

Admissibility of retarded diagonal systems with one-dimensional input space

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Abstract

We investigate infinite-time admissibility of a control operator B in a Hilbert space state-delayed dynamical system setting of the form $\dot{z}(t) = Az(t) + A_1z(t - \tau) + Bu(t)$, where A generates a diagonal C_0 -semigroup, $A_1 \in \mathcal{L}(X)$ is also diagonal and $u \in L^2(0, \infty; \mathbb{C})$. Our approach is based on the Laplace embedding between L^2 and the Hardy space $H^2(\mathbb{C}_+)$. The results are expressed in terms of the eigenvalues of A and A_1 and the sequence representing the control operator.

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1 Introduction

State-delayed differential equations arise in many areas of applied mathematics, which is related to the fact that in the real world there is an inherent input-output delay in every physical system. Among sources of delay we have the spatial character of the system in relation to signal propagation, measurements processing or hatching time in biological systems, to name a few. Whenever the delay has a considerable influence on the outcome of the process it has to be incorporated into a process's mathematical model. Hence, an understanding of a state-delayed system, even in a linear case, plays a crucial role in the analysis and control of dynamical systems, particularly when the asymptotic behaviour is concerned.

In order to cover a possibly large area of dynamical systems our analysis uses an abstract description. Hence the retarded state-delayed dynamical system we are interested in has an abstract representation given by

$$\begin{cases} \dot{z}(t) = Az(t) + A_1z(t - \tau) + Bu(t) \\ z(0) = x \\ z_0 = f, \end{cases} \quad (1)$$

where the state space X is a Hilbert space, $A : D(A) \subset X \rightarrow X$ is a closed, densely defined generator of a C_0 -semigroup $(T(t))_{t \geq 0}$ on X , $A_1 \in \mathcal{L}(X)$ and $0 < \tau < \infty$ is a fixed delay (some discussions of the difficulties inherent in taking A_1 unbounded appear in Subsection 4.2). The input function is $u \in L^2(0, \infty; \mathbb{C})$, B is the control operator, the pair $x \in D(A)$ and $f \in L^2(-\tau, 0; X)$ forms the initial condition. We also assume that X possesses a sequence of normalized eigenvectors $(\phi_k)_{k \in \mathbb{N}}$ forming a Riesz basis, with associated eigenvalues $(\lambda_k)_{k \in \mathbb{N}}$.

We analyse (1) from the perspective of infinite-time admissibility which, roughly speaking, asserts whether a solution z of (1) follows a required type of trajectory. A more detailed

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description of admissibility requires an introduction of pivot duality and some related norm inequalities. For that reason we postpone it until Subsection 2.2, where all these elements are already introduced for the setting within which we analyse (1).

With regard to previous admissibility results, necessary and sufficient conditions for infinite-time admissibility of B in the undelayed case of (1), under an assumption of diagonal generator $(A, D(A))$, were analysed e.g. using Carleson measures e.g. in [13, 14, 28]. Those results were extended to normal semigroups [29], then generalized to the case when $u \in L^2(0, \infty; t^\alpha dt)$ for $\alpha \in (-1, 0)$ in [31] and further to the case $u \in L^2(0, \infty; w(t)dt)$ in [16, 17]. For a thorough presentation of admissibility results, not restricted to diagonal systems, for the undelayed case we refer the reader to [15] and a rich list of references therein.

For the delayed case, in contrast to the undelayed one, a different setting is required. Some of the first studies in such setting are [12] and [8], and these form a basis for [5]. In this article we follow the latter one in developing a setting for admissibility analysis. We also build on [24] where a similar setting was used to present admissibility results for a simplified version of (1), that is with a diagonal generator $(A, D(A))$ with the delay in its argument (see the Examples section below).

In fact, as the system analysed in [24] is a special case of (1), the results presented here contain those of [24]. The most important drawback of results in [24] is that the conditions leading to sufficiency for infinite-time admissibility there imply also that the semigroup generator is bounded. Thus, to obtain some results for unbounded generators one is forced to go through the so-called reciprocal systems. Results presented below are free from such limitation and can be applied to unbounded diagonal generators directly, as shown in the Examples section.

This paper is organised as follows. Section 2 defines the notation and provides preliminary results. These include a general delayed equation setting, which is applied later to the problem of our interest and the problem of infinite-time admissibility. Section 3 shows how the general setting looks for a particular case of retarded diagonal case. It then shows a component-wise analysis of infinite-time admissibility and provides results for the complete system. Section 4 gives examples.

2 Preliminaries

In this paper we use Sobolev spaces (see e.g. [10, Chapter 5]) $W^{1,2}(J, X) := \{f \in L^2(J, X) : \frac{d}{dt}f \in L^2(J, X)\}$ and $W_0^{1,2}(J, X) := \{f \in W^{1,2}(J, X) : f(\partial J) = 0\}$, where $\frac{d}{dt}f$ is a weak derivative of f and J is an interval with boundary ∂J .

For any $\alpha \in \mathbb{R}$ we denote the following half-planes

$$\mathbb{C}_\alpha := \{s \in \mathbb{C} : \operatorname{Re} s < \alpha\}, \quad \underline{\mathbb{C}}_\alpha := \{s \in \mathbb{C} : \operatorname{Re} s > \alpha\},$$

with a simplification for two special cases, namely $\mathbb{C}_- := \mathbb{C}_0$ and $\mathbb{C}_+ := \underline{\mathbb{C}}_0$. We make use of the Hardy space $H^2(\mathbb{C}_+)$ that consists of all analytic functions $f : \mathbb{C}_+ \rightarrow \mathbb{C}$ for which

$$\sup_{\alpha > 0} \int_{-\infty}^{\infty} |f(\alpha + i\omega)|^2 d\omega < \infty. \quad (2)$$

If $f \in H^2(\mathbb{C}_+)$ then for a.e. $\omega \in \mathbb{R}$ the limit

$$f^*(i\omega) = \lim_{\alpha \downarrow 0} f(\alpha + i\omega) \quad (3)$$

exists and defines a function $f^* \in L^2(i\mathbb{R})$ called the *boundary trace* of f . Using boundary traces $H^2(\mathbb{C}_+)$ is made into a Hilbert space with the inner product defined as

$$\langle f, g \rangle_{H^2(\mathbb{C}_+)} := \langle f^*, g^* \rangle_{L^2(i\mathbb{R})} := \frac{1}{2\pi} \int_{-\infty}^{+\infty} f^*(i\omega) \bar{g}^*(i\omega) d\omega \quad \forall f, g \in H^2(\mathbb{C}_+). \quad (4)$$

For more information about Hardy spaces see [23], [11] or [22]. We also make use of the Paley–Wiener Theorem (see [25, Chapter 19] for the scalar version or [2, Theorem 1.8.3] for the vector-valued one)

Theorem 1 (Paley–Wiener). *Let Y be a Hilbert space. Then the Laplace transform $\mathcal{L} : L^2(0, \infty; Y) \rightarrow H^2(\mathbb{C}_+; Y)$ is an isometric isomorphism.*

2.1 The delayed equation setting

We follow a general setting for a state-delayed system from [5, Chapter 3.1], described for a diagonal case also in [24]. And so, to include the influence of the delay we extend the state space of (1). To that end consider a trajectory of (1) given by $z : [-\tau, \infty) \rightarrow X$. For each $t \geq 0$ we call $z_t : [-\tau, 0] \rightarrow X$, $z_t(\sigma) := z(t + \sigma)$ a *history segment* with respect to $t \geq 0$. With history segments we consider a so-called *history function* of z denoted by $h_z : [0, \infty) \rightarrow L^2(-\tau, 0; X)$, $h_z(t) := z_t$. In [5, Lemma 3.4] we find the following

Proposition 2. *Let $1 \leq p < \infty$ and $z \in W_{loc}^{1,p}(-\tau, \infty; X)$. Then the history function $h_z : t \rightarrow z_t$ of z is continuously differentiable from \mathbb{R}_+ into $L^p(-\tau, 0; X)$ with derivative*

$$\frac{\partial}{\partial t} h_z(t) = \frac{\partial}{\partial \sigma} z_t.$$

To remain in the Hilbert space setting we limit ourselves to $p = 2$ and take

$$\mathcal{X} := X \times L^2(-\tau, 0; X) \tag{5}$$

as the aforementioned state space extension with an inner product

$$\left\langle \begin{pmatrix} x \\ f \end{pmatrix}, \begin{pmatrix} y \\ g \end{pmatrix} \right\rangle_{\mathcal{X}} := \langle x, y \rangle_X + \langle f, g \rangle_{L^2(-\tau, 0; X)}. \tag{6}$$

Then $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ becomes a Hilbert space with the norm $\|\begin{pmatrix} x \\ f \end{pmatrix}\|_{\mathcal{X}}^2 = \|x\|_X^2 + \|f\|_{L^2}^2$. We assume that a linear and bounded *delay operator* $\Psi : W^{1,2}(-\tau, 0; X) \rightarrow X$ acts on history segments z_t and thus consider (1) in the form

$$\begin{cases} \dot{z}(t) = Az(t) + \Psi z_t + Bu(t) \\ z(0) = x, \\ z_0 = f, \end{cases} \tag{7}$$

where the pair $x \in D(A)$ and $f \in L^2(-\tau, 0; X)$ forms an initial condition. A particular choice of Ψ can be found in (21) below. Due to Proposition 2, system (7) may be written as an abstract Cauchy problem

$$\begin{cases} \dot{v}(t) = \mathcal{A}v(t) + \mathcal{B}u(t) \\ v(0) = \begin{pmatrix} x \\ f \end{pmatrix}, \end{cases} \tag{8}$$

where $v : [0, \infty) \ni t \mapsto \begin{pmatrix} z(t) \\ z_t \end{pmatrix} \in \mathcal{X}$ and \mathcal{A} is a linear operator on $D(\mathcal{A}) \subset \mathcal{X}$, where

$$D(\mathcal{A}) := \left\{ \begin{pmatrix} x \\ f \end{pmatrix} \in D(A) \times W^{1,2}(-\tau, 0; X) : f(0) = x \right\}, \tag{9}$$

$$\mathcal{A} := \begin{pmatrix} A & \Psi \\ 0 & \frac{d}{d\sigma} \end{pmatrix}, \tag{10}$$

and the control operator is $\mathcal{B} = \begin{pmatrix} B \\ 0 \end{pmatrix}$. Operator $(\mathcal{A}, D(\mathcal{A}))$ is closed and densely defined on \mathcal{X} [5, Lemma 3.6]. Note that up to this moment we do not need to know more about Ψ .

Concerning the resolvent of $(\mathcal{A}, D(\mathcal{A}))$, let

$$A_0 := \frac{d}{d\sigma}, \quad D(A_0) = \{z \in W^{1,2}(-\tau, 0; X) : z(0) = 0\},$$

be the generator of a nilpotent left shift semigroup on $L^2(-\tau, 0; X)$. For $s \in \mathbb{C}$ define $\epsilon_s : [-\tau, 0] \rightarrow \mathbb{C}$, $\epsilon_s(\sigma) := e^{s\sigma}$. Define also $\Psi_s \in \mathcal{L}(D(A), X)$, $\Psi_s x := \Psi(\epsilon_s(\cdot)x)$. Then [5, Proposition 3.19] provides

Proposition 3. For $s \in \mathbb{C}$ and for all $1 \leq p < \infty$ we have

$$s \in \rho(\mathcal{A}) \quad \text{if and only if} \quad s \in \rho(A + \Psi_s).$$

Moreover, for $s \in \rho(\mathcal{A})$ the resolvent $R(s, \mathcal{A})$ is given by

$$R(s, \mathcal{A}) = \begin{pmatrix} R(s, A + \Psi_s) & R(s, A + \Psi_s)\Psi R(s, A_0) \\ \epsilon_s R(s, A + \Psi_s) & (\epsilon_s R(s, A + \Psi_s)\Psi + I)R(s, A_0) \end{pmatrix}. \quad (11)$$

In the sequel we make use of Sobolev towers, also known as a duality with a pivot (see [26, Chapter 2] or [9, Chapter II.5]). To this end we have

Definition 4. Let $\beta \in \rho(A)$ and denote $(X_1, \|\cdot\|_1) := (D(A), \|\cdot\|_1)$ with $\|x\|_1 := \|(\beta I - A)x\|$ ($x \in D(A)$). Similarly, we set $\|x\|_{-1} := \|(\beta I - A)^{-1}x\|$ ($x \in X$). Then the space $(X_{-1}, \|\cdot\|_{-1})$ denotes the completion of X under the norm $\|\cdot\|_{-1}$. For $t \geq 0$ we define $T_{-1}(t)$ as the continuous extension of $T(t)$ to the space $(X_{-1}, \|\cdot\|_{-1})$.

The adjoint generator plays an important role in the pivot duality setting. Thus we take

Definition 5. Let $A : D(A) \rightarrow X$ be a densely defined operator. The *adjoint* of $(A, D(A))$, denoted $(A^*, D(A^*))$, is defined on

$$D(A^*) := \{y \in X : \text{the functional } X \ni x \mapsto \langle Ax, y \rangle \text{ is bounded}\}. \quad (12)$$

Since $D(A)$ is dense in X the functional in (12) has a unique bounded extension to X . By the Riesz representation theorem there exists a unique $w \in X$ such that $\langle Ax, y \rangle = \langle x, w \rangle$. Then we define $A^*y := w$ so that

$$\langle Ax, y \rangle = \langle x, A^*y \rangle \quad \forall x \in D(A) \quad \forall y \in D(A^*). \quad (13)$$

We have the following (see [26, Prop. 2.10.2])

Proposition 6. With the notation of Definition 4 let $(A^*, D(A^*))$ be the adjoint of $(A, D(A))$. Then $\beta \in \rho(A^*)$, $(X_1^d, \|\cdot\|_1^d) := (D(A^*), \|\cdot\|_1^d)$ with $\|x\|_1^d := \|(\beta I - A^*)x\|$ ($x \in D(A^*)$) is a Hilbert space and X_{-1} is the dual of X_1^d with respect to the pivot space X , that is $X_{-1} = (D(A^*))'$.

Much of our reasoning is justified by the following Proposition, which we include here for the reader's convenience (for more details see [9, Chapter II.5] or [26, Chapter 2.10]).

Proposition 7. With the notation of Definition 4 we have the following

- (i) The spaces $(X_1, \|\cdot\|_1)$ and $(X_{-1}, \|\cdot\|_{-1})$ are independent of the choice of $\beta \in \rho(A)$.
- (ii) $(T_1(t))_{t \geq 0}$ is a C_0 -semigroup on the Banach space $(X_1, \|\cdot\|_1)$ and we have $\|T_1(t)\|_1 = \|T(t)\|$ for all $t \geq 0$.
- (iii) $(T_{-1}(t))_{t \geq 0}$ is a C_0 -semigroup on the Banach space $(X_{-1}, \|\cdot\|_{-1})$ and $\|T_{-1}(t)\|_{-1} = \|T(t)\|$ for all $t \geq 0$.

In the sequel, we denote the restriction (extension) of $T(t)$ described in Definition 4 by the same symbol $T(t)$, since this is unlikely to lead to confusions.

In the sequel we also use the following result by Miyadera and Voigt [9, Corollaries III.3.15 and 3.16], that gives sufficient conditions for a perturbed generator to remain a generator of a C_0 -semigroup.

Proposition 8. Let $(A, D(A))$ be the generator of a strongly continuous semigroup $(T(t))_{t \geq 0}$ on a Banach space X and let $P \in \mathcal{L}(X_1, X)$ be a perturbation which satisfies

$$\int_0^{t_0} \|PT(r)x\| dr \leq q\|x\| \quad \forall x \in D(A) \quad (14)$$

for some $t_0 > 0$ and $0 \leq q < 1$. Then the sum $A + P$ with domain $D(A + P) := D(A)$ generates a strongly continuous semigroup $(S(t))_{t \geq 0}$ on X . Moreover, for all $t \geq 0$ the C_0 -semigroup $(S(t))_{t \geq 0}$ satisfies

$$S(t)x = T(t)x + \int_0^t S(s)PT(t-s)x ds \quad \forall x \in D(A). \quad (15)$$

2.2 The admissibility problem

The basic object in the formulation of admissibility problem is a linear system and its mild solution

$$\dot{x}(t) = Ax(t) + Bu(t); \quad x(t) = T(t)x_0 + \int_0^t T(t-s)Bu(s) ds, \quad (16)$$

where $x : [0, \infty) \rightarrow X$, $u \in V$, where V is a normed space of measurable functions from $[0, \infty)$ to U and B is a *control operator*; $x_0 \in X$ is an initial state.

In many practical examples the control operator B is unbounded, hence (16) is viewed on an extrapolation space $X_{-1} \supset X$ where $B \in \mathcal{L}(U, X_{-1})$. Introduction of X_{-1} , however, comes at a price of physical interpretation of the solution. To be more precise, a dynamical system expressed by (16) describes a physical system where one can assign a physical meaning to X , with the use of which the modelling is performed. That is not always true for X_{-1} . We would then like to study those control operators B for which the (mild) solution is a continuous X -valued function that carries a physical meaning. In a rigorous way, to ensure that the state $x(t)$ lies in X it is sufficient that $\int_0^t T_{-1}(t-s)Bu(s) ds \in X$ for all inputs $u \in V$.

Definition 9. Let $B \in \mathcal{L}(U, X_{-1})$ and $t \geq 0$. The *forcing operator* $\Phi_t \in \mathcal{L}(V, X_{-1})$ is given by

$$\Phi_t(u) := \int_0^t T(t-\sigma)Bu(\sigma) d\sigma. \quad (17)$$

Put differently, we have

Definition 10. The control operator $B \in \mathcal{L}(U, X_{-1})$ is called

- (i) *finite-time admissible* for $(T(t))_{t \geq 0}$ on a Hilbert space X if for each $t > 0$ there is a constant K_t such that

$$\|\Phi_t(u)\|_X \leq K_t \|u\|_V \quad \forall u \in V; \quad (18)$$

- (ii) *infinite-time admissible* for $(T(t))_{t \geq 0}$ if there is a constant $K \geq 0$ such that

$$\|\Phi_t\|_{\mathcal{L}(V, X)} \leq K \quad \forall t \geq 0. \quad (19)$$

For the infinite-time admissibility it is convenient to define a different version of the forcing operator, namely $\Phi_\infty : L^2(0, \infty; U) \rightarrow X_{-1}$,

$$\Phi_\infty(u) := \int_0^\infty T(t)Bu(t) dt. \quad (20)$$

The infinite-time admissibility of B follows then from the boundedness of Φ_∞ in (20) taken as an operator from $L^2(0, \infty; U)$ to X . For a more detailed discussion concerning infinite-time admissibility see also [15] and [26] with references therein.

3 The setting of retarded diagonal systems

We begin with a general setting of the previous section expressed by (8) with elements defined there. Then, consecutively specifying these elements, we reach a description of a concrete case of a retarded diagonal system.

Let the delay operator Ψ be a point evaluation i.e. define $\Psi \in \mathcal{L}(W^{1,2}(-\tau, 0; X), X)$ as

$$\Psi(f) := A_1 f(-\tau), \quad (21)$$

where boundedness of Ψ results from continuous embedding of $W^{1,2}(-\tau, 0; X)$ in $C([-\tau, 0], X)$ (see e.g. [6, Theorem 8.8], [1, Theorem III.4.10.2] or [10, Chapter 5.9.2]).

With the delay operator given by (21) we are in a position to describe pivot duality for \mathcal{X} given by (5) with $(\mathcal{A}, D(\mathcal{A}))$ given by (10)-(9) and with $\mathcal{B} = \begin{pmatrix} B \\ 0 \end{pmatrix}$. Then, using the pivot duality, we consider (8) on the completion space \mathcal{X}_{-1} where the control operator $\mathcal{B} \in \mathcal{L}(U, \mathcal{X}_{-1})$. To write explicitly all the elements of the pivot duality setting we need to determine the adjoint $(\mathcal{A}^*, D(\mathcal{A}^*))$ operator (see Proposition 6).

Proposition 11. Let X , $(A, D(A))$ and A_1 be as in (1) and $(\mathcal{A}, D(\mathcal{A}))$ be defined by (10)–(9) with Ψ given by (21). Then $(\mathcal{A}^*, D(\mathcal{A}^*))$, the adjoint of $(\mathcal{A}, D(\mathcal{A}))$, is given by

$$D(\mathcal{A}^*) = \left\{ \begin{pmatrix} y \\ g \end{pmatrix} \in D(A^*) \times W^{1,2}(-\tau, 0; X) : A_1^* y = g(-\tau) \right\}, \quad (22)$$

$$\mathcal{A}^* \begin{pmatrix} y \\ g \end{pmatrix} = \begin{pmatrix} A^* y + g(0) \\ -\frac{d}{d\sigma} g \end{pmatrix}, \quad (23)$$

where $(A^*, D(A^*))$ is the adjoint of $(A, D(A))$ and A_1^* is the adjoint of A_1 .

Proof. Let F be the set defined as the right hand side of (22). To show that $D(\mathcal{A}^*) \subset F$ we adapt the approach from [19]. Let $v = \begin{pmatrix} f(0) \\ f \end{pmatrix} \in D(\mathcal{A})$, $w = \begin{pmatrix} y \\ g \end{pmatrix} \in D(\mathcal{A}^*)$ and let

$$\mathcal{A}^* w = \begin{pmatrix} (\mathcal{A}^* w)^0 \\ (\mathcal{A}^* w)^1 \end{pmatrix}.$$

By (10), (9) and the adjoint Definition 5 we get

$$\begin{aligned} \langle \mathcal{A}^* w, v \rangle_X &= \langle (\mathcal{A}^* w)^0, f(0) \rangle_X + \int_{-\tau}^0 \langle (\mathcal{A}^* w)^1(\sigma), f(\sigma) \rangle_X d\sigma \\ &= \langle y, Af(0) \rangle_X + \langle y, A_1 f(-\tau) \rangle_X + \int_{-\tau}^0 \langle g(\sigma), \frac{d}{d\sigma} f \rangle_X d\sigma, \end{aligned} \quad (24)$$

and boundedness of the above for every $v \in D(\mathcal{A})$ implies that $y \in D(A^*)$. Observe also that

$$\begin{aligned} \int_{-\tau}^0 \langle (\mathcal{A}^* w)^1(\sigma), f(\sigma) \rangle_X d\sigma &= \int_{-\tau}^0 \left\langle (\mathcal{A}^* w)^1(\sigma), f(0) - \int_{\sigma}^0 \frac{d}{d\xi} f(\xi) d\xi \right\rangle_X d\sigma \\ &= \int_{-\tau}^0 \langle (\mathcal{A}^* w)^1(\sigma), f(0) \rangle_X d\sigma - \int_{-\tau}^0 \left\langle (\mathcal{A}^* w)^1(\sigma), \int_{\sigma}^0 \frac{d}{d\xi} f(\xi) d\xi \right\rangle_X d\sigma \\ &= \int_{-\tau}^0 \langle (\mathcal{A}^* w)^1(\sigma), f(0) \rangle_X d\sigma - \int_{-\tau}^0 \int_{-\tau}^{\xi} \left\langle (\mathcal{A}^* w)^1(\sigma), \frac{d}{d\xi} f(\xi) \right\rangle_X d\sigma d\xi \\ &= \left\langle \int_{-\tau}^0 (\mathcal{A}^* w)^1(\sigma) d\sigma, f(0) \right\rangle_X - \int_{-\tau}^0 \left\langle \int_{-\tau}^{\xi} (\mathcal{A}^* w)^1(\sigma) d\sigma, \frac{d}{d\xi} f(\xi) \right\rangle_X d\xi. \end{aligned} \quad (25)$$

Putting the result of (25) into (24) and rearranging gives that for every $v \in D(\mathcal{A})$

$$\begin{aligned} &\langle (\mathcal{A}^* w)^0 + \int_{-\tau}^0 (\mathcal{A}^* w)^1(\sigma) d\sigma - A^* y - A_1^* y, f(0) \rangle_X \\ &= \int_{-\tau}^0 \left\langle \int_{-\tau}^{\sigma} (\mathcal{A}^* w)^1(\xi) d\xi - A_1^* y + g(\sigma), \frac{d}{d\sigma} f(\sigma) \right\rangle_X d\sigma, \end{aligned} \quad (26)$$

where we used the fact that $f(-\tau) = f(0) - \int_{-\tau}^0 \frac{d}{d\sigma} f(\sigma) d\sigma$. As for every constant $f : [-\tau, 0] \rightarrow D(A)$ we have $\begin{pmatrix} f(0) \\ f \end{pmatrix} \in D(\mathcal{A})$, there is

$$(\mathcal{A}^* w)^0 = A^* y + A_1^* y - \int_{-\tau}^0 (\mathcal{A}^* w)^1(\sigma) d\sigma, \quad (27)$$

and then

$$g(\sigma) = A_1^* y - \int_{-\tau}^{\sigma} (\mathcal{A}^* w)^1(\xi) d\xi, \quad \forall \sigma \in [-\tau, 0]. \quad (28)$$

Equation (28) shows that $w = \begin{pmatrix} y \\ g \end{pmatrix} \in D(\mathcal{A}^*)$ implies $g \in W^{1,2}(-\tau, 0; X)$. Taking the limits gives

$$g(-\tau) = A_1^* y \quad (29)$$

and

$$(\mathcal{A}^*w)^0 = A^*y + g(0). \quad (30)$$

Differentiating (28) with respect to σ we also have

$$(\mathcal{A}^*w)^1 = -\frac{d}{d\sigma}g. \quad (31)$$

To show that $D(\mathcal{A}^*) \supset F$ let $w = \begin{pmatrix} y \\ g \end{pmatrix} \in F$ and $v = \begin{pmatrix} x \\ f \end{pmatrix} = \begin{pmatrix} f(0) \\ f \end{pmatrix} \in D(\mathcal{A})$. By (13) we need to show that $\langle \mathcal{A}^*w, v \rangle_{\mathcal{X}} = \langle w, \mathcal{A}v \rangle_{\mathcal{X}}$, where \mathcal{A}^*w we take as given by (23). We have

$$\begin{aligned} \langle \mathcal{A}^*w, v \rangle_{\mathcal{X}} &= \left\langle \begin{pmatrix} A^*y + g(0) \\ -\frac{d}{d\sigma}g \end{pmatrix}, \begin{pmatrix} f(0) \\ f \end{pmatrix} \right\rangle_{\mathcal{X}} = \langle A^*y + g(0), f(0) \rangle_X + \int_{-\tau}^0 \left\langle -\frac{d}{d\sigma}g, f \right\rangle_X d\sigma \\ &= \langle A^*y, f(0) \rangle_X + \langle g(0), f(0) \rangle_X + \left\langle -g, f \right\rangle_X \Big|_{-\tau}^0 - \int_{-\tau}^0 \left\langle -g, \frac{d}{d\sigma}f \right\rangle_X d\sigma \\ &= \langle A^*y, f(0) \rangle_X + \langle g(-\tau), f(-\tau) \rangle_X + \left\langle g, \frac{d}{d\sigma}f \right\rangle_{L^2} \\ &= \langle A^*y, f(0) \rangle_X + \langle A_1^*y, f(-\tau) \rangle_X + \left\langle g, \frac{d}{d\sigma}f \right\rangle_{L^2} \\ &= \langle y, Af(0) \rangle_X + \langle y, A_1f(-\tau) \rangle_X + \left\langle g, \frac{d}{d\sigma}f \right\rangle_{L^2} = \langle w, \mathcal{A}v \rangle_{\mathcal{X}}. \end{aligned}$$

□

Denoting $D(\mathcal{A}^*)'$ for the dual to $D(\mathcal{A}^*)$ with respect to the pivot space \mathcal{X} , by Proposition 6 we have

$$\mathcal{X}_{-1} = D(\mathcal{A}^*)'. \quad (32)$$

System (8) represents an abstract Cauchy problem, which is well-posed if and only if $(\mathcal{A}, D(\mathcal{A}))$ generates a C_0 -semigroup on \mathcal{X} . To show that this is the case we use a perturbation approach. We represent $\mathcal{A} = \mathcal{A}_0 + \mathcal{A}_\Psi$, where

$$\mathcal{A}_0 := \begin{pmatrix} A & 0 \\ 0 & \frac{d}{d\sigma} \end{pmatrix}, \quad (33)$$

with domain $D(\mathcal{A}_0) = D(\mathcal{A})$ and

$$\mathcal{A}_\Psi := \begin{pmatrix} 0 & \Psi \\ 0 & 0 \end{pmatrix}, \quad (34)$$

where $\mathcal{A}_\Psi \in \mathcal{L}(X \times W^{1,2}(-\tau, 0; X), \mathcal{X})$. The following proposition [5, Theorem 3.25] gives a necessary and sufficient condition for the unperturbed part $(\mathcal{A}_0, D(\mathcal{A}_0))$ to generate a C_0 -semigroup on \mathcal{X} .

Proposition 12. *Let X be a Banach space. The following are equivalent:*

- (i) *The operator $(A, D(A))$ generates a strongly continuous semigroup $(T(t))_{t \geq 0}$ on X .*
- (ii) *The operator $(\mathcal{A}_0, D(\mathcal{A}_0))$ generates a strongly continuous semigroup $(\mathcal{T}_0(t))_{t \geq 0}$ on $X \times L^p(-\tau, 0; X)$ for all $1 \leq p < \infty$.*
- (iii) *The operator $(\mathcal{A}_0, D(\mathcal{A}_0))$ generates a strongly continuous semigroup $(\mathcal{T}_0(t))_{t \geq 0}$ on $X \times L^p(-\tau, 0; X)$ for one $1 \leq p < \infty$.*

The C_0 -semigroup $(\mathcal{T}_0(t))_{t \geq 0}$ is given by

$$\mathcal{T}_0(t) := \begin{pmatrix} T(t) & 0 \\ S_t & S_0(t) \end{pmatrix} \quad \forall t \geq 0, \quad (35)$$

where $(S_0(t))_{t \geq 0}$ is the nilpotent left shift C_0 -semigroup on $L^p(-\tau, 0; X)$,

$$S_0(t)f(s) := \begin{cases} f(s+t) & \text{if } s+t \in [-\tau, 0], \\ 0 & \text{else} \end{cases} \quad (36)$$

and $S_t : X \rightarrow L^p(-\tau, 0; X)$,

$$(S_t x)(s) := \begin{cases} T(s+t)x & \text{if } -t < s \leq 0, \\ 0 & \text{if } -\tau \leq s \leq -t. \end{cases} \quad (37)$$

Proposition 8 provides now a sufficient condition for the perturbation $(\mathcal{A}_\Psi, D(\mathcal{A}))$ such that $\mathcal{A} = \mathcal{A}_0 + \mathcal{A}_\Psi$ is a generator, as given by the following

Proposition 13. *Operator $(\mathcal{A}, D(\mathcal{A}))$ generates a C_0 -semigroup $(\mathcal{T}(t))_{t \geq 0}$ on \mathcal{X} .*

Proof. We use Proposition 8 with $(\mathcal{A}, D(\mathcal{A}))$ given by (10)–(9) and represented as sum of (33) and (34) with Ψ given by (21). Thus a sufficient condition for $(\mathcal{A}, D(\mathcal{A}))$ to be a generator of a strongly continuous semigroup $(\mathcal{T}(t))_{t \geq 0}$ on \mathcal{X} is that the perturbation $\mathcal{A}_\Psi \in \mathcal{L}(X \times W^{1,2}(-\tau, 0; X), \mathcal{X})$ given by (34) satisfies

$$\int_0^{t_0} \left\| \mathcal{A}_\Psi \mathcal{T}_0(r)v \right\|_{\mathcal{X}} dr \leq q \|v\|_{\mathcal{X}} \quad \forall v \in D(\mathcal{A}_0)$$

for some $t_0 > 0$ and $0 \leq q < 1$.

Let $\begin{pmatrix} x \\ f \end{pmatrix} \in D(\mathcal{A}_0)$ and let $0 < t < 1$. Then, using the notation of Proposition 12 and defining $M := \max \{ \sup_{s \in [0,1]} \|T(s)\|, 1 \}$ we have

$$\begin{aligned} \int_0^t \left\| \mathcal{A}_\Psi \mathcal{T}_0(r)v \right\|_{\mathcal{X}} dr &= \int_0^t \|\Psi(S_r x + S_0(r)f)\|_X dr \\ &= \int_0^t \|A_1(S_r x)(-\tau) + A_1 S_0(r)f(-\tau)\|_X dr \\ &\leq \|A_1\| \int_0^t \|T(-\tau+r)x\|_X dr + \|A_1\| \int_0^t \|f(-\tau+r)\|_X dr \\ &= \|A_1\| \int_{-\tau}^{-\tau+t} \|T(s)x\|_X ds + \|A_1\| \int_{-\tau}^{-\tau+t} \|f(s)\|_X ds \\ &\leq tM \|A_1\| \|x\|_X + \|A_1\| \left(\int_{-\tau}^{-\tau+t} \|f(s)\|_X^2 ds \right)^{\frac{1}{2}} \left(\int_{-\tau}^{-\tau+t} 1^2 ds \right)^{\frac{1}{2}} \\ &\leq tM \|A_1\| \|x\|_X + t^{\frac{1}{2}} \|A_1\| \|f\|_{L^2} \leq t^{\frac{1}{2}} M \|A_1\| (\|x\|_X + \|f\|_{L^2}) \\ &\leq (2t)^{\frac{1}{2}} M \|A_1\| \|v\|_{\mathcal{X}}, \end{aligned}$$

where we used Hölder's inequality and the fact that

$$(\|x\|_X + \|f\|_{L^2(-\tau,0;X)})^2 \leq 2(\|x\|_X^2 + \|f\|_{L^2(-\tau,0;X)}^2),$$

with $\|v\|_{\mathcal{X}} = (\|x\|_X^2 + \|f\|_{L^2(-\tau,0;X)}^2)^{\frac{1}{2}}$. Setting now t_0 small enough so that

$$q := (2t_0)^{\frac{1}{2}} M \|A_1\| < 1$$

we arrive at our conclusion. \square

Remark 14. The operator Ψ defined in (21) is a special case of a much wider class of operators that satisfy (14) and thus $(\mathcal{A}, D(\mathcal{A}))$ in (10) remains a generator of a strongly continuous semigroup $(\mathcal{T}(t))_{t \geq 0}$ on \mathcal{X} . For the proof of this general case see [5, Section 3.3.3].

We obtained results in Proposition 11 and Proposition 13 only by specifying a particular type of delay operator in the general setting of Section 2. Let us now specify the state space as $X := l^2$ with the standard orthonormal basis $(e_k)_{k \in \mathbb{N}}$, $(A, D(A))$ is a diagonal generator of a C_0 -semigroup $(T(t))_{t \geq 0}$ on X with a sequence of eigenvalues $(\lambda_k)_{k \in \mathbb{N}} \subset \mathbb{C}$ such that

$$\sup_{k \in \mathbb{N}} \operatorname{Re} \lambda_k < \infty, \quad (38)$$

and $A_1 \in \mathcal{L}(X)$ is a diagonal operator with a sequence of eigenvalues $(\gamma_k)_{k \in \mathbb{N}} \subset \mathbb{C}$. In other words, we introduce a finite-time state delay into the standard setting for diagonal systems [26, Chapter 2.6]. Hence, the C_0 -semigroup generator $(A, D(A))$ is given by

$$D(A) = \left\{ z \in l^2(\mathbb{C}) : \sum_{k \in \mathbb{N}} (1 + |\lambda_k|^2) |z_k|^2 < \infty \right\}, \quad (Az)_k = \lambda_k z_k. \quad (39)$$

Making use of the pivot duality, as the space X_1 we take $(D(A), \|\cdot\|_{gr})$, where the graph norm $\|\cdot\|_{gr}$ is equivalent to

$$\|z\|_1^2 = \sum_{k \in \mathbb{N}} (1 + |\lambda_k|^2) |z_k|^2.$$

The adjoint generator $(A^*, D(A^*))$ has the form

$$D(A^*) = D(A), \quad (A^*z)_k = \bar{\lambda}_k z_k. \quad (40)$$

The space X_{-1} consists of all sequences $z = (z_k)_{k \in \mathbb{N}} \subset \mathbb{C}$ for which

$$\sum_{k \in \mathbb{N}} \frac{|z_k|^2}{1 + |\lambda_k|^2} < \infty, \quad (41)$$

and the square root of the above series gives an equivalent norm on X_{-1} . By Proposition 6 the space X_{-1} can be written as $(D(A^*))'$. Note also that the operator $B \in \mathcal{L}(\mathbb{C}, X_{-1})$ is represented by the sequence $(b_k)_{k \in \mathbb{N}} \subset \mathbb{C}$ as $\mathcal{L}(\mathbb{C}, X_{-1})$ can be identified with X_{-1} .

This completes the description of the setting for a diagonal retarded system. From now on we consider system (1) reformulated as (7) and its Cauchy problem representation (8) as defined with the diagonal elements described in this section.

3.1 Analysis of a single component

Let us now focus on the k -th component of (1), namely

$$\begin{cases} \dot{z}_k(t) = \lambda_k z_k(t) + \gamma_k z_k(t - \tau) + b_k u(t) \\ z_k(0) = x_k, \\ z_{0_k} = f_k, \end{cases} \quad (42)$$

where $\lambda_k, \gamma_k, b_k, x_k \in \mathbb{C}$, $f_k := \langle f, l_k \rangle_{L^2(-\tau, 0; X)} l_k$ with l_k being the k -th component of an orthonormal basis in $L^2(-\tau, 0; X)$ (see [4, Chapter 3.5, p.138] for a description of such bases). Here b_k is the k th component of B .

For clarity of notation, until the end of this subsection, we drop the subscript k and rewrite (42) in the form

$$\begin{cases} \dot{z}(t) = \lambda z(t) + \Psi z_t + bu(t) \\ z(0) = x, \\ z_0 = f, \end{cases} \quad (43)$$

where the delay operator $\Psi \in \mathcal{L}(W^{1,2}(-\tau, 0; \mathbb{C}), \mathbb{C})$ is given by

$$\Psi(f) = \gamma f(-\tau) \quad \forall f \in W^{1,2}(-\tau, 0; \mathbb{C}). \quad (44)$$

The setting for the k -th component now includes the extended state space

$$\mathcal{X} := \mathbb{C} \times L^2(-\tau, 0; \mathbb{C}) \quad (45)$$

with an inner product

$$\left\langle \begin{pmatrix} x \\ f \end{pmatrix}, \begin{pmatrix} y \\ g \end{pmatrix} \right\rangle_{\mathcal{X}} := x\bar{y} + \langle f, g \rangle_{L^2(-\tau, 0; \mathbb{C})} \quad \forall \begin{pmatrix} x \\ f \end{pmatrix}, \begin{pmatrix} y \\ g \end{pmatrix} \in \mathcal{X}. \quad (46)$$

The Cauchy problem for the k -th component is

$$\begin{cases} \dot{v}(t) = \mathcal{A}v(t) + \mathcal{B}u(t) \\ v(0) = \begin{pmatrix} x \\ f \end{pmatrix}, \end{cases} \quad (47)$$

where $v : [0, \infty) \ni t \mapsto \begin{pmatrix} z \\ z_t \end{pmatrix} \in \mathcal{X}$ and \mathcal{A} is an operator on $D(\mathcal{A}) \subset \mathcal{X}$ defined as

$$D(\mathcal{A}) := \left\{ \begin{pmatrix} x \\ f \end{pmatrix} \in \mathbb{C} \times W^{1,2}(-\tau, 0; \mathbb{C}) : f(0) = x \right\}, \quad (48)$$

$$\mathcal{A} := \begin{pmatrix} \lambda & \Psi \\ 0 & \frac{d}{d\sigma} \end{pmatrix} \quad (49)$$

and $\mathcal{B} := \begin{pmatrix} b \\ 0 \end{pmatrix} \in \mathcal{L}(\mathbb{C}, \mathcal{X}_{-1})$. By Proposition 6 and Proposition 11 for the k -th component we have

$$\mathcal{X}_{-1} = D(\mathcal{A}^*)', \quad (50)$$

where

$$D(\mathcal{A}^*) = \left\{ \begin{pmatrix} y \\ g \end{pmatrix} \in \mathbb{C} \times W^{1,2}(-\tau, 0; \mathbb{C}) : \bar{\gamma}y = g(-\tau) \right\}, \quad (51)$$

$$\mathcal{A}^* \begin{pmatrix} y \\ g \end{pmatrix} = \begin{pmatrix} \bar{\lambda}y + g(0) \\ -\frac{d}{d\sigma}g \end{pmatrix}, \quad (52)$$

and $D(\mathcal{A}^*)'$ is the dual to $D(\mathcal{A}^*)$ with respect to the pivot space \mathcal{X} in (45). As the proof is essentially the same, we only state a k -th component version of Proposition 13, namely

Proposition 15. *The operator $(\mathcal{A}, D(\mathcal{A}))$ given by (48)–(49) generates a strongly continuous semigroup $(\mathcal{T}(t))_{t \geq 0}$ on \mathcal{X} given by (45). \square*

Now that we know that the k -th component Cauchy problem (47) is well-posed we can formally write its \mathcal{X}_{-1} -valued mild solution as

$$v(t) = \mathcal{T}(t)v(0) + \int_0^t \mathcal{T}(t-s)\mathcal{B}u(s) ds, \quad (53)$$

where the control operator is $\mathcal{B} = \begin{pmatrix} b \\ 0 \end{pmatrix} \in \mathcal{L}(\mathbb{C}, \mathcal{X}_{-1})$ and $\mathcal{T}(t) \in \mathcal{L}(\mathcal{X}_{-1})$ is the extension of the C_0 -semigroup generated by $(\mathcal{A}, D(\mathcal{A}))$ in (48)–(49).

The following, being a corollary from Proposition 3, gives the form of the k -th component resolvent $R(s, \mathcal{A})$.

Proposition 16. *For $s \in \mathbb{C}$ and for all $1 \leq p < \infty$ there is*

$$s \in \rho(\mathcal{A}) \text{ if and only if } s \in \rho(\lambda + \Psi_s). \quad (54)$$

Moreover, for $s \in \rho(\mathcal{A})$ the resolvent $R(s, \mathcal{A})$ is given by

$$R(s, \mathcal{A}) = \begin{pmatrix} R(s, \lambda + \Psi_s) & R(s, \lambda + \Psi_s)\Psi R(s, A_0) \\ \epsilon_s R(s, \lambda + \Psi_s) & (\epsilon_s R(s, \lambda + \Psi_s)\Psi + I)R(s, A_0) \end{pmatrix}, \quad (55)$$

where $R(s, \lambda + \Psi_s) \in \mathcal{L}(\mathbb{C})$,

$$R(s, \lambda + \Psi_s) = \frac{1}{s - \lambda - \gamma e^{-s\tau}} \quad \forall s \in \underline{\mathbb{C}}_{|\lambda|+|\gamma|} \quad (56)$$

and $R(s, A_0) \in \mathcal{L}(L^2(-\tau, 0; \mathbb{C}))$,

$$R(s, A_0)f(r) = \int_r^0 e^{s(r-t)} f(t) dt \quad r \in [-\tau, 0] \quad \forall s \in \underline{\mathbb{C}}_{|\lambda|+|\gamma|}. \quad (57)$$

Proof. The proof runs along the lines of [24, Proposition 3.3] with necessary adjustments for the forms of diagonal operators involved. \square

By Proposition 16 the resolvent component $R(s, \lambda + \Psi_s)$ is analytic in $\underline{\mathbb{C}}_{|\lambda|+|\gamma|}$. To ensure analyticity of $R(s, \lambda + \Psi_s)$ in \mathbb{C}_+ , as required to apply $H(\mathbb{C}_+)$ -based approach, we introduce the following sets.

Remark 17. We take the principal argument of λ to be $\text{Arg } \lambda \in (-\pi, \pi]$.

Let $\mathbb{D}_r \subset \mathbb{C}$ be an open disc centred at 0 with radius $r > 0$. We shall require the following subset of the complex plane, depending on $\tau > 0$ and $a \in (-\infty, \frac{1}{\tau}]$ and shown in Fig. 1, namely:

- for $a < 0$:

$$\Lambda_{\tau,a} := \left\{ \eta \in \mathbb{C} \setminus \mathbb{D}_{|a|} : \operatorname{Re} \eta + a < 0, |\eta| < |\eta_\pi|, \right. \\ \left. |\operatorname{Arg} \eta| > \tau \sqrt{|\eta|^2 - a^2} + \arctan \left(-\frac{1}{a} \sqrt{|\eta|^2 - a^2} \right) \right\} \cup \mathbb{D}_{|a|}, \quad (58)$$

where η_π is such that

$$\sqrt{|\eta_\pi|^2 - a^2} \tau + \arctan \left(-\frac{1}{a} \sqrt{|\eta_\pi|^2 - a^2} \right) = \pi;$$

- for $a = 0$:

$$\Lambda_{\tau,a} := \left\{ \eta \in \mathbb{C} \setminus \{0\} : \operatorname{Re} \eta < 0, |\eta| < \frac{\pi}{2\tau}, |\operatorname{Arg} \eta| > \tau |\eta| + \frac{\pi}{2} \right\}; \quad (59)$$

- for $0 < a \leq \frac{1}{\tau}$

$$\Lambda_{\tau,a} := \left\{ \eta \in \mathbb{C} : \operatorname{Re} \eta + a < 0, |\eta| < |\eta_\pi|, \right. \\ \left. |\operatorname{Arg} \eta| > \tau \sqrt{|\eta|^2 - a^2} + \arctan \left(-\frac{1}{a} \sqrt{|\eta|^2 - a^2} \right) + \pi \right\}, \quad (60)$$

where η_π is such that $|\eta_\pi| > a$ and

$$\sqrt{|\eta_\pi|^2 - a^2} \tau + \arctan \left(-\frac{1}{a} \sqrt{|\eta_\pi|^2 - a^2} \right) = 0.$$

The analyticity of $R(s, \lambda + \Psi_s)$ in \mathbb{C}_+ follows now from the following [18]

Proposition 18. *Let $\tau > 0$ and let $\lambda, \gamma, \eta \in \mathbb{C}$ such that $\lambda = a + ib \in \underline{\mathbb{C}}_{\frac{1}{\tau}}$. Then*

- (i) *every solution of the equation $s - a - \eta e^{-s\tau} = 0$ belongs to \mathbb{C}_- if and only if $\eta \in \Lambda_{\tau,a}$;*
- (ii) *every solution of*

$$s - \lambda - \gamma e^{-s\tau} = 0 \quad (61)$$

and its version with conjugate coefficients

$$s - \bar{\lambda} - \bar{\gamma} e^{-s\tau} = 0 \quad (62)$$

belongs to \mathbb{C}_- if and only if $\gamma e^{-ib\tau} \in \Lambda_{\tau,a}$.

In relation to the form of $R(s, \lambda + \Psi_s)$ consider the following technical result based on [27], originally stated for real coefficients, that for complex ones becomes

Lemma 19. *Let $\tau > 0$ and $\lambda, \gamma \in \mathbb{C}$ such that $\lambda = a + ib$ with $a \leq \frac{1}{\tau}$, $b \in \mathbb{R}$ and $\gamma e^{-ib\tau} \in \Lambda_{\tau,a}$. Then*

$$J := \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{|i\omega - \lambda - \gamma e^{-i\omega\tau}|^2} = \begin{cases} J_a, & |\gamma| < |a| \\ J_e, & |\gamma| = |a| \\ J_\gamma, & |\gamma| > |a| \end{cases} \quad (63)$$

where

$$J_a := \frac{1}{2\sqrt{a^2 - |\gamma|^2}} \times \\ \times \frac{e^{\sqrt{a^2 - |\gamma|^2}\tau} (a - \sqrt{a^2 - |\gamma|^2}) + e^{-\sqrt{a^2 - |\gamma|^2}\tau} (-a - \sqrt{a^2 - |\gamma|^2})}{2 \operatorname{Re}(\gamma e^{-ib\tau}) + e^{\sqrt{a^2 - |\gamma|^2}\tau} (a - \sqrt{a^2 - |\gamma|^2}) - e^{-\sqrt{a^2 - |\gamma|^2}\tau} (-a - \sqrt{a^2 - |\gamma|^2})}, \quad (64)$$

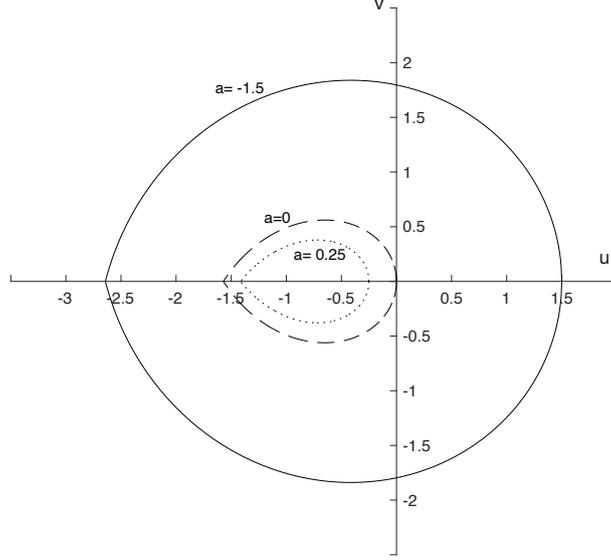


Figure 1: Outer boundaries for some $\Lambda_{\tau,a}$ sets, defined in (58)–(60) with $\eta = u + iv$, for $\tau = 1$ and different values of a : solid for $a = -1.5$, dashed for $a = 0$ and dotted for $a = 0.25$.

$$J_e := \frac{1}{2} \frac{a\tau - 1}{\operatorname{Re}(\gamma e^{-ib\tau}) + a}, \quad (65)$$

and

$$J_\gamma := \frac{1}{2\sqrt{|\gamma|^2 - a^2}} \times \frac{a \sin(\sqrt{|\gamma|^2 - a^2}\tau) - \sqrt{|\gamma|^2 - a^2} \cos(\sqrt{|\gamma|^2 - a^2}\tau)}{\operatorname{Re}(\gamma e^{-ib\tau}) + a \cos(\sqrt{|\gamma|^2 - a^2}\tau) + \sqrt{|\gamma|^2 - a^2} \sin(\sqrt{|\gamma|^2 - a^2}\tau)}. \quad (66)$$

The proof of Lemma 19 is a rather technical one and so it is in the Appendix section. We easily obtain

Corollary 20. *Let $\tau > 0$, $\lambda = 0$ and $\gamma \in \Lambda_{\tau,0}$. Then*

$$J_0 := \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{|i\omega - \gamma e^{-i\omega\tau}|^2} = -\frac{\cos(|\gamma|\tau)}{2(\operatorname{Re}(\gamma) + |\gamma| \sin(|\gamma|\tau))}. \quad (67)$$

□

Referring to (20) and the mild solution of the k -th component (53) the infinite-time forcing operator $\Phi_\infty \in \mathcal{L}(L^2(0, \infty; \mathbb{C}), \mathcal{X}_{-1})$ is given by

$$\Phi_\infty(u) := \int_0^\infty \mathcal{T}(t)\mathcal{B}u(t) dt, \quad (68)$$

where

$$\mathcal{T}(t)\mathcal{B} = \begin{pmatrix} \mathcal{T}_{11}(t) & \mathcal{T}_{12}(t) \\ \mathcal{T}_{21}(t) & \mathcal{T}_{22}(t) \end{pmatrix} \begin{pmatrix} b \\ 0 \end{pmatrix} = \begin{pmatrix} b\mathcal{T}_{11}(t) \\ b\mathcal{T}_{21}(t) \end{pmatrix}.$$

Hence the forcing operator (68) becomes

$$\Phi_\infty(u) = \begin{pmatrix} \int_0^\infty \mathcal{T}_{11}(t)bu(t) dt \\ \int_0^\infty \mathcal{T}_{21}(t)bu(t) dt \end{pmatrix} \in \mathcal{X}_{-1} = D(\mathcal{A}^*)'. \quad (69)$$

We can represent formally a similar product with the resolvent $R(s, \mathcal{A})$ from (55), namely

$$R(s, \mathcal{A})\mathcal{B} = \begin{pmatrix} R_{11}(s) & R_{12}(s) \\ R_{21}(s) & R_{22}(s) \end{pmatrix} \begin{pmatrix} b \\ 0 \end{pmatrix} = \frac{b}{s - \lambda - \gamma e^{-s\tau}} \begin{pmatrix} 1 \\ \epsilon_s \end{pmatrix}, \quad (70)$$

where the correspondence of sub-indices with elements of (55) is the obvious one and will be used from now on to shorten the notation.

The connection between the C_0 -semigroup $\mathcal{T}(t)$ and the resolvent $R(s, \mathcal{A})$ is given by the Laplace transform, whenever the integral converges, and

$$R(s, \mathcal{A})\mathcal{B} = \int_0^\infty e^{-sr} \mathcal{T}(r)\mathcal{B} dr = b \begin{pmatrix} \mathcal{L}(\mathcal{T}_{11})(s) \\ \mathcal{L}(\mathcal{T}_{21})(s) \end{pmatrix} \in \mathcal{L}(\mathbb{C}, \mathcal{X}_{-1}). \quad (71)$$

Theorem 21. *Suppose that for a given delay $\tau > 0$ there is $\lambda = a + i\beta \in \underline{\mathbb{C}}_{\frac{1}{\tau}}$ and $\gamma e^{-i\beta\tau} \in \Lambda_{\tau, a}$. Then the control operator $\mathcal{B} = \begin{pmatrix} b \\ 0 \end{pmatrix}$ for the system (47) is infinite-time admissible for every $u \in L^2(0, \infty; \mathbb{C})$ and*

$$\|\Phi_\infty(u)\|_{\mathcal{X}}^2 \leq (1 + \tau)|b|^2 J \|u\|_{L^2(0, \infty; \mathbb{C})}^2,$$

where J is given by (63).

Proof. 1. Let the standard inner product on $L^2(0, \infty; \mathbb{C})$ be given by $\langle f, g \rangle_{L^2(0, \infty; \mathbb{C})} = \int_0^\infty f(t)\bar{g}(t) dt$ for every $f, g \in L^2(0, \infty; \mathbb{C})$. Using (69) and (50) we may write for the first component of $\Phi_\infty(u)$

$$\int_0^\infty \mathcal{T}_{11}(t)bu(t) dt = b\langle \mathcal{T}_{11}, \bar{u} \rangle_{L^2(0, \infty; \mathbb{C})} \quad (72)$$

assuming that $\mathcal{T}_{11} \in L^2(0, \infty; \mathbb{C})$. This assumption is equivalent, due to Theorem 1, to $\mathcal{L}(\mathcal{T}_{11}) \in H^2(\mathbb{C}_+)$, where the last inclusion holds. Indeed, using (70) and (71) we see that $\mathcal{L}(\mathcal{T}_{11})(s) = bR_{11}(s) = \frac{b}{s - \lambda - \gamma e^{-s\tau}}$. The assumptions on λ and γ give that R_{11} is analytic in \mathbb{C}_+ . The boundary trace $R_{11}^* = \mathcal{L}(\mathcal{T}_{11})^*$ is given a.e. as

$$\mathcal{L}(\mathcal{T}_{11})^*(i\omega) = \frac{1}{i\omega - \lambda - \gamma e^{-i\omega\tau}}.$$

Lemma 19 now gives that $\mathcal{L}(\mathcal{T}_{11})^* \in L^2(i\mathbb{R})$ and thus, by (4), $R_{11} \in H^2(\mathbb{C}_+)$.

2. Again by Theorem 1 and definition of the inner product on $H^2(\mathbb{C})_+$ in (4) we have

$$b\langle \mathcal{T}_{11}, \bar{u} \rangle_{L^2(0, \infty; \mathbb{C})} = \frac{b}{2\pi} \int_{-\infty}^{+\infty} \frac{1}{i\omega - \lambda - \gamma e^{-i\omega\tau}} \overline{\mathcal{L}(\bar{u})^*}(i\omega) d\omega.$$

The Cauchy–Schwarz inequality now gives

$$\begin{aligned} & \left| b \left| \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{1}{i\omega - \lambda - \gamma e^{-i\omega\tau}} \overline{\mathcal{L}(\bar{u})^*}(i\omega) d\omega \right| \right| \\ & \leq |b| \left(\frac{1}{2\pi} \int_{-\infty}^{+\infty} \left| \frac{1}{i\omega - \lambda - \gamma e^{-i\omega\tau}} \right|^2 d\omega \right)^{\frac{1}{2}} \left(\frac{1}{2\pi} \int_{-\infty}^{+\infty} |\overline{\mathcal{L}(\bar{u})^*}(i\omega)|^2 d\omega \right)^{\frac{1}{2}} \\ & = |b| J^{\frac{1}{2}} \|u\|_{L^2(0, \infty; \mathbb{C})}, \end{aligned}$$

with J given by (63). Combining this result with point 1 we obtain

$$\left| \int_0^\infty \mathcal{T}_{11}(t)bu(t) dt \right|^2 \leq |b|^2 J \|u\|_{L^2(0, \infty; \mathbb{C})}^2. \quad (73)$$

3. Consider now the second element of the forcing operator (69), namely

$$\int_0^\infty \mathcal{T}_{21}(t)bu(t) dt \in W,$$

where we denote by W the second component of $\mathcal{X}_{-1} = D(\mathcal{A}^*)'$. If we assume that $\mathcal{T}_{21} \in L^2(0, \infty; W)$ then using the vector-valued version of Theorem 1 this is equivalent to $\mathcal{L}(\mathcal{T}_{21}) \in H^2(\mathbb{C}_+, W)$, but the last inclusion holds. Indeed, to show it notice that $R_{21} = \epsilon_s R_{11}$ where

$$\epsilon_s(\sigma) := e^{s\sigma}, \quad \sigma \in [-\tau, 0],$$

is, as a function of s , analytic everywhere for every value of σ , and follow exactly the reasoning in point 1.

4. We introduce an auxiliary function $\phi : [0, \infty) \rightarrow \mathbb{C}$. For that purpose fix $\mathcal{T}_{21} \in L^2(0, \infty; W)$ and $x_0 \in W$ and define $\phi(t) := \langle \mathcal{T}_{21}(t), x_0 \rangle_W$. Then $\phi \in L^2(0, \infty; \mathbb{C})$, as the Cauchy–Schwarz inequality gives

$$\int_0^\infty |\langle \mathcal{T}_{21}(t), x_0 \rangle_W|^2 dt \leq \int_0^\infty \|\mathcal{T}_{21}(t)\|_W^2 dt \|x_0\|_W^2 < \infty.$$

5. Consider now the following:

$$b \int_0^\infty \phi(t)u(t) dt = b \int_0^\infty \langle \mathcal{T}_{21}(t), x_0 \rangle_W u(t) dt = b \left\langle \int_0^\infty \mathcal{T}_{21}(t)u(t) dt, x_0 \right\rangle_W.$$

We also have

$$b \int_0^\infty \phi(t)u(t) dt = b \langle \phi, \bar{u} \rangle_{L^2(0, \infty; \mathbb{C})} = b \langle \mathcal{L}(\phi)^*, \mathcal{L}(\bar{u})^* \rangle_{L^2(i\mathbb{R})}.$$

To obtain the boundary trace $\mathcal{L}(\phi)^*$ notice that

$$\begin{aligned} \mathcal{L}(\phi)(s) &= \int_0^\infty e^{-sr} \langle \mathcal{T}_{21}(r), x_0 \rangle_W dr = \left\langle \int_0^\infty e^{-sr} \mathcal{T}_{21}(r) dr, x_0 \right\rangle_W \\ &= \langle \mathcal{L}(\mathcal{T}_{21})(s), x_0 \rangle_W = \langle R_{21}(s), x_0 \rangle_W. \end{aligned}$$

Using now (70) yields the result

$$\mathcal{L}(\phi)^*(i\omega) = \langle R_{21}^*(i\omega), x_0 \rangle_W = \left\langle \frac{\epsilon_{i\omega}}{i\omega - \lambda - \gamma e^{-i\omega\tau}}, x_0 \right\rangle_W.$$

Finally we obtain

$$\left\langle \int_0^\infty \mathcal{T}_{21}(t)u(t) dt, x_0 \right\rangle_W = \left\langle \frac{1}{2\pi} \int_{-\infty}^{+\infty} R_{21}^*(i\omega) \overline{\mathcal{L}(\bar{u})^*}(i\omega) d\omega, x_0 \right\rangle_W$$

and

$$\int_0^\infty \mathcal{T}_{21}(t)u(t) dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} R_{21}^*(i\omega) \overline{\mathcal{L}(\bar{u})^*}(i\omega) d\omega \in W. \quad (74)$$

6. By the definition of the norm on $L^2(-\tau, 0; \mathbb{C})$ we have

$$\begin{aligned} \|R_{21}^*(i\omega)\|_{L^2(-\tau, 0; \mathbb{C})}^2 &= \int_{-\tau}^0 \left| \frac{e^{i\omega t}}{i\omega - \lambda - \gamma e^{-i\omega\tau}} \right|^2 dt = \frac{1}{|i\omega - \lambda - \gamma e^{-i\omega\tau}|^2} \int_{-\tau}^0 |e^{i\omega t}|^2 dt \\ &= \frac{\tau}{|i\omega - \lambda - \gamma e^{-i\omega\tau}|^2}. \end{aligned}$$

The Cauchy–Schwarz inequality gives

$$\begin{aligned}
& |b| \left\| \frac{1}{2\pi} \int_{-\infty}^{+\infty} R_{21}^*(i\omega) \overline{\mathcal{L}(\bar{u})}^*(i\omega) d\omega \right\|_{L^2(-\tau, 0; \mathbb{C})} \\
& \leq |b| \frac{1}{2\pi} \int_{-\infty}^{+\infty} \|R_{21}^*(i\omega)\|_{L^2(-\tau, 0; \mathbb{C})} |\overline{\mathcal{L}(\bar{u})}^*(i\omega)| d\omega \\
& = |b| \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\tau^{\frac{1}{2}}}{|i\omega - \lambda - \gamma e^{-i\omega\tau}|} |\overline{\mathcal{L}(\bar{u})}^*(i\omega)| d\omega \\
& \leq |b| \left(\frac{1}{2\pi} \int_{-\infty}^{+\infty} \left(\frac{\tau^{\frac{1}{2}}}{|i\omega - \lambda - \gamma e^{-i\omega\tau}|} \right)^2 d\omega \right)^{\frac{1}{2}} \left(\frac{1}{2\pi} \int_{-\infty}^{+\infty} |\overline{\mathcal{L}(\bar{u})}^*(i\omega)|^2 d\omega \right)^{\frac{1}{2}} \\
& = |b| (\tau J)^{\frac{1}{2}} \|u\|_{L^2(0, \infty; \mathbb{C})},
\end{aligned}$$

with J given by (63). Combining this result with point 5 gives

$$\left\| \int_0^\infty \mathcal{T}_{21}(t) b u(t) dt \right\|_{L^2(-\tau, 0; \mathbb{C})}^2 \leq |b|^2 \tau J \|u\|_{L^2(0, \infty; \mathbb{C})}^2. \quad (75)$$

7. Taking now the norm $\|\cdot\|_{\mathcal{X}}$ resulting from (46) and using (69), (73), (75) and Lemma 19 we arrive at

$$\begin{aligned}
\|\Phi_\infty(u)\|_{\mathcal{X}}^2 &= \left\| \int_0^\infty \mathcal{T}_{11}(t) b u(t) dt \right\|_{\mathcal{X}}^2 + \left\| \int_0^\infty \mathcal{T}_{21}(t) b u(t) dt \right\|_{L^2(-\tau, 0; \mathbb{C})}^2 \\
&= (1 + \tau) |b|^2 J \|u\|_{L^2(0, \infty; \mathbb{C})}^2.
\end{aligned} \quad (76)$$

□

3.2 Analysis of the whole retarded delay system

Let us return to the diagonal system (1) reformulated as (8) with the extended state space $\mathcal{X} = l^2 \times L^2(-\tau, 0; l^2)$ and the control operator $\mathcal{B} \in \mathcal{L}(\mathbb{C}, \mathcal{X}_{-1})$. We also return to denoting the k -th component of the extended state space with the subscript. By Proposition 15 a mild solution of (42) is given by (53), that is $v_k : [0, \infty) \rightarrow \mathcal{X}$,

$$v_k(t) = \begin{pmatrix} z_k(t) \\ z_{t_k} \end{pmatrix} = \mathcal{T}_k(t) v_k(0) + \int_0^t \mathcal{T}_k(t-s) \mathcal{B}_k u(s) ds. \quad (77)$$

Given the structure of the Hilbert space $\mathcal{X} = l^2 \times L^2(-\tau, 0; l^2)$ in (6) the mild solution (77) has values in the subspace of \mathcal{X} spanned by the k -th element of its basis. Hence, defining $v : [0, \infty) \rightarrow \mathcal{X}$,

$$v(t) := \sum_{k \in \mathbb{N}} v_k(t), \quad (78)$$

we obtain the unique mild solution of (8). Using (78) and (6) we have

$$\begin{aligned}
\|v(t)\|_{\mathcal{X}}^2 &= \left\| \begin{pmatrix} z(t) \\ z_t \end{pmatrix} \right\|_{\mathcal{X}}^2 = \|z(t)\|_{l^2}^2 + \|z_t\|_{L^2(-\tau, 0; l^2)}^2 \\
&= \sum_{k \in \mathbb{N}} |z_k(t)|^2 + \sum_{k \in \mathbb{N}} |\langle z_t, l_k \rangle_{L^2(-\tau, 0; l^2)}|^2 \\
&= \sum_{k \in \mathbb{N}} \left(|z_k(t)|^2 + \|z_{t_k}\|_{L^2(-\tau, 0; \mathbb{C})}^2 \right) \\
&= \sum_{k \in \mathbb{N}} \|v_k(t)\|_{\mathcal{X}}^2,
\end{aligned} \quad (79)$$

where we used again (45) and notation from (42). We can formally write the mild solution (78) as a function $v : [0, \infty) \rightarrow \mathcal{X}_{-1}$,

$$v(t) = \mathcal{T}(t)v(0) + \int_0^t \mathcal{T}(t-s)\mathcal{B}u(s) ds. \quad (80)$$

where the control operator $\mathcal{B} \in \mathcal{L}(\mathbb{C}, \mathcal{X}_{-1})$ is given by $\mathcal{B} = \begin{pmatrix} (b_k)_{k \in \mathbb{N}} \\ 0 \end{pmatrix}$. We may now state the main theorem of this subsection.

Theorem 22. *Let for the given delay $\tau \in (0, \infty)$ sequences $(\lambda_k)_{k \in \mathbb{N}}$ and $(\gamma_k)_{k \in \mathbb{N}}$ be such that*

$$\lambda_k = a_k + i\beta_k \in \underline{\mathbb{C}}_{\frac{1}{\tau}} \quad \text{and} \quad \gamma_k e^{-i\beta_k \tau} \in \Lambda_{\tau, a_k} \quad \forall k \in \mathbb{N},$$

with Λ_{τ, a_k} defined in (58)–(60). Then the control operator $\mathcal{B} \in \mathcal{L}(\mathbb{C}, \mathcal{X}_{-1})$ given by $\mathcal{B} = \begin{pmatrix} (b_k)_{k \in \mathbb{N}} \\ 0 \end{pmatrix}$ is infinite-time admissible for system (8) if the sequence $(C_k)_{k \in \mathbb{N}} \in l^1$, where

$$C_k := |b_k|^2 J_k \quad (81)$$

and J_k is given by (63) for every (λ_k, γ_k) , $k \in \mathbb{N}$.

Proof. Define the infinite-time forcing operator for (80) as $\Phi_\infty : L^2(0, \infty) \rightarrow \mathcal{X}_{-1}$,

$$\Phi_\infty(u) := \int_0^\infty \mathcal{T}(t)\mathcal{B}u(t) dt.$$

From (78) it can be represented as

$$\Phi_\infty(u) = \sum_{k \in \mathbb{N}} \Phi_{\infty_k}(u), \quad (82)$$

where $\Phi_{\infty_k}(u)$ is given by

$$\Phi_{\infty_k}(u) := \int_0^\infty \mathcal{T}_k(t)\mathcal{B}_k u(t) dt, \quad k \in \mathbb{N}.$$

Then, similarly as in (79) and using the assumption we see that

$$\|\Phi_\infty(u)\|_{\mathcal{X}}^2 = \sum_{k \in \mathbb{N}} \|\Phi_{\infty_k}(u)\|_{\mathcal{X}}^2 \leq (1 + \tau) \left(\sum_{k \in \mathbb{N}} |C_k| \right) \|u\|_{L^2(0, \infty; \mathbb{C})}^2 < \infty.$$

□

Condition $(C_k)_{k \in \mathbb{N}} \in l^1$ of Theorem 22 may not be easy to verify given the form of J_k in (63). However, in certain situations the required condition follows from relatively simple relations between generator eigenvalues and a control sequence - see Example 4.1 below.

The l^1 -convergence condition was also used in [24], where the results are, in fact, a special case of the present reasoning. This can be seen in Sections 4.2 and 4.3 below.

4 Examples

A motivating example of a dynamical system is the heat equation with delay [21], [20] (or a diffusion model with a delay in the reaction term [30, Section 2.1]). Consider a homogeneous rod with zero temperature imposed on its both ends and its temperature change described by the following model

$$\begin{cases} \frac{\partial w}{\partial t}(x, t) = \frac{\partial^2 w}{\partial x^2}(x, t) + g(w(x, t - \tau)), & x \in (0, \pi), t \geq 0, \\ w(0, t) = 0, \quad w(\pi, t) = 0, & t \in [0, \infty), \\ w(x, 0) = w_0(x), & x \in (0, \pi), \\ w(x, t) = \varphi(x, t) & x \in (0, \pi), t \in [-\tau, 0], \end{cases} \quad (83)$$

where the temperature profile $w(\cdot, t)$ belongs to the state space $X = L^2(0, \pi)$, initial condition is formed by the initial temperature distribution $w_0 \in W^{2,2}(0, \pi) \cap W_0^{1,2}(0, \pi)$ and the

initial history segment $\varphi_0 \in W^{1,2}(-\tau, 0; X)$, the action of g is such that it can be considered as a linear and bounded diagonal operator on X . More precisely, consider first (83) without the delay term i.e. the classical one-dimensional heat equation setting [26, Chapter 2.6]. Define

$$D(A) := W^{2,2}(0, \pi) \cap W_0^{1,2}(0, \pi), \quad Az := \frac{d^2}{dx^2}z. \quad (84)$$

Note that $0 \in \rho(A)$. For $k \in \mathbb{N}$ let $\phi_k \in D(A)$, $\phi_k(x) := \sqrt{\frac{2}{\pi}} \sin(kx)$ for every $x \in (0, \pi)$. Then $(\phi_k)_{k \in \mathbb{N}}$ is an orthonormal Riesz basis in X and

$$A\phi_k = -k^2\phi_k \quad \forall k \in \mathbb{N}. \quad (85)$$

Introduce now the delay term $g : X \rightarrow X$, $g(z) := A_1z$ where $A_1 \in \mathcal{L}(X)$ is such that $A_1\phi_k = \gamma_k\phi_k$ for every $k \in \mathbb{N}$. We can now, using history segments, reformulate (83) into an abstract setting

$$\dot{z}(t) = Az(t) + A_1z_t(-\tau), \quad z(0) = w_0, \quad z_0 = \varphi_0. \quad (86)$$

Using standard Hilbert space methods and transforming system (86) into the l^2 space (we use the same notation for the l^2 version of (86)) and introducing control signal we obtain a retarded system of type (1). The most important aspect of the above example is the sequence of eigenvalues $(\lambda_k)_{k \in \mathbb{N}} = (-k^2)_{k \in \mathbb{N}}$, a characteristic feature of the heat equation. Although the above heat equation is expressed using a specific Riesz basis, the idea behind remains the same. More precisely - one can redo the reasoning leading to a version of Theorem 22 based on a general Riesz basis instead of the standard orthonormal basis in X . Such approach, however, would be based on the same ideas and would inevitably suffer from a less clear presentation, and so we refrain from it.

4.1 Eigenvalues with unbounded real part

Consider initially generators with unbounded real parts of their eigenvalues. For a given delay $\tau > 0$ let a diagonal generator $(A, D(A))$ have a sequence of eigenvalues $(\lambda_k)_{k \in \mathbb{N}}$ such that

$$\lambda_k = a_k + i\beta_k \in \mathbb{C}_- \quad \text{and} \quad a_k \rightarrow -\infty \text{ as } k \rightarrow \infty. \quad (87)$$

Let the operator $A_1 \in \mathcal{L}(X)$ be diagonal with a sequence of eigenvalues $(\gamma_k)_{k \in \mathbb{N}}$. Boundedness of A_1 implies that there exists $M < \infty$ such that $|\gamma_k| \leq M$ for every $k \in \mathbb{N}$. As A_1 is diagonal we easily get $|\gamma_k| \leq \|A_1\| \leq M$. Let the control operator B be represented by the sequence $(b_k)_{k \in \mathbb{N}} \subset \mathbb{C}$.

To use Theorem 22 we need to assure additionally that $\gamma_k e^{i\beta_k\tau} \in \Lambda_{\tau, a_k}$ for every $k \in \mathbb{N}$ and that the sequence $(C_k)_{k \in \mathbb{N}} = (|b_k|^2 J_{a_k})_{k \in \mathbb{N}} \in l^1$. However, for the former part we note that the boundedness of A_1 implies that there exists $N \in \mathbb{N}$ such that

$$|\gamma_k| < |a_k| \quad \forall k > N. \quad (88)$$

Fix such N . By the definition of $\Lambda_{\tau, a}$ in (58) we see that $(\gamma_k)_{k > N} \subset \Lambda_{\tau, a_N}$. Thus the only additional assumption on operator A_1 we need is

$$\gamma_k e^{-i\beta_k\tau} \in \Lambda_{\tau, a_k} \quad \forall k \leq N. \quad (89)$$

Assume that (87) and (89) hold. Then the sequence $(C_k)_{k \in \mathbb{N}} \in l^1$ if and only if

$$\sum_{k \geq N} |C_k| = \sum_{k \geq N} |b_k|^2 J_{a_k} < \infty,$$

where J_{a_k} is given by (64) for every $k \geq N$. Let us denote $r_k := \sqrt{a_k^2 - |\gamma_k|^2}$. As $k \rightarrow \infty$ we have

$$r_k \rightarrow \infty, \quad a_k - r_k \rightarrow -\infty, \quad a_k + r_k \rightarrow 0.$$

and thus we obtain

$$\begin{aligned}
\lim_{k \rightarrow \infty} \frac{|C_{k+1}|}{|C_k|} &= \lim_{k \rightarrow \infty} \frac{|b_{k+1}|^2 J_{a_{k+1}}}{|b_k|^2 J_{a_k}} = \lim_{k \rightarrow \infty} \frac{|b_{k+1}|^2}{|b_k|^2} \frac{r_k}{r_{k+1}} \times \\
&\quad \times \frac{e^{r_{k+1}\tau} (a_{k+1} - r_{k+1}) + e^{-r_{k+1}\tau} (-a_{k+1} - r_{k+1})}{2 \operatorname{Re}(\gamma_{k+1} e^{-ib_{k+1}\tau}) + e^{r_{k+1}\tau} (a_{k+1} - r_{k+1}) - e^{-r_{k+1}\tau} (-a_{k+1} - r_{k+1})} \times \\
&\quad \times \frac{2 \operatorname{Re}(\gamma_k e^{-ib_k\tau}) + e^{r_k\tau} (a_k - r_k) - e^{-r_k\tau} (-a_k - r_k)}{e^{r_k\tau} (a_k - r_k) + e^{-r_k\tau} (-a_k - r_k)} \\
&= \lim_{k \rightarrow \infty} \frac{|b_{k+1}|^2}{|b_k|^2} \frac{|a_k| \sqrt{1 - \frac{|\gamma_k|^2}{a_k^2}}}{|a_{k+1}| \sqrt{1 - \frac{|\gamma_{k+1}|^2}{a_{k+1}^2}}} \times \\
&\quad \times \frac{1 - e^{-2r_{k+1}\tau} \frac{a_{k+1} + r_{k+1}}{a_{k+1} - r_{k+1}}}{1 + e^{-r_{k+1}\tau} \frac{2 \operatorname{Re}(\gamma_{k+1} e^{-ib_{k+1}\tau})}{a_{k+1} - r_{k+1}} + e^{-2r_{k+1}\tau} \frac{a_{k+1} + r_{k+1}}{a_{k+1} - r_{k+1}}} \times \\
&\quad \times \frac{1 + e^{-r_k\tau} \frac{2 \operatorname{Re}(\gamma_k e^{-ib_k\tau})}{a_k - r_k} + e^{-2r_k\tau} \frac{a_k + r_k}{a_k - r_k}}{1 - e^{-2r_k\tau} \frac{a_k + r_k}{a_k - r_k}} \\
&= \lim_{k \rightarrow \infty} \frac{|b_{k+1}|^2}{|b_k|^2} \frac{|a_k|}{|a_{k+1}|},
\end{aligned}$$

provided that at least one of these limits exists. The above results clearly depends on a particular set of eigenvalues.

Let us now look at the abstract heat equation (86). The sequence of eigenvalues in (85) i.e. $(\lambda_k)_{k \in \mathbb{N}} = (-k^2)_{k \in \mathbb{N}}$ clearly satisfies (87). For such $(\lambda_k)_{k \in \mathbb{N}}$ we have

$$\lim_{k \rightarrow \infty} \frac{|C_{k+1}|}{|C_k|} = \lim_{k \rightarrow \infty} \frac{|b_{k+1}|^2}{|b_k|^2}, \quad (90)$$

provided that at least one limit exists. By the d'Alembert series convergence criterion $\lim_{k \rightarrow \infty} \frac{|C_{k+1}|}{|C_k|} < 1$ implies $(C_k)_{k \in \mathbb{N}} \in l^1$.

Take the delay $\tau = 1$ and assume that A_1 in (86) is such that (89) holds, i.e. there exists $N \in \mathbb{N}$ such that $\gamma_k \in \Lambda_{1, -k^2}$ for every $k \leq N$ and $\gamma_k \in \Lambda_{1, -N^2}$ for every $k > N$. Then, by Theorem 22 for $\mathcal{B} = \binom{(b_k)_{k \in \mathbb{N}}}{0}$ to be infinite-time admissible it is sufficient to take any $(b_k)_{k \in \mathbb{N}} \in l^2$ such that

$$\lim_{k \rightarrow \infty} \frac{|b_{k+1}|^2}{|b_k|^2} < 1.$$

Note the role of the "first" eigenvalues γ_k of A_1 which need to be inside consecutive $\Lambda_{\tau, -k^2}$ regions. As A_1 is a structural part of retarded system (86) it may not always be possible to apply Theorem 22.

4.2 Direct state-delayed diagonal systems

With small additional effort we can show that the so-called direct (or pure, see e.g. [3]) delayed system, where the delay is in the argument of the generator, is a special case of the problem analysed here. Thus we apply our admissibility results to a dynamical system analysed in [24] and given by

$$\begin{cases} \dot{z}(t) = Az(t - \tau) + Bu(t) \\ z(0) = x, \\ z_0 = f, \end{cases} \quad (91)$$

where $(A, D(A))$ is a diagonal generator of a C_0 -semigroup $(T(t))_{t \geq 0}$ on l^2 , B is a control operator, $0 < \tau < \infty$ is a delay and the control signal $u \in L^2(0, \infty; \mathbb{C})$. Let the sequence $(\lambda_k)_{k \in \mathbb{N}}$ of the eigenvalues of $(A, D(A))$ be such that $\sup_{k \in \mathbb{N}} \operatorname{Re} \lambda_k < 0$.

We construct a setting as the one in Section 3 and proceed with analysis of a k -th component, with a delay operator given again by point evaluation as $\Psi_k \in \mathcal{L}(W^{1,2}(-\tau, 0; \mathbb{C}), \mathbb{C})$, $\Psi_k(f) := \lambda_k f(-\tau)$ (we leave the index k on purpose) and it is bounded as λ_k is finite. The equivalent of (42) now reads

$$\begin{cases} \dot{z}_k(t) = \lambda_k z_k(t - \tau) + b_k u(t) \\ z_k(0) = x_k, \\ z_{0_k} = f_k, \end{cases} \quad (92)$$

where the role of γ_k in (42) is played by λ_k in (92), while λ_k of (42) is 0 in (92), and this holds for every k . Thus, instead of a collection $\{\Lambda_{\tau, a_k}\}_{k \in \mathbb{N}}$, we are concerned only with $\Lambda_{\tau, 0}$. Using now Corollary 20 instead of Lemma 19, the equivalent of Theorem 21 in the direct state-delayed setting takes the form

Theorem 23. *Let $\tau > 0$ and take $\lambda_k \in \Lambda_{\tau, 0}$. Then the control operator $\mathcal{B} = \begin{pmatrix} b_k \\ 0 \end{pmatrix}$ for the system based on (92) is infinite-time admissible for every $u \in L^2(0, \infty; \mathbb{C})$ and*

$$\|\Phi_\infty u\|_{\mathcal{X}}^2 \leq (1 + \tau) |b_k|^2 \frac{-\cos(|\lambda_k| \tau)}{2(\operatorname{Re}(\lambda_k) + |\lambda_k| \sin(|\lambda_k| \tau))} \|u\|_{L^2(0, \infty; \mathbb{C})}^2.$$

□

As Theorem 23 refers only to k -component it is an immediate consequence of Theorem 21. Using the same approach of summing over components the equivalent of Theorem 22 takes the form

Theorem 24. *Let for the given delay $\tau \in (0, \infty)$ the sequence $(\lambda_k)_{k \in \mathbb{N}} \subset \Lambda_{\tau, 0}$. Then the control operator $\mathcal{B} \in \mathcal{L}(\mathbb{C}, \mathcal{X}_{-1})$ for the system based on (91) and given by $\mathcal{B} = \begin{pmatrix} b_k \\ 0 \end{pmatrix}_{k \in \mathbb{N}}$ is infinite-time admissible if the sequence $(C_k)_{k \in \mathbb{N}} \in l^1$, where*

$$C_k := |b_k|^2 \frac{-\cos(|\lambda_k| \tau)}{2(\operatorname{Re}(\lambda_k) + |\lambda_k| \sin(|\lambda_k| \tau))}. \quad (93)$$

□

Note that the assumption that $\lambda_k \in \Lambda_{\tau, 0}$ for every $k \in \mathbb{N}$, due to boundedness of the $\Lambda_{\tau, 0}$ set, implies that A is in fact a bounded operator. While the result of Theorem 24 is correct, it is not directly useful in analysis of unbounded operators. Instead, its usefulness follows from the so-called reciprocal system approach. For a detailed presentation of the reciprocal system approach see [7], while for its application see [24]. We note here only that as there is some sort of symmetry in admissibility analysis of a given undelayed system and its reciprocal, introduction of a delay breaks this symmetry. In the current context consider the example of the next section.

Remark 25. In [24] the result corresponding to Theorem 24 uses a sequence $(C_k)_{k \in \mathbb{N}}$ which based not only on a control operator and eigenvalues of the generator, but also on some constants δ_k and m_k so that $C_k = C_k(b_k, \lambda_k, \delta_k, m_k)$. As δ_k and m_k originate from the proof of the result corresponding to Theorem 23, it requires additional effort to make the condition based on them useful. In the current form of Theorem 24 this problem does not exist and the convergence of (93) depends only on the relation between eigenvalues of the generator and the control operator.

4.3 Bounded real eigenvalues

In a diagonal framework of Example 4.2 let us consider, for a given delay τ , a sequence $(\lambda_k)_{k \in \mathbb{N}} \subset \mathbb{R} \cap \Lambda_{\tau, 0}$ such that $\lambda_k \rightarrow 0$ as $k \rightarrow \infty$. In particular, let $\lambda_k := (-\frac{\pi}{2} + \varepsilon)\tau^{-1}k^{-2}$ for some sufficiently small $0 < \varepsilon < \frac{\pi}{2}$. Such sequence of λ_k typically arises when considering a reciprocal system of a undelayed heat equation, as is easily seen by (85). The ratio of absolute values of two consecutive coefficients (93) is

$$\frac{|C_{k+1}|}{|C_k|} = \frac{|b_{k+1}|^2}{|b_k|^2} \frac{|\cos(|\lambda_{k+1}| \tau)|}{|\cos(|\lambda_k| \tau)|} \frac{|\lambda_k|}{|\lambda_{k+1}|} \frac{|1 - \sin(|\lambda_k| \tau)|}{|1 - \sin(|\lambda_{k+1}| \tau)|}.$$

It is easy to see that

$$\lim_{k \rightarrow \infty} \frac{|C_{k+1}|}{|C_k|} = \lim_{k \rightarrow \infty} \frac{|b_{k+1}|^2}{|b_k|^2}, \quad (94)$$

provided that at least one of these limits exists. By the d'Alembert series convergence criterion $\lim_{k \rightarrow \infty} \frac{|C_{k+1}|}{|C_k|} < 1$ implies $(C_k)_{k \in \mathbb{N}} \in l^1$. Thus, by (94) for $\mathcal{B} = \left(\binom{b_k}{0} \right)_{k \in \mathbb{N}}$ to be infinite-time admissible for system (91) it is sufficient to take any $(b_k)_{k \in \mathbb{N}} \in l^2$ such that

$$\lim_{k \rightarrow \infty} \frac{|b_{k+1}|^2}{|b_k|^2} < 1. \quad (95)$$

5 Appendix

5.1 Proof of Lemma 19

We rewrite J as

$$\begin{aligned} J &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{|i\omega - \lambda - \gamma e^{-i\omega\tau}|^2} \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{(i\omega - \lambda - \gamma e^{-i\omega\tau})(-i\omega - \bar{\lambda} - \bar{\gamma} e^{i\omega\tau})} \\ &= \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{ds}{(s - \lambda - \gamma e^{-s\tau})(-s - \bar{\lambda} - \bar{\gamma} e^{s\tau})} \\ &= \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} E_1(s)E_2(s) ds, \end{aligned} \quad (96)$$

where

$$E_1(s) := \frac{1}{s - \lambda - \gamma e^{-s\tau}}, \quad E_2(s) := \frac{1}{-s - \bar{\lambda} - \bar{\gamma} e^{s\tau}}. \quad (97)$$

Note that writing explicitly E_1 and E_2 as functions of s and parameters λ , γ and τ we have

$$E_1(s, \lambda, \gamma, \tau) = E_2(-s, \bar{\lambda}, \bar{\gamma}, \tau).$$

Let \mathcal{E}_1 be the set of poles of E_1 and \mathcal{E}_2 be the set of poles of E_2 . As, by assumption, $\gamma e^{-ib\tau} \in \Lambda_{\tau, a}$ Proposition 18 states that $\mathcal{E}_1 \subset \mathbb{C}_-$ and $\mathcal{E}_2 \subset \mathbb{C}_+$. Thus we have that E_1 is analytic in $\mathbb{C} \setminus \mathbb{C}_-$ while E_2 is analytic in $\mathbb{C} \setminus \mathbb{C}_+$.

Let $s_n \in \mathcal{E}_1$, i.e. $s_n - \lambda - \gamma e^{-s_n\tau} = 0$. Rearranging gives

$$\frac{\gamma}{s_n - \lambda} = e^{s_n\tau}.$$

Substituting above to E_2 gives

$$E_2(s_n) = -\frac{s_n - \lambda}{(s_n + \bar{\lambda})(s_n - \lambda) + |\gamma|^2}. \quad (98)$$

and this value is finite as $s_n \notin \mathcal{E}_2$. Rearranging (96) to account for (98) gives

$$J = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \left(E_1(s) \left(E_2(s) + \frac{s - \lambda}{(s + \bar{\lambda})(s - \lambda) + |\gamma|^2} \right) - E_1(s) \frac{s - \lambda}{(s + \bar{\lambda})(s - \lambda) + |\gamma|^2} \right) ds. \quad (99)$$

The above integrand has no poles at the roots $\{z_1, z_2\}$ of

$$(s + \bar{\lambda})(s - \lambda) + |\gamma|^2 = 0. \quad (100)$$

However, as the two parts of the integrand in (99) will be treated separately, we need to consider poles introduced by z_1 and z_2 with regard to the contour of integration. Rewrite also (100) as

$$(s + \bar{\lambda})(s - \lambda) + |\gamma|^2 = (s - z_1)(s - z_2) = 0. \quad (101)$$

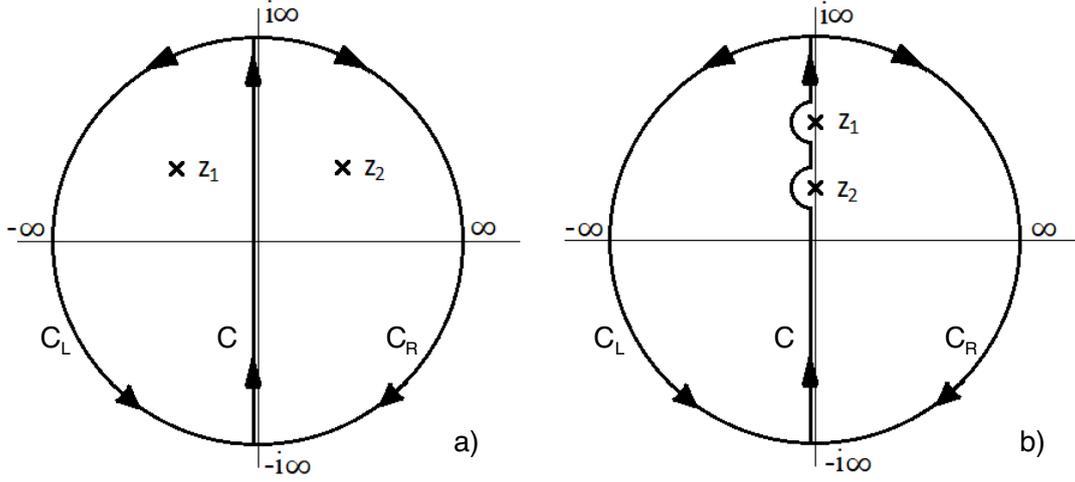


Figure 2: Contours of integration in (99): part *a*) is used for the case $|\gamma| < |a|$, part *b*) is used when $|\gamma| \geq |a|$. Both parts are drawn for a sufficiently large r so that $\Gamma_I(r) = C$, $\Gamma_L(r) = C_L$ and $\Gamma_R(r) = C_R$ and they enclose particular values of z_1 and z_2 in (102) and (110), respectively. The location of infinitesimally small semicircles around z_1 and z_2 in part *b*) is to be modified depending on the location of z_1 and z_2 on the imaginary axis.

From this point onwards we analyse three cases given by the right side of (63). Assume first that $|\gamma| < |a|$. Then

$$z_1 = -\sqrt{a^2 - |\gamma|^2} + ib, \quad z_2 = \sqrt{a^2 - |\gamma|^2} + ib. \quad (102)$$

Figure 2a shows integration contours $\Gamma_1(r) = \Gamma_I(r) + \Gamma_L(r)$ and $\Gamma_2(r) = \Gamma_I(r) + \Gamma_R(r)$ for $r \in (0, \infty)$ used for calculation of J . In particular Γ_I runs along the imaginary axis, Γ_L is a left semicircle and Γ_R is a right semicircle. Due to the above argument for a sufficiently large r we get

$$\begin{aligned} J &= \frac{1}{2\pi i} \lim_{r \rightarrow \infty} \int_{\Gamma_I(r)} \left(E_1(s) \left(E_2(s) + \frac{s - \lambda}{(s + \bar{\lambda})(s - \lambda) + |\gamma|^2} \right) - E_1(s) \frac{s - \lambda}{(s + \bar{\lambda})(s - \lambda) + |\gamma|^2} \right) ds \\ &= \frac{1}{2\pi i} \int_{C+C_L} E_1(s) \left(E_2(s) + \frac{s - \lambda}{(s - z_1)(s - z_2)} \right) ds \\ &\quad - \frac{1}{2\pi i} \int_{C+C_R} E_1(s) \frac{s - \lambda}{(s - z_1)(s - z_2)} ds. \end{aligned} \quad (103)$$

In calculation of the above we used the fact both integrals round the semicircles at infinity are zero as the integrands are, at most, of order s^{-2} and for every fixed $\varphi, \lambda, \gamma, \tau$,

$$\lim_{r \rightarrow \infty} \frac{1}{r e^{i\varphi} - \lambda - \gamma e^{-r e^{i\varphi} \tau}} = 0. \quad (104)$$

Define separate parts of (103) as

$$J_L := \frac{1}{2\pi i} \int_{C+C_L} E_1(s) \left(E_2(s) + \frac{s - \lambda}{(s - z_1)(s - z_2)} \right) ds \quad (105)$$

and

$$J_R := -\frac{1}{2\pi i} \int_{C+C_R} E_1(s) \frac{s - \lambda}{(s - z_1)(s - z_2)} ds \quad (106)$$

and consider them separately.

To calculate J_L note that from (98) it follows that for every $s_n \in \mathcal{E}_1$ the value

$$E_2(s_n) = -\frac{s_n - \lambda}{(s_n - z_1)(s_n - z_2)}$$

is finite and that implies that $\{z_1, z_2\} \cap \mathcal{E}_1 = \emptyset$. Thus the only pole of the integrand in (105) encircled by the $C + C_L$ contour is at z_1 . Denoting this integrand by f the residue formula gives

$$\text{Res}_{z_1} f(s) = \lim_{s \rightarrow z_1} (s - z_1) f(s) = E_1(z_1) \frac{z_1 - \lambda}{z_1 - z_2}.$$

As the $C + C_L$ contour is counter-clockwise we obtain

$$J_L = E_1(z_1) \frac{z_1 - \lambda}{z_1 - z_2}. \quad (107)$$

To calculate J_R note that the only pole encircled by the $C + C_R$ contour is at z_2 . Denoting the integrand of (106) by g the residue formula gives

$$\text{Res}_{z_2} g(s) = \lim_{s \rightarrow z_2} (s - z_2) g(s) = E_1(z_2) \frac{z_2 - \lambda}{z_2 - z_1}.$$

As the $C + C_R$ contour is clockwise we obtain

$$J_R = E_1(z_2) \frac{z_2 - \lambda}{z_2 - z_1}. \quad (108)$$

Thus we obtain

$$J = J_L + J_R = \frac{1}{z_2 - z_1} \left((\lambda - z_1) E_1(z_1) + (z_2 - \lambda) E_1(z_2) \right), \quad (109)$$

where z_1, z_2 are given by (102). We substitute these values for z_1 and z_2 and perform tedious calculations to obtain

$$\begin{aligned} J &= \frac{1}{2\sqrt{a^2 - |\gamma|^2}} \times \\ &\quad \frac{\gamma e^{-ib\tau} \left(e^{\sqrt{a^2 - |\gamma|^2}\tau} (a - \sqrt{a^2 - |\gamma|^2}) + e^{-\sqrt{a^2 - |\gamma|^2}\tau} (-a - \sqrt{a^2 - |\gamma|^2}) \right)}{\gamma e^{-ib\tau} \left(\bar{\gamma} e^{ib\tau} - e^{-\sqrt{a^2 - |\gamma|^2}\tau} (-\sqrt{a^2 - |\gamma|^2} - a) - e^{\sqrt{a^2 - |\gamma|^2}\tau} (\sqrt{a^2 - |\gamma|^2} - a) + \gamma e^{-ib\tau} \right)} \\ &= \frac{1}{2\sqrt{a^2 - |\gamma|^2}} \times \\ &\quad \frac{e^{\sqrt{a^2 - |\gamma|^2}\tau} (a - \sqrt{a^2 - |\gamma|^2}) + e^{-\sqrt{a^2 - |\gamma|^2}\tau} (-a - \sqrt{a^2 - |\gamma|^2})}{2 \text{Re}(\gamma e^{-ib\tau}) + e^{\sqrt{a^2 - |\gamma|^2}\tau} (a - \sqrt{a^2 - |\gamma|^2}) - e^{-\sqrt{a^2 - |\gamma|^2}\tau} (-a - \sqrt{a^2 - |\gamma|^2})}. \end{aligned}$$

Assume now that $|\gamma| > |a|$. The roots $\{z_1, z_2\}$ of (101) are

$$z_1 = i\sqrt{a^2 - |\gamma|^2} + ib, \quad z_2 = -i\sqrt{a^2 - |\gamma|^2} + ib. \quad (110)$$

To calculate J in (99) we now use the contour shown in Figure 2b. We again define J_L and J_R as in (105) and (106), respectively, but with this new contour.

As $\gamma e^{-ib\tau} \in \Lambda_{\tau, a}$ by Proposition 18 no pole of E_1 lies on the imaginary axis. Hence no pole of the integrand in (105) is encircled by the $C + C_L$ contour and this gives

$$J_L = 0. \quad (111)$$

For J_R the only poles of the integrand of (106) encircled by the $C + C_R$ contour are z_1 and z_2 . Denoting this integrand by g the residue formula gives

$$\text{Res}_{z_1} g(s) = E_1(z_1) \frac{z_1 - \lambda}{z_1 - z_2}, \quad \text{Res}_{z_2} g(s) = E_1(z_2) \frac{z_2 - \lambda}{z_2 - z_1}.$$

As the $C + C_R$ contour is clockwise we obtain

$$J_R = E_1(z_1) \frac{z_1 - \lambda}{z_1 - z_2} + E_1(z_2) \frac{z_2 - \lambda}{z_2 - z_1}. \quad (112)$$

Thus we obtain

$$J = J_L + J_R = \frac{1}{z_2 - z_1} \left((\lambda - z_1) E_1(z_1) + (z_2 - \lambda) E_1(z_2) \right), \quad (113)$$

where z_1, z_2 are given by (110). Substituting these values, again after tedious calculations, we obtain

$$\begin{aligned} J &= \frac{1}{2i\sqrt{|\gamma|^2 - a^2}} \times \\ &\quad \frac{a \left(e^{i\sqrt{|\gamma|^2 - a^2}\tau} - e^{-i\sqrt{|\gamma|^2 - a^2}\tau} \right) - i\sqrt{|\gamma|^2 - a^2} \left(e^{i\sqrt{|\gamma|^2 - a^2}\tau} + e^{-i\sqrt{|\gamma|^2 - a^2}\tau} \right)}{2 \operatorname{Re}(\gamma e^{-ib\tau}) + a \left(e^{i\sqrt{|\gamma|^2 - a^2}\tau} + e^{-i\sqrt{|\gamma|^2 - a^2}\tau} \right) - i\sqrt{|\gamma|^2 - a^2} \left(e^{i\sqrt{|\gamma|^2 - a^2}\tau} - e^{-i\sqrt{|\gamma|^2 - a^2}\tau} \right)} \\ &= \frac{1}{2\sqrt{|\gamma|^2 - a^2}} \times \\ &\quad \frac{a \sin(\sqrt{|\gamma|^2 - a^2}\tau) - \sqrt{|\gamma|^2 - a^2} \cos(\sqrt{|\gamma|^2 - a^2}\tau)}{\operatorname{Re}(\gamma e^{-ib\tau}) + a \cos(\sqrt{|\gamma|^2 - a^2}\tau) + \sqrt{|\gamma|^2 - a^2} \sin(\sqrt{|\gamma|^2 - a^2}\tau)}. \end{aligned}$$

For the last case assume that $|\gamma| = |a| > 0$, as the assumption $\gamma e^{-b\tau} \in \Lambda_{\tau,0}$ excludes the case $|a| = |\gamma| = 0$ because $0 \notin \Lambda_{\tau,0}$. Instead of $\{z_1, z_2\}$ we now have a single double root z_0 of (101) given by

$$z_0 = ib. \quad (114)$$

As z_0 lies on the imaginary axis we use the contour shown in Figure 2b tailored to the case $z_1 = z_2 = z_0$. Define J_L and J_R as in (105) and (106), respectively, but with the contour tailored for z_0 . For the same reasons as in (111) we have

$$J_L = 0. \quad (115)$$

For J_R the only pole of the integrand of (106) encircled by the $C + C_R$ contour is z_0 . Denoting this integrand by g the residue formula for a double root gives

$$\operatorname{Res}_{z_0} g(s) = \lim_{s \rightarrow z_0} \frac{d}{ds} \left((s - z_0)^2 g(s) \right) = \frac{(a\tau - 1)\gamma e^{-ib\tau}}{(a + \gamma e^{-ib\tau})^2}.$$

With the current assumption we have that $a^2 = \gamma\bar{\gamma}$. By this and the fact that the $C + C_R$ contour is clockwise we obtain

$$J_R = \frac{1}{2} \frac{a\tau - 1}{\operatorname{Re}(\gamma e^{-ib\tau}) + a}. \quad (116)$$

As $J = J_L + J_R$ this finishes the proof. \square

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7.2 Competing interests

The authors have no competing interests as defined by Springer, or other interests that might be perceived to influence the results and/or discussion reported in this paper.

7.3 Authors' contributions

JRP and RZ are responsible for the initial conception of the research and the approach method. RZ performed the research concerning every element needed for the single component as well as the whole system analysis. Examples for unbounded generators were provided by RK and RZ, while examples for direct state-delayed systems come from the work of JRP and RZ. Figures 1 - 2 were prepared by RZ. All authors participated in writing the manuscript. All authors reviewed the manuscript.

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