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Layers for zero-probability and stable coherence over Łukasiewicz events

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Abstract The notion of stable coherence has been recently introduced to characterize coherent assignments to conditional many-valued events by means of hyperreal-valued states. In a nutshell, an assignment, or book, β on a finite set of conditional events is stably coherent if there exists a coherent variant β' of β such that β' maps all antecedents of conditional events to a strictly positive hyperreal number, and such that β and β' differ by an infinitesimal. In this paper we provide a characterization of stable coherence in terms of layers of zero probability for books on Łukasiewicz logic events.

Keywords Layers of zero-probability, Conditional probability, Stable coherence, MV-algebras.

1 Motivation

If $\mathbf{A}$ is a Boolean algebra and $P : A \to [0, 1]$ is a finitely additive probability on $\mathbf{A}$, the conditional probability of "a given b" can be quantified as the ratio $P(a \mid b) = P(a \wedge b)/P(b)$, whenever $P(b) > 0$. Obviously, the conditional probability $P(\cdot \mid \cdot)$ defined in this way on $A \times A$ is only a partial function and, in particular, it is not defined on all the pairs $(a \mid b)$ with $P(b) = 0$.

Krauss [10] and Nelson [22] proposed a way to overcome this problem employing nonstandard probabilities: instead of a real-valued probability $P : A \to [0, 1]$, they

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consider a nonstandard probability measure $P^* : A \to {}^*[0,1]$, where ${}^*[0,1]$ is a nonstandard (or hyperreal) unit interval, such that only the impossible event $\perp$ keeps having probability zero, but any other (non-impossible) event takes non-zero probability, possibly infinitesimal. Then, the (standard) conditional probability of $a \mid b$ with $b \neq \perp$ is defined as $St(P^*(a \wedge b)/P^*(b))$, where $St : {}^*\mathbb{R} \to \mathbb{R}$ denotes the *standard part* function.

The problem of defining the conditional probability of "a given b" when b has probability zero has also been studied by Blum, Brandenburger and Dekel [1]. Their idea, that can be actually traced back to Rényi [25], involves lexicographic probabilities. In a nutshell, a lexicographic probability is a sequence $\langle \mu_0, \mu_1, \ldots \rangle$ of real-valued probability measures, indexed so that, given an event a, its lexicographic probability is $\mu_0(a)+\varepsilon_1\mu_1(a)+\varepsilon_2\mu_2(a)+\ldots$, where the ε_i's are infinitesimals such that the order of ε_{i+1} is strictly below the order of ε_i. Thus, a lexicographic probability again ends up with a single, nonstandard, probability measure.

The intimate relation between conditional probability, nonstandard probability and lexicographic probability has been clarified, for the case of Boolean algebras, by Halpern [8], who proved that Popper spaces (i.e. conditional probability spaces satisfying certain regularity conditions), nonstandard probability spaces and lexicographic probability spaces are, under certain conditions, interdefinable.

The connection between conditional, lexicographic and nonstandard probability becomes even more evident if we look at the foundational issues of probability theory and, in particular, de Finetti's foundation of subjective probability theory [6]. Let us recall that de Finetti defines the probability of an unknown event e as the *fair price* (between 0 and 1) which a rational Gambler is willing to pay to participate in a betting game against the Bookmaker, the payoffs of which are 1 in case e occurs, and 0 otherwise. Given a finite set of events, an assignment on them (in terms of prices) is hence said to be *coherent* if it does not ensure Bookmaker to incur in a sure loss, i.e., Bookmaker is not going to lose money independently on the truth-realization of the events involved in the game. Based on this very simple idea, de Finetti showed that all theorems of probability theory may be derived as consequences of his coherence condition. In their book [5] Coletti and Scozzafava characterize coherent conditional assignments (in the sense of de Finetti [6], see also [24]) in terms of lexicographic probabilities by employing what they called *zero-layers* (or *layers of zero probabilities*), while Krauss [10] characterizes coherent conditional probabilities in terms of nonstandard probability measures. These two approaches have been employed in [7] to provide a logical characterization of coherence for conditional assignments.

Framing probability theory in a more general algebraic setting than the Boolean one, often brings non trivial technical complications. In particular for the case of MV-algebras, although there exists a common agreement on the fact that *states* (see [19] and Section 2.3 below) represent a suitable generalization of probability measures on these structures, the case of conditional probability on MV-algebras is far from being settled. Indeed, several and often non-equivalent proposals have been made (cf. [12,16,21]) and, in particular, it is not clear yet if the above discussed characterizations of conditional probability in terms of lexicographic and nonstandard probability, can be achieved.

However, from the foundational prospective, the authors of [17] investigate coherent assignment of *conditional states* (i.e., conditional probabilities on MV-algebras) by means of nonstandard states, introducing the notion of *stable co-*

herence. Imagine a real-valued assignment (or *book*) β on both conditional events $a_1 \mid b_1, \ldots, a_n \mid b_n$ and unconditional events $b_1, \ldots, b_n$, where $a_1, \ldots, a_n, b_1, \ldots, b_n$ are elements of an MV-algebra. Then β is said to be *stably coherent* if there exists an hyperreal-valued variant β' of β such that β' is coherent and, for every $i = 1, \ldots, n$, $\beta'(b_i) > 0$ while $|\beta(a_i \mid b_i) - \beta'(a_i \mid b_i)|$ and $|\beta(b_i) - \beta'(b_i)|$ are infinitesimal. Hence, stable coherence generalizes Krauss and Nelson approaches from Boolean to MV-algebras (see Section 4 for details).

In this paper we make a step further in the understanding of conditional states and their relation with nonstandard states by employing the notion of layers of zero-probability. In doing so, we will generalize lexicographic probabilities to lexicographic states and we will characterize stable coherence in terms of lexicographic states of free MV-algebras. In particular, after collecting basic notions and preliminary results on MV-algebras and state theory in Section 2, we will first characterize coherent conditional assignments which corresponds to (faithful) hyperreal-valued states (Section 3) and then, in Section 4, stably coherent books. We end this paper presenting some conclusions and ideas for future work on this topic.

2 Preliminaries

2.1 MV-algebras and MV-algebras with product

Let $\mathsf{MV} = (\oplus, \neg, \bot, \top)$ be a signature of type $(2, 1, 0, 0)$. For every $k \in \mathbb{N}$, let Form(k) be the set of *formulas* of MV built from k variables $\{\mathsf{z}_1, \mathsf{z}_2, \ldots, \mathsf{z}_k\}$, $\bot$, $\top$, and operations $\neg, \oplus$. Further binary operation symbols are defined as follows over the signature MV: $\varphi \odot \psi$ is $\neg(\neg\varphi \oplus \neg\psi)$, $\varphi \to \psi$ is $\neg\varphi \oplus \psi$, $\varphi \leftrightarrow \psi$ is $(\varphi \to \psi) \odot (\psi \to \varphi)$, $\varphi \ominus \psi$ is $\neg(\varphi \to \psi)$, $\varphi \vee \psi$ is $(\varphi \to \psi) \to \psi$, and $\varphi \wedge \psi$ is $\neg(\neg\varphi \vee \neg\psi)$.

An MV-algebra is any algebra $\mathbf{A} = (A, \oplus, \neg, \bot, \top)$ such that:

(1) $(A, \oplus, \bot)$ is a commutative monoid
(2) $\neg\top = \bot$
(3) $\neg\neg x = x$
(4) $x \oplus \top = \top$
(5) $x \oplus (\neg(\neg y \oplus x)) = y \oplus (\neg(\neg x \oplus y))$.

The MV-algebra on the real unit interval

$$[0, 1]_{MV} = ([0, 1], \oplus, \neg, 0, 1),$$

where $x \oplus y = \min(1, x + y)$, and $\neg x = 1 - x$, is called the *standard MV-algebra*. The variety $\mathbb{MV}$ of MV-algebras is generated, as a variety and as a quasivariety, by $[0, 1]_{MV}$, see [3,4].

Every MV-algebra $\mathbf{A}$, can be equipped with an order relation $\leq$ so defined: for all $x, y \in A$,

$$x \leq y \text{ iff } x \to y = \top.$$

An MV-algebra $\mathbf{A}$ is said to be *linearly ordered*, or an *MV-chain*, provided that the order $\leq$ is linear. Moreover, in every MV-algebra $\mathbf{A}$ we can define a partial operation $+ : A \times A \to A$ in the following way: $x + y$ is defined if and only if $x \odot y = 0$ and in this case $x + y = x \oplus y$.

Example 1 (1) Every Boolean algebra is an MV-algebra. Moreover, for every MV-algebra $\mathbf{A}$, the set of its idempotent elements $B(\mathbf{A}) = \{x : x \oplus x = x\}$ is the domain of the largest Boolean subalgebra of $\mathbf{A}$. This Boolean subalgebra is called the *Boolean skeleton of* $\mathbf{A}$.

(2) Fix a $k \in \mathbb{N}$, and let $\mathcal{F}_k$ be the set of all *McNaughton functions* (cf. [4]) from the hypercube $[0,1]^k$ into $[0,1]$, in other words, the set of all functions $f : [0,1]^k \to [0,1]$ which are continuous, piecewise linear and such that each piece has integer coefficients. The following pointwise operations defined on $\mathcal{F}_k$:

$$(f \oplus g)(x) = \min\{1, f(x) + g(x)\}, \text{ and } \neg f(x) = 1 - f(x),$$

make the structure $\mathcal{F}_k = (\mathcal{F}_k, \oplus, \neg, 0, 1)$ an MV-algebra, where 0 and 1 respectively denote the functions constantly equal to 0 and 1. Actually, $\mathcal{F}_k$ is the free MV-algebra over k-free generators which, in turn, coincides with the Lindenbaum-Tarski algebra of Łukasiewicz logic in the language with k propositional variables $\mathrm{Form}(k)$.

A *PMV-algebra* is a structure $\mathbf{P} = (P, \oplus, \neg, \cdot, \bot, \top)$ such that $(P, \oplus, \neg, \bot, \top)$ is an MV-algebra, and $\cdot : P \times P \to P$ satisfies the following, for any $x, y, z \in P$

(P1) $(P, \cdot, \top)$ is a commutative monoid
(P2) If $x + y$ is defined, so it is $z \cdot x + z \cdot y$ and it coincides with $z \cdot (x + y)$,

The variety of PMV-algebras will be denoted by $\mathbb{PMV}$. A relevant example of PMV-algebras is obtained equipping the standard MV-algebra $[0,1]_{MV}$ with the ordinary product of reals on $[0,1]$. This algebra will be denoted by $[0,1]_{PMV}$. The algebra $[0,1]_{PMV}$ generates a proper sub-quasivariety of $\mathbb{PMV}$ (see [9] for details), denoted $\mathbb{PMV}^+$, namely the class of *PMV*$^+$*-algebras*, which is the class PMV-algebras that satisfy the quasi-equation [14,15]:

$$x^2 = \bot \Rightarrow x = \bot$$

An MV-algebra $\mathbf{A}$ (PMV$^+$-algebra respectively) is said to be *simple* if its unique congruences are $\{(a,a) \mid a \in A\}$ and $A \times A$ and it is called *semisimple* if $\mathbf{A}$ is a subdirect product of simple algebras in the same (quasi)variety. For the sake of a later use, let us recall that an MV-algebra (PMV$^+$-algebra) is simple iff it is a subalgebra of the algebra whose carrier is $[0,1]$. As a consequence, if an algebra (MV or PMV$^+$) is a subalgebra of $[0,1]^X$ for some nonempty set X, then is semisimple. In particular, the MV-algebra $\mathcal{F}_k$ is semisimple for every $k \in \mathbb{N}$.

If $\mathbf{A}$ and $\mathbf{B}$ are MV-algebras (resp. PMV$^+$-algebras), we will henceforth denote by $\mathscr{H}(\mathbf{A}, \mathbf{B})$ the class of MV-homomorphisms from $\mathbf{A}$ into $\mathbf{B}$ (PMV$^+$- homomorphism resp.), and we will denote $\mathscr{H}(\mathbf{A})$ the class of MV-homomorphisms (PMV$^+$-homomorphism resp.) of $\mathbf{A}$ in $[0,1]_{MV}$ ($[0,1]_{PMV}$, resp.).

2.2 McNaughton functions and triangulation of the hypercube

As we recalled in Example 1 (2), a k-ary *McNaughton function* is a continuous function $f\colon [0,1]^k \to [0,1]$ such that $f(\mathbf{x}) \in \{0,1\}$ if $\mathbf{x} \in \{0,1\}^k$, and defined as follows: there exist linear polynomials $p_1, \ldots, p_l$ with integer coefficients such that for every $\mathbf{x} \in [0,1]^k$, there is a $j \in \{1, \ldots, l\}$ such that $f(\mathbf{x}) = p_j(\mathbf{x})$. Let us identify every point $\mathbf{x} = (x_1, \ldots, x_k) \in [0,1]^k$ with a valuation $e_{\mathbf{x}}$ of the k-variables $\{\mathsf{z}_1, \ldots, \mathsf{z}_k\}$ into $[0,1]_{MV}$ such that $e_{\mathbf{x}}(\mathsf{z}_i) = x_i$. Hence a routine induction on $\varphi \in \mathrm{Form}(k)$ shows that the function $M_\varphi : [0,1]^k \to [0,1]$ defined as

$$M_\varphi(\mathbf{x}) = e_{\mathbf{x}}(\varphi)$$

is a McNaughton function. Conversely, for every k-ary McNaughton function f, there exists a formula $\varphi \in \mathrm{Form}(k)$ such that $M_\varphi = f$, [18].

In what follows we will frequently use the following construction. Let $p_1, \ldots, p_l$ be the linear pieces of a k-ary McNaughton function f. For every permutation π of $\{1, \ldots, l\}$, let:

$$P_\pi = \{\mathbf{x} \in [0,1]^k \mid p_{\pi(1)}(\mathbf{x}) \le p_{\pi(2)}(\mathbf{x}) \le \cdots \le p_{\pi(l)}(\mathbf{x})\}, \tag{1}$$

$$C = \{P_\pi \mid \pi \text{ is a permutation}\}. \tag{2}$$

Let S be a set of k-dimensional polyhedra, and denote by $\mathcal{V}(S)$ (or simply $\mathcal{V}$ when S is clear by the context) the set of vertices of the polyhedra in S. Let us observe that, since each P_π is built up from finitely many polynomial functions with integer coefficients, the set C defined in (2) is a finite set of k-dimensional polyhedra whose vertices $\mathcal{V}(C)$ are rational points. That is, every $P_\pi \in C$ is the convex hull of a finite set of rational points in $[0,1]^k$. Along the lines of [4, Proposition 3.3.1], C can be manufactured into a *unimodular triangularization* Δ of $[0,1]^k$ that *linearizes* f, that is, from C we can define a finite set Δ of k-dimensional *unimodular simplexes*[1] over the rational vertices $\mathcal{V}(\Delta)$, enjoying the following properties:

(*i*) the union of all simplexes in Δ is equal to $[0,1]^k$;
(*ii*) any two simplexes in Δ intersect in a common face;
(*iii*) for each simplex $T \in \Delta$, there exists $j = 1, \ldots, k$ such that the restriction of f to T coincides with p_j.

Notice that, while the first two conditions say that Δ is a triangularization of $[0,1]^k$, the third states that Δ linearizes f. In such a case, we also say that f is *linear* over Δ. Let $\mathbf{x}_i$ be a vertex of a simplex in Δ. The *Schauder hat* at $\mathbf{x}_i$ is the McNaughton function $\mathbf{h}_i^\circ$ linearized by S such that $\mathbf{h}_i^\circ(\mathbf{x}_i) = 1/\mathrm{den}(\mathbf{x})$ and $\mathbf{h}_i^\circ(\mathbf{x}_j) = 0$ for every vertex $\mathbf{x}_j$ distinct from $\mathbf{x}_i$ in S. The *normalized* Schauder hat at $\mathbf{x}_i$ is the function

$$\mathbf{h}_i = \mathrm{den}(\mathbf{x}_i) \cdot \mathbf{h}_i^\circ.$$

Note that $\mathbf{h}_i(\mathbf{x}_i) = 1$ and $\mathbf{h}_i(\mathbf{x}_j) = 0$ for every vertex $\mathbf{x}_j \neq \mathbf{x}_i$. Given a unimodular triangulation Δ of $[0,1]^k$, $\mathcal{H}(\Delta)$ will denote the finite set of normalized Schauder hats associated to the vertices of Δ.

The above construction can be generalized to the case of finitely many McNaughton functions. In particular, if $M = \{f_1, \ldots, f_n\}$ is a finite set of McNaughton

[1] An k-dimensional *simplex* is the convex hull of $k+1$ affinely independent vertices. The empty set $\emptyset$ is a (-1)-dimensional simplex. A l-dimensional *face* of the k-simplex T over vertices $\mathbf{x}_1, \ldots, \mathbf{x}_{k+1}$ is the k-simplex spanned by $l+1$ vertices of T.
Let T be an k-dimensional simplex over rational vertices. Let $\mathbf{x} = (a_1/d, \ldots, a_k/d)$ be a vertex of T, for uniquely determined relatively prime integers $a_1, \ldots, a_k, d$ with $d \ge 1$. Call $(a_1, \ldots, a_k, d)$ the *homogeneous coordinates* of $\mathbf{x}$, and call $\mathrm{den}(\mathbf{x}) = d$ the *denominator* of $\mathbf{x}$. Then, T is *unimodular* if the absolute value of the determinant of the integer square matrix having the homogeneous coordinates of the ith vertex as its ith row is equal to 1 for all $i = 1, \ldots, n+1$. A r-dimensional simplex ($r \le n$) is unimodular if it is a face of some unimodular k-dimensional simplex.

functions on the k-cube $[0,1]^k$, we can always find a unimodular triangulation $\varDelta_M$ of $[0,1]^k$ with vertices $\mathcal{V}_M = \{\mathbf{x}_1, \ldots, \mathbf{x}_m\}$ and linearizing M, i.e. such that each f_i is linear over each simplex of $\varDelta_M$. Then we can determine the corresponding set of Schauder hats $\mathcal{H}(\varDelta_M) = \{\mathbf{h}_1, \ldots, \mathbf{h}_m\}$, one corresponding to each vertex. As for a simpler notation, we will henceforth write $\mathcal{H}_M$ in place of $\mathcal{H}(\varDelta_M)$. We refer the reader to [20, §4] for the details.

Next lemma from [20] shows that the set of Schauder hats $\mathcal{H}_M$ (one for each vertex) is in fact an MV-partition of $[0,1]^k$ that allows to recover the McNaughton functions $f_i \in M$ as linear combinations of them.

Lemma 1 *Let $M = \{f_1, \ldots, f_n\}$, $\varDelta_M$, $\mathcal{V}_M$ and $\mathcal{H}_M$ as above. Then:*

1. *For distinct $\mathbf{h}_i, \mathbf{h}_j \in \mathcal{H}_M$, $\mathbf{h}_i \odot \mathbf{h}_j = 0$;*
2. *$\bigoplus_{t=1}^m \mathbf{h}_t = 1$;*
3. *For every $i, j \in \{1, \ldots, m\}$, $\mathbf{h}_i(\mathbf{x}_j) = 1$ if $i = j$, and $\mathbf{h}_i(\mathbf{x}_j) = 0$ otherwise;*
4. *For each $i = 1, \ldots, n$, $f_i = \bigoplus_{t=1}^m \mathbf{h}_t \cdot f_i(\mathbf{x}_t)$;*
5. *For every $i = 1, \ldots, n$, if f_i is not the function constantly equal to 0, then there is $\mathbf{x} \in \varDelta_M$ such that $f_i(\mathbf{x}) > 0$.*

Proof Claims (1)-(4) have been proved in [20, Lemma 3.4]. The proof of (5) directly follows from (3) and (4). Indeed, write f_i as $\bigoplus_{t=1}^m \mathbf{h}_t \cdot f_i(\mathbf{x}_t)$. Then, there must be a vertex $\mathbf{x}_t$ for which $f_i(\mathbf{x}_t) > 0$, for otherwise, f_i would be constantly 0 contradicting the hypothesis.

2.3 States of MV-algebras

States of MV-algebras were introduced by Mundici in [19]. The following definition recalls what a state is and introduces hyperreal states as well.

Definition 1 Let $\mathbf{A}$ be an MV-algebra. A *state* of $\mathbf{A}$ is a map $s : A \to [0,1]$ such that

(**s1**) $s(\perp) = 0$,
(**s2**) whenever $a, b \in A$ and $a \odot b = \perp$, $s(a \oplus b) = s(a) + s(b)$.

A *hyperreal state* of $\mathbf{A}$ is a map from A into a non trivial ultrapower ${}^*[0,1]$ of $[0,1]$ satisfying (**s1**) and (**s2**). A (hyperreal) state is *faithful* if $\perp$ is the unique element of A mapped to 0.

In case $\mathbf{A}$ is the MV-algebra $\mathcal{F}_k$ of k-variable McNaughton functions, and $\varDelta$ is a unimodular triangulation of $[0,1]^k$, then a (hyperreal) state s of $\mathcal{F}_k$ is said to be $\mathcal{H}(\varDelta)$-*faithful* provided that $s(\mathbf{h}) > 0$ for all $\mathbf{h} \in \mathcal{H}(\varDelta)$.

As a consequence of Lemma 1 we get the following results.

Lemma 2 *Let $M = \{f_1, \ldots, f_n\} \subseteq \mathcal{F}_k$, and further let $\varDelta_M$, $\mathcal{V}_M = \{\mathbf{x}_1, \ldots, \mathbf{x}_m\}$, and $\mathcal{H}_M = \{\mathbf{h}_1, \ldots, \mathbf{h}_m\}$ be as above. Then every state $s : \mathcal{F}_k \to [0,1]$ satisfies*

$$\sum_{t=1}^{m} s(\mathbf{h}_t) = 1. \tag{3}$$

Moreover, for every $i = 1, \ldots, n$, we have:

$$s(f_i) = \sum_{t=1}^{m} s(\mathbf{h}_t) \cdot f_i(\mathbf{x}_t). \tag{4}$$

Proof The axiom of additivity (**s2**) almost directly shows that any state is homogeneous w.r.t. rational constants: for every $r \in [0,1] \cap \mathbb{Q}$ and for every $f \in \mathcal{F}_k$, $s(r \cdot f) = r \cdot s(f)$. Hence Equation (4) follows since $f_i(\mathbf{x}_t) \in [0,1] \cap \mathbb{Q}$ for every $t = 1, \ldots, m$ and every $i = 1, \ldots, n$. As for an alternative proof of these claims, see for instance [20, Theorem 2.1]). $\square$

Definition 2 For every finite subset of k-variable McNaughton functions $M \subseteq \mathcal{F}_k$, let $\mathcal{F}_k(M)$ be the MV-subalgebra of $[0,1]^{[0,1]^k}$ generated by $\mathcal{F}_k \cup \{f_i \cdot f_j \mid f_i, f_j \in M\}$, where $f_i \cdot f_j$ denotes the pointwise product in $[0,1]$ of f_i and f_j.

Obviously, for every finite $M \subseteq \mathcal{F}_k$, $\mathcal{F}_k$ is an MV-subalgebra of $\mathcal{F}_k(M)$. Adopting the same notation used in previous section, let Δ_M be a unimodular triangulation of $[0,1]^k$ linearizing M, whose vertices are $\mathcal{V}_M = \{\mathbf{x}_1, \ldots, \mathbf{x}_m\}$ and whose corresponding Schauder hats are $\mathcal{H}_M = \{\mathbf{h}_1, \ldots, \mathbf{h}_m\}$. Now let s be a (hyperreal) state of $\mathcal{F}_k$, and define $s_M : \mathcal{F}_k(M) \to (^*)[0,1]$ by the following stipulation: for every $l \in \mathcal{F}_k(M)$,

$$s_M(l) = \sum_{t=1}^{m} s(\mathbf{h}_t) \cdot l(\mathbf{x}_t), \tag{5}$$

The following is fairly proved.

Lemma 3 *Let $M \subseteq \mathcal{F}_k$ be finite. For every (hyperreal) state s on $\mathcal{F}_k$, the map s_M defined as in (5) is a (hyperreal) state of $\mathcal{F}_k(M)$ which extends s.*

Proof First of all, $s_M(\top) = \sum_{t=1}^{m} s(\mathbf{h}_t) \cdot \top(\mathbf{x}_t) = \sum_{t=1}^{m} s(\mathbf{h}_t) = 1$. Moreover, for $f, g \in \mathcal{F}_k(M)$ such that $f \odot g = \bot$,

$$\begin{aligned} s_M(f \oplus g) &= \textstyle\sum_{t=1}^{m} s(\mathbf{h}_t)(f \oplus g)(\mathbf{x}_t) \\ &= \textstyle\sum_{t=1}^{m} s(\mathbf{h}_t)(f + g)(\mathbf{x}_t) \\ &= \textstyle\sum_{t=1}^{m} [s(\mathbf{h}_t) \cdot f(\mathbf{x}_t)] + [s(\mathbf{h}_t) \cdot g(\mathbf{x}_t)] \\ &= \textstyle\sum_{t=1}^{m} s(\mathbf{h}_t) \cdot f(\mathbf{x}_t) + \sum_{t=1}^{m} s(\mathbf{h}_t) \cdot g(\mathbf{x}_t) \\ &= s_M(f) + s_M(g). \end{aligned}$$

Hence s_M is a state of $\mathcal{F}_k(M)$. Finally, for every $f \in \mathcal{F}_k$, the very definition of s_M ensures that $s_M(f) = s(f)$. $\square$

Definition 3 Let Δ be a unimodular triangulation of the k-cube $[0,1]^k$. We call *distribution* every map $d : \mathcal{H}(\Delta) \to [0,1]$ such that

$$\sum_{\mathbf{h} \in \mathcal{H}(\Delta)} d(\mathbf{h}) = 1. \tag{6}$$

The above definition makes sense because, as we will see later on, normalized Schauder hats behave as atoms for Boolean algebras. On the other hand Schauder hats, conversely to the atoms of a Boolean algebra, depend on the chosen triangulation Δ. For every Δ, each distribution $d : \mathcal{H}(\Delta) \to [0,1]$ induces a state s^d of $\mathcal{F}_k$ essentially as in (5): for every $f \in \mathcal{F}_k$,

$$s^d(f) = \sum_{h \in \mathcal{H}(\Delta)} d(\mathbf{h}) \cdot f(\mathbf{x}), \tag{7}$$

where $\mathbf{x}$ is the unique vertex of Δ such that $\mathbf{h}(\mathbf{x}) = 1$. Notice that, whenever d is *regular*, (i.e., $d(\mathbf{h}) > 0$ for all $\mathbf{h} \in \mathcal{H}(\Delta)$), then s^d is $\mathcal{H}(\Delta)$-faithful.

3 On layers and hyperreal states on conditional events

By (unconditional) *many-valued events* (or simply *events*) we will understand any formula, up to logical equivalence, of Form(k) as defined in Section 2.1. By *many-valued conditional events* we will understand ordered pairs of events (φ, ψ) such that $\psi \neq \bot$, and we will use the traditional notation $\varphi \mid \psi$ to denote them. Since we will henceforth always deal with many-valued conditional events, we will simply name them *conditional events* without danger of confusion. Moreover, (2) of Example 1 allows us to uniquely identify any formulas $\varphi, \psi \in$ Form(k) with their respective McNaughton functions $f, g : [0,1]^k \to [0,1]$ in $\mathcal{F}_k$. Thus, we will also write $f \mid g$ to identify the conditional event $\varphi \mid \psi$ without loss of generality. Notice that the condition on conditional events $\varphi \mid \psi$ requiring $\psi \neq \bot$ ensures that the McNaughton function g is not constantly 0.

Next, we are going to generalize Coletti and Scozzafava's notion of zero layer in our framework of many-valued events.

Definition 4 Let M be a finite subset of $\mathcal{F}_k$, Δ_M, $\mathcal{V}_M = \{\mathbf{x}_1, \ldots, \mathbf{x}_m\}$ and $\mathcal{H}_M = \{\mathbf{h}_1, \ldots, \mathbf{h}_m\}$ as above. Further, let $\mathscr{D} = \langle d_1, \ldots, d_r \rangle$ be an ordered set of distributions on $\mathcal{H}$ such that, for each $\mathbf{h} \in \mathcal{H}_M$, there exists a $d_j \in \mathscr{D}$ satisfying $d_j(\mathbf{h}) > 0$. Then we define a map $\ell : \mathcal{F}_k \to \{1, \ldots, r\} \cup \{\infty\}$ as follows:

- for every $\mathbf{h} \in \mathcal{H}_M$, $\ell(\mathbf{h}) = \min\{j : d_j(\mathbf{h}) > 0\}$,
- for every function $f \in \mathcal{F}_k$, $\ell(f) = \min\{j : \exists t \leq m, d_j(\mathbf{h}_t) > 0,\ f(\mathbf{x}_t) > 0\}$, with the convention of taking $\min \emptyset = \infty$.

For each $f \in \mathcal{F}_k$, $\ell(f)$ is called the *$\mathscr{D}$-layer* (or simply the *layer*, when $\mathscr{D}$ is clear by the context) of f.

In the following, given a finite $M \subseteq \mathcal{F}_k$ as above and a hyperreal state $s^* : \mathcal{F}_k \to {}^*[0,1]$, we will denote

$$s^*(f \mid g) = \frac{s^*_M(f \cdot g)}{s^*(g)}$$

whenever $s^*(g) > 0$, and given a distribution $d : \mathcal{H}_M \to [0,1]$, we will also denote

$$s^d(f \mid g) = \frac{s^d_M(f \cdot g)}{s^d(g)}$$

whenever $s^d(g) > 0$. Here s^*_M and s^d_M are the extensions of s^* and s^d respectively to $\mathcal{F}_k(M)$ as defined in (5).

Theorem 1 *Let $\mathcal{C} = \{\varphi_1 \mid \psi_1, \ldots, \varphi_n \mid \psi_n\}$ a finite class of conditional events, let $M = \{f_1, g_1, \ldots, f_n, g_n\}$, Δ_M, $\mathcal{H}_M$, and $\mathcal{V}_M$ be as above. Further let*

$$\Lambda : \varphi_i \mid \psi_i \mapsto \beta_i, \psi_i \mapsto \alpha_i$$

be a complete book over $\mathcal{C}$, i.e., a mapping $\Lambda : \mathcal{C} \cup \{\psi_1, \ldots, \psi_n\} \to [0,1]$. Then the following are equivalent:

(i) There exists a hyperreal state $s^* : \mathcal{F}_k \to {}^*[0,1]$ *such that for every* $i = 1, \ldots, n$, $s^*(g_i) > 0$ *and*

$$\alpha_i = St(s^*(g_i)),$$
$$\beta_i = St(s^*(f_i \mid g_i)).$$

(ii) There exists a natural number $r \in \mathbb{N}$, *and distributions* $\mathscr{D} = \{d_1, \ldots, d_r\}$ *over* $\mathcal{H}_M$ *such that, for every* $i = 1, \ldots, n$, $\ell(g_i) < \infty$, *and*

$$\alpha_i = s^d(g_i),$$
$$\beta_i = s^d(f_i \mid g_i),$$

where $d = d_{\ell(g_i)} \in \mathscr{D}$.

Proof $(i) \Rightarrow (ii)$. Let $s^* : \mathcal{F}_k \to {}^*[0,1]$ be given by hypothesis and let s^*_M be its extension on $\mathcal{F}_k(M)$ as given in (5). Now, iteratively define a sequence $\mathcal{H}_1, \ldots, \mathcal{H}_q, \ldots$ of subsets of $\mathcal{H}_M$, and a sequence $d_1, \ldots, d_q, \ldots$ of distributions on $\mathcal{H}_M$ as follows:

Step 1 $\mathcal{H}_1 = \mathcal{H}_M$ and $\Phi_1 = \bigoplus\{\mathbf{h} : \mathbf{h} \in \mathcal{H}_1\} = 1$. And for every $\mathbf{h} \in \mathcal{H}$, let

$$\alpha^1_{\mathbf{h}} = St\left(\frac{s^*(\mathbf{h})}{s^*(\Phi_1)}\right) = St(s^*(\mathbf{h})).$$

Note that $s^*(\Phi_1) = 1$.

...

Step q+1 Assume we have already defined $\mathcal{H}_1, \ldots, \mathcal{H}_q$, and $\alpha^j_{\mathbf{h}_i}$ for $j = 1, \ldots, q$. Define $\mathcal{H}_{q+1} = \{\mathbf{h} \in \mathcal{H}_q \mid \alpha^q_{\mathbf{h}_i} = 0 \text{ and } s^*(\mathbf{h}) > 0\}$. If $\mathcal{H}_{q+1} = \emptyset$ then we stop the construction, otherwise let $\Phi_{q+1} = \bigoplus\{\mathbf{h} : \mathbf{h} \in \mathcal{H}_{q+1}\}$ and define for every $\mathbf{h} \in \mathcal{H}_{q+1}$,

$$\alpha^{q+1}_{\mathbf{h}} = St\left(\frac{s^*(\mathbf{h})}{s^*(\Phi_{q+1})}\right).$$

Since $s^*(\mathbf{h}) > 0$ for every $\mathbf{h} \in \mathcal{H}_{q+1}$, it follows that $s^*(\Phi_{q+1}) > 0$, and hence

$$\begin{aligned} \textstyle\sum_{\mathbf{h}\in\mathcal{H}_{q+1}} \alpha^{q+1}_{\mathbf{h}} &= \textstyle\sum_{\mathbf{h}\in\mathcal{H}_{q+1}} St\left(\frac{s^*(\mathbf{h})}{s^*(\Phi_{q+1})}\right) \\ &= St\left(\frac{\sum_{\mathbf{h}\in\mathcal{H}_{q+1}} s^*(\mathbf{h})}{s^*(\Phi_{q+1})}\right) \\ &= St\left(\frac{s^*(\bigoplus_{\mathbf{h}\in\mathcal{H}_{q+1}} \mathbf{h})}{s^*(\Phi_{q+1})}\right) \\ &= St\left(\frac{s^*(\Phi_{q+1})}{s^*(\Phi_{q+1})}\right) \\ &= 1. \end{aligned} \tag{8}$$

Therefore $\alpha^{q+1}_{\mathbf{h}} > 0$ for at least one $\mathbf{h} \in \mathcal{H}_{q+1}$.

...

Since $\mathcal{H}$ is finite, the above construction obviously ends in (say) r steps, defining the classes $\mathcal{H}_1 \supseteq \ldots \supseteq \mathcal{H}_r \neq \emptyset$. For each $j = 1, \ldots, r$, define $d_j : \mathcal{H}_M \to [0,1]$ as follows: for every $\mathbf{h} \in \mathcal{H}$,

$$d_j(\mathbf{h}) = \begin{cases} \alpha^j_{\mathbf{h}}, & \text{if } \mathbf{h} \in \mathcal{H}_j \\ 0, & \text{otherwise.} \end{cases} \tag{9}$$

Each d_j is hence a distribution on $\mathcal{H}_j$ by (8).

Claim The distributions $d_1, \ldots, d_r$ are such that, for every g_i, $\ell(g_i) < \infty$.

Proof of the Claim. Notice that, for every g_i there exists at least an index t such that $g_i(\mathbf{x}_t) > 0$ and $s^*(\mathbf{h}_t) > 0$. In fact, by Lemma 1, $g_i = \bigoplus_{t=1}^m g_i(\mathbf{x}_t) \cdot \mathbf{h}_t$, and hence

$$s^*(g_i) = s^*\left(\bigoplus_{t=1}^m g_i(\mathbf{x}_t) \cdot \mathbf{h}_t\right) = \sum_{t=1}^m g_i(\mathbf{x}_t) \cdot s^*(\mathbf{h}_t).$$

Since by hypothesis $s^*(g_i) > 0$, then $g_i(\mathbf{x}_t) > 0$ and $s^*(\mathbf{h}_t) > 0$ for at least one $1 \leq t \leq m$. Moreover, by construction of the $\mathcal{H}_q$'s, the procedure stops when for every $\mathbf{h} \in \mathcal{H}$ such that $s^*(\mathbf{h}) > 0$, there exists a $1 \leq j \leq r$ such that $\alpha^j_{\mathbf{h}} = d_j(\mathbf{h}) > 0$. So, we have that, for every g_i, there exists a t such that $g_i(\mathbf{x}_t) > 0$ and $s^*(\mathbf{h}_t) > 0$, and hence there exists at least a j such that $d_j(\mathbf{h}_t) > 0$. Therefore, for every g_i, $\ell(g_i) < \infty$. □

Turning back to the proof of Theorem 1 $(i) \Rightarrow (ii)$, we have:

$$\begin{aligned}
s^{d_{\ell(g_i)}}(f_i \mid g_i) = \frac{\sum_{t=1}^m d_{\ell(g_i)}(\mathbf{h}_t)\cdot(f_i\cdot g_i)(\mathbf{x}_t)}{\sum_{t=1}^m d_{\ell(g_i)}(\mathbf{h}_t)\cdot g_i(\mathbf{x}_t)} &= St\left(\frac{\sum_{t=1}^m \frac{s^*(\mathbf{h}_t)}{s^*(\Phi_{\ell(g_i)})}\cdot(f_i\cdot g_i)(\mathbf{x}_t)}{\sum_{t=1}^m \frac{s^*(\mathbf{h}_t)}{s^*(\Phi_{\ell(g_i)})}\cdot g_i(\mathbf{x}_t)}\right)\\
&= St\left(\frac{\frac{1}{s^*(\Phi_{\ell(g_i)})}\sum_{t=1}^m s^*(\mathbf{h}_t)\cdot(f_i\cdot g_i)(\mathbf{x}_t)}{\frac{1}{s^*(\Phi_{\ell(g_i)})}\sum_{t=1}^m s^*(\mathbf{h}_t)\cdot g_i(\mathbf{x}_t)}\right)\\
&= St\left(\frac{\sum_{t=1}^m s^*(\mathbf{h}_t)\cdot(f_i\cdot g_i)(\mathbf{x}_t)}{\sum_{t=1}^m s^*(\mathbf{h}_t)\cdot g_i(\mathbf{x}_t)}\right)\\
&= St\left(\frac{(s^*)^+(f_i\cdot g_i)}{s^*(g_i)}\right)\\
&= \beta_i.
\end{aligned}$$

$(ii) \Rightarrow (i)$. Let ε be a positive infinitesimal. In the following, for the sake of a simpler notation, we will write $\ell(t)$ for $\ell(\mathbf{h}_t)$, for every $\mathbf{h}_t \in \mathcal{H}_M$. Since $\sum_{\mathbf{h}\in\mathcal{H}_M} d_i(\mathbf{h}) = 1$ for every $i = 1, \ldots, r$, it follows that for each i there exists $\mathbf{h} \in \mathcal{H}_M$ such that $d_i(\mathbf{h}) > 0$. We define d_∞ as the function constantly equal to 0 on $\mathcal{H}_M$ (i.e. $d_\infty(\mathbf{h}) = 0$ for each $\mathbf{h} \in \mathcal{H}_M$). Let us put

$$K = \left(\sum_{t=1}^m \varepsilon^{\ell(t)} \cdot d_{\ell(t)}(\mathbf{h}_t)\right)^{-1}$$

It is clear then that the denominator is positive and K is well defined.

Further, we define $s^* : \mathcal{F}_k \to {}^*[0,1]$ by letting for every $f \in \mathcal{F}_k$,

$$s^*(f) = K \cdot \sum_{t=1}^m \varepsilon^{\ell(t)} \cdot d_{\ell(t)}(\mathbf{h}_t) \cdot f(\mathbf{x}_t). \tag{10}$$

So defined s^* is a hyperreal state on $\mathcal{F}_k$. In fact the following holds:

$$s^*(\top) = K \cdot \sum_{t=1}^m \varepsilon^{\ell(t)} \cdot d_{\ell(t)}(\mathbf{h}_t) \cdot \top(\mathbf{x}_t) = K \cdot K^{-1} = 1.$$

Moreover, if $f \odot g = \bot$ then

$$\begin{aligned} s^*(f \oplus g) &= K \cdot \textstyle\sum_{t=1}^m \varepsilon^{\ell(t)} \cdot d_{\ell(t)}(\mathbf{h}_t) \cdot (f+g)(\mathbf{x}_t) \\ &= K \cdot \textstyle\sum_{t=1}^m \varepsilon^{\ell(t)} \cdot d_{\ell(t)}(\mathbf{h}_t) \cdot f(\mathbf{x}_t) + K \cdot \textstyle\sum_{t=1}^m \varepsilon^{\ell(t)} \cdot d_{\ell(t)}(\mathbf{h}_t) \cdot g(\mathbf{x}_t) \\ &= s^*(f) + s^*(g). \end{aligned}$$

By Lemma 1 (5) and since the normalized Schauder hats in $\mathcal{H}_M$ are also generated by the g_i's, for every $i = 1, \ldots, n$, there exists at least a vertex $\mathbf{x}_{t(i)}$ of Δ such that $g_i(\mathbf{x}_{t(i)}) > 0$. Moreover $\ell(g_i) < \infty$ for every i, and hence there exists a vertex $\mathbf{x}_t$ such that, $g_i(\mathbf{x}_t) > 0$, and $d_{\ell(g_i)}(\mathbf{h}_t) > 0$. Therefore,

$$s^*(g_i) = K \cdot \sum_{t=1}^{m} \varepsilon^{\ell(t)} \cdot d_{\ell(t)}(\mathbf{h}_t) \cdot g_i(\mathbf{x}_t) > 0.$$

The definition of $s^*(f)$ according to (10) admits the following equivalent expression obtained considering layers rather than Schauder hats:

$$s^*(f) = K \cdot \sum_{j=\ell(f)}^{r} \sum_{\{\mathbf{h}_t : \ell(\mathbf{h}_t)=j\}} \varepsilon^j \cdot d_j(\mathbf{h}_t) \cdot f(\mathbf{x}_t) \tag{11}$$

Let $s^*_M : \mathcal{F}_k(M) \to {}^*[0,1]$ be defined according to (10). Then, for every $i = 1, \ldots, n$, we have:

$$\begin{aligned} s^*(f_i \mid g_i) = \frac{s^*_M(f_i \cdot g_i)}{s^*_M(g_i)} &= \frac{\sum_{t=1}^m s^*(\mathbf{h}_t)\cdot(f_i\cdot g_i)(\mathbf{x}_t)}{\sum_{t=1}^m s^*(\mathbf{h}_t)\cdot g_i(\mathbf{x}_t)} \\ &= \frac{\sum_{t=1}^m K\cdot\varepsilon^{\ell(t)}\cdot d_{\ell(t)}(\mathbf{h}_t)\cdot(f_i\cdot g_i)(\mathbf{x}_t)}{\sum_{t=1}^m K\cdot\varepsilon^{\ell(t)}\cdot d_{\ell(t)}(\mathbf{h}_t)\cdot g_i(\mathbf{x}_t)} \\ &= \frac{\sum_{t=1}^m \varepsilon^{\ell(t)}\cdot d_{\ell(t)}(\mathbf{h}_t)\cdot(f_i\cdot g_i)(\mathbf{x}_t)}{\sum_{t=1}^m \varepsilon^{\ell(t)}\cdot d_{\ell(t)}(\mathbf{h}_t)\cdot g_i(\mathbf{x}_t)} \end{aligned}$$

Now let $w = \ell(f_i \cdot g_i) = \min\{j \mid \exists t : d_j(\mathbf{h}_t) \cdot (f_i \cdot g_i)(\mathbf{x}_t) > 0\}$, and let $k = \ell(g_i)$. Clearly $w \geq k$. Then, using (11) we have:

$$\begin{aligned} K^{-1} \cdot s^*_M(f_i \cdot g_i) = &\Big(\varepsilon^w \cdot \textstyle\sum_{\{\mathbf{h}_t : d_w(\mathbf{h}_t)>0\}} d_w(\mathbf{h}_t) \cdot (f_i \cdot g_i)(\mathbf{x}_t)\Big) + \\ &\Big(\textstyle\sum_{w'>w} \ \varepsilon^{w'} \cdot \sum_{\{\mathbf{h}_t : \ell(\mathbf{h}_t)=w'\}} d_{w'}(\mathbf{h}_t) \cdot (f_i \cdot g_i)(\mathbf{x}_t)\Big) \end{aligned}$$

$$\begin{aligned} K^{-1} \cdot s^*_M(g_i) = &\Big(\varepsilon^k \cdot \textstyle\sum_{\{\mathbf{h}_t : d_k(\mathbf{h}_t)>0\}} d_k(\mathbf{h}_t) \cdot g_i(\mathbf{x}_t)\Big) + \\ &\Big(\textstyle\sum_{k'>k} \ \varepsilon^{k'} \cdot \sum_{\{\mathbf{h}_t : \ell(\mathbf{h}_t)=k'\}} d_{k'}(\mathbf{h}_t) \cdot (f_i \cdot g_i)(\mathbf{x}_t)\Big). \end{aligned}$$

Notice that the terms $\sum_{w'>w} \ \varepsilon^{w'} \cdot \sum_{\{\mathbf{h}_t : \ell(\mathbf{h}_t)=w'\}} d_{w'}(\mathbf{h}_t) \cdot (f_i \cdot g_i)(\mathbf{x}_t)$ and $\sum_{k'>k} \ \varepsilon^{k'} \cdot \sum_{\{\mathbf{h}_t : \ell(\mathbf{h}_t)=k'\}} d_{k'}(\mathbf{h}_t) \cdot (f_i \cdot g_i)(\mathbf{x}_t)$ are infinitesimals of order respectively greater than w and k. We shall henceforth denote them by I and J respectively.

We consider two cases:

$w > k$: then $\sum_{t=1}^m d_{\ell(g_i)}(\mathbf{h}_t) \cdot (f_i \cdot g_i)(\mathbf{x}_t) = 0$ because for every $\mathbf{h}_t$ either $f_i \cdot g_i(\mathbf{x}_t) = 0$, or $d_{\ell(g_i)}(\mathbf{h}_t) = 0$ (otherwise it would have been $w = k$), and hence, since by definition of $\ell(g_i)$, $\sum_{t=1}^m d_{\ell(g_i)}(\mathbf{h}_t) \cdot g_i(\mathbf{x}_t) > 0$, then $\beta_i = 0$. Moreover, $\frac{\varepsilon^w}{\varepsilon^k} = \varepsilon^{w-k}$ is infinitesimal, whence: $St\left(\frac{s^*_M(f_i \cdot g_i)}{s^*_M(g_i))}\right) = \beta_i = 0$;

$w = k$: then $\frac{\varepsilon^w}{\varepsilon^k} = 1$, and

$$
\begin{aligned}
St\left(\frac{s_M^*(f_i \cdot g_i)}{s_M^*(g_i))}\right) &= St\left(\frac{\left(\varepsilon^w \cdot \sum_{\{\mathbf{h}_t : d_w(\mathbf{h}_t)>0\}} d_w(\mathbf{h}_t) \cdot (f_i \cdot g_i)(\mathbf{x}_t)\right) + I}{\left(\varepsilon^w \cdot \sum_{\{\mathbf{h}_t : d_w(\mathbf{h}_t)>0\}} d_w(\mathbf{h}_t) \cdot g_i(\mathbf{x}_t)\right) + J}\right) \\
&= \frac{\sum_{\{\mathbf{h}_t : d_w(\mathbf{h}_t)>0\}} d_w(\mathbf{h}_t) \cdot (f_i \cdot g_i)(\mathbf{x}_t)}{\sum_{\{\mathbf{h}_t : d_w(\mathbf{h}_t)>0\}} d_w(\mathbf{h}_t) \cdot g_i(\mathbf{x}_t)} \\
&= \frac{\sum_{\mathbf{h}_t} d_{\ell(g_i)}(\mathbf{h}_t) \cdot (f_i \cdot g_i)(\mathbf{x}_t)}{\sum_{\mathbf{h}_t} d_{\ell(g_i)}(\mathbf{h}_t) \cdot g_i(\mathbf{x}_t)} \\
&= \beta_i.
\end{aligned}
$$

This ends the proof. □

Remark 1 Finally notice that for each g_i,

$$
s^{d_{\ell(g_i)}}(g_i) = \sum_{t=1}^{m} d_{\ell(g_i)}(\mathbf{h}_t) \cdot g_i(\mathbf{x}_t) = St\left(\frac{1}{s^*(\Phi_{\ell(g_i)})} \cdot s^*(g_i)\right)
$$

where $s^{d_{\ell(g_i)}}$ is the state given by the distribution $d_{\ell(g_i)}$ as in (7).

4 Stable coherence and faithful hyperstates

Following [17], for every MV-algebra $\mathbf{A}$, a (possibly trivial) ultrapower ${}^*[0,1]$ is said to be *amenable for* $\mathbf{A}$ (or $\mathbf{A}$-*amenable*) if for every $x \in A \setminus \{\bot\}$ there is a homomorphism $h \in \mathscr{H}(\mathbf{A}, {}^*[0,1]_{MV})$ such that $h(x) > 0$. Similarly, $\mathbf{A}$-amenable ultrapowers can be defined when $\mathbf{A}$ is a PMV$^+$-algebra (cf. [17, Definition 3.1]). The standard algebra $[0,1]_{MV}$ ($[0,1]_{PMV}$ respectively) is $\mathbf{A}$-amenable iff $\mathbf{A}$ is a semisimple MV-algebra (PMV$^+$-algebra respectively) [17].

The following construction has been introduced in [17] for an arbitrary MV or PMV$^+$-algebra $\mathbf{A}$ and an $\mathbf{A}$-amenable ultrapower ${}^*[0,1]$.

Notation 1 *Let $\mathbf{A}$ be an MV-algebra (or a PMV$^+$-algebra) and let ${}^*[0,1]$ be $\mathbf{A}$-amenable. Identify every $\alpha \in {}^*[0,1]$ with the function from $\mathscr{H}(\mathbf{A}, {}^*[0,1])$ to ${}^*[0,1]$ constantly equal to α, and denote, for every $a \in A$, F_a the function from $\mathscr{H}(\mathbf{A}, {}^*[0,1])$ to ${}^*[0,1]$ such that, for all $h \in \mathscr{H}(\mathbf{A}, {}^*[0,1])$, $F_a(h) = h(a)$. The algebra $\Pi(\mathbf{A}, {}^*[0,1])$ hence is defined as the PMV$^+$-subalgebra of $[0,1]^{\mathscr{H}(\mathbf{A}, {}^*[0,1])}$ generated by $\{\alpha \mid \alpha \in {}^*[0,1]\} \cup \{F_a \mid a \in A\}$.*

We will henceforth denote $\Pi(\mathcal{F}_k, [0,1]_{MV})$ simply by $\Pi(\mathcal{F}_k)$.

Remark 2 As we recalled in Subsection 2.1, for every k, $\mathcal{F}_k$ is an MV-subalgebra of $[0,1]^{[0,1]^k}$. Moreover, $\mathcal{F}_k$ is semisimple, whence $[0,1]$ is $\mathcal{F}_k$-amenable. Thus, up to identifying $[0,1]^k$ with $\mathscr{H}(\mathcal{F}_k)$ (see [4, Proposition 3.4.7] together with [19, Theorem 2.5]), $\Pi(\mathcal{F}_k)$ is a PMV$^+$-subalgebra of $[0,1]^{[0,1]^k}$. In particular, it is easy to see that for every $\mathbf{x} \in [0,1]^k$, the map $f \in \Pi(\mathcal{F}_k) \mapsto f(\mathbf{x}) \in [0,1]$ is a well-defined PMV$^+$-homomorphism.

Lemma 4 *For each finite $M \subseteq \mathcal{F}_k$, $\mathcal{F}_k(M)$ is an MV-subalgebra of $\Pi(\mathcal{F}_k)$. As a consequence, $\mathcal{F}_k$ is an MV-subalgebra of $\Pi(\mathcal{F}_k)$ as well.*

Proof In order to prove that $\mathcal{F}_k(M)$ embeds into $\Pi(\mathcal{F}_k)$, notice that the map $\Phi : a \in \mathcal{F}_k(M) \mapsto F_a \in [0,1]^{\mathscr{H}(\mathcal{F}_k)}$ is an embedding (cf. [17, Lemma 3.1]). Moreover, since the maps F_a generate $\Pi(\mathcal{F}_k)$, Φ also defines an embedding of $\mathcal{F}_k(M)$ into $\Pi(\mathcal{F}_k)$. The second claim trivially follows since $\mathcal{F}_k$ is an MV-subalgebra of $\mathcal{F}_k(M)$. □

Notice that, whenever $\mathbf{A}$ is a semisimple MV-algebra, then $\Pi(\mathbf{A}, [0,1]_{MV})$ is a subalgebra of $[0,1]^{\mathscr{H}(\mathbf{A})}$. Now, every MV-algebra of the form $[0,1]^X$ can be uniquely endowed with a pointwise product and hence regarded as PMV$^+$-algebra. Moreover, the PMV-algebra $[0,1]^X$ is semisimple. In particular, $\Pi(\mathbf{A}, [0,1]_{MV})$ is a semisimple PMV$^+$-algebra. As for the other direction, Lemma 4 immediately proves that, if $\Pi(\mathbf{A}, [0,1]_{MV})$ is a semisimple PMV$^+$-algebra, then $\mathbf{A}$ is semisimple as well. Thus, the following holds.

Proposition 1 *An MV-algebra* $\mathbf{A}$ *is semisimple iff so is* $\Pi(\mathbf{A}, [0,1]_{MV})$. *In particular,* $\Pi(\mathcal{F}_k)$ *is semisimple.*

Definition 5 ([17]) Let $\mathbf{A}$ be a MV-algebra, and let $^*[0,1]$ be $\mathbf{A}$-amenable. A *hyperstate* of $\mathbf{A}$ is a map $s^\circ : \Pi(\mathbf{A}, {}^*[0,1]) \to {}^*[0,1]$ such that

1. $s^\circ(\top) = \top$,
2. For each $a, b \in A$ with $a \odot b = \bot$, $s^\circ(a \oplus b) = s^\circ(a) + s^\circ(b)$,
3. For all $\alpha \in {}^*[0,1]$ and for all $a \in \Pi(\mathbf{A}, {}^*[0,1])$, $s^\circ(\alpha \cdot a) = \alpha \cdot s^\circ(a)$,
4. For all $a \in \Pi(\mathbf{A}, {}^*[0,1])$, there exists a PMV$^+$-homomorphism $h \in \mathscr{H}(\Pi(\mathbf{A}, {}^*[0,1]), {}^*[0,1])$ such that $h(\alpha) = \alpha$ for every $\alpha \in {}^*[0,1]$, and $h(a) \leq s^\circ(a)$.

A hyperstate is said to be *faithful* if $s^\circ(a) > 0$ for all $a \neq \bot$.

Each homomorphism as in Definition 5 (4) is called a *hypervaluation*.

Remark 3 In Definition 5 the request that $^*[0,1]$ is the $\mathbf{A}$-amenable ultrapower used in the construction of $\Pi(\mathbf{A}, {}^*[0,1])$ and, at the same time, the codomain of s° is not mandatory. Indeed, whenever $^\circ[0,1]$ is another ultrapower of $[0,1]$ such that $^*[0,1]$ is a PMV$^+$-subalgebra of $^\circ[0,1]$, then $^\circ[0,1]$ is $\mathbf{A}$-amenable as well. Hence the above definition can be relaxed by defining a hyperstate to be a map $s^\circ : \Pi(\mathbf{A}, {}^*[0,1]) \to {}^\circ[0,1]$ satisfying 1–4 such that $^*[0,1]$ is $\mathbf{A}$-amenable and a PMV$^+$-subalgebra of $^\circ[0,1]$. In what follows we will actually adopt this slightly more general definition without danger of confusion.

Now we are ready to prove that extensions of complete books on a set of conditional events with respect to hyperreal states and with respect to faithful hyperstates are equivalent in the following terms.

Theorem 2 *Let* $\mathcal{C} = \{\varphi_1 \mid \psi_1, \ldots, \varphi_n \mid \psi_n\}$ *a finite class of conditional events, let* $M = \{f_1, g_1, \ldots, f_n, g_n\}$, Δ_M, $\mathcal{H}_M$, $\mathcal{V}_M$ *be as usual, and let* $\Lambda : \varphi_i \mid \psi_i \mapsto \beta_i, \psi_i \mapsto \alpha_i$ *be a complete book over* $\mathcal{C}$. *Then the following are equivalent:*

(i) There exists a hyperreal state $s^* : \mathcal{F}_k \to {}^*[0,1]$ *such that for every* ψ_i, $s^*(g_i) > 0$, $St(s^*(g_i)) = \alpha_i$, *and for every* $i = 1, \ldots, n$,

$$\beta_i = St\left(\frac{s^*_M(f_i \cdot g_i)}{s^*(g_i)}\right).$$

(ii) *There exists a faithful hyperstate* $s^\circ : \Pi(\mathcal{F}_k) \to {}^\circ[0,1]$ *(for* ${}^\circ[0,1]$ *suitable ultrapower of* $[0,1]$*) such that for every* $i = 1, \ldots, n$, $St(s^\circ(g_i)) = \alpha_i$

$$\beta_i = St\left(\frac{s^\circ(f_i \cdot g_i)}{s^\circ(g_i)}\right).$$

Proof (i) $\Rightarrow$ (ii). The first step to prove in this direction consists of extending the hyperstate s^* from $\mathcal{F}_k$ to $\Pi(\mathcal{F}_k)$. This can be done as usual: recalling the notation we introduced in Remark 2, we define for each $f \in \Pi(\mathcal{F}_k)$,

$$(s^*)'(f) = \sum_{t=1}^{m} f(\mathbf{x}_t) \cdot s^*(\mathbf{h}_t). \tag{12}$$

Since $\mathcal{F}_k$ is an MV-subalgebra of $\Pi(\mathcal{F}_k)$ (Lemma 4) s^* and $(s^*)'$ coincide on $f_1, g_1, \ldots, f_n, g_n$, and s^*_M and $(s^*)'$ coincide on $f_1 \cdot g_1, \ldots, f_n \cdot g_n$. Therefore, for every $i = 1, \ldots, n$,

$$St((s^*)'(g_i)) = St(s^*(g_i)) = \alpha_i, \text{ and } St\left(\frac{(s^*)'(f_i \cdot g_i)}{(s^*)'(g_i)}\right) = St\left(\frac{s^*_M(f_i \cdot g_i)}{s^*(g_i)}\right) = \beta_i.$$

Let hence prove that $(s^*)'$ is a hyperstate. Conditions 1 and 2 of Definition 5 are clearly satisfied by (12). Let $\alpha \in [0,1]$, and let $f \in \Pi(\mathcal{F}_k)$. Then

$$\begin{aligned}(s^*)'(\alpha \cdot f) &= \textstyle\sum_{t=1}^{m} (\alpha \cdot f)(\mathbf{x}_t) \cdot s^*(\mathbf{h}_t)\\ &= \textstyle\sum_{t=1}^{m} \alpha(\mathbf{x}_t) \cdot f(\mathbf{x}_t) \cdot s^*(\mathbf{h}_t)\\ &= \textstyle\sum_{t=1}^{m} \alpha \cdot f(\mathbf{x}_t) \cdot s^*(\mathbf{h}_t)\\ &= \alpha \cdot \textstyle\sum_{t=1}^{m} f(\mathbf{x}_t) \cdot s^*(\mathbf{h}_t)\\ &= \alpha \cdot (s^*)'(f).\end{aligned}$$

Hence Condition 3 also holds. In order to prove Condition 4, notice that $(s^*)'$ is in fact a convex combination of finitely many hypervaluations. Indeed, since $\mathcal{F}_k$ is semisimple, every homomorphism of $\mathcal{F}_k$ in $[0,1]_{MV}$ is a hypervaluation in the sense of Definition 5. Hence 4 holds.

If $(s^*)'$ is faithful we set $s^\circ = (s^*)'$ and we are done. Otherwise we can perform the following construction which has been used in [17, Theorem 5.1] for a similar purpose. Assuming that $(s^*)'$ is not faithful, we proceed as follows: let

$$H = \{c \in \Pi(\mathcal{F}_k) : c > 0 \text{ and } (s^*)'(c) = 0\}.$$

For each finite set $F = \{c_1, \ldots, c_s\} \subseteq H$ let $\varepsilon_F \in {}^*[0,1]$ be a positive infinitesimal. Since $c_1, \ldots, c_s$ are all strictly positive and since by Proposition 1 the PMV$^+$-algebra $\Pi(\mathcal{F}_k)$ is semisimple, $[0,1]_{PMV}$ is $\Pi(\mathcal{F}_k)$-amenable, whence there are valuations $h_1, \ldots, h_s$ such that for every $j = 1, \ldots, s$, $h_j(c_j) > 0$. Hence, define $s^*_F : \Pi(\mathcal{F}_k) \to {}^*[0,1]$ such that for each $f \in \Pi(\mathcal{F}_k)$,

$$s^*_F(f) = (1 - \varepsilon_F) \cdot (s^*)'(f) + \frac{h_1(f) + \ldots + h_s(f)}{s} \cdot \varepsilon_F.$$

Let now $Fin(H)$ be the class of all finite subsets of H, and hence it is easily seen that, for each $F \in Fin(H)$, s^*_F is a hyperstate.

To conclude the proof and following the line of [17, Theorem 4.2], there exists an ultrafilter U on $Fin(\Pi(\mathcal{F}_k))$ by which we define

$${}^\circ[0,1] = ({}^*[0,1])^{Fin(\Pi(\mathcal{F}_k))}/U,$$

and $s^\circ : \Pi(\mathcal{F}_k, {}^*[0,1]) \to {}^\circ[0,1]$ as follows: for every $a \in \Pi(\mathcal{F}_k)$,

$$s^\circ(a) = \{s^*_F(a) : F \in Fin(\Pi(\mathcal{F}_k))\}/U. \tag{13}$$

Then s° is a faithful hyperstate such that, for every $i = 1, \ldots, n$, $St(s^\circ(g_i)) = \alpha_i$, and $St(\frac{s^\circ(f_i \cdot g_i)}{s^\circ(g_i)}) = \beta_i$ (see [17, Theorem 4.2]).

$(ii) \Rightarrow (i)$. As we have already proved in Lemma 4, $\mathcal{F}_k(M)$ is an MV-subalgebra of $\Pi(\mathcal{F}_k)$, and hence the hyperstate $s^*_M : \mathcal{F}_k(M) \to {}^\circ[0,1]$, defined by restriction (and in particular its restriction s^* to $\mathcal{F}_k$) satisfies our claim. □

In [17], the authors introduce the notion of *stable coherence* as a rationality criterion for complete conditional books when the assessment to some conditioning events is zero.

Definition 6 (Stable coherence, [17]) Let $\mathbf{A}$ be an MV-algebra, let $\mathcal{C} = \{a_1 \mid b_1, \ldots, a_n \mid b_n\}$ be a finite class of conditional events and let $\Lambda : a_i \mid b_i \mapsto \beta_i$, $b_i \mapsto \alpha_i$ be a complete book on $\mathcal{C}$. Then Λ is said to be *stably coherent* if there exists a hyperreal-valued complete book Λ' on $\mathcal{C}$ such that

(i) Λ' is coherent,
(ii) for all $i = 1, \ldots, n$, $\Lambda'(b_i) > 0$,
(iii) for all $i = 1, \ldots, n$, $|\beta_i - \Lambda'(a_i \mid b_i)|$ is infinitesimal, and $|\alpha_i - \Lambda'(b_i)|$ is infinitesimal.

In [17], hyperstates are proved to characterize stably coherent books on arbitrary MV-algebras.

Theorem 3 ([17, Theorem 5.1]) *Let $\mathbf{A}$ be an MV-algebra, let $\mathcal{C} = \{a_1 \mid b_1, \ldots, a_n \mid b_n\}$ be a finite set of conditional events in $\mathbf{A}$, and let $\Lambda : a_i \mid b_i \mapsto \beta_i$, $b_i \mapsto \alpha_i$ be a complete book on $\mathcal{C}$. Then the following are equivalent:*

(i) Λ is stably coherent.
(ii) There is a faithful hyperstate s° of $\Pi(\mathbf{A}, {}^*[0,1])$ such that, for every $i = 1, \ldots, n$, $s^\circ(b_i) = \alpha_i$ and

$$\beta_i = St\left(\frac{s^\circ(a_i \cdot b_i)}{s^\circ(b_i)}\right).$$

The next result, that follows from the above Lemma 2 and Theorems 1 and 3, summarizes a full characterization of stably coherent books in terms of hyperstates and layers of zero-probability. Thus it generalizes to the case of Łukasiewicz events, both Krauss's [10] and Coletti and Scozzafava's [5] theorems.

Corollary 1 *Let $\mathcal{C} = \{\varphi_1 \mid \psi_1, \ldots, \varphi_n \mid \psi_n\}$ a finite class of conditional events, let M, $\mathcal{H}_M$ as in Theorem 1, and let*

$$\Lambda : \varphi_i \mid \psi_i \mapsto \beta_i, \psi_i \mapsto \alpha_i$$

be a complete book over $\mathcal{C}$. Then the following are equivalent:

(i) Λ is stably coherent.
(ii) There exists a faithful hyperstate $s^\circ : \Pi(\mathcal{F}_k) \to {}^\circ[0,1]$ (for ${}^\circ[0,1]$ suitable ultrapower of $[0,1]$) such that for every $i = 1, \ldots, n$, $St(s^\circ(g_i)) = \alpha_i$

$$\beta_i = St\left(\frac{s^\circ(f_i \cdot g_i)}{s^\circ(g_i)}\right).$$

(iii) There exists a hyperreal state $s^*: \mathcal{F}_k \to {}^*[0,1]$ *such that for every* ψ_i, $s^*(g_i) > 0$, $St(s^*(g_i)) = \alpha_i$, *and for every* $i = 1, \ldots, n$,

$$\beta_i = St\left(\frac{s^*_M(f_i \cdot g_i)}{s^*(g_i)}\right) = St(s^*(f_i \mid g_i)).$$

(iv) There exists a natural number $r \in \mathbb{N}$, *and distributions* $\mathscr{D} = \{d_1, \ldots, d_r\}$ *over* $\mathcal{H}_M$ *such that, for every* $i = 1, \ldots, n$, $\ell(g_i) < \infty$, *and*

$$\alpha_i = s^d(g_i),$$
$$\beta_i = s^d(f_i \mid g_i),$$

where $d = d_{\ell(g_i)} \in \mathscr{D}$.

5 Conclusion

In this paper we have pushed forward an investigation aimed at providing a deeper understanding on the foundational aspects of conditional states, i.e., conditional probability on many-valued events. Generalizing Coletti and Scozzafava's notion of zero-layer [5] from Boolean to MV-algebras, we have shown that the arising notion of lexicographic state allows us to characterize stably coherent books [17]. We have framed our investigation on free MV-algebras and in our future work we plan to extend at least part of this work to the case of all MV-algebras. Further, we plan to use the characterization result we have shown in Section 4 (namely, Corollary 1) to study the computational complexity of the problem of deciding stably coherent books.

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