Smooth approximations by continuous choice-functions

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Abstract We explore the existence of rational-valued approximation processes by continuous functions of two variables, such that the output continuously depends of the imposed error-bound. To this sake we prove that the theory of densely ordered sets with generic predicates is $\aleph_0$-categorical. A model of the theory and a particular continuous choice-function are constructed. This function transfers to all other models by the respective isomorphisms. If some common-sense conditions are fulfilled, the processes are computable. As a byproduct, other functions with surprising properties can be constructed.

Keywords choice · densely ordered sets · order topology · continuous functions · roots · algebraic and transcendental numbers · $\aleph_0$-categorical theories.

1 Introduction

All sciences, including mathematics and computer science, and most of their practical applications, are supported by numeric computations. The computed numbers mostly express quantities. The numbers expressible with a finite amount of digits are rational numbers, and so they can only approximate the exact values of continuously varying quantities.

Digital expressions can exactly represent only a subset of the rational numbers, as for example the rational number $1/3 = 0.333\dots$ needs already an infinity of digits to be exactly represented. That is why we choose to focus on exactly represented rational numbers only. They are expressions m/n where m is an integer and n is a strictly positive integer. As every rational number has an infinity of such representations, we choose to always refer to the shortest of them, which satisfies the condition $\gcd(m,n) = 1$. Such a representation is an irreducible fraction. Of course, an irreducible fraction can be always approximated by a decimal expression up to the desired error.

This paper deals with the delicate problem of approximating continuous quantities by rational numbers. Approximation is a subject that lies in the focus of this Journal. When both measurement and input are fuzzy, the approximation is the only truth we are dealing with. Various methods, borrowed and adapted from all areas, numeric analysis and algebra to functional analysis and differential equations, are put to work - see for example (Coroianu & all, 2019), (Dawaz, 2008), (Ishibuchi & all, 2006) and (Wang, Li, 2019). But remarkably, most methods used to achieve good approximations are based essentially on the continuum of the real numbers and, as I believe, do not use enough the intimate properties of the rational numbers.

The set of rational numbers $\mathbb{Q}$ builds an ordered field, which is enumerable and lies densely in the set $\mathbb{R}$ of real numbers. Information obtained by measurements, as like the digital signals sent by us in order to control a process, are essentially rational numbers. Of course, instead of the rationals, one can use other rings, like that consisting of the numbers $m/2^k$, where m is an integer and k is a natural number. This ring is also dense in $\mathbb{R}$, has the advantage that all elements are expressed by finitely many digits in the bases 10 or 2, but does not build a field. The choice of the set used for approximations and numeric computations depends finally of the practical scope. We keep $\mathbb{Q}$ in focus, but as it will turn out, we are interested specially in the fact that the set is dense and countable.

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In this article we deal with the problem of finding approximation processes with rational values, which are *continuous* as function of their error bound. Consider the following usual problem of numeric computation:

Find a good rational approximation of $\sqrt{3.7}$.

The term *good* is meaning-less without a specification. In fact one may consider as good an approximation whose square is no farther than 0.1 from the target value, and can ask precisely:

Find $z \in \mathbb{Q}$ *such that* $3.6 < z^2 < 3.8$, or may pretend a better approximation, whose square is no farther as 0.01 from the target value, and can ask precisely:

Find $z \in \mathbb{Q}$ *such that* $3.69 < z^2 < 3.79$.

In general we are interested of a process to get rational numbers as follows:

For every $\varepsilon \in \mathbb{Q}$ *with* $\varepsilon > 0$, *find* $z(\varepsilon) \in \mathbb{Q}$ *such that* $3.7 - \varepsilon < z(\varepsilon)^2 < 3.7 + \varepsilon$.

So far we formulated the request to find a process to produce good approximations. Such processes are well-known, as every algorithm computing approximations of $\sqrt{u}$ for $u \in \mathbb{Q}$ is an answer. But now, we supplementary ask for the function $z(\varepsilon)$ to be continuous in ε. This condition is motivated by practical reasons, as follows.

Suppose that some device controls a process that has as goal to reach some target. The target is a specific value, and as the time goes by, the approximation error must become smaller. The process can be the flight of a rocket to a physical target, filling a recipient with a fluid, accelerating or slowing down a moving train to a given speed, or anything else. The function to be computed can be considered to be algebraic or analytic and to depend of several parameters, which can be constant or can vary continuously during the process. Most important, the error bound of the approximation is one of the parameters. The physical nature of the process, as also practical considerations depending on the resistance and the endurance of material devices, or conditions given by applications in connection with human users, request the computed value $z(\dots, \varepsilon, \dots)$ to continuously depend on the error bound ε.

Now we come back to the square root example.

Consider ordered fields K that satisfy the statement:

$$\forall x, y \;\; 0 < x < y \rightarrow \exists z \; x < z^2 < y. \qquad (1)$$

The statement is true in the field of real numbers $\mathbb{R}$ and in the field of rational numbers $\mathbb{Q}$. It turns out that not all ordered fields have this property. For the sake of completeness, a counterexample is displayed in the Appendix.

Recall that an ordered field has a canonical order topology τ_1, defined such that the open intervals build a basis for the open sets. The affine space K^n gets the product topology τ_n generated by τ_1. Here the open boxes $I_1 \times I_2 \times \dots \times I_n$ produce a basis for the open sets. Every I_k is an open interval in K. Let K_+ be the set of strictly positive elements of K. For an ordered set $(A, <)$, let $I(A)$ denote the open set $\{(x, y) \in A^2 \,|\, x < y\}$. We look for *continuous* functions $z(x, y)$ defined as $z : I(K_+) \rightarrow K_+$ such that:

$$\forall x, y \;\; 0 < x < y \rightarrow x < z(x, y)^2 < y. \qquad (2)$$

In $\mathbb{R}$ there are plenty of such continuous functions. For every $\alpha \in (0, 1)$ the following functions are examples:

$z(x, y) = \sqrt{\alpha x + (1 - \alpha) y}$,

$z(x, y) = \alpha \sqrt{x} + (1 - \alpha)\sqrt{y}$,

$z(x, y) = \sqrt{x^{\alpha} y^{1-\alpha}}$, etc.

Question 1: *Is there any continuous* $z : I(\mathbb{Q}_+) \rightarrow \mathbb{Q}_+$ *satisfying the condition (2)?*

Question 2: *If the Question 1 has a positive answer, is there any algorithm to compute a continuous function* $z : I(\mathbb{Q}_+) \rightarrow \mathbb{Q}_+$ *satisfying the condition (2)?*

Both questions are given positive answers in this paper. Moreover, instead of $z \rightsquigarrow z^2$ we may consider any continuous (computable) function $z \rightsquigarrow f(z)$ which is strictly monotone on some interval and eventually has a computable inverse. Technically, we consider dense and co-dense sets of values that might be assumed by the function. So, as a byproduct, our approach leads also to some strange continuous functions, as a function that continuously associates to any open interval (x, y) whose endpoints are algebraic numbers, a rational inner point $z(x, y)$.

None of the algorithms used so far to produce rational approximations for square roots, does satisfy any uniformity property as the continuity in $\mathbb{Q}$ coupled with the condition (2).

The algorithm presented here to positively answer Question 2 is too slow for applications, being slightly over-exponential. This is not surprising, as the technique used is to tame a countable version of the Axiom of Choice.

However, the fact that both questions have positive answers shares a new light on the possibility to get smooth control of a process while executing only classical computations with integers and rational numbers. It turns out that the dense countable sets, without being the continuum, are rich enough to enable us a smooth control of processes.

2 Prerequisites

First we will introduce the general notion of Skolem function, which is usually used in Logic and Model Theory, and we will argue that some natural approximation problems are in fact the quest for continuous Skolem functions associated with some formal statements.

Definition 1 Let L be a set of function symbols, relation symbols and individual constant symbols, and let M be an algebraic structure interpreting these symbols. Suppose that M satisfies an L-statement of the form:

$$A = \forall x_1, \ldots, x_n \, \exists y \; \varphi(x_1, \ldots, x_n, y),$$

where $\varphi(x_1, \ldots, x_n, y)$ is a formula containing symbols from L, logical connectives, parentheses, the individual variables $x_1, \ldots, x_n, y$, but no other individual variables and no quantifiers. A function $f : M^n \to M$ such that the structure M satisfies the statement:

$$\forall x_1, \ldots, x_n \; \varphi(x_1, \ldots, x_n, f(x_1, \ldots, x_n)),$$

is called a Skolem function associated with the given statement.

A Skolem function is the result of a choice-process. As for given $x_1, \ldots, x_n \in M$, by the truth of the statement A, the set:

$$F_{x_1,\ldots,x_n} = \{y \in M \,|\, \varphi(x_1, \ldots, x_n, y) \text{ is true in } M\} \neq \emptyset,$$

it is enough to arbitrarily choose $f(x_1, \ldots, x_n) \in F_{x_1,\ldots,x_n}$. In this sense, every Skolem function is a choice-function, and its existence is assured by the Axiom of Choice. It is also remarkable that in countable structures, the full Axiom of Choice is not needed, as one always can put the hand on the smaller index of an element satisfying some condition, if it is sure that such elements do exist.

We go further and we suppose that the set M is endowed not only with an algebraic L-structure, but also with a topological structure (M, τ_1). There is always a product topology induced by the topology τ_n on M^n, the topology generated by the cartesian products of open sets. A natural condition is to produce not only a choice-function, but a *continuous choice-function* $f : M^n \to M$ for the statement A.

The problem presented in the Introduction is a problem of this kind. Indeed, the ordered fields in question satisfy the condition (1):

$$\forall (x, y) \in I(K_+) \;\; \exists z \;\; x < z^2 < y. \tag{3}$$

We ask for a continuous function $z : I(K_+) \to K_+$ satisfying condition (2):

$$\forall (x, y) \in I(K_+) \;\;\; x < z(x, y)^2 < y. \tag{4}$$

How do we define continuity? Let $(S, <)$ be a totally ordered set. A typical neighborhood of some element x is an open interval $(x_1, x_2) = \{t \,|\, x_1 < t < x_2\}$ defined by two elements x_1 and x_2 such that $x \in (x_1, x_2)$. For the product topology on $S^2 = S \times S$ the fundamental open sets are the open rectangles $(x_1, x_2) \times (y_1, y_2)$. If $D \subseteq S^2$ is an open set and $z : D \to S$ is some function, we say that z is continuous on D if and only if:

$$\forall (x, y) \in D \, \forall z_1, z_2 \in S \; z_1 < z(x, y) < z_2 \to$$

$$\to \exists x_1, x_2, y_1, y_2 \;\; \{x_1 < x < x_2 \wedge y_1 < y < y_2 \wedge$$

$$\wedge \forall (x', y') \in D \; [x_1 < x' < x_2 \wedge y_1 < y' < y_2 \to$$

$$\to z_1 < z(x', y') < z_2]\}.$$

Finally, we recall the basic facts about the dense total orders without endpoints. For the whole Model Theory below we refer to the book (Prestel, 1986). The language $L = \{<\}$ consists of a binary relation $<$ only. Let T be the deductive closure of the following axioms:

1. $\forall x \;\; \neg(x < x)$.
2. $\forall x, y \;\; x < y \vee x = y \vee y < x$.
3. $\forall x, y, z \;\; x < y \wedge y < z \to x < z$.
4. $\forall x \, \exists y, z \;\; y < x \wedge x < z$.
5. $\forall x, y \, \exists z \;\; x < y \to x < z < y$.

The first three axioms say that $<$ is a total transitive order, the fourth axiom is the absence of endpoints and the fifth axiom is the density.

Definition 2 Let the language L be finite or countable. An L-theory T is called complete if for every L-statement φ, the statement φ or its negation $\neg \varphi$ belongs to T. T is called consistent if for no formal statement φ, both φ and $\neg \varphi$ belong to T. T is called $\aleph_0$-categorical if every two models A and B of cardinal $\leq \aleph_0$ of T, are isomorphic.

Lemma 1 *Let the language L be finite or countable, and let T be a consistent L-theory that has only infinite models. If T is $\aleph_0$-categorical, then T is complete.*

Proof: If T is not compete, there is a statement φ such that none of φ and $\mathbb{N}eg\,\varphi$ belongs to T. As T is consistent and deductively closed, both theories $T_1 = T \cup \{\varphi\}$ and $T_2 = T \cup \{\neg \varphi\}$ are consistent. By the Löwenheim-Skolem Theorem, both theories have models of cardinal $\leq \aleph_0$, A and respectively B. As models of T, A and B are isomorphic. But this is impossible, because there is a formal statement φ which is true in A and false in B. □

Theorem 1 *Let $L = \{<\}$ and let T be the theory of dense total orders without endpoints. Then T is consistent, $\aleph_0$-categorical and complete.*

Proof: T is consistent, because it has models, like the structure $(\mathbb{Q}, <)$. According to any of the axioms 4 or 5, T has no finite models. So it is sufficient to show that T is $\aleph_0$-categorical and the completeness will follow.

In order to show that any two countable models $(A, <)$ and $(B, <)$ are isomorphic, one applies a classical method called *back and forth procedure*. The method will be applied also in the next sections and is essential for this article, so it will be exposed here with details. The total dense orders without endpoints are maybe the least difficult context to explain it.

Let $a : \mathbb{N} \to A$ and $b : \mathbb{N} \to B$ be arbitrary bijective enumerations of those countable sets. An isomorphism $\iota : A \to B$

is a one-to-one mapping from A onto B that respects order, such that for all $a', a'' \in A$, $\iota(a') < \iota(a'')$ if and only if $a' < a''$. The isomorphism ι is constructed by induction as follows:

Step 0: We define $\iota(a(0)) = b(0)$.

Step 1: Consider the element $b(1)$. If $b(1) > b(0)$, find the least $k \in \mathbb{N}$ such that $a(k) > a(0)$ and define $\iota(a(k)) = b(1)$. If $b(1) < b(0)$, find the least $k \in \mathbb{N}$ such that $a(k) < a(0)$ and define $\iota(a(k)) = b(1)$.

Take $n \geq 1$.

Step $2n$: We suppose that $2n-1$ pairs $(a_i, b_i) \in A \times B$ have been already found, such that $a_0 < a_1 < \ldots < a_{2n-2}$, $b_0 < b_1 < \ldots < b_{2n-2}$ and a partial isomorphism have been constructed so far, such that $\iota(a_i) = b_i$ for $i = 0, \ldots, 2n-2$. Let k' be the least natural number such that $b(k') \neq b_i$ for all $i = 0, \ldots, 2n-2$. The element $b(k')$ belongs to exactly one of the following intervals: $J_0 = (-\infty, b_0)$, $J_1 = (b_0, b_1)$, ..., $J_{2n-1} = (b_{2n-2}, +\infty)$ of B. Consider the corresponding intervals $I_0 = (-\infty, a_0)$, $I_1 = (a_0, a_1)$, ..., $I_{2n-1} = (a_{2n-2}, +\infty)$ of A. If $b(k')$ belongs to the interval J_m, take k to be the least natural number such that $a(k) \in I_m$ and define $\iota(a(k)) = b(k')$.

Step $2n+1$: We suppose that $2n$ pairs $(a_i, b_i) \in A \times B$ have been already found, such that $a_0 < a_1 < \ldots < a_{2n-1}$, $b_0 < b_1 < \ldots < b_{2n-1}$ and a partial isomorphism have been constructed so far, such that $\iota(a_i) = b_i$ for $i = 0, \ldots, 2n-1$. Let k be the least natural number such that $a(k) \neq a_i$ for all $i = 0, \ldots, 2n-1$. The element $a(k)$ belongs to exactly one of the following intervals: $I_0 = (-\infty, a_0)$, $I_1 = (a_0, a_1)$, ..., $I_{2n} = (b_{2n-1}, +\infty)$ of A. Consider the corresponding intervals $J_0 = (-\infty, b_0)$, $J_1 = (b_0, b_1)$, ..., $J_{2n} = (b_{2n-1}, +\infty)$ of B. If $a(k)$ belongs to the interval I_m, take k' to be the least natural number such that $b(k') \in I_m$ and define $\iota(a(k)) = b(k')$.

□

3 Generic predicates in countable dense orderings

Definition 3 Let $(A, <)$ be a totally ordered set. A subset $P \subset A$ is called a generic predicate if P is dense in A and $A \setminus P$ is dense in A. Instead of $A \setminus P$ dense in A we may also say that P is codense in A.

Lemma 2 *Let $\mathbb{Q}_+ = \{x \in \mathbb{Q} \,|\, x > 0\}$, and let P be the set $\{x \in \mathbb{Q} \,|\, \exists y \in \mathbb{Q},\, x = y^2\}$. Then $(\mathbb{Q}_+, <)$ is a dense countable ordered set without endpoints and $P \subset \mathbb{Q}_+$ is a generic predicate.*

Proof: $\mathbb{Q}_+$ is countable and is of course a model of the theory T of dense ordered sets without endpoints. All axioms are easily verified.

For rational numbers $0 < x < y$, there is some B such that $B^2 > 2y$. For $w, \delta \in \mathbb{Q}$ with $0 < \delta < 1$ and $0 < w < B$,
$(w+\delta)^2 - w^2 < (2B+1)\delta$.
If $y - x = d$, take $\delta < d/(2B+1)$. The sequence $x_k = k\delta$ for $k = 0, \ldots, [B/\delta] - 1$ has the property that $x_{k+1}^2 - x_k^2 < d$. So at least one x_k^2 lies between x and y. It follows that P is dense in $\mathbb{Q}_+$.

Also $\mathbb{Q}_+ \setminus P$ is dense in $\mathbb{Q}_+$: Let $a \in \mathbb{Q}$. For every $k \in \mathbb{N}$, $k \geq 2$, the rational numbers u_k and v_k defined as:

$$u_k = \frac{k-1}{k}\, a^2 < a^2 < v_k = \frac{k+1}{k}\, a^2,$$

are not squares. For big values of k, the numbers u_k and v_k come arbitrarily close to a^2. □

Remark 1 The proof uses only the fact that $\mathbb{Q}$ is a densely ordered archimedean ring. One can take $\delta = 2^{-M}$ to be small enough, in order to show that the squares are dense, and the subsequence u_{2^k} instead of u_k in order to show that the non-squares are dense. So the result is true also for the ring $\mathbb{T} = \{m/2^k \,|\, k \geq 0, m = 2s+1 \in \mathbb{Z}\}$ instead of $\mathbb{Q}$.

Definition 4 For an ordered set $(A, <)$, denote with $I(A)$ the set $\{(x,y) \in A^2 \,|\, x < y\}$. This set can be identified with the set of open intervals in A.

Lemma 3 *There is a continuous function $z : I(\mathbb{Q}_+) \to \mathbb{Q}_+$ such that:*

$$\forall (x,y) \in I(\mathbb{Q}_+) \quad x < z(x,y)^2 < y, \tag{5}$$

if and only if there is a continuous function $Z : I(\mathbb{Q}_+) \to P$ such that:

$$\forall (x,y) \in I(\mathbb{Q}_+) \quad x < Z(x,y) < y.$$

Proof: If the function $z(x,y)$ is continuous, so will be also the function $Z(x,y) = z(x,y)^2$. All the values taken by the function $Z(x,y)$ are squares by definition. On the other hand if the function $Z(x,y)$ is continuous and takes only squares as values, the function $z(x,y) = \sqrt{Z(x,y)}$ is well defined and continuous. □

It turns out that the densely ordered sets with generic predicates and without endpoints have a complete theory, which is introduced below:

Definition 5 Let $L = \{P, <\}$ be the formal language consisting of a unary predicate $P(x)$ and a binary relation $<$. Let T_g be the deductive closure of the following first order statements:

1. $\forall x \quad \neg(x < x)$.
2. $\forall x, y \quad x < y \vee x = y \vee y < x$.
3. $\forall x, y, z \quad x < y \wedge y < z \to x < z$.
4. $\forall x \, \exists y, z \quad y < x \wedge x < z$.
5. $\forall x, y \, \exists z \quad x < y \to x < z < y \wedge P(z)$.
6. $\forall x, y \, \exists z \quad x < y \to x < z < y \wedge \neg P(z)$.

The theory is given by the axioms of the densely ordered set without endpoints, excepting the axiom of density, which is replaced by two new axioms stating the density and the codensity of the generic predicate P.

Theorem 2 *The theory T_g is consistent and does not have finite models. Moreover, every two countable models of T_g are isomorphic (T_g is $\aleph_0$-categorical). Consequently, T_g is complete.*

Proof: As the structure $(\mathbb{Q}_+, <, P)$ is a model of T_g by Lemma 2, T_g is consistent. If we forget the predicate P, the axioms say that every model is a densely ordered set without endpoints, so the theory has only infinite models.

The proof that every two countable models are isomorphic works similarly with the proof of Theorem 1. The step 0 is to see if one has $P(a(0))$ or $\neg P(a(0))$ and to take for $f(a(0))$ the element $b(k')$ of smallest index, which satisfies the same condition. For the step 1, one looks for the element $b(k')$ of smallest index, which is different of $f(a(0))$. This element is bigger or smaller than $f(a(0))$, and satisfies or does not satisfy P. One finds the element of smaller index $a(k)$ which respects the same conditions relatively to $a(0)$ and relatively to P and defines $f(a(k)) = b(k')$.

For the general even step, after defining $f(u_1) < f(u_2) < \ldots < f(u_m)$ for finitely many elements $u_1 < u_2 < \ldots < u_m$, one takes the element u of smallest index in the domain of f for which f has not been defined yet, and one verifies if $u < u_1$, or there is an i with $u_i < u < u_{i+1}$, or $u_m < u$ and also if $P(u)$ or if $\neg P(u)$ is true. Then one defines $f(u)$ to be the element of smallest index in the co-domain of f which satisfies the same conditions. The axioms of T_g assure the existence of such an element. The odd step is similar. $\square$

Remark 2 The theory of densely ordered sets with no first and last element and with n generic predicates $P_1, \ldots, P_n$ has the same properties. To write it down, replace the last two axioms with 2^n axioms of the form:

$$\forall x, y \, \exists z \quad x < y \to x < z < y \wedge \bigwedge_{i=1}^{n} \varepsilon_i P_i(z)$$

for every $\varepsilon \in \{0,1\}^n$. Here $\varepsilon_k = 1$ means that P_k occurs not negated, while $\varepsilon_k = 0$ means that P_k occurs negated.

Definition 6 Let $(A, <, P)$ be a model of T_g. Let $I(A) = \{(x,y) \in A^2 \,|\, x < y\}$. A admits a continuous choice function if there is a continuous function $Z : I(A) \to P$ such that:

$$(A, <, P, Z) \models \forall (x,y) \in I(A) \;\; x < Z(x,y) < y.$$

Theorem 3 *There is a countable model of T_g that admits a continuous choice-function if and only if all countable models admit continuous choice-functions.*

Proof: Suppose that the model $(B, <, P)$ admits the continuous choice-function $Z : I(B) \to P_B$ and consider a model A of T_g. Let $f : A \to B$ be the isomorphism constructed in the proof of Theorem 2. Then $f^{-1}(I(B)) = I(A)$ and $Z'(x,y) = f^{-1}(Z(f(x), f(y)))$ is a continuous choice-function on A. Observe that the isomorphism f is also a homeomorphism between the two topological spaces defined by the orders. $\square$

It turns out that in cardinal $2^{\aleph_0}$ there are models of T_g which do not admit continuous choice-functions for generic predicates.

Theorem 4 *Let $(\mathbb{R}, <, P)$ where $<$ is the usual order in the real numbers and P is some generic predicate in the reals. $(\mathbb{R}, <, P)$ does not admit continuous choice-functions.*

Proof: A continuous function $Z : I(\mathbb{R}) \to \mathbb{R}$ has a connected image. But $Z(I(\mathbb{R})) \subset P$ (totally disconnected), so the function Z is constant. Let c be this constant value. We take $x, y \in \mathbb{R}$ with $c < x < y$. It follows that $c < x < Z(x,y) = c$, which is a contradiction. $\square$

4 Continuous choice-functions

In this section a particular countable dense ordering with generic predicate is constructed, together with a continuous choice-function. According to the Theorem 3, we will conclude that all countable dense orderings with generic predicates admit continuous choice-functions.

Lemma 4 *There is a subset $D \subset \mathbb{R}$ with the following properties:*

1. *D is countable.*
2. *D is dense in $\mathbb{R}$.*
3. *D is linearly independent over $\mathbb{Q}$.*

Proof: Let $(p_n)_{n\in\mathbb{N}}$ be a bijective enumeration of the prime natural numbers, and let $(U_n)_{n\in\mathbb{N}}$ be a bijective enumeration of the open intervals (a,b) with rational ends. The family $(\log_2 p_n)_{n\in\mathbb{N}}$ is linear independent over $\mathbb{Q}$. We take $d_n = c_n \log_2 p_n$ where for every $n \in \mathbb{N}$ we choose and fix a $c_n \in \mathbb{Q} \setminus \{0\}$ such that $d_n \in U_n$. Then $D = \{d_n \,|\, n \in \mathbb{N}\}$. $\square$

Definition 7 Let $C \subset \mathbb{R}$ be the smallest set such that:

1. $D \subset C$.
2. For all $x, y \in C$ with $x < y$,
$$Z(x,y) = \frac{x+y}{2} \in C.$$

Let $P = C \setminus D$. The order $<$ on C is the order induced by $\mathbb{R}$.

Lemma 5 *The structure $(C, <, P = C \setminus D)$ is a countable model of T_g.*

Proof: We associate to every $d \in D$ a symbol $\underline{d}$. The set Λ of all Z-terms over D is inductively defined as follows:

1. For every $d \in D$, the symbol $\underline{d} \in \Lambda$.
2. For all $t_1, t_2 \in \Lambda$, $Z(t_1, t_2) \in \Lambda$.

The set Λ is countable. As every element in C has at least a name in Λ, C is countable as well.

$D \subset C \subset \overline{D} = \mathbb{R}$ so C is densely ordered and D is dense in C.

$P = C \setminus D$ is dense in C as well. Indeed, let $x, y \in C$ with $x < y$. D is dense in $\mathbb{R}$ so there is $u \in D$ with $x < u < y$. Also, there is a $v \in D$ such that $x < u < v < y$. But this implies that:

$$x < Z(u, v) < y,$$

and $w = Z(u, v) \notin D$ because this would imply a non-trivial relation of $\mathbb{Q}$-linear dependence $2w - u - v = 0$ between elements of D. □

Lemma 6 *The structure $(C, <, P = C \setminus D)$ admits a continuous choice-function.*

Proof: The function $Z : \mathbb{R}^2 \to \mathbb{R}$ given by $Z(x, y) = (x + y)/2$ is continuous and satisfies $x < Z(x, y) < y$ if $x < y$. We consider the restriction $Z : I(C) \to C$. This function is well defined, continuous and $Z(I(C)) \subseteq P$.

Indeed, if $x, y \in C$, then:

$$x = \sum_{d \in D} x_d d, \;\; y = \sum_{d \in D} y_d d,$$

are both linear combinations of elements of D with rational coefficients. Only finitely many of the coefficients x_d, y_d are not 0. The coefficients which are not 0, are strictly positive. Moreover, if $x \mathbb{N}ot \in D$ then at least two of the coefficients x_d are $\neq 0$, and the same is true for y. It follows that:

$$Z(x, y) = \frac{1}{2} \sum_{d \in D} x_d d + \frac{1}{2} \sum_{d \in D} y_d d = \sum_{d \in D} Z(x_d, y_d) d$$

is a linear combination containing at least two different elements $u, v \in D$. If $Z(x, y) = w \in D$, this implies the existence of some $\mathbb{Q}$-linear relation of dependence with coefficients in $\mathbb{Q}$ between at least three elements of D. This is a contradiction with the linear independence of D over $\mathbb{Q}$. So $Z(x, y) \in C \setminus D = P$.

In conclusion:

$$\forall (x, y) \in I(C) \;\; x < Z(x, y) < y \wedge P(Z(x, y)).$$

□

Theorem 5 *All countable models of T_g admit continuous choice-functions for the generic predicate P.*

Proof: This is a direct consequence of the Theorem 3 and of Lemmas 5 and 6. □

As interesting densely ordered sets with generic predicates, we mention: the rational $2n$-powers in the positive rationals $(\mathbb{Q}_+, \{x^{2n} \,|\, x \in \mathbb{Q}_+\})$ for some fixed n, the rational $2n+1$-powers in the rationals $(\mathbb{Q}, \{x^{2n+1} \,|\, x \in \mathbb{Q}\})$ for some fixed n, the set $\mathscr{O}_p$ of rational numbers that do not contain the prime p in their denominator $(\mathbb{Q}, \mathscr{O}_p)$, the set m_p of rational numbers that contain p in their numerator $(\mathbb{Q}, m_p)$, the rationals in some irrational field-extension $(\mathbb{Q}(\vartheta), \mathbb{Q})$, the rationals in the real algebraic numbers $(\mathbb{A}, \mathbb{Q})$. All these pairs admit continuous choice-functions.

For example, there is a continuous function $Z : I(\mathbb{A}) \to \mathbb{Q}$ such that:

$$\forall \, (x, y) \in I(\mathbb{A}) \;\; x < Z(x, y) < y.$$

But more interesting are maybe the applications announced in the Introduction:

Corollary 1 *There is a continuous function $z : I(\mathbb{Q}_+) \to \mathbb{Q}_+$ such that for all $x < y$,*

$$x < z(x, y)^2 < y.$$

Proof: As shown in the Lemma 3, the continuous function with the property $x < z(x, y)^2 < y$ exists if the structure $(\mathbb{Q}_+, <, P)$ admits a continuous choice-function, where P is the set of perfect rational squares. But this has been proven in Theorem 5. □

This can be immediately generalized as follows:

Corollary 2 *Let $f : \mathbb{Q} \to \mathbb{Q}$ be a continuous function and $J = (a, b) \subset \mathbb{Q}$ a (possibly unbounded) interval, such that $f \,|\, J$ is strictly monotone. Then there is a continuous function $z : I(J) \to J$ such that for all $x < y$,*

$$x < f(z(x, y)) < y.$$

Proof: The proof works identical with the case $f(x) = x^2$, provided that for $J = (a, b) \subset \mathbb{Q}$, the set $f(J)$ is generic in $(f(a), f(b)) \cap \mathbb{Q}$. The set $f(J)$ is dense in $(f(a), f(b)) \cap \mathbb{Q}$ by the continuity of f but is not necessarily codense. In the case that $f(J)$ is not codense in $(f(a), f(b)) \cap \mathbb{Q}$, we consider some generic predicate P in J, and we apply the construction for the predicate $f(P)$ which is generic in $(f(a), f(b)) \cap \mathbb{Q}$. The predicate P might be $\mathscr{O}_2$ or may consist of the cubes of rational numbers. Also, the inverse of f, defined on the image of f, is a continuous function because f is continuous and monotone. This can be proved by considering the extension of f by continuity, which is a strictly monotone continuous function $\tilde{f} : (a, b) \subset \mathbb{R} \to \mathbb{R}$ with $\tilde{f} \,|\, \mathbb{Q} = f$. □

If instead of $\mathbb{Q}$ we consider the ring $\mathbb{T}$, the whole construction works: $\mathbb{T}$ is a dense ordering without endpoints, the squares build a generic predicate in $\mathbb{T}_+$, the cubes build a generic predicate in $\mathbb{T}$, and both Corollaries 1 and 2 work with $\mathbb{T}$ instead of $\mathbb{Q}$. $\mathbb{T}$ has the advantage that all elements have a finite binary or decimal expansion, but has of course a smaller power of expression than $\mathbb{Q}$.

Corollary 3 *The continuous choice-function on*

$$(C, <, P = C \setminus D)$$

has the property that:

$$\forall u \;\; [P(u) \to \exists x, y \;\; x < y \wedge u = Z(x, y)].$$

So all choice-functions obtained by transfer on other models of T_g have this property as well. Moreover, the set of pairs (x, y) with $Z(x, y) = u$ contains at most one element.

5 Effective computability

For the following considerations we fix a computable bijection $c : \mathbb{N} \to \mathbb{Q}$ of the rational numbers. Examples can be easily derived from (Calkin, Wilf, 2000).

Lemma 7 *There is a computable function that enumerates bijectively the set $(U_n)_{n\in\mathbb{N}}$ of all intervals with rational ends.*

Proof: The function lists the intervals (a,b) with $a,b \in \mathbb{Q}$. For $j < k$ let us denote with I_{jk} the interval

$(\min(c(j),c(k)),\max(c(j),c(k)))$.

The first interval on the list is $I_{0,1}$, followed by $I_{0,2}$ and $I_{1,2}$. After writing down all intervals $I_{j,k}$ with $0 \le j < k \le n$, the function lists the intervals $I_{0,n+1}$, ..., $I_{n,n+1}$. □

Lemma 8 *There is a computable function that enumerates bijectively a set D which is dense in $\mathbb{R}$ and linearly independent over $\mathbb{Q}$.*

Proof: The function lists the elements of D writing down expressions:

$$\frac{i}{j}\log_2 p$$

where $i \in \mathbb{Z}$, $j,p \in \mathbb{N}$, $\gcd(i,j) = 1$ and p is a prime number. In order to compute the element x_n, let $U_n = (a_n,b_n)$ be the corresponding open interval with rational ends and let p_n be the n-th prime number. For all rational numbers $c = c(u)$ one checks if:

$$a_n < c(u)\log_2 p_n < b_n.$$

As $\mathbb{Q}$ is dense in $\mathbb{R}$, this will finally happen for some first $u \in \mathbb{N}$. In this case one takes $x_n = c(u)\log_2 p_n$. The inequalities are effectively decidable because by multiplying with a common denominator they are equivalent with:

$$A_n < C\log_2 p_n < B_n$$

where $A_n,C,B_n \in \mathbb{Z}$, which is true if and only if:

$$2^{A_n} < p_n^C < 2^{B_n}.$$

This is solved by direct computation. □

Lemma 9 *There is a computable function that enumerates bijectively the set of terms Λ generated by a set of constant symbols $\underline{D}$ representing the elements of D and by a binary function symbol $Z(x,y)$.*

Proof: This list of terms can be inductively generated as follows. We start with $\underline{d}_0$ and continue with $Z(\underline{d}_0,\underline{d}_0)$. The third term is $\underline{d}_1$. Now all combinations with already done terms are done using the function Z, but every new candidate is compared with already written down terms and is written down only if it is not in the list. The process continues by adding a new constant term and repeating the generation of new terms. □

Lemma 10 *Let C be the smallest set of real numbers containing D and closed to the operation $z(x,y) = (x+y)/2$. Then there is a computable function that enumerates bijectively the set C.*

Proof: Let $\lambda : \mathbb{N} \to \Lambda$ the function that bijectively enumerates Λ, whose existence was proven in Lemma 9 and let $d : \mathbb{N} \to D$ be the computable bijective enumeration of D given by Lemma 8. We list bijectively the elements of C in the following way. The function λ lists the terms. Every constant symbol $\underline{d}_k$ is replaced by the expression $d(k)$. After the substitution, one checks if the term represents a already listed element. If it represents a new element, then it is is brought in the normed form:

$$\frac{m_1}{n_1}\log_2 p_{i_1} + \ldots + \frac{m_k}{n_k}\log_2 p_{i_k},$$

where $m_j \in \mathbb{Z}$, $n_j, p_{i_j} \in \mathbb{N}$, $\gcd(m_i,n_i) = 1$ and all p_{i_j} are pairwise different primes, written in increasing order. In this form, the new element is written down on the list.

The comparisons can be done effectively, as they reduce to decide the truth of equalities of the form:

$$\frac{m_1}{n_1}\log_2 p_{i_1} + \ldots + \frac{m_k}{n_k}\log_2 p_{i_k} =$$

$$= \frac{r_1}{s_1}\log_2 q_{j_1} + \ldots + \frac{r_l}{s_l}\log_2 q_{j_l}.$$

Such equations are ultimately equivalent with equations of the shape $p_{i_1}^{M_1} \ldots p_{i_k}^{M_k} = q_{j_1}^{R_1} \ldots q_{j_l}^{R_l}$, easily checked by the unicity of prime number decomposition in natural numbers. □

Lemma 11 *Let $(A_1,P_1,<_1)$ and $(A_2,P_2,<_2)$ be two countable models of T_g such that there are computable bijective enumerations of A_1 and A_2 and such that the predicates P_1 and P_2, as like the order relations $<_1$ and $<_2$, are effectively decidable. Then there is a computable isomorphism $f : A_1 \to A_2$ between the structures $(A_1,P_1,<_1)$ and $(A_2,P_2,<_2)$.*

Proof: It is obvious that under these assumptions the back and forth proof method used for the proof of Theorem 1 and Theorem 2 leads to an effective procedure. □

Theorem 6 *Let $(A,P,<)$ be a countable model of T_g such that there is computable bijective enumeration of A and the predicate P, as like the order relation $<$ is effectively decidable. Then the structure A has an effectively computable continuous choice-function $Z : H(A) \to A$.*

Proof: The structure $(A,P,<)$ satisfies the conditions of Lemma 11 and it is easy to check that the structure $(C,C \setminus D,<)$ satisfies those conditions as well. The set C has a computable bijective enumeration by Lemma 10. The predicate $C \setminus D$ is decidable by the fact that the elements of C are expressible by unique normed expressions and they belong to

D if and only if just one $\log_2 p$ appears in the corresponding expression. The order can be decided because:

$$\frac{m_1}{n_1}\log_2 p_{i_1} + \ldots + \frac{m_k}{n_k}\log_2 p_{i_k} <$$

$$< \frac{r_1}{s_1}\log_2 q_{j_1} + \ldots + \frac{r_l}{s_l}\log_2 q_{j_l}$$

is ultimately equivalent with $p_{i_1}^{M_1} \ldots p_{i_k}^{M_k} < q_{j_1}^{R_1} \ldots q_{j_l}^{R_l}$ with powers in $M_i, R_j \in \mathbb{Z}$ and this can be decided by direct computation.

By Lemma 11 there is a computable isomorphism $f : A \to C$ between $(A, P, <)$ and $(C, C \setminus D, <)$. It follows that

$$Z(a,b) = f^{-1}(Z(f(a), f(b)))$$

is a computable continuous choice-function in A. The point is that in order to compute $Z(a,b)$ we do not need the whole isomorphism f. Its list of values must be computed until $f(a)$ and $f(b)$ are known, and then only until

$$f^{-1}(Z(f(a), f(b)))$$

is known. $\square$

In conclusion, all structures mentioned in Section 4 admit computable continuous choice-functions. Also, there are smooth approximation functions satisfying the Corollary 1. Also, assuming that the function f from Corollary 2 is computable and has a computable inverse, there is a smooth approximation function satisfying the Corollary 2. Moreover, as the ring $\mathbb{T}$ of rational numbers with powers of 2 as denominators admits computable bijective enumerations, all the results work again with $\mathbb{T}$ instead of $\mathbb{Q}$. Remarkable, a lot of approximation methods, like Newton's Method, do not work on $\mathbb{T}$ because $\mathbb{T}$ is not a field.

6 Suounitnoc functions

In this section we show another application of the method, to illustrate how one can use this method to construct a lot of examples in Analysis.

Edward Nelson defined in (Nelson, 1977) the notion of suounitnoc function by reversing the roles of ε and δ in the definition of the continuous function. Nelson shown that a function $f : \mathbb{R} \to \mathbb{R}$ is suounitnoc in one point if and only if it is suounitnoc everywhere. He also shown that a function $f : \mathbb{R} \to \mathbb{R}$ is suounitnoc if and only if the function, seen as standard object in a model of its axiom system called IST, has limited values for all limited arguments. This is equivalent with the standard statement that the images of bounded sets are bounded.

We translate here the definitions of continuity and suounitnocity in the language containing only the order relation symbol and a symbol for the one-variable function f.

Definition 8 Let $(A, <)$ be an ordered set. The function $f : A \to A$ is called continuous in x if $(A, <, f)$ fulfills the statement:

$$C(x): \ \forall f_1, f_2 \ f_1 < f(x) < f_2 \to \exists x_1, x_2 [x_1 < x < x_2 \wedge$$

$$\forall x' \{x_1 < x' < x_2 \to f_1 < f(x') < f_2\}].$$

It is called continuous if $\forall x \, C(x)$.

Definition 9 Let $(A, <)$ be an ordered set. The function $f : A \to A$ is called suounitnoc in x if $(A, <, f)$ fulfills the statement:

$$S(x): \ \forall x_1, x_2 \ x_1 < x < x_2 \to \exists f_1, f_2 [f_1 < f(x) < f_2 \wedge$$

$$\wedge \forall x' \{x_1 < x' < x_2 \to f_1 < f(x') < f_2\}].$$

One observes from the beginning that after the existential quantifier, $f_1 < f(x) < f_2$ becomes superfluous, and the formula becomes:

$$S(x): \ \forall x_1, x_2 \ x_1 < x < x_2 \to \exists f_1, f_2 \, \forall x' \ (x_1 < x' < x_2 \to$$

$$\to f_1 < f(x') < f_2).$$

The function is called suounitnoc if $\forall x \, C(x)$.

The following generalization of Nelsons results is immediate:

Theorem 7 *Let A be an ordered set without endpoints.*

1. *If a function $f : A \to A$ is suounitnoc in a point x, then it is suounitnoc in every x.*
2. *$f : A \to A$ is suounitnoc if and only if every bounded interval has a bounded image.*

Proof: For (1), take some $x' \neq x$ and $x'_1 < x' < x'_2$. Choose x''_1 and x''_2 such that all points x, x', x'_1 and x'_2 are bounded by them, and apply the hypothesis. The proof of (2) becomes trivial. $\square$

We recall that:

$$\mathbb{R} \models (\ f \text{ continuous } \to \ f \text{ suounitnoc }).$$

Indeed, in $\mathbb{R}$ continuous functions lead compact sets to compact sets. As every bounded interval has a compact closure by the Heine-Borel Theorem, its image must be also bounded, and so the continuous function is suounitnoc. $\square$

We will show now that we cannot get the implication (f continuous $\to$ f suounitnoc) without using the second order fact that $\mathbb{R}$ is a complete ordered field. Consider the language of ordered fields expanded with one unary function symbol f. It is $L_f = \{+, -, \cdot, 0, 1, <, f(\cdot)\}$.

Definition 10 Let OF_f be the deductive closure of the axioms of ordered fields in the language L_f. Let RCF_f the deductive closure of the axioms of the real closed fields in the language L_f.

As the symbol f does not occur in any of the axioms of OF_f or RCF_f, it is not to expect that any of these theories was complete.

Theorem 8 *The theory OF_f does not prove the statement (f continuous $\to$ f suounitnoc).*

Proof: Consider the structure

$(\mathbb{Q}, +, -, \cdot, 0, 1, <, f(\cdot))$

with $f : \mathbb{Q} \to \mathbb{Q}$ defined by:

$$f(x) = \frac{1}{x^3 - 2}.$$

The function f is continuous on $\mathbb{Q}$ because it is continuous on $\mathbb{R}$, excepting in $x = \sqrt[3]{2}$. But $\sqrt[3]{2}$ is an irrational number. The function f is not suounitnoc because if $a, b \in \mathbb{Q}$ such that $a < \sqrt[3]{2} < b$ then $f((a,b))$ is unbounded. □

Theorem 9 *The theory RCF_f does not prove the statement (f continuous $\to$ f suounitnoc).*

Proof: Consider the ordered sets $(\mathbb{Q}, <)$ and $(\mathbb{A}, <)$. $\mathbb{A}$ is the field of all real algebraic numbers. These sets are countable densely ordered sets without endpoints, so there is an order isomorphism $h : \mathbb{Q} \to \mathbb{A}$. Now consider the function $f_1 : \mathbb{A} \to \mathbb{A}$ given by $f_1(x) = h(f(h^{-1}(x)))$, where f has been defined in the proof of the Theorem 8. As both properties (f continuous) and (f suounitnoc) are definable using only the order, and h is an isomorphism of ordered sets, it follows that f_1 is continuous on $\mathbb{A}$ but not suounitnoc on $\mathbb{A}$. □

7 Conclusions

1. Only values of countable dense sets can be expressed and communicated in digital processes.
2. There are practical reasons to look for approximation processes whose results depend continuously of the intended error bound.
3. As the values of a process of approximation belong always to a dense countable set, it is natural to study the continuity of a process of approximation, considered as function with values in a dense countable set.
4. We proved that such functions exist. With some natural restrictions, these functions are also computable.
5. Countable densely ordered sets, like the rational numbers or the rational numbers whose denominator is a power of 2, are a good support for algorithms of continuous approximation.
6. The fact that all countable dense orderings with generic predicate are isomorphic allows us a structural approach to this problem. A special continuous choice-function is sufficient for solving a wide class of problems.
7. The algorithms displayed here are significant only as an existence proof. They are not fast enough for practical applications.
8. The set of real numbers does not allow any continuous choice-function.

8 Appendix

Not all ordered fields satisfy the statement:

$\forall x, y \exists z \quad 0 < x < y \to x < z^2 < y.$

For example in the field of rational functions with rational coefficients $\mathbb{Q}(T)$, ordered such that $T > \mathbb{Q}$, there is no square between $x = T$ and $y = 2T$. Indeed, if:

$$T < \frac{f^2(T)}{g^2(T)} < 2T,$$

for some polynomials $f, g \in \mathbb{Z}[T]$, then:

$$1 < \frac{f^2(T)}{Tg^2(T)} < 2,$$

must be fulfilled. We may let $T \to \infty$. As the numerator and the denominator have different degrees, we get $1 \leq 0$ or $\infty \leq 2$, which are both contradictions.

Compliance with Ethical Standards

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