; \forall\, t \in [0, \omega) \;:\; \|x_2(t)\| \leq M \mathrm{e}^{-\lambda t} \|x_2^0\| + \int_0^t M \mathrm{e}^{-\lambda(t-s)} \|A_{21}\|\, \|y\|_\infty \, \mathrm{d}s \;\leq& \\
M\, \|x_2^0\| + \frac{M}{\lambda} \|A_{21}\| \|y\|_\infty\,.&
\end{aligned}$$

Therefore, $x_2 \in L^\infty([0, \omega), \mathbb{R}^{n_2})$.
Since $k(t) \geq \hat{k} > \|A_{11}\|$ by assumption, we may apply the theory of Neumann series to conclude, for all $t \in [0, \omega)$,

$$\|(k(t)I_m - A_{11})^{-1}\| = k(t)^{-1} \|(I_m - k(t)^{-1}A_{11})^{-1}\| \leq k(t)^{-1} \frac{1}{1 - k(t)^{-1}\|A_{11}\|} \leq k(t)^{-1} \frac{\hat{k}}{\hat{k} - \|A_{11}\|},$$

and (8.23) gives

$$\forall\, t \in [0, \omega): \; \|e(t)\| \le k(t)^{-1} \frac{\hat{k}}{\hat{k} - \|A_{11}\|} \Big( \|A_{12}\| \|x_2\|_\infty + \|A_{11}\| \|y_{\rm ref}\|_\infty \Big). \tag{8.29}$$

Suppose $k \notin L^\infty([0,\omega), \mathbb{R})$, i.e. there exists a sequence $(t_i)$ such that $t_i \nearrow \omega$ and $k(t_i) \nearrow \infty$ for $i \to \infty$. Then (8.29) yields $\lim_{i\to\infty} e(t_i) = 0$ and therefore, due to boundedness of $\varphi(\cdot)$, $\lim_{i\to\infty} \varphi(t_i)^2 \|e(t_i)\|^2 = 0$. This shows $\lim_{i\to\infty} k(t_i) = \hat{k}$, a contradiction; and hence $k \in L^\infty([0,\omega), \mathbb{R})$.

*Step 5*: We show $\omega = \infty$ and equation (6.4).
By definition and boundedness of $k(\cdot)$ we have,

$$\forall\, t \in [0, \omega): \; \hat{k}(1 - \varphi(t)^2 \|e(t)\|^2)^{-1} = k(t) \le \|k\|_\infty\,,$$

or, equivalently,

$$\forall\, t \in [0, \omega) \; : \; \varphi(t)\, \|e(t)\| \le \left( 1 - \frac{\hat{k}}{\|k\|_\infty} \right)^{1/2}. \tag{8.30}$$

This implies (6.4). Invoking (8.23), neither $(x_2(\cdot), k(\cdot))$ has a finite escape time nor does it tend to the boundary of $\mathcal{D}$; therefore [17, §10, Thm. VI] yields $\omega = \infty$.

*Step 6*: We show that for $x_3(\cdot)$ as defined in (8.22c) holds $x_3 \in \mathcal{B}^1(\mathbb{R}_{\ge 0}, \mathbb{R}^{n_3})$. In view of $y_{\rm ref} \in \mathcal{B}^{\nu+1}(\mathbb{R}_{\ge 0}; \mathbb{R}^m)$, it suffices to show that $e \in \mathcal{B}^{\nu+1}(\mathbb{R}_{\ge 0}; \mathbb{R}^m)$. Let, as before, $(x_2(\cdot), k(\cdot))$ be the solution component of (8.22a), (8.22b), (8.22e).
*6a*) By definition and boundedness of $k(\cdot)$ we have

$$\begin{aligned} &\exists\, \delta > 0 \;\forall\, t \ge 0: \quad \|(k(t) I_m - A_{11})^{-1}\| \ge \delta \\ &\exists\, \tilde{\delta} > 0 \;\forall\, t \ge 0: \quad 1 - \varphi(t)^2 \|e(t)\|^2 \; > \; \tilde{\delta}\,. \end{aligned} \tag{8.31}$$

By (8.25a), (8.31) and since $y_{\rm ref} \in \mathcal{C}^{\nu+1}(\mathbb{R}_{\ge 0}; \mathbb{R}^m)$, a straightforward calculation gives, for any multi index $(i_1, i_2, i_3) \in \{0, \ldots, \nu\} \times \mathbb{N}_0 \times \mathbb{N}_0$, that the map

$$\frac{\partial^{i_1}}{\partial t^{i_1}} \frac{\partial^{i_2}}{\partial x_2^{i_2}} \frac{\partial^{i_3}}{\partial k^{i_3}} f(t, x_2, k) \colon \mathcal{D} \to \mathbb{R}^{n_2} \tag{8.32}$$

is well defined and $\frac{\partial^{i_1}}{\partial t^{i_1}} \frac{\partial^{i_2}}{\partial x_2^{i_2}} \frac{\partial^{i_3}}{\partial k^{i_3}} f(\cdot, x_2(\cdot), k(\cdot))$ is bounded. Similarly, we conclude from (8.25a), (8.31), $y_{\rm ref} \in \mathcal{B}^{\nu+1}(\mathbb{R}_{\ge 0}; \mathbb{R}^m)$, and $\varphi \in \mathcal{B}^{\nu+1}(\mathbb{R}_{\ge 0}; \mathbb{R})$ that for any multi index $(i_1, i_2, i_3) \in \{0, \ldots, \nu\} \times \mathbb{N}_0 \times \mathbb{N}_0$, the map

$$\frac{\partial^{i_1}}{\partial t^{i_1}} \frac{\partial^{i_2}}{\partial x_2^{i_2}} \frac{\partial^{i_3}}{\partial k^{i_3}} h(t, x_2, k) \colon \mathcal{D} \to \mathbb{R} \tag{8.33}$$

is well defined and $\frac{\partial^{i_1}}{\partial t^{i_1}} \frac{\partial^{i_2}}{\partial x_2^{i_2}} \frac{\partial^{i_3}}{\partial k^{i_3}} h(\cdot, x_2(\cdot), k(\cdot))$ is bounded.
*6b*) We may now differentiate (8.28) for $i = 0, \ldots, \nu$ and use the findings of Step 6a) to conclude successively that $x_2^{(i+1)}(\cdot)$ and $k^{(i+1)}(\cdot)$ are continuous and bounded. Hence $x_2 \in \mathcal{B}^{\nu+1}(\mathbb{R}_{\ge 0}; \mathbb{R}^{n_2})$ and $k \in \mathcal{B}^{\nu+1}(\mathbb{R}_{\ge 0}; \mathbb{R})$ and so $e \in \mathcal{B}^{\nu+1}(\mathbb{R}_{\ge 0}; \mathbb{R}^m)$ follows.

*Step 7*: In view of Step 1 and $T$ as defined in Corollary 2.6, it remains to show boundedness of $x(\cdot) = T^{-1}\, (y(\cdot)^\top, x_2(\cdot)^\top, x_3(\cdot)^\top, x_4(\cdot)^\top)^\top$; this follows from Steps 4-6 and boundedness of $y(\cdot)$.
Therefore, the proof of the theorem is complete. □

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