Expanding the propositional logic of a t-norm with truth-constants: completeness results for rational semantics

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Abstract

In this paper we consider the expansions of logics of a left-continuous t-norm with truth-constants from a subalgebra of the rational unit interval. From known results on standard semantics, we study completeness for these propositional logics with respect to chains defined over the rational unit interval with a special attention to the completeness with respect to the canonical chain, i.e. the algebra over $[0,1] \cap \mathbb{Q}$ where each truth-constant is interpreted in its corresponding rational truth-value. Finally, we study rational completeness results when we restrict ourselves to deductions between the so-called *evaluated formulae.*

Keywords: Mathematical Fuzzy Logic, Left-continuous t-norms, T-norm based logics, Truth-constants, Evaluated formulae, Real and Rational completeness

1 Introduction

Fuzzy logics primarily deal with a notion of *comparative truth* in the sense that their typical semantics consist of algebras of totally ordered, and hence *comparable*, truth-values. Moreover, their general algebraic semantics is usually such that every member is decomposable as a subdirect product of linearly ordered ones. Therefore, linearly ordered algebras of truth-values play a crucial *semantical* role in fuzzy logics.[1] By contrast, in the most popular fuzzy logic systems only the classical truth-values (1 and 0, for definitive truth and definitive falsity respectively) have a *syntactical* counterpart in the form of truth-constants $\overline{1}$ and $\overline{0}$, and thus the notion of *intermediate* or *partial* truth (corresponding to intermediate truth-values between the classical ones) is not syntactically stressed in these systems. Nevertheless, more expressive fuzzy logic systems with an explicit (i.e. syntactic) usage of intermediate truth-values may be required for some applications and they have been developed as well. The first one was proposed by Pavelka in [25]: a propositional many-valued logical system which turned out to be equivalent to the expansion of Łukasiewicz Logic Ł by adding into the language a truth-constant $\overline{r}$ for each *real* $r \in (0,1)$, together with a number of additional axioms. Although the resulting logic is not strongly complete with respect to the intended semantics defined by the Łukasiewicz t-norm (as it already happens in the original Łukasiewicz logic without truth-constants), Pavelka proved that his logic, denoted here PL, is indeed strongly

[1] See [1] for some rationale supporting the idea of fuzzy logics as the logics of chains.

complete but in a different sense. Namely, he defined the *truth degree* of a formula φ in a theory T as $\|\varphi\|_T = \inf\{e(\varphi) \mid e \text{ is a PL-evaluation model of } T\}$, and the *provability degree* of φ in T as $|\varphi|_T = \sup\{r \mid T \vdash_{\mathrm{PL}} \overline{r} \to \varphi\}$ and proved that these two degrees coincide. This kind of completeness is usually known as *Pavelka-style completeness*, and strongly relies on the continuity of Łukasiewicz truth-functions. Novák extended Pavelka's approach to a first-order logic [22]. Furthermore, Łukasiewicz logic extended with real truth-constants has been extensively developed by Nóvak and colleagues in the frame of the so-called *fuzzy logics with evaluated syntax* [23]. In this context, we also mention [27], where Turunen extended Pavelka's approach by expanding Łukasiewicz propositional logic with truth-constants from an arbitrary complete injective MV-algebra.

Hájek showed in [17] that Pavelka's logic PL could be significantly simplified while keeping the Pavelka-style completeness results. Indeed he showed that it is enough to extend the language only by a countable number of truth-constants, one constant $\overline{r}$ for each *rational* in $r \in (0, 1)$, and by adding to Łukasiewicz Logic the two following additional axiom schemata, called *book-keeping axioms*:

$$\overline{r}\&\overline{s} \leftrightarrow \overline{r *_{\text{Ł}} s}$$
$$(\overline{r} \to \overline{s}) \leftrightarrow \overline{r \Rightarrow_{\text{Ł}} s}$$

where $*_{\text{Ł}}$ and $\Rightarrow_{\text{Ł}}$ are the Łukasiewicz t-norm and its residuum respectively. He called this new system *Rational Pavelka Logic*, RPL for short. Moreover, he proved that RPL is strongly complete (in the usual sense) for finite theories.

Similar expansions for a big class of other propositional t-norm based fuzzy logics (and even for a few distinguished first-order t-norm based fuzzy logics) have been analogously defined in [12, 26, 9, 14, 4, 15], but Pavelka-style completeness could not be obtained, as Łukasiewicz Logic is the only t-norm based logic whose truth-functions are continuous. Thus, in these papers rather than Pavelka-style completeness the authors have focused on the usual notion of completeness of a logic with respect to a significant class of linearly ordered algebras. It has been shown that all those logics are algebraizable in the sense of [2] and they are complete with respect to the linearly ordered members of their equivalent algebraic semantics. Moreover, the completeness properties with respect to standard chains (chains defined over the real unit interval $[0, 1]$) for those logics and their restriction to the evaluated formulae (formulae $\overline{r} \to \varphi$ where no constant corresponding to an intermediate truth-value occurs on φ; they correspond to Novák's evaluated syntax) have been studied.

However, a coherent commitment with the notion of fuzzy logics as the logics of chains leads to the consideration of alternative semantics of linearly ordered algebras. There are actually several works (see [6, 8, 13]) where these semantics and the completeness properties that they yield have been studied. Some very significant results support the consideration of semantics based on rational chains as more powerful and versatile than the traditional standard semantics. For instance, it has been proved that whenever standard completeness holds for finite theories, the logic enjoys rational completeness for arbitrary theories (even when this is not true for the standard semantics). Furthermore, it has been shown that the completeness with respect to rational semantics is equivalent to completeness with respect to hyperreal semantics. On the other hand, in Hájek's setting Pavelka logics were seen as an expansion of a fuzzy logic with truth-constants for rational values, which allows an effective representation of truth-constants (that would not be possible if they would be irrational) and the study of the computational complexity of the logics as it has been done in [18, 14]. Therefore, the general consideration of t-norm based logics expanded with *rational* truth-constants and their *rational* completeness properties appears as a very natural development of the theory. This is what we intend to do in this paper.

The paper is structured as follows. After this introduction, in Sections 2.1 and 2.2 we recall some known standard completeness results about the logic of a (left-continuous) t-norm belonging to two particular families (denoted by **CONT-fin** and **WNM-fin**) and their expansions with a countable set of truth-constants. In Section 2.3 we focus on the completeness properties when restricting the deductions to evaluated formulae, and in particular we amend some results which were erroneously generalized in [9]. Sections 3 and 4 contain our new results about completeness of the expanded logics with respect to rational semantics. Namely, in Section 3 we study general completeness properties, with respect to both the semantics given by all rational chains and the semantics given by the canonical rational chain. Finally, in Section 4 we restrict ourselves to deductions among evaluated formulae and study again their (canonical) rational completeness properties.

2 Preliminaries

2.1 The logic of a t-norm L_*

The basic logic in this framework is the propositional Monoidal T-norm based Logic MTL [10], with primitive connectives & (multiplicative conjunction), $\to$ (implication), $\wedge$ (additive conjunction) and the truth-constant $\overline{0}$. MTL is in fact the logic of all left-continuous t-norms and their residua [19], in the sense that the set of its theorems is exactly $\bigcap\{Taut(*) \mid *$ is a left-continuous t-norm$\}$, where $Taut(*)$ denotes the set of tautologies when interpreting respectively &, $\to$ and $\wedge$ by the left-continuous t-norm $*$, its residuum $\Rightarrow$ and the min operation. This implies that MTL is the most general t-norm based fuzzy logic since left-continuity is the necessary and sufficient condition for a t-norm to have a residuum. Moreover, this logic is a generalization of Basic Fuzzy Logic BL previously defined by Hájek in [17] and proved to be the logic of all continuous t-norms and their residua in [5].

However, the expansion with truth-constants of a fuzzy logic is typically defined for a logic complete with respect to some particular standard algebra defined over the real unit interval by a t-norm and its residuum. Thus, we describe next the kinds of logics we need. In the following we will denote the standard algebra defined over the real unit interval by a t-norm $*$ and its residuum $\Rightarrow$ by $[0,1]_* = \langle [0,1], *, \Rightarrow, \min, \max, 0, 1\rangle$, and by L_* the axiomatic extension of MTL whose equivalent algebraic semantics is the variety generated by $[0,1]_*$. The members of this variety are called L_*-*algebras*.

There are three main examples of continuous t-norms:

1. Łukasiewicz t-norm: $a *_{\text{Ł}} b = \max\{0, a+b-1\}$
2. Product t-norm: $a *_{\Pi} b = a \cdot b$ (the usual product of real numbers)
3. Minimum (or Gödel) t-norm: $a *_{\mathrm{G}} b = \min\{a, b\}$

Their residua are respectively the following operations:

$$a \Rightarrow_{\text{Ł}} b = \begin{cases} 1, & \text{if } a \leq b, \\ 1-a+b, & \text{otherwise.} \end{cases}$$

$$a \Rightarrow_{\Pi} b = \begin{cases} 1, & \text{if } a \leq b, \\ b/a, & \text{otherwise.} \end{cases}$$

$$a \Rightarrow_{G} b = \begin{cases} 1, & \text{if } a \leq b, \\ b, & \text{otherwise.} \end{cases}$$

They are the most prominent examples of continuous t-norms because it is possible to describe all continuous t-norms in terms of ordinal sums of these three distinguished ones (as proved in [21, 20]). Amongst them, we denote as **CONT-fin** the set of continuous t-norms which are decomposable as a *finite* ordinal sum of the three basic ones. Relevant examples of these logics are Ł $\oplus$ G and Ł $\oplus$ Π, which correspond, respectively, to the ordinal sum of $*_{Ł}$ and $*_{G}$, and the ordinal sum of $*_{Ł}$ and $*_{\Pi}$. In [11] it is proved that for each $* \in$ **CONT-fin** the logic L_* is finitely axiomatizable.

Interesting examples of left-continuous (and in general non-continuous) t-norms are the weak nilpotent minimum t-norms (WNM-t-norms from now on) introduced in [10]. Given a negation function n (a function $n : [0,1] \to [0,1]$, such that $n(1) = 0$, it is order-reversing, and $a \leq n(n(a))$ for every a, as defined in [7]) and $a, b \in [0,1]$, the WNM-t-norm $*_n$ is defined as:

$$a *_n b = \begin{cases} \min\{a, b\}, & \text{if } a > n(b), \\ 0, & \text{otherwise.} \end{cases}$$

and its residuum by:

$$a \Rightarrow_n b = \begin{cases} 1, & \text{if } a \leq b, \\ \max\{n(a), b\}, & \text{otherwise.} \end{cases}$$

Notice that $*_G$ is a WNM-t-norm (the only continuous one). Other important particular cases are the so-called *nilpotent minimum t-norm*, $*_{\mathrm{NM}} = *_n$ when n is the involutive negation $n(x) = 1 - x$, and the WNM-t-norms $*_{n_1}$ and $*_{n_2}$ defined respectively by the negation functions[2]:

$$n_1(a) = \begin{cases} 1, & \text{if } a = 0, \\ \frac{1}{2}, & \text{if } 0 < a \leq \frac{1}{2}, \\ 0, & \text{if } \frac{1}{2} < a \leq 1. \end{cases}$$

$$n_2(a) = \begin{cases} 1 - x, & \text{if } 0 \leq a \leq \frac{1}{3}, \\ \frac{2}{3}, & \text{if } \frac{1}{3} < a \leq \frac{2}{3}, \\ 1 - x, & \text{if } \frac{2}{3} < a \leq 1. \end{cases}$$

Weak nilpotent minimum t-norms and their associated logics have been intensively studied in [24] with special attention to a particular subclass with a nice structure: those satisfying the Finite Partition Property (FPP, for short) which, roughly speaking, means that their defining negation functions admit a piecewise description by means of finite sets of intervals in which it is either constant or involutive, as in the mentioned four examples. We denote by **WNM-fin** the set of all WNM-t-norms satisfying the FPP. In [24] a method to axiomatize a large family of logics L_* for $* \in$ **WNM-fin** is given.

[2]Notice that we could consider parameterized isomorphic t-norms $*^c_{n_1}$ or $*^c_{n_2}$: in the first the case by defining the negation using any $c \in (0,1)$ instead of $\frac{1}{2}$, and in the second case by using any $c \in (\frac{1}{2}, 1)$ instead of $\frac{2}{3}$ (and $1 - c$ instead of $\frac{1}{3}$).

Given a logic L_* and a class $\mathbb{K}$ of L_*-chains, one defines three completeness properties:

- L_* has the property of *strong* $\mathbb{K}$-*completeness*, S$\mathbb{K}$C for short, if for every set Γ of formulae and every formula φ, $\Gamma \vdash_{L_*} \varphi$ iff $\Gamma \models_{\mathbb{K}} \varphi$.
- L_* has the property of *finite strong* $\mathbb{K}$-*completeness*, FS$\mathbb{K}$C for short, if for every *finite* set Γ of formulae and every formula φ, $\Gamma \vdash_{L_*} \varphi$ iff $\Gamma \models_{\mathbb{K}} \varphi$.
- L_* has the property of $\mathbb{K}$-*completeness*, $\mathbb{K}$C for short, if for every formula φ, $\vdash_{L_*} \varphi$ iff $\models_{\mathbb{K}} \varphi$.

Obviously, S$\mathbb{K}$C implies FS$\mathbb{K}$C, and this in turn implies $\mathbb{K}$C. If $\mathbb{K}$ is the class of all chains over the real unit interval $[0,1]$ we use the notation $\mathcal{R}$C and call the properties *standard completeness*, while if it is the class of all chains over the rational unit interval $[0,1]^{\mathbb{Q}} = [0,1] \cap \mathbb{Q}$ we use the notation $\mathcal{Q}$C and call the properties *rational completeness.* Observe that for every left-continuous t-norm $*$, the logic L_* satisfies the $\mathcal{R}$C by definition. Moreover, there is the following general result connecting standard and rational completeness:

Theorem 2.1 ([8])**.** *Let $*$ be a left-continuous t-norm. If* L_* *has the* FS$\mathcal{R}$C*, then it has the* S$\mathcal{Q}$C.

A summary of completeness properties for the logics mentioned so far is given in Table 1.

Logic	FS$\mathcal{R}$C	S$\mathcal{R}$C	S$\mathcal{Q}$C
MTL	Yes	Yes	Yes
BL	Yes	No	Yes
Ł	Yes	No	Yes
Π	Yes	No	Yes
G	Yes	Yes	Yes
Ł $\oplus$ G	Yes	No	Yes
Ł $\oplus$ Π	Yes	No	Yes
WNM	Yes	Yes	Yes
NM	Yes	Yes	Yes
$L_{*_{n_1}}$	Yes	Yes	Yes
$L_{*_{n_2}}$	Yes	Yes	Yes

Table 1: Standard and rational completeness properties for some prominent propositional fuzzy logics.

2.2 Adding truth-constants

Let $*$ be a left-continuous t-norm and $\Rightarrow$ its residuum. Let $\mathcal{C} = \langle C, *, \Rightarrow, \min, \max, 0, 1\rangle$ be a *countable* subalgebra of $[0,1]_*$. Then $L_*(\mathcal{C})$ is the propositional fuzzy logic defined as follows:

(i) the language of $L_*(\mathcal{C})$ is that of L_* expanded with a new propositional constant $\overline{r}$ for each $r \in C \setminus \{0,1\}$,

(ii) the axioms of $L_*(\mathcal{C})$ are those of L_* plus the book-keeping axioms:

$$\overline{r}\&\overline{s} \leftrightarrow \overline{r * s}$$
$$(\overline{r} \to \overline{s}) \leftrightarrow \overline{r \Rightarrow s}$$

for each $r, s \in C$.

(iii) the only inference rule of $L_*(\mathcal{C})$ is Modus Ponens.

Its algebraic counterpart, called $L_*(\mathcal{C})$-*algebras*, are the expansions of L_*-algebras with nullary functions $\overline{r}^{\mathcal{A}}$ (one for each $r \in C$) satisfying the book-keeping axioms, i.e. for every $r, s \in C$ the following identities hold:

$$\overline{r}^{\mathcal{A}} \&^{\mathcal{A}} \overline{s}^{\mathcal{A}} = \overline{r * s}^{\mathcal{A}}$$
$$\overline{r}^{\mathcal{A}} \to^{\mathcal{A}} \overline{s}^{\mathcal{A}} = \overline{r \Rightarrow s}^{\mathcal{A}}.$$

$L_*(\mathcal{C})$-chains defined over the real unit interval $[0, 1]$ are called *standard*. Among them there is one which reflects the intended standard semantics, the so-called *canonical standard* $L_*(\mathcal{C})$-chain $[0, 1]_{L_*(\mathcal{C})}$ which is the standard chain over $[0, 1]_*$ where the truth-constants are interpreted by their defining values, i.e. if $\mathcal{A}$ denotes $[0, 1]_{L_*(\mathcal{C})}$ one has $\overline{r}^{\mathcal{A}} = r$ for all $r \in C$. It is worth to point out that for a logic $L_*(\mathcal{C})$ there may exist multiple standard chains as long as there exist different ways of interpreting the truth-constants on $[0, 1]$ respecting the book-keeping axioms. The mapping that sends each truth-constant to its interpretation in an $L_*(\mathcal{C})$-chain $\mathcal{A}$ is a homorphism from $\mathcal{C}$ to $\mathcal{A}$ and thus it is univocally determined by its kernel, which is the filter $\{r \in \mathcal{C} \mid \overline{r}^{\mathcal{A}} = \overline{1}^{\mathcal{A}}\}$. Thus, the existence of different non-trivial filters allows the existence of different standard $L_*(\mathcal{C})$-chains. For instance, if $\mathcal{A} = [0, 1]_{L_*}$ is the standard chain of the Ł, Π, G or NM logics, a possible way of expanding such a chain with truth-constants from $\mathcal{C}$ is interpreting them as follows:

$$\overline{r}^{\mathcal{A}} = \begin{cases} 1, & \text{if } r \in F \\ 0, & \text{if } \neg r \in F \\ r, & \text{otherwise} \end{cases}$$

where F is any proper filter of $\mathcal{C}$. The resulting standard chain is denoted $[0, 1]^F_{L_*(\mathcal{C})}$. In the case of Ł the only proper filter is the trivial one $\{1\}$ and hence the only standard chain is the canonical one. In the case of Π there are two filters, $(0, 1]$ and $\{1\}$, and thus, besides the canonical chain there is another one where all truth-constants $\overline{r}$ with $r > 0$ are interpreted as 1. For G (resp. NM) there are infinitely many filters: all the intervals $[c, 1]$ for every $c > 0$ (resp. $c > \frac{1}{2}$) and $(c, 1]$ for every $c \geq 0$ (resp. $c \geq \frac{1}{2}$). In that case there are infinitely-many standard chains. The situation for arbitrary left-continuous t-norms can be much more complex, but the next proposition shows that the construction above still remains valid for a large family of left-continuous t-norms.

Proposition 2.2 ([14]). *Let $* \in$ **CONT-fin**$\cup$**WNM-fin** and let $\mathcal{C}$ be a countable subalgebra of $[0, 1]_*$. Then, for every filter F of $\mathcal{C}$, there exists a standard $L_*(\mathcal{C})$-chain $\mathcal{A}$ such that for every $r \in C$, $\overline{r}^{\mathcal{A}} = 1$ if $r \in F$, $\overline{r}^{\mathcal{A}} = 0$ if $\neg r \in F$, and $\overline{r}^{\mathcal{A}} = r$ otherwise. This chain will be denoted as $[0, 1]^F_{L_*(\mathcal{C})}$ and called the* standard chain of type F.

Completeness properties w.r.t. both the class of all standard chains and the canonical standard chain (denoted as Can$\mathcal{R}$C, CanFS$\mathcal{R}$C and CanS$\mathcal{R}$C) have been studied and solved in the literature for all those logics $L_*(\mathcal{C})$ such that:

(1) $* \in$ **CONT-fin** $\cup$ **WNM-fin**, and

(2) $\mathcal{C}$ is a countable subalgebra of $[0,1]_*$ such that C has elements in the interior of each component of the ordinal sum (in the case of continuous t-norms) or in the interior of each interval of the partition (in the case of WNM-t-norms).

The results are gathered in Table 2. Note that, although not shown in the table, none of the logics $\mathrm{L}_*(\mathcal{C})$ enjoys the CanS$\mathcal{R}$C. Moreover, the strong standard completeness for weak nilpotent minimum logics can be refined in the following way:

Proposition 2.3 ([14]). *For every* $* \in$ **WNM-fin** *and every countable* $\mathcal{C} \subseteq [0,1]_*$*, the logic* $\mathrm{L}_*(\mathcal{C})$ *has the* S$\mathbb{K}$C*, where* $\mathbb{K} = \{[0,1]^F_{\mathrm{L}_*(\mathcal{C})} \mid F$ *is a filter of* $\mathcal{C}\}$.

2.3 About real completeness properties restricted to evaluated formulae

In the cases where the standard completeness fails[3] one can try to improve the situation by restricting deductions among to the so-called *evaluated formulae*, i.e. formulae $\overline{r} \to \varphi$ where no additional truth-constant occurs in φ. Actually, one must also require an additional constraint on the values r as next proposition shows.

Proposition 2.4. *For any logic* $\mathrm{L}_*(\mathcal{C})$ *with* $* \in$ **CONT-fin** $\cup$ **WNM-fin***, if there exists* $r \in C \setminus \{0\}$ *such that* r *is negative (i.e.* $r \leq \neg r$*) and there is a filter* F *such that* $\neg r \in F$*, then* $\mathrm{L}_*(\mathcal{C})$ *does not enjoy the Can*FS$\mathcal{R}$C.

Proof. Under the hypotheses of the proposition, it is easy to check that

$$\overline{r} \to \neg(\varphi \to \psi) \models_{[0,1]_{\mathrm{L}_*(\mathcal{C})}} \psi \to \varphi$$

holds true but

$$\overline{r} \to \neg(\varphi \to \psi) \not\models_{[0,1]^F_{\mathrm{L}_*(\mathcal{C})}} \psi \to \varphi$$

since $\neg\overline{r}$ is interpreted as 1 (and hence $\overline{r}$ interpreted to 0) on the algebra $[0,1]^F_{\mathrm{L}_*(\mathcal{C})}$, and thus the premise is always true while this is not the case for $\psi \to \varphi$. $\square$

Indeed, this proposition does not put any restriction in the case of logics $\mathrm{L}_*(\mathcal{C})$ with $* \in$ **CONT-fin** since $C \setminus \{0\}$ either does not have negative elements or the negation of the negative elements belong to a Łukasiewicz component, and thus they cannot belong to a non-trivial filter. On the other hand, when $* \in$ **WNM-fin** the proposition restricts the search for completeness results to deductions among *positively evaluated* formulae, i.e. formulae $\overline{r} \to \varphi$ where $r > \neg r$, as done in [12, 14].

A second (negative) result that deserves some comments is about the CanS$\mathcal{R}$C property. When $\mathcal{C} = [0,1] \cap \mathbb{Q}$, it is easy to notice that all the logics $\mathrm{L}_*(\mathcal{C})$ under our scope fail to satisfy the CanS$\mathcal{R}$C restricted to evaluated formulae as it can be seen with the following counterexample (already used in [9] in the continuous t-norm case). Let $\Gamma = \{\overline{(\frac{n}{n+1})} \to \varphi \mid n \in \mathbb{N}\}$. For every logic $\mathrm{L}_*(\mathcal{C})$ we have $\Gamma \models_{[0,1]^{\mathbb{Q}}_{\mathrm{L}_*(\mathcal{C})}} \varphi$. But if $\Gamma \vdash_{\mathrm{L}_*(\mathcal{C})} \varphi$ then, since the logic is finitary, there would exist $n_0 \in \mathbb{N}$ such that $\overline{(\frac{n_0}{n_0+1})} \to \varphi \vdash_{\mathrm{L}_*(\mathcal{C})} \varphi$, hence, we would have $\overline{(\frac{n_0}{n_0+1})} \to \varphi \models_{[0,1]^{\mathbb{Q}}_{\mathrm{L}_*(\mathcal{C})}} \varphi$; a contradiction. However, in [9] it is erroneously claimed that

[3] Looking at Table 2, this is the case of the S$\mathcal{R}$C and the Canonical completeness properties for some of the logics under our scope.

Logic	FS$\mathcal{R}$C	S$\mathcal{R}$C	Can$\mathcal{R}$C	Can(FS/S)$\mathcal{R}$C
Ł($\mathcal{C}$)	Yes	No	Yes	Yes / No
$\Pi(\mathcal{C})$	Yes	No	Yes	No
G($\mathcal{C}$)	Yes	Yes	Yes	No
(Ł $\oplus$ G)($\mathcal{C}$), $a \notin C$	Yes	No	Yes	No
(Ł $\oplus$ Π)($\mathcal{C}$), $a \notin C$	Yes	No	Yes	No
Other $\mathrm{L}_*(\mathcal{C})$ with $* \in$ **CONT-fin**	Yes	No	No	No
NM($\mathcal{C}$)	Yes	Yes	Yes	No
$\mathrm{L}_{*_{n_1}}(\mathcal{C})$	Yes	Yes	Yes	No
$\mathrm{L}_{*_{n_2}}(\mathcal{C})$	Yes	Yes	Yes	No
Other $\mathrm{L}_*(\mathcal{C})$ with $* \in$ **WNM-fin**	Yes	Yes	No	No

Table 2: Standard completeness properties for propositional fuzzy logics with truth-constants (a denotes the element separating both components in the ordinal sums Ł $\oplus$ G and Ł $\oplus$ Π).

this counterexample applies for all logics $\mathrm{L}_*(\mathcal{C})$, independently of the particular algebra of truth-values $\mathcal{C}$. Indeed, one can use an analogous counterexample for the case the algebra $\mathcal{C}$ has an accumulation point r which is the supremum of a strictly increasing sequence $\langle r_i \rangle_{i \in \mathbb{N}}$ of points of C. We call sup-*accessible* such an accumulation point r. But the CanS$\mathcal{R}$C may hold for logics $\mathrm{L}_*(\mathcal{C})$ with $* \in$ **WNM-fin** where $\mathcal{C}$ does not have sup-accessible elements as it will be proved in Theorem 4.4 in the case of rational semantics; the same proof applies to the real semantics as well.

The obtained results in [9, 14] are summarised in Table 3, but notice that we have purposely left empty some slots of the CanS$\mathcal{R}$C property for the logics G($\mathcal{C}$), NM($\mathcal{C}$), $\mathrm{L}_{*_{n_1}}(\mathcal{C})$ and $\mathrm{L}_{*_{n_2}}(\mathcal{C})$, which were wrongly reported in [14] papers for some kinds of truth-constants algebras $\mathcal{C}$. The table can be fully completed by taking into account that the missing cases coincide with the ones given in Table 5 for CanS$\mathcal{Q}$C.

3 Rational completeness results: the general case

Let $*$ be a left-continuous t-norm and $\Rightarrow$ its residuum such that the rational unit interval $[0,1]^{\mathbb{Q}}$ is closed under the operations $*$ and $\Rightarrow$. Let $[0,1]^{\mathbb{Q}}_*$ be the L_*-chain defined by the restriction of $*$ and $\Rightarrow$ to $[0,1]^{\mathbb{Q}}$. Let $\mathcal{C}$ be a subalgebra of $[0,1]^{\mathbb{Q}}_*$ and consider the logic $\mathrm{L}_*(\mathcal{C})$. Now $\mathrm{L}_*(\mathcal{C})$-chains defined over the rational unit interval are called *rational chains* and among them, the one which reflects the intended rational semantics is the so-called *canonical rational* $\mathrm{L}_*(\mathcal{C})$-chain

$$[0,1]^{\mathbb{Q}}_{\mathrm{L}_*(\mathcal{C})} = \langle [0,1]^{\mathbb{Q}}, *, \Rightarrow, \min, \max, \langle r : r \in C \rangle \rangle,$$

i.e. the rational chain over $[0,1]^{\mathbb{Q}}_*$ where the truth-constants are interpreted by their defining values.

The possible interpretations of truth-constants over rational chains can be described in the same way as in the real case and, hence, algebras $([0,1]^{\mathbb{Q}})^F_{\mathrm{L}_*(\mathcal{C})}$ are defined analogously.

Logic	S$\mathcal{R}$C	Can$\mathcal{R}$C	CanFS$\mathcal{R}$C	CanS$\mathcal{R}$C
Ł($\mathcal{C}$)	No	Yes	Yes	No
Π($\mathcal{C}$)	No	Yes	Yes	No
G($\mathcal{C}$)	Yes	Yes	Yes	–
(Ł ⊕ G)($\mathcal{C}$), $a \notin C$	No	Yes	Yes	No
(Ł ⊕ Π)($\mathcal{C}$), $a \notin C$	No	Yes	Yes	No
Other $L_*(\mathcal{C})$ with $* \in$ **CONT-fin**	No	No	No	No
NM($\mathcal{C}$)	Yes	Yes	Yes	–
$L_{*_{n_1}}(\mathcal{C})$	Yes	Yes	Yes	–
$L_{*_{n_2}}(\mathcal{C})$	Yes	Yes	Yes	–
Other $L_*(\mathcal{C})$ $* \in$ **WNM-fin**	Yes	No	No	No

Table 3: Standard completeness properties for propositional fuzzy logics with truth-constants restricted to (positively) evaluated formulae (again a denotes the element separating both components in the ordinal sums Ł ⊕ G and Ł ⊕ Π).

We will consider logics $L_*(\mathcal{C})$ with the same restrictions on $*$ and $\mathcal{C}$ stated in the previous section and their completeness properties with respect to all rational $L_*(\mathcal{C})$-chains ($\mathcal{Q}$C, FS$\mathcal{Q}$C and S$\mathcal{Q}$C) and with respect to the canonical rational $L_*(\mathcal{C})$-chain (denoted as Can$\mathcal{Q}$C, CanFS$\mathcal{Q}$C and CanS$\mathcal{Q}$C). The properties in the first group obviously hold for all the logics appearing in Table 2 by virtue of Theorem 2.1, since all of them enjoy the FS$\mathcal{R}$C. We start the study of canonical completeness properties for the Łukasiewicz-based logics Ł($\mathcal{C}$).

Theorem 3.1. *For every $\mathcal{C} \subseteq [0,1]^{\mathbb{Q}}_{Ł}$, the logic Ł($\mathcal{C}$) enjoys the* CanFS$\mathcal{Q}$C.

Proof. Suppose that for some arbitrary finite set of formulae we have $\varphi_1, \ldots, \varphi_n \nvdash_{Ł(\mathcal{C})} \psi$. We must prove that $\varphi_1, \ldots, \varphi_n \not\models_{[0,1]^{\mathbb{Q}}_{Ł(\mathcal{C})}} \psi$. On one hand, by the FS$\mathcal{R}$C [17], there is an evaluation e over $[0,1]_{Ł(\mathcal{C})}$ such that $e(\varphi_1) = \ldots = e(\varphi_n) = 1$ and $e(\psi) < 1$. On the other hand, by using a result in [3],[4] we know that $[0,1]_{Ł(\mathcal{C})}$ is partially embeddable into $[0,1]^{\mathbb{Q}}_{Ł(\mathcal{C})}$ (and hence, preserving truth-constants), i.e. every partial finite subalgebra of $[0,1]_{Ł(\mathcal{C})}$ is embeddable into $[0,1]^{\mathbb{Q}}_{Ł(\mathcal{C})}$. Therefore there is an embedding f of the partial algebra $\{e(\eta) \mid \eta$ a subformula of $\psi, \varphi_1, \ldots, \varphi_n\}$ into $[0,1]^{\mathbb{Q}}_{Ł(\mathcal{C})}$. Then any evaluation e' on $[0,1]^{\mathbb{Q}}_{Ł(\mathcal{C})}$ such that $e'(p) = f(e(p))$ for every propositional variable p appearing in $\psi, \varphi_1, \ldots, \varphi_n$ is a model of $\{\varphi_1, \ldots, \varphi_n\}$ while it is not a model of ψ. □

However, the CanFS$\mathcal{Q}$C fails for the remaining logics we are considering. Indeed, given $* \in$ **CONT-fin** ∪ **WNM-fin** non-isomorphic to Łukasiewicz t-norm and $\mathcal{C} \subseteq [0,1]^{\mathbb{Q}}_*$, there exists a non-trivial proper filter F of $\mathcal{C}$ and we can take $r \in F \setminus \{1\}$. Then:

- $(p \to q) \to \overline{r} \models_{[0,1]^{\mathbb{Q}}_{L_*(\mathcal{C})}} q \to p$
- $(p \to q) \to \overline{r} \not\models_{([0,1]^{\mathbb{Q}})^F_{L_*(\mathcal{C})}} q \to p$

[4] Actually, the result we use from [3] comes from a translation of the paper in Russian [16].

Therefore, there is an entailment which holds for the canonical rational chain, but not for all chains. Thus, none of these logics enjoy the CanFS$\mathcal{Q}$C. This implies, of course, the failure of CanS$\mathcal{Q}$C for all these logics as well; moreover, the failure of CanS$\mathcal{Q}$C for Ł$(\mathcal{C})$ will follow from the results in the next section.

We turn now to the Can$\mathcal{Q}$C. Several positive results are proved in the following theorems.

Theorem 3.2. *For every $\mathcal{C} \subseteq [0,1]_{\Pi}^{\mathbb{Q}}$, the logic $\Pi(\mathcal{C})$ enjoys the* Can$\mathcal{Q}$C.

Proof. For this proof we need to introduce some notation. Given $x = \langle x_1, \dots, x_n\rangle \in \mathbb{R}^n$ and $\delta = \langle \delta_1, \dots, \delta_n\rangle \in (\mathbb{R}_+)^n$, we define the set $E_\delta(x) = \{\langle y_1, \dots, y_n\rangle \in ([0,1]_{\mathbb{Q}})^n \mid x_i = y_i$ if $x_i \in \mathbb{Q}$, and $y_i \in (x_i - \delta_i, x_i + \delta_i)$ if $x_i \notin \mathbb{Q}\}$.

Suppose that $\nvdash_{\Pi(\mathcal{C})} \varphi$. Assume further that the variables of φ are among $\{p_1, \dots, p_n\}$. By the Can$\mathcal{R}$C, there is an evaluation e on $[0,1]_{\Pi(\mathcal{C})}$ such that $e(\varphi) < 1$. We will prove by induction that for every subformula ψ of φ the following two conditions hold:

1. If $e(\psi) = 0$, then there is $E_\delta(e(p_1), \dots, e(p_n))$ such that for every evaluation v on $[0,1]^{\mathbb{Q}}_{\Pi(\mathcal{C})}$, if $\langle v(p_1), \dots, v(p_n)\rangle \in E_\delta(e(p_1), \dots, e(p_n))$ then $v(\psi) = 0$.

2. If $e(\psi) \neq 0$, then for every $\varepsilon > 0$ there is $E_\delta(e(p_1), \dots, e(p_n))$ such that for every evaluation v on $[0,1]^{\mathbb{Q}}_{\Pi(\mathcal{C})}$, if $\langle v(p_1), \dots, v(p_n)\rangle \in E_\delta(e(p_1), \dots, e(p_n))$ then $\mid v(\psi) - e(\psi) \mid < \varepsilon$.

Indeed:

- Assume $\psi = p_i$. If $e(p_i) = 0$, then any δ does the job as $e(p_i) \in \mathbb{Q}$ and we will have $v(p_i) = 0$ for every v. If $e(p_i) \neq 0$, for every $\varepsilon > 0$ it is enough to take a δ such that $\delta_i = \varepsilon$.

- If $\psi = \overline{r}$ for some $r \in C$, it is trivial.

- Assume $\psi = \alpha \& \beta$. If $e(\alpha \& \beta) = 0$, then one of the two conjuncts must be evaluated to 0; assume for instance that $e(\alpha) = 0$. The induction hypothesis for α gives a $E_\delta(e(p_1), \dots, e(p_n))$ such that for every evaluation v on $[0,1]^{\mathbb{Q}}_{\Pi(\mathcal{C})}$, if $\langle v(p_1), \dots, v(p_n)\rangle \in E_\delta(e(p_1), \dots, e(p_n))$ then $v(\alpha) = 0$, and hence $v(\alpha \& \beta) = 0$. Suppose now that $e(\alpha \& \beta) \neq 0$. Then $e(\alpha) \neq 0$ and $e(\beta) \neq 0$. It is enough to use the continuity of product function and the induction hypothesis to prove that for every $\varepsilon > 0$ there is $E_\delta(e(p_1), \dots, e(p_n))$ such that for every evaluation v on $[0,1]^{\mathbb{Q}}_{\Pi(\mathcal{C})}$ if $\langle v(p_1), \dots, v(p_n)\rangle \in E_\delta(e(p_1), \dots, e(p_n))$, $\mid v(\alpha) \cdot v(\beta) - e(\alpha) \cdot e(\beta) \mid < \varepsilon$.

- Assume $\psi = \alpha \to \beta$. If $e(\alpha \to \beta) = 0$, then $e(\alpha) \neq 0$ and $e(\beta) = 0$. The induction hypothesis on α and β does the job. Suppose now that $e(\alpha \to \beta) \neq 0$. If $e(\alpha) = 0$, the induction hypothesis on α does the job, so assume that $e(\alpha) \neq 0$ and $e(\beta) \neq 0$. Now, again, it is enough to use the induction hypothesis and the continuity of the implication function in any point $\langle x, y\rangle$ such that $x \neq 0$.

From this it easily follows that there is an evaluation v on $[0,1]^{\mathbb{Q}}_{\Pi(\mathcal{C})}$ such that $v(\varphi) < 1$, and thus the theorem is proved. □

Theorem 3.3. *Let* $\mathrm{L}_* \in \{\text{Ł} \oplus \mathrm{G}, \text{Ł} \oplus \Pi\}$*. Let a be the element separating the two components of* $*$.[5] *Then, for every* $\mathcal{C} \subseteq [0,1]_*^{\mathbb{Q}}$ *such that* $a \notin C$*, the logic* $\mathrm{L}_*(\mathcal{C})$ *enjoys the* Can$\mathcal{Q}$C.

Proof. The proof of this theorem is the same in both cases and it runs parallel to the previous one. Suppose, for instance, that $\nvdash_{(\text{Ł}\oplus\Pi)(\mathcal{C})} \varphi$. By the Can$\mathcal{R}$C, we know that there is an evaluation e on $[0,1]_{(\text{Ł}\oplus\Pi)(\mathcal{C})}$ such that $e(\varphi) < 1$. Assume further that the variables of φ are among $\{p_1, \ldots, p_n\}$. What can be proved now by induction is slightly different, namely that for every subformula ψ of φ:

1. If $e(\psi) = a$, then there is $E_\delta(e(p_1), \ldots, e(p_n))$ such that for every evaluation v on $[0,1]^{\mathbb{Q}}_{(\text{Ł}\oplus\Pi)(\mathcal{C})}$, if $\langle v(p_1), \ldots, v(p_n)\rangle \in E_\delta(e(p_1), \ldots, e(p_n))$ then $v(\psi) = a$.

2. If $e(\psi) \neq a$, then for every $\varepsilon > 0$ there is $E_\delta(e(p_1), \ldots, e(p_n))$ such that for every evaluation v on $[0,1]^{\mathbb{Q}}_{(\text{Ł}\oplus\Pi)(\mathcal{C})}$, if $\langle v(p_1), \ldots, v(p_n)\rangle \in E_\delta(e(p_1), \ldots, e(p_n))$ then $\mid v(\psi) - e(\psi) \mid < \varepsilon$.

From this it easily follows that there is an evaluation v on $[0,1]^{\mathbb{Q}}_{(\text{Ł}\oplus\Pi)(\mathcal{C})}$ such that $v(\varphi) < 1$, and thus the theorem is proved. □

Theorem 3.4. *The logics* $\mathrm{G}(\mathcal{C})$*,* $\mathrm{NM}(\mathcal{C})$*,* $\mathrm{L}_{*_{n_1}}(\mathcal{C})$ *and* $\mathrm{L}_{*_{n_2}}(\mathcal{C})$ *enjoy the Can*$\mathcal{Q}$C.

To prove this theorem we need to recall some characterizations of the four t-norms related to the logics involved in the theorem.

(i) $*_{\mathrm{G}}$*,* $*_{\mathrm{NM}}$*,* $*_{n_1}$ *and* $*_{n_2}$ *are the only four t-norms (up to isomorphisms) of* **WNM-fin** *such that their corresponding negation on the set of positive elements is either both involutive and continuous, or it is identically* 0.

(ii) Let $* \in$ **WNM-fin** and let $F_a = [a, 1]$ be a filter of $[0,1]_*$ for some positive element $a \in [0,1]$. Then $*_{\mathrm{G}}$*,* $*_{\mathrm{NM}}$*,* $*_{n_1}$ *and* $*_{n_2}$ *are the only four t-norms* $*$ *of* **WNM-fin** *(up to isomorphisms) of* **WNM-fin** *such that the quotient algebra* $[0,1]_*/F_a$ *is isomorphic to* $[0,1]_*$.

(i) follows from a simple inspection of the graph of the negation functions defining the t-norms in **WNM-fin**. As for (ii), notice that the classes of $[0,1]_*/F_a$ are $[1]_{F_a} = F_a$, $[0]_{F_a} = \{x \mid \neg x \in F_a\}$ and $[x]_{F_a} = \{x\}$ otherwise, hence $[0,1]_*/F_a$ can be viewed as the restriction of $*$ on the interval $[\neg a, a]$, and so isomorphic to a t-norm $*'$ on $[0,1]$. Now, one can check that only in the considered four cases, the t-norm $*'$ is isomorphic to the initial one $*$. See [14] for details. Finally, let us notice that these characterizations remain valid when considering the t-norms defined on the rational unit interval $[0,1]^{\mathbb{Q}}$.

Proof of Theorem 3.4 This theorem can be proved in a similar way we proved in [14] that these logics enjoy the Can$\mathcal{R}$C, but we detail the proof here for the reader's convenience. By virtue of the rational analogue of Proposition 2.3, we know that the theorems of $\mathrm{L}_*(\mathcal{C})$ are the common tautologies over $([0,1]^{\mathbb{Q}})^F_{\mathrm{L}_*(\mathcal{C})}$ for each filter F of $\mathcal{C}$. Therefore, it is enough to prove that if φ is a tautology with respect to $[0,1]^{\mathbb{Q}}_{\mathrm{L}_*(\mathcal{C})}$, it is also a tautology of $([0,1]^{\mathbb{Q}})^F_{\mathrm{L}_*(\mathcal{C})}$ for each filter F of $\mathcal{C}$.

[5] Notice that the particular choice of an element $a \in (0,1)^{\mathbb{Q}}$ is not important, as all the resulting algebras are isomorphic and hence they yield the same logic.

Let e be an interpretation over the chain $([0,1]^{\mathbb{Q}})^F_{Ł_*(\mathcal{C})}$. Suppose that $\mathcal{A}$ is the finite algebra generated by $\{e(\psi) \mid \psi$ subformula of $\varphi\}$ and let $\alpha = \min\{r \in F \mid \overline{r}$ occurs in $\varphi\}$. Without loss of generality we can assume $F = F_\alpha = [\alpha, 1]$ since defining the evaluation e' on $([0,1]^{\mathbb{Q}})^{F_\alpha}_{Ł_*(\mathcal{C})}$ by $e'(p) = e(p)$ for all propositional variables p, we have that e and e' coincide on φ and all its subformulae (hence from now on we will only speak about F_α and e').

Let $\overline{F_\alpha} = \{r \in C \mid \neg r \in F_\alpha\}$ and let $f : [0,1]^{\mathbb{Q}} \to [0,1]^{\mathbb{Q}}$ a mapping such that: (1) $f(1) = 1$ and $f(0) = 0$, and (2) f restricted to $(0,1)^{\mathbb{Q}}$ is a bijection on $(\neg\alpha, \alpha)^{\mathbb{Q}}$ satisfying $f(r) = r$ for all $r \notin F_\alpha \cup \overline{F_\alpha}$ such that $\overline{r}$ occurs in φ. Then define an evaluation v on the canonical rational chain by setting $v(p) = f(e'(p))$ for any propositional variable p. Since by hypothesis φ is a tautology of the canonical rational chain, $v(\varphi) = 1$. Take the algebra $[0,1]^{\mathbb{Q}}_*/F_\alpha$. By characterization (ii) above this algebra is isomorphic to $[0,1]^{\mathbb{Q}}_*$. Define the evaluation v' on the quotient algebra $[0,1]^{\mathbb{Q}}_*/F_\alpha$ obtained from v. It is obvious that $v'(\varphi) = [1]_{F_\alpha}$. But a simple computation shows that the algebra $\mathcal{B}$ generated by $\{v'(\psi) \mid \psi$ subformula of $\varphi\}$ is isomorphic to $\mathcal{A}$: observe that $v'(\varphi)$ over the quotient algebra corresponds to $e'(\varphi)$ over the chain $([0,1]^{\mathbb{Q}})^F_{Ł_*(\mathcal{C})}$. Therefore $e'(\varphi) = 1$ and the theorem is proved. □

For the remaining logics this completeness property fails. In the case of continuous t-norms it can be seen by means of the same counterexamples used in [9] to show that these logics do not enjoy the Can$\mathcal{R}$C. Namely, in each case we find a suitable formula of the form $\overline{r} \to \varphi$ (which is, actually, an evaluated formula) such that it is a tautology of the canonical rational algebra $[0,1]^{\mathbb{Q}}_{Ł_*(\mathcal{C})}$ (i.e. it holds that $r \leq e(\varphi)$ for every evaluation e), but it is not a tautology of the algebra $([0,1]^{\mathbb{Q}})^F_{Ł_*(\mathcal{C})}$ for some proper filter F of $\mathcal{C}$ containing r (i.e. it holds that $v(\varphi) < 1$ for some evaluation v, since as in this algebra r is interpreted to 1). In the following we assume that the first component of $[0,1]_*$ is defined on the interval $[0,a]$. We provide first the required counterexamples for the cases when the first component is product or Gödel, and then we consider the case when the first component is Łukasiewicz taking also into account whether $a \in C$ or not ($\mathcal{A}$ stands for an arbitrary standard BL-chain):

1. If $[0,1]_* = [0,a]_\Pi \oplus \mathcal{A}$, then take $b \in C \cap (0,a)$, the filter $F = (0,1] \cap C$, the formula $\overline{b} \to \neg p \vee ((p \to p\&p) \to p)$ and an evaluation v such that $v(p) = a$.

2. If $[0,1]_* = [0,a]_G \oplus \mathcal{A}$ with $\mathcal{A} \not\cong [0,1]_G$, then we take b as any element of $C \cap (0,a)$, the filter $F = [b,1] \cap C$, the formula $\overline{b} \to (p \to p\&p)$ and $v(p)$ any non idempotent element from $\mathcal{A}$.

3. If $[0,1]_* = [0,a]_Ł \oplus \mathcal{A}$ and $a \in C$, then take the filter $F = [a,1] \cap C$, the formula $\overline{a} \to (\neg\neg p \to p)$ and $v(p) \in A \setminus \{1\}$.

4. If $[0,1]_* = [0,a]_Ł \oplus [a,b]_G \oplus \mathcal{A}$ with $\mathcal{A} \not\cong [0,1]_G$ and $a \notin C$, then take any element $d \in (a,b) \cap C$, the filter $F = [d,1] \cap C$, the formula $\overline{d} \to (\neg\neg p \to p) \vee (p \to p\&p)$ and $v(p)$ any non idempotent element from $\mathcal{A}$.

5. If $[0,1]_* = [0,a]_Ł \oplus [a,b]_\Pi \oplus \mathcal{A}$ and $a \notin C$, then take any element $d \in (a,b) \cap C$, the filter $F = (a,1] \cap C$, the formula $\overline{d} \to [(\neg\neg p\&\neg\neg q\&((p \to p\&q) \to q)\&(q \to p)\&(p \to p\&p)) \to p]$ and $v(p) = b$ and $v(q) \in (a,b)$.

Observe that for a t-norm $*$ whose decomposition begins with two copies of Łukasiewicz t-norm, the idempotent element a separating them has to belong to the truth-constants subalgebra $\mathcal{C}$. Indeed, take into account that, by assumption, C must contain a non idempotent element c of the second component and for this element there exists a natural number m such that $c^m = c * \overset{m}{\cdots} * c = a$, and thus $a \in C$. Hence this case is subsumed in the above third item.

In the case of WNM-t-norm based logics we can provide counterexamples in the same fashion for all t-norms from **WNM-fin** except those isomorphic to $*_{\rm G}$, $*_{\rm NM}$, $*_{n_1}$ and $*_{n_2}$ (already proved to enjoy the Can$\mathcal{Q}$C in Theorem 3.4). Let n be a negation function, I its first positive interval, and let $*_n$ be the associated WNM-t-norm to n .

1. If I is constant (and hence n has no fix point) and $*_n$ is not Gödel t-norm, then take $r \in I \cap C$, the filter $F = [r, 1] \cap C$, the formula $\overline{r} \to \neg\neg(p \vee \neg p)$ and an evaluation v such that $v(p) \in I$. See (a) in Figure 1.

2. If I is involutive and there is a positive constant interval J, then take $r \in I \cap C$, the filter $F = [r, 1] \cap C$, the formula $\overline{r} \to (\neg\neg(p \vee \neg p) \to (p \vee \neg p))$ and $v(p) \in J$. See (b) in Figure 1.

3. If I is involutive, there are no positive constant intervals and $*_n$ is not isomorphic to $*_{\rm NM}$, then there must be a discontinuity point in the positive part, and symmetrically a constant interval J in the negative part. Then take $r \in I \cap C$, the filter $F = [r, 1] \cap C$, the formula $\overline{r} \to (\neg\neg(p \wedge \neg p) \to (p \wedge \neg p))$ and $v(p) \in J$. See (c) in Figure 1.

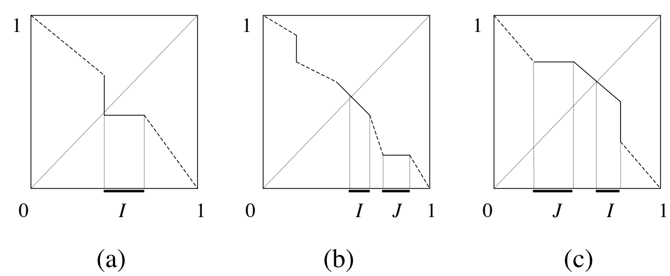

Figure 1: Negation functions used in the counterexamples for **WNM-fin**.

See the new results we have obtained on rational completeness properties in Table 4.

4 Rational completeness results: the case of positively evaluated formulae

In this section we focus on restricted completeness properties for the different logics when only (positively) evaluated formulae are involved with the aim of improving the negative results got for arbitrary formulae.

As regards to completeness properties with respect to the class of all rational chains there is nothing to add: the S$\mathcal{Q}$C holds for evaluated formulae because it already holds in general for all formulae. Thus, we only need to examine the restricted canonical completeness properties.

To start with, let us consider the property of CanS$\mathcal{Q}$C. A first negative result shows that CanS$\mathcal{Q}$C restricted to evaluated formulae fails for any logic $\mathrm{L}_*(\mathcal{C})$ such that L_* does not enjoy the S$\mathcal{R}$C.

Logic	S$\mathcal{Q}$C	Can$\mathcal{Q}$C	CanFS$\mathcal{Q}$C	CanS$\mathcal{Q}$C
Ł$(\mathcal{C})$	Yes	Yes	Yes	No
$\Pi(\mathcal{C})$	Yes	Yes	No	No
$\mathrm{G}(\mathcal{C})$	Yes	Yes	No	No
$(Ł \oplus \mathrm{G})(\mathcal{C})$, $a \notin C$	Yes	Yes	No	No
$(Ł \oplus \Pi)(\mathcal{C})$, $a \notin C$	Yes	Yes	No	No
Other $\mathrm{L}_*(\mathcal{C})$ $* \in$ **CONT-fin**	Yes	No	No	No
$\mathrm{NM}(\mathcal{C})$	Yes	Yes	No	No
$\mathrm{L}_{*_{n_1}}(\mathcal{C})$	Yes	Yes	No	No
$\mathrm{L}_{*_{n_2}}(\mathcal{C})$	Yes	Yes	No	No
Other $\mathrm{L}_*(\mathcal{C})$ $* \in$ **WNM-fin**	Yes	No	No	No

Table 4: Rational completeness properties for propositional fuzzy logics with truth-constants (a denotes the element separating both components in the ordinal sums Ł $\oplus$ G and Ł $\oplus$ Π).

Proposition 4.1. *Let $*$ be any left-continuous t-norm closed on $[0,1]^{\mathbb{Q}}$ such that L_* does not enjoy the S$\mathcal{R}$C. Then, for any $\mathcal{C} \subseteq [0,1]^{\mathbb{Q}}_*$, the logic $\mathrm{L}_*(\mathcal{C})$ does not enjoy the Can*S$\mathcal{Q}$C *restricted to evaluated formulae.*

Proof. In [6] it is shown that, given a class $\mathbb{K}$ of L-algebras, a necessary and sufficient condition for a core fuzzy logic L to have the S$\mathbb{K}$C is that any countable L-chain be embedded into an L-chain from the class $\mathbb{K}$. Now, if L_* does not have the S$\mathcal{R}$C, there is a countable L_*-chain $\mathcal{A}$ that cannot be embedded into any L_*-chain over the real interval $[0,1]$, in particular it cannot be embedded into the (canonical real) L_*-chain $[0,1]_*$. Therefore, since the rational interval algebra $[0,1]^{\mathbb{Q}}_*$ is a subalgebra of $[0,1]_*$, the algebra $\mathcal{A}$ cannot be embedded either into the rational interval algebra $[0,1]^{\mathbb{Q}}_*$. Therefore, L_* does not enjoy strong completeness with respect to the $[0,1]^{\mathbb{Q}}_*$. Finally, since L_*-formulae are also evaluated $\mathrm{L}_*(\mathcal{C})$-formulae, it is clear that the CanS$\mathcal{Q}$C restricted to evaluated formulae does not hold either. $\square$

Corollary 4.2. *For any t-norm $* \in$* **CONT-fin** $\setminus \{*_{\mathrm{G}}\}$ *and any $\mathcal{C} \subseteq [0,1]^{\mathbb{Q}}_*$, the logic $\mathrm{L}_*(\mathcal{C})$ does not enjoy the Can*S$\mathcal{Q}$C *restricted to evaluated formulae.*

Analogously to the case of real semantics, a second negative result is linked to a particular topological property of the algebras of truth-constants $\mathcal{C}$ used when expanding the original logic L_* with truth-constants. The same counterexample mentioned in Section 2.3 for the case the algebra of truth-values C has an accumulation point r which is the supremum of a strictly increasing sequence $\langle r_i \rangle_{i \in \mathbb{N}}$ of points of C yields the following proposition.

Proposition 4.3. *Let $*$ be a left-continuous t-norm and let $\mathcal{C} \subseteq [0,1]^{\mathbb{Q}}_*$ containing at least one* sup*-accessible point. Then $\mathrm{L}_*(\mathcal{C})$ does not enjoy the Can*S$\mathcal{Q}$C *restricted to evaluated formulae.*

However, we are able to prove that the CanS$\mathcal{Q}$C for evaluated formulae does hold for the expansions of four distinguished WNM logics with algebras of truth-constants $\mathcal{C}$ without (positive) sup-accessible points.

Theorem 4.4. *Let $*$ be any of the following four t-norms from* **WNM-fin***: $*_{\mathrm{G}}$, $*_{\mathrm{NM}}$, $*_{n_1}$ or $*_{n_2}$. Let $\mathcal{C} \subseteq [0,1]_*^{\mathbb{Q}}$ be such that it contains no positive* sup*-accessible point. Then $\mathrm{L}_*(\mathcal{C})$ enjoys the Can*S$\mathcal{Q}$C *for (positively) evaluated formulae.*

Proof.
(i) First we prove the CanS$\mathcal{Q}$C for $\mathrm{G}(\mathcal{C})$. Soudness is obvious as usual. Thus, for completeness we have to prove that if a (possibly infinite) family of evaluated formulae $\{\overline{r}_i \to \varphi_i \mid i \in I\}$ does not prove an evaluated formula $\overline{s} \to \psi$ then there is an evaluation v over the canonical chain such that for every $i \in I$, $v(\overline{r}_i \to \varphi_i) = 1$ and $v(\overline{s} \to \psi) < 1$.

By the algebraizability of the logic with truth-constants if the syntactical deduction is not valid there is a countable $\mathrm{G}(\mathcal{C})$-chain $\mathcal{A}$ and an evaluation e over it such that for every $i \in I$ $e(\overline{r}_i \to \varphi_i) = \overline{1}^{\mathcal{A}}$ and $e(\overline{s} \to \psi) < \overline{1}^{\mathcal{A}}$. Suppose this is a chain of type F, that is, F is a filter of $\mathcal{C}$ such that for every $r \in F$, $\overline{r}^{\mathcal{A}} = \overline{1}^{\mathcal{A}}$. Observe that, since the elements of C are not sup-accessible, for each point $r \in C$ there is an interval $I_r^- = (r - \delta, r) \cap \mathbb{Q}$ (with countably many elements) such that $I_r^- \cap C = \emptyset$. To built the desired evaluation v we need to study two cases:

(i-1) Suppose that $s \in F$. In such a case define the mapping $f : A \to [0,1]^{\mathbb{Q}}$ as follows: $f(\overline{1}^{\mathcal{A}}) = 1, f(\overline{0}^{\mathcal{A}}) = 0$ and f restricted to $A \setminus \{\overline{0}^{\mathcal{A}}, \overline{1}^{\mathcal{A}}\}$ is an embedding into I_s^-. An easy computation shows that f is a morphism of G-chains (without truth-constants). Define the $[0,1]_{\mathrm{G}(\mathcal{C})}^{\mathbb{Q}}$-evaluation v as $v(p) = f(e(p))$ for every propositional variable p. Such v satisfies the required conditions since: If $r_i \in F$, then $v(\varphi_i) = e(\varphi_i) = 1 \geq r_i$ and if $r_i \notin F$, then $v(\varphi_i) \in \{1\} \cup I_s^-$ and thus greater or equal than r_i. Moreover, since $e(\psi) < 1$, $v(\psi) \in I_s^- \cup \{0\}$ and thus $v(\psi) < s$.

(i-2) Suppose that $s \notin F$. In such a case define the mapping $f : A \to [0,1]^{\mathbb{Q}}$ as follows: $f(\overline{1}^{\mathcal{A}}) = 1, f(\overline{0}^{\mathcal{A}}) = 0$ and $f(\overline{s}^{\mathcal{A}}) = s$ and f restricted to $(\overline{s}^{\mathcal{A}}, \overline{1}^{\mathcal{A}})$ is an embedding into I_1^- and f restricted to $(\overline{0}^{\mathcal{A}}, \overline{s}^{\mathcal{A}})$ is an embedding into I_s^-. An easy computation shows that f is a morphism of the G-chains (without truth-constants). Define the $[0,1]_{\mathrm{G}(\mathcal{C})}^{\mathbb{Q}}$-evaluation v as $v(p) = f(e(p))$ for every propositional variable p. Such v satisfies the required conditions since: If $r_i \in F$, then $v(\varphi_i) = e(\varphi_i) = 1 \geq r_i$; if $r_i \notin F$, $r_i > s$, then $e(\varphi_i) > \overline{s}^{\mathcal{A}}$ and thus $v(\varphi_i) \in I_1^- \cup \{1\}$, which implies $v(\varphi_i) \geq r_i$; if there is some $r_i = s$ obviously $v(\varphi_i) \geq s$; if $r_i < s$, then $v(\varphi_i) \in \{1\} \cup I_1^- \cup I_s^-$ which implies $v(\varphi_i) \geq r_i$. Finally, since $e(\psi) < \overline{s}^{\mathcal{A}}$, $v(\psi) \in I_s^- \cup \{0\}$ and thus $v(\psi) < s$.

(ii) The proof for $\mathrm{NM}(\mathcal{C})$ is similar but we detail it for the sake of completeness. Again, we have to prove that if a family of evaluated formulae $\{\overline{r}_i \to \varphi_i \mid i \in I\}$ does not prove an evaluated formula $\overline{s} \to \psi$ (where all the r_i's and s are positive elements of C), then there is an evaluation v over the canonical $\mathrm{NM}(\mathcal{C})$-chain $[0,1]_{\mathrm{NM}(\mathcal{C})}$ such that $v(\overline{r}_i \to \varphi_i) = 1$ for all i and $v(\overline{s} \to \psi) < 1$. Since $\mathrm{NM}(\mathcal{C})$ is algebraizable, there is countable $\mathrm{NM}(\mathcal{C})$-chain $\mathcal{A}$ and an evaluation e over it such that $e(\overline{r}_i \to \varphi_i) = \overline{1}^{\mathcal{A}}$ and $e(\overline{s} \to \psi) < \overline{1}^{\mathcal{A}}$. Assume $\mathcal{A}$ is a chain of type F, that is, F is a filter of $\mathcal{C}$ such that for all $r \in F$, then $\overline{r}^{\mathcal{A}} = \overline{1}^{\mathcal{A}}$ and $\overline{\neg r}^{\mathcal{A}} = \overline{0}^{\mathcal{A}}$. By hyphotesis the positive elements of C are not sup-accessible, therefore for each positive point $r \in C$ there is an interval $I_r^- = (r - \delta, r) \cap \mathbb{Q}$ such that $I_r^- \cap C = \emptyset$. To build the desired evaluation v we need to consider two cases:

(ii-1) Suppose $s \in F$. In such a case define the mapping $f : A \to [0,1]$ as follows: $f(\overline{1}^{\mathcal{A}}) = 1, f(\overline{0}^{\mathcal{A}}) = 0, f(\overline{\tfrac{1}{2}}^{\mathcal{A}}) = \frac{1}{2}$, f restricted to $(\overline{\tfrac{1}{2}}^{\mathcal{A}}, \overline{1}^{\mathcal{A}})$ is an embedding into I_s^- and

f restricted to $(\overline{0}^{\mathcal{A}}, \overline{\frac{1}{2}}^{\mathcal{A}})$ is the dual embedding into $(1-r, 1-r+\delta) \cap \mathbb{Q}$, i.e. satisfying $f(x) = 1 - f(\neg x)$. An easy computation shows that f is a morphism of NM-chains (without truth-constants). Define the $[0,1]_{\mathrm{NM}(\mathcal{C})}$-evaluation v as $v(p) = f(e(p))$ for each propositional variable p. Such an evaluation v satisfies the required conditions since: if $r_i \in F$, then $v(\varphi_i) = e(\varphi_i) = 1 \geq r_i$; if $r_i \notin F$ and $r_i > \frac{1}{2}$, then $v(\varphi_i) \in \{1\} \cup I_s^-$ and thus greater or equal than r_i; and finally, if $r_i \notin F$ and $r_i = \frac{1}{2}$ (if any) is obvious. Moreover, since $e(\psi) < 1$, $v(\psi) \in I_s^- \cup \{0\}$ and thus $v(\psi) < s$.

(ii-2) Suppose now $s \notin F$. In such a case define the mapping $f : A \to [0,1]$ as follows: $f(\overline{1}^{\mathcal{A}}) = 1, f(\overline{0}^{\mathcal{A}}) = 0, f(\overline{s}^{\mathcal{A}}) = s, f(\overline{\frac{1}{2}}^{\mathcal{A}}) = \frac{1}{2}, f(\overline{1-s}^{\mathcal{A}}) = 1 - s$, and f restricted to $(\overline{s}^{\mathcal{A}}, \overline{1}^{\mathcal{A}})$ is an embedding (α) into I_1^-, f restricted to $(\overline{\frac{1}{2}}^{\mathcal{A}}, \overline{s}^{\mathcal{A}})$ is an embedding (β) into I_s^-, f restricted to $(\overline{1-s}^{\mathcal{A}}, \overline{\frac{1}{2}}^{\mathcal{A}})$ is the dual embedding of (α) and f restricted to $(\overline{0}^{\mathcal{A}}, \overline{1-s}^{\mathcal{A}}))$ is the dual embedding of (β). An easy computation shows that f is a morphism of NM-chains (without truth-constants). Define the $[0,1]_{\mathrm{NM}(\mathcal{C})}$-evaluation v as $v(p) = f(e(p))$ for any propositional variable p. Such v satisfies the required conditions since: if $r_i \in F$, then $v(\varphi_i) = e(\varphi_i) = 1 \geq r_i$; if $r_i \notin F$ and $r_i > s$, then $v(\varphi_i) \in I_1^- \cup \{1\}$, which implies $v(\varphi_i) \geq r_i$; if $r_i = s$, then obviously $v(\varphi_i) \geq s$; if $\frac{1}{2} < r_i < s$, then $v(\varphi_i) \in \{1\} \cup I_1^- \cup I_s^-$, which implies $v(\varphi_i) \geq r_i$. The case $r_i = \frac{1}{2}$ (if any) is obvious. Finally, since $e(\psi) < \overline{s}^{\mathcal{A}}$, in any case we have $v(\psi) < s$.

(iii) The proofs for the cases of $\mathrm{L}_{*_{n_1}}(\mathcal{C})$ and $\mathrm{L}_{*_{n_2}}(\mathcal{C})$ easily follow from the proofs of $\mathrm{G}(\mathcal{C})$ and $\mathrm{NM}(\mathcal{C})$ taking into account that the negations in the (rational) canonical algebras $[0,1]_{*_{n_1}}$ and $[0,1]_{*_{n_2}}$ (the functions n_1 and n_2 described in Section 2), when restricted to the set of positive elements, coincide with the ones of the canonical G and NM-algebras, $[0,1]_{\mathrm{G}}$ and $[0,1]_{\mathrm{NM}}$ respectively. □

Although no logic $\mathrm{L}_*(\mathcal{C})$ enjoys the Can$\mathcal{SQ}$C for $* \in$ **CONT-fin** $\setminus \{*_{\mathrm{G}}\}$, we can still show several positive results of restricted CanFS$\mathcal{Q}$C for some of these logics apart from $Ł_*(\mathcal{C})$, which already enjoys the unrestricted CanFS$\mathcal{Q}$C.

Theorem 4.5. *For every $\mathcal{C} \subseteq [0,1]_{\Pi}^{\mathbb{Q}}$, the logic $\Pi(\mathcal{C})$ enjoys the* CanFS$\mathcal{Q}$C *restricted to evaluated formulae.*

Proof. Assume that $\{\overline{r_i} \to \varphi_i \mid i = 1,\dots,n\} \cup \{\overline{s} \to \psi\}$ is a finite set of evaluated formulae such that $\{\overline{r_i} \to \varphi_i \mid i = 1,\dots,n\} \nvdash_{\Pi(\mathcal{C})} \overline{s} \to \psi$. We must prove that $\{\overline{r_i} \to \varphi_i \mid i = 1,\dots,n\} \not\models_{[0,1]_{\Pi(\mathcal{C})}^{\mathbb{Q}}} \overline{s} \to \psi$. By the CanFS$\mathcal{R}$C restricted to evaluated formulae, there is an evaluation e on $[0,1]_{\Pi(\mathcal{C})}$ such that for every $i \in \{1,\dots,n\}$, $e(\overline{r_i} \to \varphi_i) = 1$ and $e(\overline{s} \to \psi) \neq 1$, i.e. $s > e(\psi)$ and $r_i \leq \varphi_i$ for every i. Without loss of generality we can assume that $r_i < e(\varphi_i)$ for every i (if it is not the case, we choose any positive real number α such that for every i, $r_i \leq e(\varphi_i) < e(\varphi_i)^\alpha$ and $s > e(\psi)^\alpha > e(\psi)$ and take instead of e the evaluation $e'(p) = e(p)^\alpha$). Then we use the same trick as in the proof of Theorem 3.2 showing by induction that for every subformula η of $\psi, \varphi_1, \dots, \varphi_n$ (we assume that the variables in η are among $\{p_1,\dots,p_k\}$):

1. If $e(\eta) = 0$, then there is $E_\delta(e(p_1),\dots,e(p_k))$ such that for every evaluation v on $[0,1]_{\Pi(\mathcal{C})}^{\mathbb{Q}}$, if $\langle v(p_1),\dots,v(p_k)\rangle \in E_\delta(e(p_1),\dots,e(p_k))$ then $v(\eta) = 0$.
2. If $e(\eta) \neq 0$, then for every $\varepsilon > 0$ there is $E_\delta(e(p_1),\dots,e(p_k))$ such that for every evaluation v on $[0,1]_{\Pi(\mathcal{C})}^{\mathbb{Q}}$, if $\langle v(p_1),\dots,v(p_k)\rangle \in E_\delta(e(p_1),\dots,e(p_k))$ then $\mid v(\eta) - e(\eta) \mid < \varepsilon$.

Therefore, there is an evaluation v on $[0,1]^{\mathbb{Q}}_{\Pi(\mathcal{C})}$ that maps to 1 the premises while maps the conclusion to some lower value and hence $\{\overline{r_i} \to \varphi_i \mid i = 1,\ldots,n\} \not\models_{[0,1]^{\mathbb{Q}}_{\Pi(\mathcal{C})}} \overline{s} \to \psi$. $\square$

Theorem 4.6. *Let* $\mathrm{L}_* \in \{\text{Ł} \oplus \mathrm{G}, \text{Ł} \oplus \Pi\}$. *Let a be the element separating the two components of* $*$. *Then, for every $\mathcal{C} \subseteq [0,1]^{\mathbb{Q}}_*$ such that $a \notin C$, the logic* $\mathrm{L}_*(\mathcal{C})$ *enjoys the* CanFS$\mathcal{Q}$C *restricted to evaluated formulae.*

Proof. Both cases are analogously proved. Suppose, for instance, that for some finite set for evaluated formulae $\{\overline{r_i} \to \varphi_i \mid i = 1,\ldots,n\} \not\vdash_{(\text{Ł}\oplus\Pi)(\mathcal{C})} \overline{s} \to \psi$. By the CanFS$\mathcal{R}$C restricted to evaluated formulae, there is an evaluation e on $[0,1]_{(\text{Ł}\oplus\Pi)(\mathcal{C})}$ such that for every $i \in \{1,\ldots,n\}$ $e(\overline{r_i} \to \varphi_i) = 1$ and $e(\overline{s} \to \psi) \neq 1$, i.e. $s > e(\psi)$ and $r_i \leq \varphi_i$ for every i. Using the fact that $[0,1]_{\text{Ł}(\mathcal{C})}$ is partially embeddable into $[0,1]^{\mathbb{Q}}_{\text{Ł}(\mathcal{C})}$, we can assume that e is an evaluation on the canonical $(\text{Ł}\oplus\Pi)(\mathcal{C})$-chain defined over $[0,a]^{\mathbb{Q}}_{\text{Ł}} \oplus [a,1]_{\Pi}$. Moreover, as in the previous proof, we can assume that $r_i < e(\varphi_i)$ for every i. Finally, using an analogous claim proved by induction we obtain an evaluation over $[0,1]^{\mathbb{Q}}_{(\text{Ł}\oplus\Pi)(\mathcal{C})}$ showing that $\{\overline{r_i} \to \varphi_i \mid i = 1,\ldots,n\} \not\models_{[0,1]^{\mathbb{Q}}_{(\text{Ł}\oplus\Pi)(\mathcal{C})}} \overline{s} \to \psi$. $\square$

On the other hand, the counterexamples exhibited at the end of Section 3.1 to refute the unrestricted Can$\mathcal{Q}$C for a number of cases, including the logics corresponding to the remaining **WNM-fin** t-norms, are actually counterexamples for the Can$\mathcal{Q}$C restricted to evaluated formulae as well since the formulae involved were indeed (positively) evaluated formulae.

Corollary 4.7. *(i) For* $* \in$ **WNM-fin** $\setminus \{*_{\mathrm{G}}, *_{\mathrm{NM}}, *_{n_1}, *_{n_2}\}$ *and any $\mathcal{C} \subseteq [0,1]^{\mathbb{Q}}_*$, the logic* $\mathrm{L}_*(\mathcal{C})$ *does not enjoy the Can$\mathcal{Q}$C restricted to evaluated formulae.*

(ii) For $* \in$ **CONT-fin** $\setminus \{*_{\text{Ł}}, *_{\Pi}, *_{\mathrm{G}}, *_{\text{Ł}} \oplus *_{\Pi}, *_{\text{Ł}} \oplus *_{\mathrm{G}}\}$ *and any $\mathcal{C} \subseteq [0,1]^{\mathbb{Q}}_*$, the logic* $\mathrm{L}_*(\mathcal{C})$ *does not enjoy the Can$\mathcal{Q}$C restricted to evaluated formulae.*

(iii) If $* \in$ **CONT-fin** *is such that $[0,1]_* = [0,a]_{\text{Ł}} \oplus [a,1]_{\Pi}$ or $[0,1]_* = [0,a]_{\text{Ł}} \oplus [a,1]_{\mathrm{G}}$, and $\mathcal{C} \subseteq [0,1]^{\mathbb{Q}}_*$ is such that $a \in C$, then the logic* $\mathrm{L}_*(\mathcal{C})$ *does not enjoy the Can$\mathcal{Q}$C restricted to evaluated formulae.*

With these results we have covered all the logics under our scope. As a summary, the obtained rational completeness results restricted to evaluated formulae are gathered in Table 5, where C^+ denotes the set of positive elements of the algebra $\mathcal{C}$, and $\mathcal{P}_{\sup-acc}([0,1])$ denotes the set of subsets of $[0,1]$ containing at least one sup-accessible point.

5 Conclusions

In this paper we have discussed the rational semantics for fuzzy logics expanded with truth-constants as a new topic for research. The results we have presented for the propositional case show the interest of the approach, as the rational semantics has demonstrated to provide better completeness properties for the main propositional fuzzy logics.

An interesting issue to comment is about rational completeness results for expansions of logics $\mathrm{L}_*(\mathcal{C})$ with the Baaz's projection connective Δ, as it was done in [9, 14] for the real semantics. The additional axioms of $\mathrm{L}_{*\Delta}(\mathcal{C})$are the well-known five axioms and rule for Δ (see e.g. [17] for details) together with an additional book-keeping axiom

$$\Delta\overline{r} \leftrightarrow \overline{\delta(r)}$$

Logic	Can$\mathcal{Q}$C	CanFS$\mathcal{Q}$C	CanS$\mathcal{Q}$C
Ł($\mathcal{C}$)	Yes	Yes	No
$\Pi(\mathcal{C})$	Yes	Yes	No
G($\mathcal{C}$), $C^+ \in \mathcal{P}_{\sup-acc}([0,1])$	Yes	Yes	No
G($\mathcal{C}$), $C^+ \notin \mathcal{P}_{\sup-acc}([0,1])$	Yes	Yes	Yes
(Ł $\oplus$ G)($\mathcal{C}$), $a \notin C$	Yes	Yes	No
(Ł $\oplus$ Π)($\mathcal{C}$), $a \notin C$	Yes	Yes	No
Other $\mathrm{L}_*(\mathcal{C})$, $* \in$ **CONT-fin**	No	No	No
NM($\mathcal{C}$), $C^+ \in \mathcal{P}_{\sup-acc}([0,1])$	Yes	Yes	No
NM($\mathcal{C}$), $C^+ \notin \mathcal{P}_{\sup-acc}([0,1])$	Yes	Yes	Yes
$\mathrm{L}_{*_{n_1}}(\mathcal{C})$, $C^+ \in \mathcal{P}_{\sup-acc}([0,1])$	Yes	Yes	No
$\mathrm{L}_{*_{n_1}}(\mathcal{C})$, $C^+ \notin \mathcal{P}_{\sup-acc}([0,1])$	Yes	Yes	Yes
$\mathrm{L}_{*_{n_2}}(\mathcal{C})$, $C^+ \in \mathcal{P}_{\sup-acc}([0,1])$	Yes	Yes	No
$\mathrm{L}_{*_{n_2}}(\mathcal{C})$, $C^+ \notin \mathcal{P}_{\sup-acc}([0,1])$	Yes	Yes	Yes
Other $\mathrm{L}_*(\mathcal{C})$, $* \in$ **WNM-fin**	No	No	No

Table 5: Canonical rational completeness properties for propositional fuzzy logics with truth-constants restricted to (positively) evaluated formulae (again a denotes the element separating both components in the ordinal sums Ł $\oplus$ G and Ł $\oplus$ Π).

where $\delta(r) = 1$ if $r = 1$ and $\delta(r) = 0$ otherwise. The logics $\mathrm{L}_{*\Delta}(\mathcal{C})$ are also algebraizable with equivalent algebraic semantics given by the the variety of $\mathrm{L}_{*\Delta}(\mathcal{C})$-algebras, defined in the natural way. It is also easy to check that $\mathrm{L}_{*\Delta}(\mathcal{C})$-algebras are representable as subdirect product of chains. A key observation is that, due to the presence of the Δ operator, all $\mathrm{L}_{*\Delta}(\mathcal{C})$-chains are simple, and this fact simplifies the analysis. A quick inspection of the above mentioned results about real completeness show that they remain valid for the rational semantics as well, namely, we have that all logics $\mathrm{L}_{*\Delta}(\mathcal{C})$ for $* \in$ **CONT-fin** $\cup$ **WNM-fin** enjoy the CanFS$\mathcal{Q}$C, and they enjoy the CanS$\mathcal{Q}$C in the case of $* \in$ **WNM-fin**.

Future research includes, among others: (i) extending the investigation on rational completeness properties to wider classes of logics based on left-continuous t-norms; and (ii) studying rational completeness properties for first-order predicate fuzzy logics, as it has been done for the standard semantics in [15].

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