ON NORMAL-VALUED BASIC PSEUDO HOOPS

MICHAL BOTUR[1], ANATOLIJ DVUREČENSKIJ[2], AND TOMASZ KOWALSKI[3]

[1] Department of Algebra and Geometry
Faculty of Natural Sciences, Palacký University
17. listopadu 12, CZ-771 46 Olomouc, Czech Republik
[2] Mathematical Institute, Slovak Academy of Sciences
Štefánikova 49, SK-814 73 Bratislava, Slovakia
[3] Department of Mathematics and Statistics
University of Melbourne
Parkville, VIC 3010, Australia
E-mail: botur@inf.upol.cz, dvurecen@mat.savba.sk,
kowatomasz@gmail.com

Abstract. We show that every pseudo hoop satisfies the Riesz Decomposition Property. We visualize basic pseudo hoops by functions on a linearly ordered set. Finally, we study normal-valued basic pseudo hoops giving a countable base of equations for them.

1. Introduction

The Romanian algebraic school during the last decade contributed a lot to noncommutative generalizations of many-valued reasoning which generalizes MV-algebras by C.C. Chang [Cha]. They introduced pseudo MV-algebras, [GeIo] (independently introduced also in [Rac] as generalized MV-algebras), pseudo BL-algebras, [DGI1, DGI2], pseudo hoops, [GLP]. We recall that pseudo BL-algebras are also a noncommutative generalization of P. Hájek's BL-algebras: a variety that is an algebraic counterpart of fuzzy logic, [Haj].

However, as it was recently recognized, many of these notions have a very close connections with notions introduced already by B. Bosbach in his pioneering papers on various classes of semigroups: among others he introduced complementary semigroups (today known as pseudo-hoops). A deep investigation of these structures can be found in his papers [Bos1, Bos2]; more information are available in his recent papers [Bos3, Bos4]. Nowadays, all these structures can be also studied under one common roof, as residuated lattices, [GaTs].

Now all these structures are intensively studied by many experts. Very important results were presented in [JiMo]. In the paper [Dvu4], it was proved that every linearly ordered pseudo hoop is an ordinal sum of negative cones or intervals

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of lattice-ordered groups, see also [AgMo]. The paper [DGK] introduced interesting classes of pseudo hoops, like systems $\mathcal{MPH}$ and $\mathcal{MPH}_b$ of all pseudo hoops (bounded pseudo hoops) M such that every maximal filter of M is normal, and the system $\mathcal{NVPH}$ of normal-valued basic pseudo-hoops M such that every value in M is normal in its cover. The latter one is inspired by analogous notions from theory of ℓ-groups. In [DGK], there was proved that $\mathcal{NVPH} \subset \mathcal{MPH}$, $\mathcal{MPH}_b \subset \mathcal{MPH}$ and $\mathcal{NVPH}$, $\mathcal{MPH}_b$ are varieties but $\mathcal{MPH}$ is not a variety, [DGK, Rem 4.2].

The main aim is to continue in the study of pseudo hoops, focusing on normal-valued ones. We present an equational basis of normal-valued basic pseudo hoops. In addition, we show that every pseudo hoop satisfies the Riesz Decomposition Property (RDP) and we present also a Holland's type representation of basic pseudo hoops.

The paper is organized as follows. Section 2 gathers the basic notions and properties of pseudo hoops and Section 3 deals with basic pseudo hoops. Section 4 proves the Riesz Decomposition Property for pseudo hoops, and presents some results on filters. Some kind of the Holland Representation Theorem for basic pseudo hoops which enables us to visualize them by functions on a linearly ordered set is presented in Section 5. Finally, Section 6 studies normal-valued basic pseudo hoops and presents a countable base of equations characterizing them. In addition two open questions are formulated.

2. Basic Facts and Properties

We recall that according to [GLP], a *pseudo hoop* is an algebra $(M; \odot, \to, \rightsquigarrow, 1)$ of type $\langle 2, 2, 2, 0\rangle$ such that, for all $x, y, z \in M$,

(i) $x \odot 1 = x = 1 \odot x$;
(ii) $x \to x = 1 = x \rightsquigarrow x$;
(iii) $(x \odot y) \to z = x \to (y \to z)$;
(iv) $(x \odot y) \rightsquigarrow z = y \rightsquigarrow (x \rightsquigarrow z)$;
(v) $(x \to y) \odot x = (y \to x) \odot y = x \odot (x \rightsquigarrow y) = y \odot (y \rightsquigarrow x)$.

We recall that $\odot$ have higher priority than $\to$ or $\rightsquigarrow$, and those higher than $\wedge$ and $\vee$, and $\wedge$ is higher than $\vee$.

If $\odot$ is commutative (equivalently $\to=\rightsquigarrow$), M is said to be a *hoop*. If we set $x \leq y$ iff $x \to y = 1$ (this is equivalent to $x \rightsquigarrow y = 1$), then $\leq$ is a partial order such that $x \wedge y = (x \to y) \odot x$ and M is a $\wedge$-semilattice.

We say that a pseudo hoop M

(i) is *bounded* if there is a least element 0, otherwise, M is *unbounded*,
(ii) satisfies *prelinearity* if, given $x, y \in M$, $(x \to y) \vee (y \to x)$ and $(x \rightsquigarrow y) \vee (y \rightsquigarrow x)$ are defined in M and they are equal 1,
(iii) is *cancellative* if $x \odot y = x \odot z$ and $s \odot x = t \odot x$ imply $y = z$ and $s = t$,
(iv) is a *pseudo BL-algebra* if M is a bounded lattice satisfying prelinearity.

For a pseudo BL-algebra, we define $x^- = x \to 0$ and $x^\sim = x \rightsquigarrow 0$. A pseudo BL-algebra is said to be a pseudo MV-algebra if $x^{-\sim} = x = x^{\sim -}$ for every $x \in M$.

From (v) of the definition of pseudo hoops we have that a pseudo hoop is cancellative iff $x \odot y \leq x \odot z$ and $s \odot x \leq t \odot x$ imply $y \leq z$ and $s \leq t$.

Many examples of pseudo hoops can be made from ℓ-groups. Now let G be an ℓ-group (written multiplicatively and with a neutral element e). On the negative cone $G^- = \{g \in G : \ g \leq e\}$ we define: $x \odot y := xy$, $x \to y := (yx^{-1}) \wedge e$,

$x \rightsquigarrow y := (x^{-1}y) \wedge e$, for $x, y \in G^-$. Then $(G^-; \odot, \to, \rightsquigarrow, e)$ is an unbounded (whenever $G \neq \{e\}$) cancellative pseudo hoop. Conversely, according to [GLP, Prop 5.7], every cancellative pseudo hoop is isomorphic to some $(G^-; \odot, \to, \rightsquigarrow, e)$.

If $u \geq e$ is a strong unit unit (= order unit) in G, we define on $[-u, e]$ operations $x \odot y := (xy) \vee (-u)$, $x \to y := (yx^{-1}) \wedge e$, $x \rightsquigarrow y := (x^{-1}y) \wedge e$, for $x, y \in [-u, e]$. Then $([-u, e]; \odot, \to, \rightsquigarrow, -u, e)$ is a bounded pseudo hoop (= pseudo MV-algebra). By [Dvu1], every pseudo MV-algebra is of the form $([-u, e]; \odot, \to, \rightsquigarrow, -u, e)$.

For any $x \in M$ and any integer $n \geq 0$ we define x^n inductively: $x^0 := 1$ and $x^n := x^{n-1} \odot x$ for $n \geq 1$.

A subset F of a pseudo hoop is said to be a *filter* if (i) $x, y \in F$ implies $x \odot y \in F$, and (ii) $x \leq y$ and $x \in F$ imply $y \in F$. We denote by $\mathcal{F}(M)$ the set of all filters of M. According to [GLP, Prop 3.1], a subset F is a filter iff (i) $1 \in F$, and (ii) $x, x \to y \in F$ implies $y \in F$ ($x, x \rightsquigarrow y \in F$ implies $y \in F$), i.e., F is a *deductive system*. If $a \in M$, then the filter, $F(a)$, generated by a is the set

$$F(a) = \{x \in M : x \geq a^n \text{ for some } n \geq 1\}.$$

A filter F is normal if $x \to y \in F$ iff $x \rightsquigarrow y \in F$. This is equivalent $a \odot F = F \odot a$ for any $a \in M$; here $a \odot F = \{a \odot h : \ h \in F\}$ and $F \odot a = \{h \odot a : \ h \in F\}$. If F is a normal filter, we define $x \theta_F y$ iff $x \to y \in F$ and $y \to x \in F$, then θ_F is a congruence on M, [GLP, Prop 3.13], and $M/F = \{x/\theta_F : \ x \in M\}$ is again a pseudo hoop, where x/θ_F is an equivalence class corresponding to the element $x \in M$, we write also $x/F = x/\theta_F$. Moreover, there is a one-to-one correspondence, [GLP, Prop 3.15], among the set of normal filters, F, and the set of congruences.

We recall that a filter F of a pseudo hoop M is called *maximal* if it is a proper subset of M and not properly contained in any proper filter of M. We recall that if M is not bounded, then it can happen that M has no maximal filter; for example this is true for the real interval $(0, 1]$ equipped with $s \odot t = \min\{s, t\}$, and $s \to t = 1$ iff $s \leq t$, otherwise $s \to t = t$ ($s, t \in (0, 1]$). In [Dvu3], it was proved that every linear pseudo BL-algebra admits a unique maximal filter, and this filter is normal.

3. Basic Pseudo Hoops

A pseudo hoop M is said to be *basic* if, for all $x, y, z \in M$,

(B1) $(x \to y) \to z \leq ((y \to x) \to z) \to z$;
(B2) $(x \rightsquigarrow y) \rightsquigarrow z \leq ((y \rightsquigarrow x) \rightsquigarrow z) \rightsquigarrow z$.

It is straightforward to verify that any linearly ordered pseudo hoop and hence any representable pseudo hoop (= a subdirect product of linearly ordered pseudo hoops) is basic.

By [GLP, Prop 4.6], every basic pseudo hoop is a distributive lattice. By [GLP, Prop 4.6], M is a distributive lattice with prelinearity.

We note, see [GLP, Lem 2.6], that if $\bigvee_i b_i$ exists, then so do $\bigvee_i (a \odot b_i)$ and $\bigvee_i (b_i \odot a)$, moreover, $a \odot (\bigvee_i b_i) = \bigvee_i (a \odot b_i)$ and $(\bigvee_i b_i) \odot a = \bigvee_i (b_i \odot a)$.

Proposition 3.1. *If a pseudo hoop M satisfies prelinearity, then $\odot$ distributes $\wedge$ from both sides, i.e. for all $x, y, z \in M$, we have*

(i) $z \odot (x \wedge y) = (z \odot x) \wedge (z \odot y)$,
(ii) $(x \wedge y) \odot z = (z \odot z) \wedge (y \odot z)$.

Proof. First of all, if $a \leq b$, then $a \leq c \rightsquigarrow b$ and $a \leq c \to b$ for any $c \in M$. Indeed, $a \leq b \leq c \rightsquigarrow b$.

Second, for all $a,b,c \in M$, $(a \rightsquigarrow b) \rightsquigarrow (a \rightsquigarrow c) = (b \rightsquigarrow a) \rightsquigarrow (b \rightsquigarrow c)$ and $(a \to b) \to (a \to c) = (b \to a) \to (b \to c)$. In fact, by [GLP, Thm 2.2], $(a \rightsquigarrow b) \rightsquigarrow (a \rightsquigarrow c) = (a \odot (a \rightsquigarrow b)) \rightsquigarrow c = (a \wedge b) \rightsquigarrow c = (b \wedge a) \rightsquigarrow c = (b \odot (b \rightsquigarrow a)) \rightsquigarrow c = (b \rightsquigarrow a) \rightsquigarrow (b \rightsquigarrow c)$. In the same way we prove the second equality.

By [GLP, Lem 2.5(19)], we have $x \rightsquigarrow y = x \rightsquigarrow (x \wedge y) \leq z \odot x \rightsquigarrow z \odot (x \wedge y)$. Hence, by the first part, $x \rightsquigarrow y \leq (z \odot x \rightsquigarrow z \odot y) \rightsquigarrow (z \odot x \rightsquigarrow z \odot (x \wedge y))$. In a similar way, $y \rightsquigarrow x \leq (z \odot y \rightsquigarrow z \odot x) \rightsquigarrow (z \odot y \rightsquigarrow z \odot (x \wedge y))$. By the second remark of the proof, the right-hand sides of the last two inequalities are the same, we denote it by s. Hence, $x \rightsquigarrow y, y \rightsquigarrow x \leq s$ and prelinearity implies $s = 1$. Therefore, $z \odot x \rightsquigarrow z \odot y \leq z \odot x \rightsquigarrow z \odot (x \wedge y)$ and $(z \odot x) \odot (z \odot x \rightsquigarrow z \odot y) \leq z \odot (x \wedge y)$, i.e., $(z \odot x) \wedge (z \odot y) \leq z \odot (x \wedge y)$. The converse inequality, $z \odot (x \wedge y) \leq (z \odot x) \wedge (z \odot y)$ is obvious. Hence, (i) holds.

The proof of (ii) is similar. □

According to [GLP], we define, for all $x, y \in M$:

$$x \vee_1 y := ((x \rightsquigarrow y) \to y) \wedge ((y \rightsquigarrow x) \to x),$$
$$x \vee_2 y := ((x \to y) \rightsquigarrow y) \wedge ((y \to x) \rightsquigarrow x).$$

Then $x, y \leq x \vee_i y$ for $i = 1, 2$.

Proposition 3.2. *If M is a pseudo hoop with prelinearity, then M is basic, M is a lattice, and*

$$((x \rightsquigarrow y) \to y) \wedge ((y \rightsquigarrow x) \to x) = x \vee y = ((x \to y) \rightsquigarrow y) \wedge ((y \to x) \rightsquigarrow x) \quad (3.1)$$

for all $x, y \in M$.

Proof. Since every pseudo hoop is a $\wedge$-semilattice, we have to show that $x \vee y$ exists in M. Let a be the left-hand side of (3.1). Due to [GLP, Prop 2.11], $a \geq x, y$. Now let $x, y \leq c$. We have $a = a \odot 1 = a \odot ((x \rightsquigarrow y) \vee (y \rightsquigarrow x)) = (a \odot (x \rightsquigarrow y)) \vee (a \odot (y \rightsquigarrow x))$. On the other hand, $a \odot (x \rightsquigarrow y) = [((x \rightsquigarrow y) \to y) \wedge ((y \rightsquigarrow x) \to x)] \odot (x \rightsquigarrow y) \leq ((x \rightsquigarrow y) \to y) \odot (x \rightsquigarrow y) = (x \rightsquigarrow y) \wedge y \leq y \leq c$. In a similar way, we have $a \odot (y \rightsquigarrow x) \leq x \leq c$. Hence, $a \leq c$.

The second equality can be proved in a similar approach.

Now applying [GLP, Prop 4.7], we have that M is basic. □

Remark 3.3. Proposition 3.2 generalizes [GLP, Prop 4.7] where it was proved that a pseudo hoop M is basic iff $\vee_1$ and $\vee_2$ are associative and $(x \rightsquigarrow y) \vee_1 (y \rightsquigarrow x) = 1$ for all $x, y \in M$.

Proposition 3.4. *The variety of bounded pseudo hoops with prelinearity is termwise equivalent to the variety of pseudo BL-algebras.*

Proof. If M is a bounded pseudo hoop with prelinearity, according to Proposition 3.2, M is basic and due to [GLP, Prop 4.10], M is termwise equivalent to a pseudo BL-algebra.

Now let M be a pseudo BL-algebra, then it is a bounded pseudo hoop with prelinearity. □

4. Filters, Prime Filters and the Riesz Decomposition Property

In this section, we extend some results on filters and we show that every pseudo hoop satisfies the Riesz Decomposition Property. This property was known only for pseudo MV-algebras, [Dvu1].

We are saying that a pseudo hoop M satisfies the *Riesz decomposition property* ((RDP) for short) if $a \ge b \odot c$ implies that there are two elements $b_1 \ge b$ and $c_1 \ge c$ such that $a = b_1 \odot c_1$. For example, (i) every pseudo MV-algebra satisfies (RDP), (ii) every cancellative pseudo hoop ($\cong G^-$ for some ℓ-group G) satisfies (RDP), (iii) if M_0 and M_1 satisfies (RDP), so does $M_0 \oplus M_1$, (iv) every linearly ordered pseudo hoop (thanks to the Aglianò-Montagna decomposition of linearly ordered pseudo hoops [Dvu4]) satisfies (RDP), (v) if G is an ℓ-group, then the kite pseudo BL-algebra $G^\dagger$ satisfies (RDP) (for kites see e.g. [JiMo, DGK]). In what follows, we show that all the latter examples are special cases of a more general result saying that every pseudo hoop satisfies (RDP).

Theorem 4.1. *Every pseudo hoop M satisfies* (RDP).

Proof. Let $a, b, c \in M$ be such that $b \odot c \le a$. Then we denote

$$b' := ((c \to a) \rightsquigarrow a) \to a, \quad c' := (c \to a) \rightsquigarrow a.$$

Clearly $c \le (c \to a) \rightsquigarrow a = c'$. Moreover, $b \odot c \le a$ yields $b \le c \to a$. Thus also $(c \to a) \rightsquigarrow a \le b \rightsquigarrow a$ holds. Because pseudo hoops are residuated structures, $b \odot ((c \to a) \rightsquigarrow a) \le a$ and $b \le ((c \to a) \rightsquigarrow a) \to a = b'$ holds. Finally, we have

$$\begin{aligned} b' \odot c' &= (((c \to a) \rightsquigarrow a) \to a) \odot ((c \to a) \rightsquigarrow a) \\ &= ((c \to a) \rightsquigarrow a) \wedge a \\ &= a. \end{aligned}$$

$\square$

If M is a pseudo hoop and $a, b \in M$, then

$$F(a \odot b) = F(a) \vee F(b) = F(b \odot a), \tag{4.1}$$

If $a \vee b$ exists in M, then, [GLP, Prop 3.4],

$$F(a \vee b) = F(a) \cap F(b). \tag{4.2}$$

Let F be a filter of a pseudo hoop M. We say that two elements $a, b \in M$ are in a relation $a \cong_F b$ iff $a \to b, b \to a \in F$. Due to [GLP, Prop 3.6], $\cong_F$ is an equivalence relation. Moreover, $a \cong_F b$ iff $x \odot a = y \odot b$ for some $x, y \in F$. We denote by $Fa := a/F$ the equivalent class corresponding to the element $a \in M$ with respect to $\cong_F$, hence $F \odot a = \{x \odot a : x \in F\} \subseteq Fa$ and $F \odot 1 = F1 = F$. We can introduce a partial binary operation $\le := \le_F$ on $M/F = \{Fa : a \in M\}$ via $Fa \le Fb$ iff $a \to b \in F$. This is equivalent to $x \odot a \le b$ for some $x \in F$. Indeed, let $Fa \le Fb$, set $x = a \to b \in F$ and then $a \wedge b = (a \to b) \odot a \le b$. Conversely, let $x \odot a \le b$ for some $x \in F$. Then $1 = x \odot a \to b = x \to (a \to b)$ which yields $x \le a \to b$ so that $a \to b \in F$.

Hence, the relation $\le := \le_F$ is a partial ordering on the set of M/F : (i) clearly $Fa \le Fa$, (ii) if $Fa \le Fb$ and $Fb \le Fa$, then $Fa = Fb$, and if $Fa \le Fb$, $Fb \le Fc$, then $Fa \le Fc$ because we have $v_1 \odot a \le b$ and $v_2 \odot b \le c$ for some $v_1, v_2 \in F$. Then $v_2 \odot v_1 \odot a \le v_2 \odot b \le c$.

These quotient classes are so-called the *right classes*. We can define also the left classes under the equivalence relation $a\,{}_F\!\cong b$ iff $a \rightsquigarrow b, b \rightsquigarrow a \in F$, and let aF be the equivalence class with respect to ${}_F\!\cong$. Then $aF \le bF$ iff $a \odot f \le b$ for some $f \in F$.

Let $\mathcal{F}(M)$ be the system of all filters of a pseudo hoop M.

Proposition 4.2. *The system of all filters, $\mathcal{F}(M)$, of a pseudo hoop M is a distributive lattice under the set-theoretical inclusion. In addition, $F \cap \bigvee_i F_i = \bigvee_i (F \cap F_i)$.*

Proof. If $\{F_i\}$ is a system of filters, then $\bigvee_i F_i = \{x \in M : x \ge f_1 \odot \cdots \odot f_n,\ f_1 \in F_{i_1}, \ldots, f_n \in F_{i_n}$, for some $i_1, \ldots, i_n, n \ge 1\} \in \mathcal{F}(M)$, and $\bigcap_i F_i \in \mathcal{F}(M)$.

It is clear that $F \cap \bigvee_i F_i \supseteq \bigvee_i (F \cap F_i)$. Let $x \in F \cap \bigvee_i F_i$. Then $x \ge f_1 \odot \cdots \odot f_n$ where $f_1 \in F_{i_1}, \ldots, f_n \in F_{i_n}$. Because every pseudo hoop satisfies (RDP), Theorem 4.1, $x = f_1^0 \odot \cdots \odot f_n^0$ where $f_j^0 \ge f_j$. Therefore, $x \le f_j^0$ so that $f_j^0 \in F \cap F_{i_j}$ and $x \in \bigvee_{j=1}^n (F \cap F_{i_j}) \subseteq \bigvee_i (F \cap F_i)$.

The lattice distributivity is clear from the first part of the present proof. □

A filter F of a pseudo hoop M is said to be *prime* if, for two filters F_1, F_2 on M, $F_1 \cap F_2 \subseteq F$ entails $F_1 \subseteq F$ or $F_2 \subseteq F$. We denote by $\mathcal{P}(M)$ the system of all prime filters of a pseudo hoop M.

We note a prime filter F is *minimal prime* if it does not contains properly another prime filter of M. We stress that a minimal prime filter exists always in any basic pseudo hoop M which admits a maximal lattice ideal of the lattice reduct of M.

Proposition 4.3. *Let F be a filter of a basic pseudo hoop M. Let us define the following statements:*

(i) *F is prime.*
(ii) *If $f \vee g = 1$, then $f \in F$ or $g \in F$.*
(iii) *For all $f, g \in M$, $f \to g \in F$ or $g \to f \in F$.*
(iii') *For all $f, g \in M$, $f \rightsquigarrow g \in F$ or $g \rightsquigarrow f \in F$.*
(iv) *If $f \vee g \in F$, then $f \in F$ or $g \in F$.*
(v) *If $f, g \in M$, then there is $c \in F$ such that $c \odot f \le g$ or $c \odot g \le f$.*
(vi) *If F_1 and F_2 are two filters of M containing F, then $F_1 \subseteq F_2$ or $F_2 \subseteq F_1$.*
(vii) *If F_1 and F_2 are two filters of M such that $F \subsetneq F_1$ and $F \subsetneq F_2$, then $F \subsetneq F_1 \cap F_2$.*
(viii) *If $f, g \notin F$, then $f \vee g \notin F$.*

Then all statements (i)–(viii) *are equivalent.*

Proof. (i) $\Rightarrow$ (ii). By (4.2), $F(f) \cap F(g) = F(f \vee g) = F(1) = \{1\}$, so that $F(f) \subseteq F$ or $F(g) \subseteq F$, and whence $f \in F$ or $g \in G$.

(ii) $\Rightarrow$ (iii), and (ii) $\Rightarrow$ (iii'). They follow from prelinearity.

(iii) $\Rightarrow$ (iv). Let $f \vee g \in F$. Let $f \to g \in F$ or $g \to f \in F$. Since $(f \vee g) \to g = f \to g$, in the first case we have $g = g \wedge (f \vee g) = ((f \vee g) \to g) \odot (f \vee g) \in F$ and similarly in the second one. In the same manner, we have (iii') $\Rightarrow$ (iv).

(iv) $\Rightarrow$ (v). From prelinearity, let e.g. $c := f \to g \in F$. Then $c \odot f = (f \to g) \odot f = f \wedge g \le g$.

(v) $\Rightarrow$ (i). Let $F_1 \cap F_2 \subseteq F$ and let $F_1 \subsetneq F$ and $F_2 \subsetneq F$. There are $f \in F_1 \setminus F$ and $g \in F_2 \setminus F$. By (v), there is $c \in F$ such that, say $c \odot f \le g$. By (4.2), we have $F(f \vee g) = F(f) \cap F(g) \subseteq F_1 \cap F_2 \subseteq F$ so that $f \vee g \in F$. Therefore, $F \ni c \odot (f \vee g) = c \odot f \vee c \odot g \le g \in F$, a contradiction.

(v) $\Rightarrow$ (vi). Suppose that $f \in F_1 \setminus F_2$ and $g \in F_2 \setminus F_1$. Then there is $c \in F$ such that e.g. $c \odot f \leq g$ giving a contradiction $g \in F_1$.

(vi) $\Rightarrow$ (vii). Due to the assumption, $F_1 \subseteq F_2$ or $F_2 \subseteq F_1$ thus $F \subsetneq F_1 \cap F_2$.

Because every pseudo hoop satisfies (RDP), we have the following implications.

(vii) $\Rightarrow$ (viii). By Proposition 4.2 and (4.2), we have $F \subsetneq (F \vee F(f)) \cap (F \vee F(g)) = F \vee F(f \vee g)$ giving $f \vee g \notin F$.

(viii) $\Rightarrow$ (iv). This is evident. $\square$

Now we present the Prime Filter Theorem for basic pseudo hoops.

Lemma 4.4. *Let M be a basic pseudo hoop. If A is a lattice ideal of M and F is a filter of M such that $F \cap A = \emptyset$, then there is a prime filter P of M containing F and disjoint with A.*

Proof. According to Zorn's Lemma, there is a maximal filter of M containing F and disjoint with A. Applying criterion Proposition 4.3(iii), we show that P is prime. If not, there are two elements f and g such that $f \to g, g \to f \notin P$.

Let $P_1 = P \vee F(f \to g)$ and $P_2 = P \vee F(g \to f)$. Due to the choice of P, there are $c_1 \in P_1 \cap A$ and $c_2 \in P_2 \cap A$. Hence, $c_1 \geq \prod_{i=1}^n (s_i \odot (f \to g))$ and $c_2 \geq \prod_{i=1}^n (t_i \odot (g \to f))$, where $s_i, t_j \in P$.

Set $s = s_1 \odot \cdots \odot s_n$, $t = t_1 \odot \cdots \odot t_n$, and $u = s \odot t \in P$.

We recall an easy equality $g \vee (h \odot k) \geq (g \vee h) \odot (g \vee k)$.

Then $c_1 \vee c_2 \geq \prod_{i=1}^n (s_i \odot (f \to g)) \vee \prod_{i=1}^n (t_i \odot (g \to f)) \geq \prod_i (\prod_j (u \odot (f \to g)) \vee (u \odot (g \to f))) \geq \prod_{i,j} (u \odot (f \to g) \vee u \odot (g \to f)) = u^{2n} \in P$ Hence, $c_1 \vee c_2 \in P$ that gives a contradiction. $\square$

We recall that an element u of M is said to be a *strong unit* in M if the filter of M generated by u is equal to M.

Remark 4.5. Let M be a basic pseudo hoop.

(1) The *value* of an element $g \in M \setminus \{1\}$ is any filter V of M that is maximal with respect to the property $g \notin V$. Due to Lemma 4.4, a value V exists and it is prime. Let $\mathrm{Val}(g)$ be the set of all values of $g < 1$. The filter V^* generated by a value V of g and by the element g is said to be the *cover* of V.

(2) We recall that a filter F is *finitely meet-irreducible* if, for each two filters F_1, F_2 such that $F \subsetneq F_1$ and $F \subsetneq F_2$, we have $F \subsetneq F_1 \cap F_2$. Due to Proposition 4.3(vii), the finite meet-irreducibility is a sufficient and necessary condition for a filter F to be prime.

(3) Proposition 4.3(iii) says that F is prime iff the set of quotient classes $\{Fa : a \in M\}$ is linearly ordered.

(4) Proposition 4.3(vi) says that the system of prime filters, $\mathcal{P}(M)$, is a *root system.*

(5) M has a maximal filter iff M admits a strong unit u.

Importance of values can be seen from the following characterization.

Lemma 4.6. *Let M be a basic pseudo hoop. Then $f \leq g$ if and only if $Vf \leq Vg$ for all values V in M. Moreover, let given $a \in M \setminus \{1\}$, V_a be a fixed value of a. Then $f \leq g$ if and only if $V_a f \leq V_a g$ for each $a \in M$.*

Proof. First we show that given a value V, we have $V(f \wedge g) = Vf \wedge Vg$ and if $f \vee g$ exists in M then $V(f \vee g) = Vf \vee Vg$.

It is clear that $V(f \wedge g) \le Vf, Vg$ and assume $Vh \le Vf, Vg$. By definition of right classes, there are $c_1, c_2 \in V$ such that $c_1 \odot h \le f$ and $c_2 \odot h \le g$. Hence, $c_1 \le h \to f$, $c_2 \le h \to g$ and $c_1 \wedge c_2 \le (h \to f) \wedge (h \to g) = h \to (f \wedge g)$ giving $(c_1 \wedge c_2) \odot h \le f \wedge g$.

Similarly, if $Vh \ge Vf, Vg$, there are $c_1, c_2 \in V$ such that $c_1 \odot f \le h$ and $c_2 \odot g \le h$. Then $c_1 \le f \to h$ and $c_2 \le g \to h$ giving $c_1 \wedge c_2 \le (f \to h) \wedge (g \to h) = (f \vee g) \to h$. Whence, $(c_1 \wedge c_2) \odot (f \vee g) \le h$.

Now suppose $Vf \le Vg$ for all values V in M and let $f \not\le g$. Then $f \to g < 1$ and there is a value V' of $f \to g$. Then $V'(f \to g) < V'1 = V'$ and $V'(f \wedge g) = V'((f \to g) \odot f) \le V'f$. We note that $V'f \not\le V'(f \wedge g)$ because then $c \odot f \le (f \wedge g)$ for some $c \in V'$ and $c \le f \to (f \wedge g) = f \to g$ giving a contradiction $f \to g \in V'$. By the first part of the proof, $V'f = V'f \wedge V'g = V'(f \wedge g) < V'f$ that is a contradiction.

The converse statement is obvious.

The proof of the second statement is the same as that of the first one. $\square$

5. Visualization

This section will visualize basic pseudo hoops in a Holland's Representation Theorem type, see e.g. [Dar] which says that every ℓ-group can be embedded into the system of automorphisms of a linearly ordered set. We show that this result can be extended also for basic pseudo hoops. We will visualize a basic pseudo hoop by a system of nondecreasing mapping of a linearly ordered set where $\odot$-operation corresponds to composition of functions, and the arrows $\to$ and $\rightsquigarrow$ are defined in a special way.

Let Ω be a linearly ordered set. A mapping $f : \Omega \to \Omega$ is said to be *residutaed* provided there exists a mapping $f^* : \Omega \to \Omega$ such that $(x)f \le y$ iff $x \le (y)f^*$, for all $x, y \in \Omega$, and we refer to f^* as the *residual* of f.

Let $e = \mathrm{id}_\Omega$. Since $(x)f \le (x)f$ we have $x \le (x)f \circ f^*$ i.e., $e \le f \circ f^*$ and similarly $f^* \circ f \le e$. In addition, $f = f \circ f^* \circ f$ and $f^* = f^* \circ f \circ f^*$.

If f_1^* and f_2^* are residuals of f, then $f_1^* = f_2^*$. Indeed, we have $f_1^* = f_1^* \circ e \le f_1^* \circ f \circ f_2^* \le f_2^*$ and by symmetry, $f_1^* = f_2^*$. Therefore, $(f \circ g)^* = g^* \circ f^*$.

For example, if P is a prime filter of a basic pseudo hoop, set $\Omega = M/P$ and given $a \in M$, let $f_a : M/P \to M/P$ be a mapping defined by $(Px)f_a := Px \odot a$, $Px \in \Omega_P$. Then the residual of f_a is a mapping f_a^* such that $(Px)f_a^* = P(a \to x)$, $Px \in \Omega$.

Let $\mathrm{Mon}(\Omega)$ be the set of all mappings $\alpha : \Omega \to \Omega$ such that $\omega_1 \le \omega_2$ entails $(\omega_1)\alpha \le (\omega_2)\alpha$. We say that $\alpha \le \beta$ iff $(\omega)\alpha \le (\omega)\beta$ for each $\omega \in \Omega$. Then $\mathrm{Mon}(\Omega)$ is a lattice ordered semigroup with the neutral element $e = \mathrm{id}_\Omega$.

This is the main result of the present section:

Theorem 5.1. *Let M be a basic pseudo hoop. Then there is a linearly ordered set Ω and a subsystem $\mathrm{M}(M)$ of $\mathrm{Mon}(\Omega)$ such that $\mathrm{M}(M)$ is a sublattice of $\mathrm{Mon}(\Omega)$ containing e and each element of it is residuated. Moreover, $\mathrm{M}(M)$ can be converted into a basic pseudo hoop where the operations are defined pointwise and is isomorphic to M with the $\odot$-operation corresponding to composition of functions.*

Proof. Let $\{V_g : g < 1\}$ be a system of values, where V_g is a fixed value of $g < 1$. We define a mapping $\phi_g : M \to \mathrm{Mon}(\Omega_g)$, where $\Omega_g = M/V_g$, by

$$(V_g x)\phi_g(a) := V_g x \odot a, \quad V_g x \in \Omega_g \quad (a \in M).$$

Then (i) if $a \le b$, then $\phi_g(a) \le \phi_g(b)$, (ii) $\phi_g(a) \circ \phi_g(b) = \phi_g(a \odot b)$, (iii) $\phi_g(a \vee b) = \phi_g(a) \vee \phi_g(b)$, (iv) $\phi_g(a \wedge b) = \phi_g(a) \wedge \phi_g(b)$. Let $M_0 = \prod\{\mathrm{Mon}(\Omega_g) : g < 1\}$ and order M_0 by coordinates. Define a mapping $f : M \to M_0$ by

$$f(a) = \{\phi_g(a) : g < 1\}, \quad a \in M.$$

By Lemma 4.6, $f(a) \le f(b)$ iff $a \le b$ and f is injective.

Let us totally order the elements of $M \setminus \{1\}$ by $\{g_t : t \in T\}$, where T is a totally ordered set. Let us set $\Omega_t := M/V_{g_t}$ and without loss of generality we can assume $\Omega_s \cap \Omega_t = \emptyset$ for all $s, t \in T$ such that $s \neq t$. Let $\Omega = \bigcup_{t\in T} \Omega_t$, and define a partial order $\preccurlyeq$ on Ω by $\omega_1 \preccurlyeq \omega_2$ iff $\omega_1 \in \Omega_s$ and $\omega_2 \in \Omega_t$ and $s < t$ or $s = t$ and $\omega_1 \le \omega_2$ in Ω_s. Then Ω is totally ordered with respect to $\preccurlyeq$.

Define a mapping $f_0 :\ M \to \mathrm{Mon}(\Omega)$ by: given $\omega \in \Omega$, there is a unique $t \in T$ such that $\omega \in \Omega_t$. Let $(\omega)f_0(a) = (\omega)(\phi_{g_t})(a) \in \Omega_t$. Hence, if $a \in M$, then $f_0(a)|_{\Omega_t}$ maps Ω_t into Ω_t for all $t \in T$. Similarly as for f, f_0 is injective and it maps M onto $\mathrm{M}(M) := f_0(M)$. We have (i) $f_0(1) = \mathrm{id}_\Omega =: e$, (ii) $f_0(a) \le f_0(b)$ iff $a \le b$, (iii) $f_0(a) \circ f_0(b) = f_0(a \odot b)$, (iv) $f_0(a \vee b) = f_0(a) \vee f_0(b)$, (v) $f_0(a \wedge b) = f_0(a) \wedge f_0(b)$. The residual of $f_0(a)$, $f_0^*(a)$, is defined as follows: if $\omega \in \Omega_t$ then $\omega = V_{g_t}x$ for some $x \in M$ and then we set $(\omega)f^*(a) = V_g(x \to a)$.

Now we endow $\mathrm{M}(M)$ with the operations: $f_0(a) \odot f_0(b) := f_0(a) \circ f_0(b) = f_0(a \odot b)$ and $f_0(a) \to f_0(b) := f_0(a \to b)$ and $f_0(a) \rightsquigarrow f_0(b) := f_0(a \rightsquigarrow b)$ for all $a, b \in M$. Then $\mathrm{M}(M)$ is a basic pseudo hoop that is an isomorphic image of M under the isomorphism $a \mapsto f_0(a)$, $a \in M$. $\square$

Question 1. How we can define $\to$ and $\rightsquigarrow$ in Theorem 5.1 to be defined by points ? We recall that in [Dvu5], we have a representation of pseudo MV-algebras by automorphisms defined on a linearly ordered sets where all operations, $\odot, \to, \rightsquigarrow$ are defined by points.

6. Normal-Valued Basic Pseudo Hoops

This is the main part of the this article, where we will study normal-valued basic pseudo hoops. In particular, we present a countable system of equations which completely characterize them.

Given $f \in M$, we define the left and right conjugates, λ_f and ρ_f, of $x \in M$ by f as follows

$$\lambda_f(x) := f \rightsquigarrow (x \odot f), \quad \rho_f(x) := f \to (f \odot x).$$

Then a filter V is normal iff $\lambda_f(V) \subseteq V$ and $\rho_f(V) \subseteq V$ for any $f \in M$.

By [BlTs, Lem 5.2],

$$\lambda_f(x \odot y) \le \lambda_f(x) \odot \lambda_f(y), \quad \rho_f(x \odot y) \le \rho_f(x) \odot \rho_f(y)$$

for all $x, y \in M$.

Let V be a filter and $f \in M$. We define

$$f^{-1}Vf := \{f \rightsquigarrow (v \odot f) : v \in V\} = \lambda_f(V),$$
$$fVf^{-1} := \{f \to (f \odot v) : v \in V\} = \rho_f(V).$$

Then a value V of a basic pseudo hoop is normal in V^* iff $Vf = fV$ for each $f \in V^*$ iff $f^{-1}Vf \subseteq V$ and $fVf^{-1} \subseteq V$ for each $f \in V^*$. We say that a basic pseudo-hoop M is *normal-valued* if every value V of M is normal in its cover V^*.

According to Wolfenstein, [Dar, Thm 41.1], an ℓ-group G is normal-valued iff every $a,b\in G^-$ satisfy $b^2a^2\le ab$, or in our language

$$b^2\odot a^2\le a\odot b. \tag{6.1}$$

Hence, every cancellative pseudo hoop M is normal-valued iff (6.1) holds for all $a,b\in M$. Moreover, every representable pseudo hoop satisfies (6.1).

Similarly, a pseudo MV-algebra is normal-valued iff (6.1) holds, see [Dvu2, Thm 6.7].

If (6.1) holds in a pseudo hoop M, then given $n\ge 1$ there is an integer $k_n\ge 1$ such that for all $a,b\in M$

$$(a\odot b)^n\ge a^{k_n}\odot b^{k_n}. \tag{6.2}$$

Indeed, by induction, we have $(a\odot b)^{n+1}=(a\odot b)^n\odot a\odot b\ge a^{k_n}\odot b^{k_n}\odot a\odot b\ge a^{k_n+2}\odot b^{2k_n+1}\ge a^{2k_n+2}\odot b^{2k_n+2}$.

If A,B are two subsets of M, we denote by $A\odot B=\{a\odot b: a\in A,\ b\in B\}$.

Proposition 6.1. *Let M be a pseudo hoop. Then* (i) *implies* (ii)*, and* (ii) *and* (iii) *are equivalent, where*

(i) *Condition* (6.1) *holds.*
(ii) *$F(a)\odot F(b)=F(a\odot b)=F(b\odot a)=F(b)\odot F(a)$ for $a,b\in M$.*
(iii) *$F\odot G=F\vee G=G\odot F$ for all filters $F,G\in\mathcal{F}(M)$.*

Proof. (i) $\Rightarrow$ (ii). Let $x\in F(a\odot b)$. There exists $n\ge 1$ and $k_n\ge 1$ such $x\ge (a\odot b)^n\ge a^{k_n}\odot b^{k_n}$. (RDP) yields that $x=a_1\odot b_1$ where $a_1\ge a^{k_n}$ and $b_1\ge b^{k_n}$ so that $x=a_1\odot b_1\in F(a)\odot F(b)$.

Conversely, let $x\in F(a)\odot F(b)$. Then $x=a_1\odot b_1$ for some $a_1\in F(a)$ and $b_1\in F(b)$. But then $x\in F(a)\vee F(b)=F(a\odot b)$ when we have used (4.1). Similarly, $F(b)\odot F(a)=F(b\odot a)$.

(ii) $\Rightarrow$ (iii). It is clear that $F\odot G\subseteq F\vee G$. Now take $x\in F\vee G$. Then $x\ge a_1\odot b_1\odot\cdots\odot a_n\odot b_n$ where $a_i\in F$ and $b_i\in G$. (RDP) yields $x=a_1^0\odot b_1^0\odot\cdots\odot a_n^0\odot b_n^0$ for $a_i^0\ge a_i$ and $b_i^0\ge b_i$. Then $x\in F(a_1)\odot F(b_1)\odot\cdots\odot F(a_n)\odot F(b_n)=F(a_1)\vee F(b_1)\vee\cdots\vee F(a_n)\vee F(b_n)=F(a_1)\vee\cdots\vee F(a_n)\vee F(b_1)\vee\cdots\vee F(b_n)=F(a_1\odot\cdots\odot a_n)\vee F(b_1\odot\cdots\odot b_n)\subseteq F\vee G$.

(iii) $\Rightarrow$ (ii). We have $F(a)\odot F(b)=F(a)\vee F(b)$. (4.1) entails $F(a\odot b)=F(a)\vee F(b)=F(b\odot a)$. ☐

Lemma 6.2. *Let M be a basic pseudo hoop. Then, for any $X\subseteq M$, the set $X^\perp=\{x:\ x\vee a=1\ \forall a\in X\}$ is a filter of M.*

Proof. The set $X^\perp$ is clearly closed with respect to upper bounds. Let $x,y\in X^\perp$ and $a\in X$. The equalities $x\vee a=y\vee a=1$ hold. Now we can compute: $(x\odot y)\vee a=(x\odot y)\vee(x\odot a)\vee a=(x\odot(y\vee a))\vee a=(x\odot 1)\vee a=1$. Thus also $x\odot y\in X^\perp$. ☐

Lemma 6.3. *Let M be a basic pseudo hoop with a strong unit $u\in M$. Then the inclusion*

$$\bigcap \mathrm{Val}(u)\subseteq\{a: a^n\ge u \text{ for all } n\in\mathbb{N}\}$$

holds.

Proof. Let $a \in M$ be such an element that there is an integer $n \in \mathbb{N}$ with the property $a^n \not\geq u$. Thus the inequality $u \to a^n < 1$ holds and the filter $\{u \to a^n\}^\perp$ is nontrivial (more precisely $u \to a^n \notin \{u \to a^n\}^\perp$). Prelinearity yields $a^n \to u \in \{u \to a^n\}^\perp$. Because u is a strong unit and $\{u \to a^n\}^\perp$ is nontrivial, also $u \notin \{u \to a^n\}^\perp$ holds. Due to Zorn's Lemma, there is a value $V \in \mathrm{Val}(u)$ such that $\{u \to a^n\}^\perp \subseteq V$.

Let us assume to contrary that $a \in V$. Clearly also $a^n, a^n \to u \in V$ which gives $(a^n \to u) \odot a^n \leq u \in V$ which is a contradiction. Finally, $a \notin V \supseteq \bigcap \mathrm{Val}(u)$ and this finishes the proof. □

We recall the following folklore result on prime filters.

Remark 6.4. *Let M be a basic pseudo hoop. Then*

$$\bigcap\{F : F \textit{ is a minimal prime filter}\} = \{1\}.$$

Proof. If $x \in M \setminus \{1\}$, then $\mathrm{Val}(x) \neq \emptyset$ and any $V \in \mathrm{Val}(x)$ contains a minimal prime filter V_M. This yields $x \notin V \supseteq V_M \supseteq \bigcap\{F : F \text{ is a minimal prime filter}\}$. □

Lemma 6.5. *Let M be a basic pseudo hoop and $a, b, x \in M$ be such that $V(a \odot b) \leq Vx$ for any $V \in \mathrm{Val}(x)$. Then $a^2 \odot b^2 \leq x$.*

Proof. We are going to prove that $(a^2 \odot b^2) \to x$ belongs to any minimal prime filter F. Let F be a minimal prime filter. If $x \in F$, then clearly $(a^2 \odot b^2) \to x \in F$.

We suppose that $x \notin F$. Thus there exists a value $V \in \mathrm{Val}(x)$ such that $F \subseteq V$. There are two cases:

(i) Let $a \notin V$. Clearly, $V(a \odot b^2) \leq V(a \odot b) \leq Vx$. Hence, $(a \odot b^2) \to x \in V$ holds. Because $a \notin V$ also $((a \odot b^2) \to x) \to a \notin V$ and, moreover, $((a \odot b^2) \to x) \to a \notin F$. Prelinearity of M gives $(a^2 \odot b^2) \to x = a \to ((a \odot b^2) \to x) \in F$.

(ii) Let $a \in V$. We can compute $Vb = V(a \odot b) \leq Vx$ and thus $b \to x \in V$. We assert that $b \notin V$, otherwise, $V1 = V(a \odot b) \leq Vx$ yields $x = 1 \to x \in V$, which is absurd. Therefore also $a^2 \odot b \notin V$. Altogether $(b \to x) \to (a^2 \odot b) \notin V$ and consequently $(b \to x) \to (a^2 \odot b) \notin F$. Analogously to the previous part, prelinearity gives $(a^2 \odot b) \to (b \to x) = (a^2 \odot b^2) \to x \in F$.

We have shown that $(a^2 \odot b^2) \to x$ belongs to any minimal prime filter. Due to Remark 6.4, we obtain $(a^2 \odot b^2) \to x = 1$ and $a^2 \odot b^2 \leq x$. □

We recall that a pseudo hoop M is *simple* if it contains a unique proper filter.

Theorem 6.6. *Let M be a normal-valued basic pseudo hoop, then the following inequalities hold.*

(i) $x^2 \odot y^2 \leq y \odot x$.

(ii) $((x \to y)^n \rightsquigarrow y)^2 \leq (x \rightsquigarrow y)^{2n} \to y$ *for any* $n \in \mathbb{N}$.

(iii) $((x \rightsquigarrow y)^n \to y)^2 \leq (x \to y)^{2n} \rightsquigarrow y$ *for any* $n \in \mathbb{N}$.

Proof. (i) For arbitrary $a, b \in M$, let $x := b \odot a$. If $V \in \mathrm{Val}(x)$, then clearly $a, b \geq x$ yields $a, b \in V^*$. Because V^*/V is simple (see [DGK, Prop 2.3]), it is commutative [DGK, Thm 2.4]. Then $V(a \odot b) = V(b \odot a) = Vx$. Due to Lemma 6.5, we obtain $a^2 \odot b^2 \leq x = b \odot a$.

(ii), (iii) For all $x, y \in M$ and each $n \in \mathbb{N}$, we denote

$$a := (x \to y)^n \rightsquigarrow y,$$
$$b := (x \rightsquigarrow y)^n, \quad b' := (x \to y)^n.$$

If $y = 1$, (ii) and (iii) trivially hold. Let $y < 1$ and let us have $V \in \mathrm{Val}(y)$. Commutativity of the algebra V^*/V and $y, x \vee y \in V^*$ yield $V((x \to y)^n) = V((((x \vee y) \to y)^n) = V((((x \vee y) \rightsquigarrow y)^n) = V((x \rightsquigarrow y)^n)$. Consequently, $Vb = Vb'$ and, moreover, $V(a \odot b) = V(b' \odot a) \leq Vy$. Due to Lemma 6.5, we obtain $a^2 \odot b^2 \leq y$ and also $a^2 \leq b^2 \to y$. The second part of the theorem can be proved analogously. □

Lemma 6.7. *Let M be a pseudo hoop satisfying the inequality $x^2 \odot y^2 \leq y \odot x$ and let F be and $a \in M$ be a fixed filter and an element of M, respectively. Then both sets $\{x \geq f \odot a^n : n \in \mathbb{N}, f \in F\}$ and $\{x \geq a^n \odot f : n \in \mathbb{N}, f \in F\}$ are equal to the filter generated by F and a.*

Proof. If $x \geq f_1 \odot a^n$ and $y \geq f_2 \odot a^m$ are such that $f_1, f_2 \in F$ and $m, n \in \mathbb{N}$ then $x \odot y \geq f_1 \odot a^n \odot f_2^2 \odot a^m \geq f_1 \odot f_2 \odot a^{2n+m}$. Clearly, also $f_1 \odot f_2^2 \in F$ and hence presented sets are filters. Moreover, the given sets are contained in the filter generated by F and a. This proves the lemma. □

Let us have a pseudo hoop with inequality $x^2 \odot y^2 \leq y \odot x$. If V is a value, then for any $x \in V^* \setminus V$, we have $F(V, x) = V^*$ and thus, for any $y \in V^*$, there are $n \in \mathbb{N}$ and $v \in V$ such that $v \odot x^n \leq y$ ($x^n \odot v \leq y$, respectively). Hence, for any $x \in V \setminus V^*$ and any $y \in V^*$, there is $n \in \mathbb{N}$ such that $V(x^n) \leq Vy$ (or $(x^n)V \leq yV$).

Theorem 6.8. *If a basic pseudo hoop M satisfies inequalities* (i)–(iii) *from Theorem* 6.6*, then M is normal-valued.*

Proof. Let (i)–(iii) hold and let V be a value. Let $x, y \in V^*$ be such that $x \to y \notin V$ (and hence $y \notin V$). Then there is $n \in \mathbb{N}$ such that $V(x \to y)^n \leq Vy$. Hence, $(x \to y)^n \to y \in V$ and also $((x \to y)^n \to y)^2 \in V$. Due to inequality (ii), $(x \rightsquigarrow y)^{2n} \to y \in V$ holds. Hence, we assert $x \rightsquigarrow y \notin V$. If not, $x \rightsquigarrow y \in V$ yields $y \geq ((x \rightsquigarrow y)^{2n} \to y) \odot (x \rightsquigarrow y)^{2n} \in V$ which is a contradiction. Altogether $x \to y \notin V$ yields $x \rightsquigarrow y \notin V$.

The converse implication $x \rightsquigarrow y \notin V$ yields $x \to y \notin V$ can be proved in an analogous way. This implies M is normal-valued. □

Combining the results of Theorem 6.8 and Theorem 6.6, we have the following corollary.

Corollary 6.9. *Let M be a basic pseudo hoop. The following statements are equivalent*

(i) *M is normal-valued.*
(ii) (i)–(iii) *from Theorem* 6.6 *hold.*

Lemma 6.10. *If a basic pseudo hoop M satisfies the inequality $x^2 \odot y^2 \leq y \odot x$, then any value V and any $x \in V^* \setminus V$ such that $Vx > V(x^2)$ satisfy $Vx \subseteq xV$.*

Proof. Let us have $x \in V^*$ such that $Vx > Vx^2$ and, moreover, let $f \in V$ be such that $\lambda_x(f) = x \rightsquigarrow (f \odot x) \notin V$. The divisibility clearly yields the equality $x \odot (\lambda_x(f))^n = f^n \odot x$ for any $n \in \mathbb{N}$. Because $f \in V$, we can interpret the last equality as $V((\lambda_x(f))^n) \geq V(x \odot (\lambda_x(f))^n) = V(f^n \odot x) = Vx$. Hence, $\lambda_x(f) \in V^* \setminus V$ and $x \in V^*$. There is $n \in \mathbb{N}$ such that $Vx^2 \geq V((\lambda_x(f))^n)$ and altogether $Vx^2 < Vx$ which is a contradiction. We have proved that for any $f \in V$ also $\lambda_x(f) \in V$.

One can easily check that for any $y \in Vx$, the inequality $Vy^2 < Vy$ holds (more precisely, if $Vy^2 = Vy$, then Vy is a least element and also $Vx = Vy$ is minimal which gives a contradiction $Vx = Vx^2$). Due to $y \in Vx$, we obtain the equality $f_1 \odot x = f_2 \odot y$ and thus also $x \odot \lambda_x(f_1) = y \odot \lambda_y(f_2)$. In the previous part, we have proved that $\lambda_x(f_1), \lambda_y(f_2) \in V$ and thus $y \in xV$. □

Question 2. Does inequality $x^2 \odot y^2 \leq y \odot x$ characterize the class of (basic) normal-valued pseudo hoops ? For example, let G be an ℓ-group and let $G^\dagger$ be the kite corresponding to G (for kites see [JiMo, DGK]). By [DGK, Lem 4.11], the kite $G^\dagger$ is a normal-valued pseudo BL-algebra iff G is a normal-valued ℓ-group. Hence, inequality (6.1) completely characterizes a kite to be normal-valued.

In what follows, we present a variety of basic pseudo hoops satisfying a single equation such that the inequality $x^2 \odot y^2 \leq y \odot x$ is a necessary and sufficient condition for M to be normal-valued.

We say that a bounded pseudo hoop M is *good* if

$$x^{-\sim} = x^{\sim -}, \quad x \in M, \tag{6.3}$$

where $x^- := x \to 0$ and $x^\sim = x \rightsquigarrow 0$. For example, every pseudo MV-algebra is good as well as every representable pseudo hoop is good, see [Dvu4]. On the other hand, a kite $G^\dagger$ is a pseudo BL-algebra which is not good whenever $G \neq \{e\}$, [DGK, Lem 4.11].

We present a stronger equality than (6.3):

$$(x \to y) \rightsquigarrow y = (x \rightsquigarrow y) \to y \tag{6.4}$$

for all $x, y \in M$.

For example, every negative cone of an ℓ-group and the negative interval of an ℓ-group with strong unit satisfies (6.4). If M is a linearly ordered pseudo hoop, due to [Dvu4, Cor 4.2], M is an ordinal sum of a system whose each component is either the negative cone of a linearly ordered ℓ-group or the negative interval of a linearly ordered ℓ-group with strong unit. Therefore, it satisfies (6.4), consequently every representable bounded pseudo hoop satisfies (6.4). On the other side, no nontrivial kite satisfies (6.4).

Lemma 6.11. *Let M be a basic pseudo hoop satisfying* (6.4). *Then M satisfies the identity*

$$(x \to y)^n \rightsquigarrow y = (x \rightsquigarrow y)^n \to y \tag{6.5}$$

for all $x, y \in M$ and for any $n \in \mathbb{N}$.

Proof. Assume for induction that (6.5) holds for any integer k with $1 \leq k \leq n$. We have

$$\begin{aligned}
(x \to y)^{n+1} \rightsquigarrow y &= (x \to y)^n \rightsquigarrow ((x \to y) \rightsquigarrow y)\\
&= (x \to y)^n \rightsquigarrow ((x \rightsquigarrow y) \to y)\\
&= (x \rightsquigarrow y) \to ((x \to y)^n \rightsquigarrow y)\\
&= (x \rightsquigarrow y) \to ((x \rightsquigarrow y)^n \to y\\
&= (x \rightsquigarrow y)^{n+1} \to y,
\end{aligned}$$

where in the third equality we have used the identity $a \rightsquigarrow (b \to c) = b \to (a \rightsquigarrow c)$, $a, b, c \in M$, see [BlTs, Lem 3.2(6)]. □

Theorem 6.12. *Let M be a basic pseudo hoop satisfying* (6.4). *Then M is normal-valued if and only if $x^2 \odot y^2 \leq y \odot x$ for all $x, y \in M$.*

Proof. The "if" condition holds by Theorem 6.6, so only need to show that if M satisfies $x^2 \odot y^2 \leq y \odot x$, then M is normal-valued. Assume M satisfies $x^2 \odot y^2 \leq y \odot x$. By Theorem 6.8, it suffices to show that M satisfies

$$((x \to y)^n \rightsquigarrow y)^2 \leq (x \rightsquigarrow y)^{2n} \to y$$

and

$$((x \rightsquigarrow y)^n \to y)^2 \leq (x \to y)^{2n} \rightsquigarrow y$$

for any $n \in \mathbb{N}$. By Lemma 6.11, it is enough to show that the first inequality of the latter two hods, which is, by residuation, equivalent to

$$((x \to y)^n \rightsquigarrow y)^2 \odot (x \rightsquigarrow y)^{2n} \leq y.$$

Now, consider

$$\begin{aligned} ((x \to y)^n \rightsquigarrow y)^2 \odot (x \rightsquigarrow y)^{2n} &= ((x \to y)^n \rightsquigarrow y) \odot ((x \to y)^n \rightsquigarrow y) \odot (x \rightsquigarrow y)^{2n} \\ &= ((x \to y)^n \rightsquigarrow y) \odot ((x \rightsquigarrow y)^n \to y) \odot (x \rightsquigarrow y)^{2n} \\ &\leq ((x \to y)^n \rightsquigarrow y) \odot y \odot (x \rightsquigarrow y)^n \\ &\leq y \end{aligned}$$

showing that the desired inequality holds. □

The following statement was proved in [DGK, Thm 3.2] in a different way, here we use Theorem 6.12.

Corollary 6.13. *Every representable pseudo hoop is normal-valued.*

Proof. If M is a linearly ordered pseudo hoop, then according to the remark just after (6.4), M satisfies (6.4). Every linearly ordered ℓ-group is normal-valued [Dar], so is its negative cone as well as its negative interval with strong unit satisfies the inequality $x^2 \odot x^2 \leq y \odot x$. Consequently, every ordinal sum of such linear components satisfies the inequality which by Theorem 6.12 entails, M is normal-valued. □

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