Generalized Hukuhara Conformable Fractional Derivative and its Application to Fuzzy Fractional Partial DifferentialEquations

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Abstract The main focus of this paper is to develop an efficient analytical method to obtain the traveling wave fuzzy solution for the fuzzy generalized Hukuhara conformable fractional equations by considering the type of generalized Hukuhara conformable fractional differentiability of the solution. To achieve this, the fuzzy conformable fractional derivative based on the generalized Hukuhara differentiability is defined, and several properties are brought on the topic, such as switching points and the fuzzy chain rule. After that, a new analytical method is applied to find the exact solutions for two famous mathematical equations: the fuzzy fractional Wave equation and the fuzzy fractional Diffusion equation. The present work is the first report in which the fuzzy traveling wave method is used to design an analytical method to solve these fuzzy problems. The final examples are asserted that our new method is applicable and efficient.

Keywords Generalized Hukuhara conformable fractional derivative · Fuzzy Traveling wave solution · Generalized partial Hukuhara differentiability · The fuzzy fractional Wave equation · The fuzzy fractional Diffusion equation.

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1 Introduction

During the last decade, the interest of mathematicians in fuzzy differential equations has been rapidly increasing. The main reason for this development is that using these problems will lead to a much more effective and elegant way of treating real-life issues. A particular subgroup of fuzzy differential equations is described with operators of fractional nature. Fractional calculus is a set of methods and hypotheses that extend the concept of a derivative operator from integer-order n to arbitrary order α. Modeling like biological population models, the predator-prey models and infectious diseases models, etc are generalized to fractional order. Fractional calculus is not only a productive and emerging field, but it also represents a new philosophy, how to construct and apply a certain type of nonlocal operator to real-world problems [17, 18, 19, 26, 27, 32, 33].

The interest in fractional fuzzy differential equations aroused in 2012 with a paper by Agarwal et al. [5]. The existence and uniqueness of a fuzzy solution for fractional differential equations are proved in [7]. The concepts of fuzzy fractional integral and Caputo partial differentiability based on generalized Hukuhara differentiability for the fuzzy multivariable functions have been introduced by H. Viet Long et al. [37]. Hoa et al. [20] introduced the fuzzy Caputo-Katugampola fractional differential equations in fuzzy space, and under generalized Lipschitz condition, the existence and uniqueness of the solution proved. The analytical solutions to some linear fractional partial fuzzy differential equations under certain conditions were investigated in [31]. The perturbation-iteration algorithm was developed for numerical solutions of some types of fuzzy fractional partial differential equations with generalized Hukuhara derivative [35]. H. Zureigat et al.[38] analyzed the compact Crank-Nicholson scheme to solve the fuzzy time diffusion equation with fractional order $0 < \alpha \leq 1$. Some new methods and useful materials concerning fuzzy fractional differential and fuzzy fractional partial differential equations are introduced in [1].

Recently, a new well-behaved simple fractional derivative called ”the conformable fractional derivative” is defined by [25, 6]. This new definition seems to be a natural extension of the usual derivative. Researchers started to combine this new definition with fuzzy calculus [22, 23, 29]. They used the concept of H-differentiability or strongly generalized differentiability. But it is well-known that the usual Hukuhara difference between two fuzzy numbers exists only under very restrictive conditions [16, 9]. To overcome this shortcoming, we will introduce the fuzzy conformable fractional derivative under generalized Hukuhara differentiability, and prove some important properties for this kind of differentiability.

Consider the following generic form of second order fuzzy fractional partial differential equation defined based on generalized Hukuhara conformable fractional derivative

$$\mathscr{D}^{\alpha}_{t_{gH}} \upsilon = F\left(\upsilon, \upsilon_{x_{gH}}, \upsilon_{xx_{gH}}, \mathscr{D}^{2\alpha}_{tt_{gH}} \upsilon\right), \tag{1}$$

with $0 < \alpha \leq 1$. The main contribution of this paper is to find the wave traveling solutions for the problem (1). For this purpose, the concept of generalized Hukuhara conformable fractional differentiability is introduced thoroughly in the fuzzy functions.

Next, the fuzzy fractional wave equation and fuzzy Diffusion equation are introduced based on the generalized Hukuhara conformable fractional differentiability. Finally, we discuss the fuzzy traveling wave solutions for these equations by considering the type of α_{gH}-differentiability.

We now give a brief outline of the main sections of the paper and state the aims and objectives of each section. Section 3 deals with aspects of background knowledge in fuzzy mathematics and fuzzy derivatives with emphasis on the generalized Hukuhara differentiability. In Section 4, generalized Hukuhara conformable fractional differentiability are studied some properties for this concept of differentiability are proved. A fuzzy fractional Wave equation and a fuzzy fractional diffusion equation under generalized Hukuhara conformable fractional differentiability are introduced in Sections 5 and 6, respectively. Moreover the fuzzy traveling wave solutions of these equations are investigated in different scenarios. Finally in Section 7, the conclusions are given.

2 Related Works

The concept of the fuzzy derivative was first introduced by Chang and Zadeh [15]. The starting point of the topic in the set-valued differential equation and also fuzzy differential equation is Hukuhara's paper [21]. Puri and Ralescu [30] suggested the fuzzy differential equations modeling with uncertainty under the concept of H-differentiability. Subsequently, Kaleva in [24] proposed fuzzy differential equations using the Hukuhara derivative, and some other authors developed it. But for some fuzzy differential equations in this framework, the diameter of the solution is unbounded as the time t increases [16].

To overcome this shortcoming, Bede and Gal introduced the weakly generalized differentiability and the strongly generalized differentiability for the fuzzy functions [9]. Moreover, they presented a more general definition of derivatives for the fuzzy functions and their applications for solving fuzzy differential equations [9,10]. Stefanini and Bede, by the concept of generalization of the Hukuhara difference of compact convex set, introduced generalized Hukuhara differentiability [36] for interval-valued functions. They showed that this concept of differentiability has relationships with weakly generalized differentiability and strongly generalized differentiability. The disadvantage of the strongly generalized differentiability of a function compared to H-differentiability is that the fuzzy differential equation has no unique solution [9]. Also, in [13] the authors studied relationships between the strongly generalized differentiability and the generalized Hukuhara differentiability, showing the equivalence between these two concepts when the set of switching points of the interval-valued function is finite.

Table 1 Related Fuzzy Works

Article	Achievements /Advantages	Disadvantages
Chang et al. [15] (1972)	Define a fuzzy function and its inverse, fuzzy parametric functions, fuzzy observation, and control.	
Hukuhara [21] (1967)	The first definitions of Hukuhara difference, and Hukuhara derivative	Hukuhara difference between two fuzzy numbers is not always a fuzzy number.
Puri et al.[30] (1986)	Prove the Rådström embedding theorem and define the concept of the differential of a fuzzy function	The diameter of the solution is unbounded as the time t increases.
Kaleva [24] (1987)	Define a fuzzy differential equations using the Hukuhara derivative	The diameter of the solution is unbounded as the time t increases.
Bede et al. [9] (2005)	Define the strongly generalized differentiability and the weakly generalized differential of a fuzzy function using the Hukuhara derivative	The fuzzy differential equations may not have a unique.
Stefanini et al. [36] (2009)	Define the generalized Hukuhara difference and generalized Hukuhara differentiability for interval-valued functions	
Allahviranloo et al. [2] (2015)	Define the fuzzy generalized Hukuhara partial differentiability and solve the fuzzy heat equation	
Harir et al. [22] (2020)	Define the fuzzy Generalized Conformable Fractional Derivative using the Hukuhara derivative	Hukuhara difference between two fuzzy numbers is not always a fuzzy number.
Harir et al. [23] (2021)	Prove the existence and uniqueness theorem for a solution to a fuzzy fractional differential equation by using the concept of conformable differentiability	The diameter of the solution is unbounded as the time t increases.
Martynyuk et al. [29] (2020)	Define the fractional-like Hukuhara-type derivatives	Hukuhara difference between two fuzzy numbers is not always a fuzzy number.
Chalco-Cano et al. [14] (2020)	A new characterization of the switching points for generalized Hukuhara differentiability	

In this way, they use the LU-parametric representation of fuzzy numbers and fuzzy valued functions to obtain valid approximations of fuzzy generalized Hukuhara derivative and solve fuzzy differential equations [12]. Allahviranloo [2] introduced the fuzzy generalized Hukuhara partial differentiability and solved the fuzzy heat equation under generalized Hukuhara differentiability. Moreover, in [28] the authors obtained the fuzzy solutions of the fuzzy Poisson equation under generalized Hukuhara differentiability. Recently, Y.Chalco-Cano et al. [14] provided a new characterization of the switching points for generalized Hukuhara differentiability and shown that the set of all switching points is at most countable.

3 Preliminaries

In the following, we focus on the basic definitions and the necessary notation which will be used throughout the paper. Let $\mathbb{E}$ is the set of fuzzy numbers and $\mathbb{T} \subset \mathbb{E}$ shows the set of all triangular fuzzy numbers.

Let $a = \Big(a_1, a_2, a_3\Big)$ and $b = \Big(b_1, b_2, b_3\Big)$ are two triangular fuzzy numbers, so the generalized Hukuhara difference, $a \ominus_{gH} b$, is defined as follows [11]

$$a \ominus_{gH} b = c \Longleftrightarrow \begin{cases} (i).\ c = (a_1 - b_1, a_2 - b_2, a_3 - b_3), \\ or, \\ (ii).\ c = (a_3 - b_3, a_2 - b_2, a_1 - b_1). \end{cases}$$

Actually

$$a \ominus_{gH} b = \Big(\min\{a_1 - b_1, a_3 - b_3\}, a_2 - b_2, \max\{a_1 - b_1, a_3 - b_3\}\Big).$$

In this article, we assume that $a \ominus_{gH} b \in \mathbb{T}$.

Let $f : [a,b] \to \mathbb{T}$ and it's first k generalized Hukuhara derivatives are continuous fuzzy triangular functions without any switching points on domain $\mathbb{I} := [a,b]$ [11].

Definition 31 *(See [8]). Let $f : \mathbb{I} \to \mathbb{E}$ be a fuzzy function and $t_0 \in \mathbb{I}$. If*

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall t \big(0 < |t - t_0| < \delta \Rightarrow D(f(t), L) < \varepsilon\big),$$

Here, D is the Hausdorff distance . Then, we say that $L \in \mathbb{E}$ is limit of f in t_0, which is denoted by $\lim_{t\to t_0} f(t) = L$. Also the fuzzy function f is said to be fuzzy continuous if

$$\lim_{t\to t_0} f(t) = f(t_0),$$

Theorem 32 *(See [3]) Let $f, g : \mathbb{I} \to \mathbb{E}$ be two fuzzy functions. If $\lim_{t\to c} f(t) = L_1$ and $\lim_{t\to c} g(t) = L_2$, $L_1, L_2 \in \mathbb{E}$ then*

$$\lim_{t\to c}[f(t) \ominus_{gH} g(t)] = L_1 \ominus_{gH} L_2.$$

Definition 33 *(See [11]) The fuzzy function $f(t)$ is generalized Hukuhara differentiable ($[gH]$-differentiable) at $t_0 \in \mathbb{I}$ if*

$$f'_{gH}(t_0) = \lim_{h \to 0} \frac{f(t_0+h) \ominus_{gH} f(t_0)}{h},$$

belongs to $\mathbb{E}$. In addition we can say that $f(t)$ is

- $[(i)-gH]$*-differentiable function if and only if for all $t \in \mathbb{I}$*

$$f'_{i.gH}(t) = \Big(f'_1(t), f'_2(t), f'_3(t)\Big),$$

defines a triangular fuzzy number.

- $[(ii)-gH]$*-differentiable function if and only if for all $t \in \mathbb{I}$*

$$f'_{ii.gH}(t) = \Big(f'_3(t), f'_2(t), f'_1(t)\Big),$$

is a triangular fuzzy number.

Proposition 34 *(See [34]) Let λ_1 and λ_2 are two real constants such that $\lambda_1, \lambda_2 \geq 0$ (or $\lambda_1, \lambda_2 \leq 0$). If $f(t)$ is a triangular fuzzy function, then*

$$\lambda_1 f(t) \ominus_{gH} \lambda_2 f(t) = (\lambda_1 - \lambda_2) f(t). \tag{2}$$

Definition 35 *(See [11]) Let $f : \mathbb{I} \to \mathbb{T}$ is a fuzzy function and $f(t) = \Big(f_1(t), f_2(t), f_3(t)\Big)$ and $t_0 \in \mathbb{I}$ then*

$$\int_a^b f(t)\mathrm{d}t = \Big(\int_a^b f_1(t)\mathrm{d}t, \int_a^b f_2(t)\mathrm{d}t, \int_a^b f_3(t)\mathrm{d}t\Big).$$

Theorem 36 *(See [11]) If f is a gH-differentiable fuzzy function with no switching point in the interval $\mathbb{I}$, then we have*

$$\int_a^b f'_{gH}(t)\mathrm{d}t = f(b) \ominus_{gH} f(a).$$

Lemma 37 *(See [37]) If $f : \mathbb{I} \to \mathbb{T}$ be a triangular fuzzy function with no switching point in interval $\mathbb{I}$, then we have*

1. *If $f(t)$ is $[i-gH]$-differentiable , then*

$$\int_a^b f'_{i.gH}(t)\mathrm{d}t = f(b) \ominus f(a).$$

2. *If $f(t)$ is $[ii-gH]$-differentiable , then*

$$\int_a^b f'_{ii.gH}(t)\mathrm{d}t = (-1)f(a) \ominus (-1)f(b).$$

Lemma 38 *(See [28]) $\int_b^a f(t)\mathrm{d}t = \ominus \int_a^b f(t)\mathrm{d}t$; where $\ominus$ denote Hukuhara difference and $f(t)$ be a fuzzy function.*

Definition 39 *(See [2]) Let* $(x_0,t_0) \in \mathbb{D} \subseteq \mathbb{R}^2$*, then the first generalized Hukuhara partial derivative ($[gH-p]$-derivative for short) of a fuzzy value function* $\upsilon(x,t)$: $\mathbb{D} \to \mathbb{E}$ *at* (x_0,t_0) *with respect to variables* x,t *are the fuzzy functions* $\partial_{x_{gH}}\upsilon(x_0,t_0)$ *and* $\partial_{t_{gH}}\upsilon(x_0,t_0)$ *given by*

$$\partial_{x_{gH}}\upsilon(x_0,t_0) = \lim_{h\to 0} \frac{\upsilon(x_0+h,t_0) \ominus_{gH} \upsilon(x_0,t_0)}{h},$$

$$\partial_{t_{gH}}\upsilon(x_0,t_0) = \lim_{k\to 0} \frac{f(x_0,t_0+k) \ominus_{gH} \upsilon(x_0,t_0)}{k},$$

provided that $\partial_{x_{gH}}\upsilon(x_0,t_0)$ *and* $\partial_{t_{gH}}\upsilon(x_0,t_0) \in \mathbb{E}$.

Definition 310 *(See [2]) A triangular fuzzy function* $\upsilon(x,t) = \Big(\upsilon_1(x,t), \upsilon_2(x,t), \upsilon_3(x,t)\Big)$, *without any switching points on* $\mathbb{D}$ *is called*

- $[(i)-p]$*-differentiable with respect to t at* (x_0,t_0) *if and only if*

$$\upsilon_{t_{i.gH}}(x_0,t_0) = \Big(\frac{\partial \upsilon_1(x,t)}{\partial t}, \frac{\partial \upsilon_2(x,t)}{\partial t}, \frac{\partial \upsilon_3(x,t)}{\partial t}\Big)\Big|_{x=x_0,t=t_0},$$

defines a triangular fuzzy number, and

- *it's* $[(ii)-p]$*-differentiable if and only if*

$$\upsilon_{t_{ii.gH}}(x_0,t_0) = \Big(\frac{\partial \upsilon_3(x,t)}{\partial t}, \frac{\partial \upsilon_2(x,t)}{\partial t}, \frac{\partial \upsilon_1(x,t)}{\partial t}\Big)\Big|_{x=x_0,t=t_0},$$

defines a triangular fuzzy number.

Moreover, if $\upsilon_x(x,t)$ *is* $[gH-p]$*-differentiable at* (x,t) *with respect to x without any switching point on* $\mathbb{D}$ *and*

- *if the type of* $[gH-p]$*-differentiability of both* $\upsilon(x,t)$ *and* $\upsilon_x(x,t)$ *are the same, then* $\upsilon_x(x,t)$ *is* $[(i)-p]$*-differentiable w.r.t x and*

$$\upsilon_{xx_{i.gH}}(x_0,t_0) = \Big(\frac{\partial^2 \upsilon_1(x,t)}{\partial x^2}, \frac{\partial^2 \upsilon_2(x,t)}{\partial x^2}, \frac{\partial^2 \upsilon_3(x,t)}{\partial x^2}\Big)\Big|_{x=x_0,t=t_0},$$

- *if the type of* $[gH-p]$*-differentiability* $\upsilon(x,t)$ *and* $\upsilon_x(x,t)$ *are different, therefore* $\upsilon_x(x,t)$ *is* $[(ii)-p]$*-differentiable w.r.t x and*

$$\upsilon_{xx_{ii.gH}}(x_0,t_0) = \Big(\frac{\partial^2 \upsilon_3(x,t)}{\partial x^2}, \frac{\partial^2 \upsilon_2(x,t)}{\partial x^2}, \frac{\partial^2 \upsilon_1(x,t)}{\partial x^2}\Big)\Big|_{x=x_0,t=t_0}.$$

4 Generalized Hukuhara Conformable Fractional Derivative

In this section, we are going to introduce conformable fractional derivative based on the generalized Hukuhara derivative. Moreover, we will prove several properties for this kind of differentiability.

Definition 41 *Let $f:[0,\infty)\to\mathbb{E}$ be a triangular fuzzy function. The generalized Hukuhara conformable fractional derivative of f of order $\alpha\in(0,1)$ is defined by*

$$T^{\alpha}_{gH}(f)(t)=\lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{1-\alpha})\ominus_{gH} f(t)}{\varepsilon}, \tag{3}$$

provided that $T^{\alpha}_{gH}(f)(t)\in\mathbb{E}$. If the generalized Hukuhara conformable fractional derivative of f of order α exists, then we simply say f is α_{gH}-differentiable.

Theorem 42 *If a fuzzy function $f:[0,\infty)\to\mathbb{E}$ is α_{gH}-differentiable at $t_0>0,\ \alpha\in(0,1]$, then f is continuous at t_0.*

Proof We have $f(t+\varepsilon t^{1-\alpha})\ominus_{gH} f(t)=\frac{f(t+\varepsilon t^{1-\alpha})\ominus_{gH} f(t)}{\varepsilon}\odot\varepsilon$. By using Theorem 32, we conclude that

$$\lim_{\varepsilon\to 0}[f(t+\varepsilon t^{1-\alpha})\ominus_{gH} f(t)]=\lim_{\varepsilon\longrightarrow 0}\frac{f(t+\varepsilon t^{1-\alpha})\ominus_{gH} f(t)}{\varepsilon}\odot\lim_{\varepsilon\longrightarrow 0}\varepsilon,$$

then

$$\lim_{\varepsilon\to 0}[f(t+\varepsilon t^{1-\alpha})\ominus_{gH} f(t)]=T^{\alpha}_{gH}(f)(t)\odot 0.$$

Now, let $h=\varepsilon t_0^{1-\alpha}$, therefore

$$\lim_{h\to 0}[f(t+h)\ominus_{gH} f(t)]=0.$$

Therefore, according to Definition 31, it can be concluded that the function f is fuzzy continuous. □

Definition 43 *Let $f:[0,\infty)\to\mathbb{E}$ be a triangular fuzzy function. The generalized Hukuhara conformable fractional derivative of f of order $\beta\in(1,2)$ is defined by*

$$T^{\beta}_{gH}(f)(t)=\lim_{\varepsilon\to 0}\frac{f'_{gH}(t+\varepsilon t^{2-\beta})\ominus_{gH} f'_{gH}(t)}{\varepsilon}, \tag{4}$$

provided that $T^{\beta}_{gH}(f)(t)\in\mathbb{E}$. If the generalized Hukuhara conformable fractional derivative of f of order β exists, then we simply say f is β_{gH}-differentiable.

Definition 44 *Let $\alpha\in(0,1)$ and f is α_{gH}-differentiable at a point $t>0$. We can say that $f(t)$ is*

- *$\alpha_{i.gH}$-differentiable function if and only if for all $t>0$*

$$T^{\alpha}_{i.gH}(f)(t)=\Big(T^{\alpha}(f_1)(t),T^{\alpha}(f_2)(t),T^{\alpha}(f_3)(t)\Big), \tag{5}$$

  defines a triangular fuzzy number.

– *$\alpha_{ii.gH}$-differentiable function if and only if for all $t > 0$*

$$T^{\alpha}_{ii.gH}(f)(t) = \Big(T^{\alpha}(f_3)(t), T^{\alpha}(f_2)(t), T^{\alpha}(f_1)(t)\Big), \tag{6}$$

be a triangular fuzzy number.

Here, $T^{\alpha}(f_i)(t)$, $i = 1,2,3$ is the conformable fractional derivative for the real valued function $f_i(t)$ [25].

Definition 45 *We say that a point $\xi_0 \in (0,\infty)$, is a switching point for the differentiability of f, if in any neighborhood $\mathscr{V}$ of ξ_0 there exist points $\xi_1 < \xi_0 < \xi_2$ such that*

Type I. at ξ_1 (5) holds while (6) does not hold and at ξ_2 (6) holds and (5) does not hold, or

Type II. at ξ_1 (6) holds while (5) does not hold and at ξ_2 (5) holds and (6) does not hold.

Theorem 46 *Let $\alpha \in (0,1)$ and f be α_{gH}-differentiable at a point $t > 0$. Then*

$$T^{\alpha}_{gH}(f)(t) = t^{1-\alpha} f'_{gH}(t).$$

Proof In Definition 41, let $h = \varepsilon t^{1-\alpha}$ and then $\varepsilon = t^{\alpha-1}h$. Hence

$$\begin{aligned} T^{\alpha}_{gH}(f)(t) &= \lim_{\varepsilon\to 0} \frac{f(t+\varepsilon t^{1-\alpha}) \ominus_{gH} f(t)}{\varepsilon} \\ &= \lim_{h\to 0} \frac{f(t+h) \ominus_{gH} f(t)}{h t^{\alpha-1}} \\ &= t^{1-\alpha} \lim_{h\to 0} \frac{f(t+h) \ominus_{gH} f(t)}{h} \\ &= t^{1-\alpha} f'_{gH}(t). \end{aligned}$$

So the desired result was obtained. □

Remark 47 *Using Theorem 46 and Definition 45 , it can be similarly easily shown that for $\beta \in (1,2)$*

$$T^{\beta}_{gH}(f)(t) = t^{2-\beta} f''_{gH}(t),$$

where f is gH-differentiable of second order.

Example 48 *Consider the fuzzy function $f : [0,\pi] \to \mathbb{E}$ defined by*

$$f(t) = (1.3\sin(t), 5.2\sin(t), 9.6\sin(t)).$$

We have the following α_{gH}-derivatives of $f(t)$

$$\begin{cases} T^{\frac{1}{2}}_{gH}(f)(t) = (1.3t^{\frac{1}{2}}\cos(t), 5.2t^{\frac{1}{2}}\cos(t), 9.6t^{\frac{1}{2}}\cos(t)) \; t \in [0, \frac{\pi}{2}], \\ \\ T^{\frac{1}{2}}_{gH}(f)(t) = (9.6t^{\frac{1}{2}}\cos(t), 5.2t^{\frac{1}{2}}\cos(t), 1.3t^{\frac{1}{2}}\cos(t)) \; t \in [\frac{\pi}{2}, \pi]. \end{cases}$$

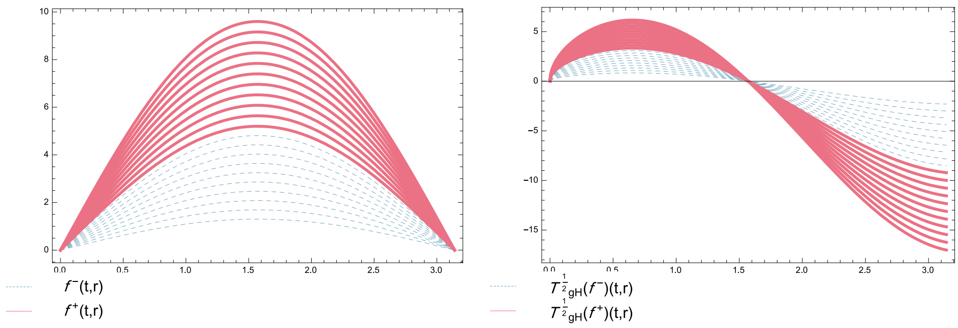

Fig. 1 r-cut representation of $f(t)$ (left) and its $T^{\frac{1}{2}}_{gH}(f)(t)$ (right) for $r \in [0,1]$ of Example 48

*Therefore, the fuzzy function $f(t)$ is $\alpha_{i.gH}$-differentiable function on $t \in [0, \frac{\pi}{2}]$. This function is switched to $\alpha_{ii.gH}$-differentiable at $t = \frac{\pi}{2}$. Hence, the point $t = \frac{\pi}{2}$ is a switching point of **Type I** for the the differentiability of f.*

Theorem 49 *Let $g : \mathbb{I} \to \zeta$ is real valued differentiable at t, and $f : \zeta \to \mathbb{E}$ be a fuzzy function such that f is gH-differentiable at the point $g(t)$ without any switching points, and $\alpha \in (0,1)$.*

- *Assume $f(t)$ is $[(i) - gH]$-differentiable at $g(t)$, then function $(fog)(t)$ is $\alpha_{i.gH}$-differentiable if*

$$T^{\alpha}_{i.gH}(fog)(t) = \begin{cases} t^{1-\alpha} g'(t) \odot f'_{i.gH}(g(t)), & \text{If } g'(t) > 0, \\ \ominus(-1) t^{1-\alpha} g'(t) \odot f'_{i.gH}(g(t)), & \text{If } g'(t) < 0. \end{cases}$$

- *Let $f(t)$ is $[(ii) - gH]$-differentiable at $g(t)$, then the function $(fog)(t)$ is $\alpha_{ii.gH}$-differentiable if*

$$T^{\alpha}_{ii.gH}(fog)(t) = \begin{cases} t^{1-\alpha} g'(t) \odot f'_{ii.gH}(g(t)), & \text{If } g'(t) > 0, \\ \ominus(-1) t^{1-\alpha} g'(x) \odot f'_{ii.gH}(g(x)), & \text{If } g'(t) < 0. \end{cases}$$

Proof First let $f(t)$ is $[(i) - gH]$-differentiable at $g(t)$. We have the following cases

i. If $g'(t) > 0$. Hence by attention to Theorem 46 we have

$$\begin{aligned} & t^{1-\alpha} g'(t) \odot \Big(f'_1(g(t)), f'_2(g(t)), f'_3(g(t)) \Big) \\ & \quad = \Big(t^{1-\alpha} g'(t) f'_1(g(t)), t^{1-\alpha} g'(t) f'_2(g(t)), t^{1-\alpha} g'(t) f'_3(g(t)) \Big) \\ & \quad = T^{\alpha}_{i.gH}(fog)(t). \end{aligned}$$

ii. If $g'(t) < 0$, then

$$\begin{aligned}
&\ominus(-1)t^{1-\alpha}g'(t)\odot\Big(f_1'(g(t)),f_2'(g(t)),f_3'(g(t))\Big)\\
&\quad=\ominus(-1)\Big(t^{1-\alpha}g'(t)f_3'(g(t)),t^{1-\alpha}g'(t)f_2'(g(t)),t^{1-\alpha}g'(t)f_1'(g(t))\Big)\\
&\quad=\ominus\Big(-t^{1-\alpha}g'(t)f_1'(g(t)),-t^{1-\alpha}g'(x)f_2'(g(t)),-t^{1-\alpha}g'(t)f_3'(g(t))\Big)\\
&\quad=\Big(t^{1-\alpha}g'(t)f_1'(g(t)),t^{1-\alpha}g'(t)f_2'(g(t)),t^{1-\alpha}g'(t)f_3'(g(t))\Big)\\
&\quad=T^{\alpha}_{i.gH}(fog)(t).
\end{aligned}$$

We can use the same procedure when $f(t)$ is $[(ii)-gH]$-differentiable at $g(t)$. □

5 Fuzzy Traveling Wave Solution of The Fuzzy Fractional Wave Equation

We want to find traveling wave fuzzy solution of the fuzzy one-dimensional homogeneous fractional wave equation. Consider this problem as follows

$$\begin{cases}\mathscr{D}^{2\alpha}_{tt_{gH}}\upsilon\ominus_{gH}\kappa^2\odot\upsilon_{xx_{gH}}=0, & (x,t)\in\mathbb{R}\times[0,\infty),\\[2ex] \upsilon(x,0)=f(x),\quad \mathscr{D}^{\alpha}_{t_{gH}}\upsilon(x,0)=g(x)\end{cases}\tag{7}$$

Where $\alpha\in(\frac{1}{2},1)$ and $\mathscr{D}^{\alpha}_{t_{gH}}$ is the generalized Huhuhara conformable fractional partial derivatives with respect to t and $\upsilon_{x_{gH}}$ is the generalized Hukuhara partial derivative with respect to x. Here, $f(x),g(x)$ are given continuous fuzzy functions. We will find the triangular analytical fuzzy solutions of Eq.(7) by using traveling wave method provided that the types of α_{gH}-differentiability of $\upsilon(x,t)$ with respect to t and $[gH-p]$-differentiability with respect to x are the same. By considering the type of α_{gH}-differentiability of $\upsilon(x,t)$ with respect to t, we have different cases as follow:

1-1-gH . Let $\upsilon(x,t)$ and $\mathscr{D}^{\alpha}_{t_{gH}}\upsilon$ are $\alpha_{i.gH}$-differentiable with respect to t, and $\upsilon(x,t)$ and $\upsilon_{x_{gH}}$ are $[i-p]$-differentiable with respect to x. Then $\mathscr{U}(\xi)$ is a $[(i)-gH]$-differentiable fuzzy. Here, we outline the main steps of traveling wave method. function

Step 1. Consider the fuzzy one-dimensional homogeneous fractional wave equation (7)

$$\upsilon(x,t)=\mathscr{U}(\xi),\quad\text{where}\quad \xi=x-\gamma\frac{t^{\alpha}}{\alpha}.$$

which can be analyzed through a change of variables $\upsilon(x,t)=\mathscr{U}(\xi)$. Here, $\mathscr{U}$ is a continuous function and gH-differentiable in ξ and γ is a positive real constant.

Step 2. We have

$$\frac{\partial\xi}{\partial t}=-\gamma t^{\alpha-1}<0,\qquad \frac{\partial\xi}{\partial x}=1>0,$$

therefore, by using the Theorem 49, the fuzzy multivariate chain rule [28], we have

$$\mathscr{D}^{\alpha}_{t_{i.gH}}\upsilon = t^{1-\alpha}\odot\frac{d_{i.gH}\mathscr{U}}{d\xi}\odot\frac{\partial\xi}{\partial t} = \ominus\gamma\odot\frac{d_{i.gH}\mathscr{U}}{d\xi}, \quad \Rightarrow \quad \mathscr{D}^{2\alpha}_{tt_{i.gH}}\upsilon = \gamma^2\odot\frac{d^2_{i.gH}\mathscr{U}}{d\xi^2},$$
$$\upsilon_{x_{i.gH}} = \frac{d_{i.gH}\mathscr{U}}{d\xi}\odot\frac{\partial\xi}{\partial x} = \frac{d_{i.gH}\mathscr{U}}{d\xi}, \quad \Rightarrow \quad \upsilon_{xx_{i.gH}} = \frac{d^2_{i.gH}\mathscr{U}}{d\xi^2}.$$

Hence the equation (7) is reduced to the following fuzzy ordinary differential equations of ξ

$$\gamma^2\frac{d^2_{i.gH}\mathscr{U}}{d\xi^2}\ominus_{gH}\kappa^2\frac{d^2_{i.gH}\mathscr{U}}{d\xi^2} = 0. \tag{8}$$

Step 3. To find fuzzy solutions for ordinary differential equations (8) and (17), we need some auxiliary boundary conditions. Which in this article, we consider the following auxiliary boundary conditions

$$\lim_{\xi\to\pm\infty}\mathscr{U}(\xi) = 0, \quad \lim_{\xi\to\pm\infty}\frac{d\mathscr{U}}{d\xi} = 0, \quad \lim_{\xi\to\pm\infty}\frac{d^2\mathscr{U}}{d\xi^2} = 0. \tag{9}$$

By using Proposition 34, Eq.(17) can also be written as follows

$$(\gamma^2-\kappa^2)\frac{d^2_{i.gH}\mathscr{U}}{d\xi^2} = 0 \tag{10}$$

One possibility is for $\frac{d^2_{i.gH}\mathscr{U}}{d\xi^2} = 0$. In which case we have

$$\mathscr{U}(\xi) = \mathscr{C}_1\oplus\mathscr{C}_2\xi \Rightarrow \upsilon(x,t) = \mathscr{C}_1\oplus\mathscr{C}_2(x-\gamma\frac{t^\alpha}{\alpha})$$

where $\mathscr{C}_1$ and $\mathscr{C}_2$ are fuzzy integral constants. But the boundary conditions (9) cannot be satisfied unless $\mathscr{C}_2 = 0$. Thus the only traveling solution is a fuzzy constant.
Another possibility is for $\gamma^2 = \kappa^2$. In this case

$$\upsilon(x,t) = \mathscr{U}(x-\kappa\frac{t^\alpha}{\alpha}), \quad \upsilon(x,t) = \mathscr{U}(x+\kappa\frac{t^\alpha}{\alpha}) \tag{11}$$

are traveling wave solutions of thefuzzy fractional wave equation and $\mathscr{U}$ can be any two gH-differentiable function. In general, it follows that any solution to the fuzzy fractional wave equation can be obtained as a superposition of two traveling waves,

$$\upsilon(x,t) = \mathscr{F}(x+\kappa\frac{t^\alpha}{\alpha})\oplus\mathscr{G}(x-\kappa\frac{t^\alpha}{\alpha}) \tag{12}$$

Since equation (12) is a fuzzy solution for equation (7), then it must apply to the initial conditions of the equation (7)

$$\upsilon(x,0) = f(x), \quad D^{\alpha}_{t_{gH}}\upsilon(x,0) = g(x). \tag{13}$$

Hence, the initial condition $\upsilon(x,0) = f(x)$ concludes

$$\mathscr{F}(x) \oplus \mathscr{G}(x) = f(x). \tag{14}$$

By considering Theorem 49 we have

$$D^{\alpha}_{t_{gH}} \upsilon(x,t) = \kappa \odot \mathscr{F}'_{i.gH}(x+\kappa\frac{t^{\alpha}}{\alpha}) \ominus \kappa \odot \mathscr{G}'_{i.gH}(x-\kappa\frac{t^{\alpha}}{\alpha}),$$

By the initial condition $D^{\alpha}_{t_{gH}} \upsilon(x,0) = g(x)$, we can write

$$\mathscr{F}'_{i.gH}(x) \ominus \mathscr{G}'_{i.gH}(x) = \frac{1}{\kappa} g(x)$$

After integration by using Lemma 37

$$\Big(\mathscr{F}(x) \ominus \mathscr{F}(0)\Big) \ominus \Big(\mathscr{G}(x) \ominus \mathscr{G}(0)\Big) = \frac{1}{\kappa}\int_0^x g(s)ds \Rightarrow \mathscr{F}(x) \ominus \mathscr{G}(x) = \Big(\mathscr{F}(0) \ominus \mathscr{G}(0)\Big) \oplus \frac{1}{\kappa}\int_0^x g(s)ds \tag{15}$$

The following system of equations is obtained by Eqs.(19) and (15)

$$\begin{cases} \mathscr{F}(x) \oplus \mathscr{G}(x) = f(x), \\ \mathscr{F}(x) \ominus \mathscr{G}(x) = \Big(\mathscr{F}(0) \ominus \mathscr{G}(0)\Big) \oplus \frac{1}{\kappa}\int_0^x g(s)ds, \end{cases}$$

such that this system of equations has the following fuzzy solutions

$$\mathscr{F}(x) = \frac{1}{2} f(x) \oplus \frac{1}{2}\Big(\mathscr{F}(0) \ominus \mathscr{G}(0)\Big) \oplus \frac{1}{2\kappa}\int_0^x g(s)ds,$$
$$\mathscr{G}(x) = \frac{1}{2} f(x) \ominus \frac{1}{2}\Big(\mathscr{F}(0) \ominus \mathscr{G}(0)\Big) \ominus \frac{1}{2\kappa}\int_0^x g(s)ds,$$

On the other hand, according to Lemma 38, $\mathscr{G}(x)$ can be rewritten as follows

$$\mathscr{G}(x) = \frac{1}{2} f(x) \ominus \frac{1}{2}\Big(\mathscr{F}(0) \ominus \mathscr{G}(0)\Big) \oplus \frac{1}{2\kappa}\int_x^0 g(s)ds$$

By substituting these equations for $\mathscr{F}$ and $\mathscr{G}$ into the general solution (12), the fuzzy traveling wave solution is obtained as follow

$$\upsilon(x,t) = \frac{1}{2}\Big(f(x+\kappa\frac{t^{\alpha}}{\alpha}) \oplus f(x-\kappa\frac{t^{\alpha}}{\alpha})\Big) \oplus \frac{1}{2\kappa}\int_{x-\kappa\frac{t^{\alpha}}{\alpha}}^{x+\kappa\frac{t^{\alpha}}{\alpha}} g(s)ds \tag{16}$$

Here, $\upsilon(x,t)$ and $\mathscr{D}^{\alpha}_{t_{gH}}\upsilon$ are $\alpha_{i.gH}$-differentiable with respect to t, and $\upsilon(x,t)$ and $\upsilon_{x_{gH}}$ are $[(i)-gH]$-differentiable with respect to x.

2-2-gH . Let $\upsilon(x,t)$ and $\mathscr{D}^{\alpha}_{t_{gH}}\upsilon$ are $\alpha_{ii.gH}$-differentiable with respect to t and $\upsilon(x,t)$ and $\upsilon_{x_{gH}}$ are $[(ii)-gH]$-differentiable with respect to x then $\mathscr{U}(\xi)$ is a $[(ii)-gH]$-differentiable fuzzy function. In this case, the main steps of the fuzzy traveling wave method are as follows

Step 1. Let we can analyzed he fuzzy one-dimensional homogeneous fractional wave equation (7) through the following change variables

$$\upsilon(x,t) = \mathscr{U}(\xi), \quad \text{where} \quad \xi = x - \gamma\frac{t^{\alpha}}{\alpha},$$

where $\mathscr{U}$ is a continuous function and gH-differentiable in ξ and γ is a positive real constant.

Step 2. We have

$$\frac{\partial\xi}{\partial t} = -\gamma t^{\alpha-1} < 0, \qquad \frac{\partial\xi}{\partial x} = 1 > 0,$$

therefore, by using the Theorem 49, the fuzzy multivariate chain rule [28], we have

$$\mathscr{D}^{\alpha}_{t_{ii.gH}}\upsilon = t^{1-\alpha} \odot \frac{d_{ii.gH}\mathscr{U}}{d\xi} \odot \frac{\partial\xi}{\partial t} = \ominus\gamma \odot \frac{d_{ii.gH}\mathscr{U}}{d\xi}, \quad \Rightarrow \quad \mathscr{D}^{2\alpha}_{tt_{ii.gH}}\upsilon = \gamma^2 \odot \frac{d^2_{ii.gH}\mathscr{U}}{d\xi^2}.$$

$$\upsilon_{x_{ii.gH}} = \frac{d_{ii.gH}\mathscr{U}}{d\xi} \odot \frac{\partial\xi}{\partial x} = \frac{d_{ii.gH}\mathscr{U}}{d\xi}, \quad \Rightarrow \quad \upsilon_{xx_{ii.gH}} = \frac{d^2_{ii.gH}\mathscr{U}}{d\xi^2}.$$

Hence the equation (7) is reduced to the following fuzzy ordinary differential equations of ξ

$$\gamma^2\frac{d^2_{ii.gH}\mathscr{U}}{d\xi^2} \ominus_{gH} \kappa^2\frac{d^2_{ii.gH}\mathscr{U}}{d\xi^2} = 0. \tag{17}$$

Step 3. As we explained in case **1-1-gH**, any solution of the fuzzy fractional wave equation can be obtained as follow

$$\upsilon(x,t) = \mathscr{F}(x + \kappa\frac{t^{\alpha}}{\alpha}) \oplus \mathscr{G}(x - \kappa\frac{t^{\alpha}}{\alpha}). \tag{18}$$

Equation (18) is a fuzzy solution for equation (7), then the initial condition $\upsilon(x,0) = f(x)$ yields

$$\mathscr{F}(x) \oplus \mathscr{G}(x) = f(x). \tag{19}$$

On the other hand, using Theorem 49 and the initial value $D^{\alpha}_{t_{gH}}\upsilon(x,0) = g(x)$, we have

$$\upsilon_{t_{ii.gH}}(x,t) = \kappa \odot \mathscr{F}'_{ii.gH}(x + \kappa\frac{t^{\alpha}}{\alpha}) \ominus \kappa \odot \mathscr{G}'_{ii.gH}(x - \kappa\frac{t^{\alpha}}{\alpha}), \quad \Rightarrow \quad \kappa\mathscr{F}'_{ii.gH}(x) \ominus \kappa\mathscr{G}'_{ii.gH}(x) = g(x).$$

Integrate each side of the above equation by using Lemma 37, therefore

$$\Big((-1)\mathscr{F}(0) \ominus (-1)\mathscr{F}(x)\Big) \ominus \Big((-1)\mathscr{G}(0) \ominus (-1)\mathscr{G}(x)\Big) = \frac{1}{\kappa}\int_0^x g(s)ds,$$

and

$$\mathscr{G}(x) \ominus \mathscr{F}(x) = \Big(\mathscr{G}(0) \ominus \mathscr{F}(0)\Big) \oplus \frac{(-1)}{\kappa}\int_0^x g(s)ds.$$

Consequently, we find that

$$\begin{cases} \mathscr{F}(x) \oplus \mathscr{G}(x) = f(x), \\ \mathscr{G}(x) \ominus \mathscr{F}(x) = \Big(\mathscr{G}(0) \ominus \mathscr{F}(0)\Big) \oplus \frac{(-1)}{\kappa} \int_0^x g(s)ds. \end{cases}$$

By solving this system and using Lemma 38, the following results are obtained

$$\begin{aligned} \mathscr{G}(x) &= \frac{1}{2} f(x) \oplus \frac{1}{2}\Big(\mathscr{G}(0) \ominus \mathscr{F}(0)\Big) \ominus \frac{(-1)}{2\kappa} \int_x^0 g(s)ds, \\ \mathscr{F}(x) &= \frac{1}{2} f(x) \ominus \frac{1}{2}\Big(\mathscr{G}(0) \ominus \mathscr{F}(0)\Big) \ominus \frac{(-1)}{2\kappa} \int_0^x g(s)ds. \end{aligned}$$

So the final solution of Eq. (7) is

$$\upsilon(x,t) = \frac{1}{2}\Big(f(x+\kappa\frac{t^\alpha}{\alpha}) \oplus f(x-\kappa\frac{t^\alpha}{\alpha})\Big) \ominus \frac{(-1)}{2\kappa} \int_{x-\kappa\frac{t^\alpha}{\alpha}}^{x+\kappa\frac{t^\alpha}{\alpha}} g(s)ds. \tag{20}$$

Where $\upsilon(x,t)$ and $\mathscr{D}^{\alpha}_{t_{gH}}\upsilon$ are $\alpha_{ii.gH}$-differentiable with respect to t, and $\upsilon(x,t)$ and $\upsilon_{x_{gH}}$ are $[(ii)-gH]$-differentiable with respect to x.

Example 51 *Consider the following fuzzy fractional wave equation*

$$\begin{cases} \mathscr{D}^{\frac{7}{4}}_{tt_{gH}}\upsilon \ominus_{gH} \upsilon_{xx_{gH}} = 0, & (x,t) \in \mathbb{R}\times(0,\infty), \\ \upsilon(x,0) = \Big(3.9x, 6.7x, 9.5x\Big), \quad \mathscr{D}^{\frac{7}{8}}_{t_{gH}}\upsilon(x,0) = \Big(3.9, 6.7, 9.5\Big). \end{cases}$$

To find a $1-1-gH$-differentiable solution for this problem, we use the equation (16)

$$\begin{aligned} \upsilon(x,t) &= \frac{1}{2}\Big(f(x+\kappa\frac{t^\alpha}{\alpha}) \oplus f(x-\kappa\frac{t^\alpha}{\alpha})\Big) \oplus \frac{1}{2\kappa}\int_{x-\kappa\frac{t^\alpha}{\alpha}}^{x+\kappa\frac{t^\alpha}{\alpha}} g(s)ds \\ &= \Bigg(\frac{3.9}{2}\Big((x+\frac{8}{7}t^{\frac{7}{8}})+(x-\frac{8}{7}t^{\frac{7}{8}})\Big), \frac{6.7}{2}\Big((x+\frac{8}{7}t^{\frac{7}{8}})+(x-\frac{8}{7}t^{\frac{7}{8}})\Big), \frac{9.5}{2}\Big((x+\frac{8}{7}t^{\frac{7}{8}})+(x-\frac{8}{7}t^{\frac{7}{8}})\Big)\Bigg) \\ &\quad \oplus \Bigg(\frac{1}{2}\int_{x-\frac{8}{7}t^{\frac{7}{8}}}^{x+\frac{8}{7}t^{\frac{7}{8}}} 3.9\mathrm{d}s, \frac{1}{2}\int_{x-\frac{8}{7}t^{\frac{7}{8}}}^{x+\frac{8}{7}t^{\frac{7}{8}}} 6.7\mathrm{d}s, \frac{1}{2}\int_{x-\frac{8}{7}t^{\frac{7}{8}}}^{x+\frac{8}{7}t^{\frac{7}{8}}} 9.5\mathrm{d}s\Bigg) \\ &= \Big(\frac{31.2}{7}t^{\frac{7}{8}} + 3.9x, \frac{53.6}{7}t^{\frac{7}{8}} + 6.7x, \frac{76}{7}t^{\frac{7}{8}} + 9.5x\Big) \end{aligned}$$

Example 52 *Consider the following fuzzy fractional wave equation*

$$\begin{cases} \mathscr{D}^{\frac{3}{2}}_{tt_{gH}}\upsilon \ominus_{gH} \upsilon_{xx_{gH}} = 0, & (x,t) \in \mathbb{R}\times(0,\infty), \\ \upsilon(x,0) = (2.1x^2, 5x^2, 7.9x^2), \quad \mathscr{D}^{\frac{3}{4}}_{t_{gH}}\upsilon(x,0) = 0. \end{cases}$$

We want to find a $1-1-gH$-differentiable solution for this problem. By equation (16) we have

$$
\begin{aligned}
\upsilon(x,t) &= \frac{1}{2}\Big(f(x+\frac{4}{3}t^{\frac{3}{4}})\oplus f(x-\frac{4}{3}t^{\frac{3}{4}})\Big)\\
&= \Big(1.05, 2.5, 3.95\Big)\Big(\Big(x+\frac{4}{3}t^{\frac{3}{4}}\Big)^2+\Big(x-\frac{4}{3}t^{\frac{3}{4}}\Big)^2\Big).
\end{aligned}
$$

We plot this solution in Figure 2.

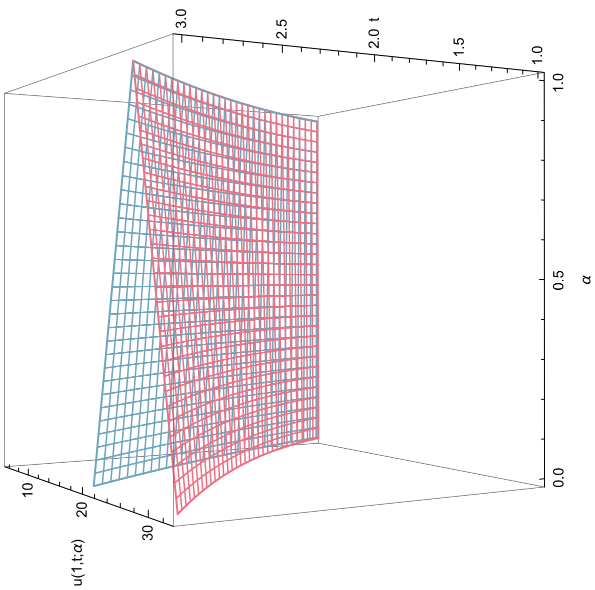

Fig. 2 Representation of $\upsilon(x,t)$ for all $r\in[0,1]$ of Example 52

Example 53 *Consider the following fuzzy fractional wave equation*

$$
\begin{cases}
\mathscr{D}^{\frac{14}{8}}_{tt_{gH}}\upsilon\ominus_{gH}\upsilon_{xx_{gH}}=0, & (x,t)\in\mathbb{R}\times(0,\infty),\\
\upsilon(x,0)=\Big(2e^{-x},3.6e^{-x},6.5e^{-x}\Big), \quad \mathscr{D}^{\frac{7}{8}}_{t_{gH}}\upsilon(x,0)=0
\end{cases}
$$

We want to find a $1-1-gH$-differentiable solution for this problem. By equation (16) we have

$$
\begin{aligned}
\upsilon(x,t) &= \frac{1}{2}\Big(f(x+\frac{8}{7}t^{\frac{7}{8}}))\oplus f(x-\frac{8}{7}t^{\frac{7}{8}})\Big)\\
&= \Big(e^{-\frac{8t^{\frac{7}{8}}}{7}-x}\Big(1+e^{\frac{16t^{\frac{7}{8}}}{7}}\Big), 1.8e^{-\frac{8t^{\frac{7}{8}}}{7}-x}\Big(1+e^{\frac{16t^{\frac{7}{8}}}{7}}\Big), 3.25e^{-\frac{8t^{\frac{7}{8}}}{7}-x}\Big(1+e^{\frac{16t^{\frac{7}{8}}}{7}}\Big)\Big)
\end{aligned}
$$

This solution is plotted in Figure 3.

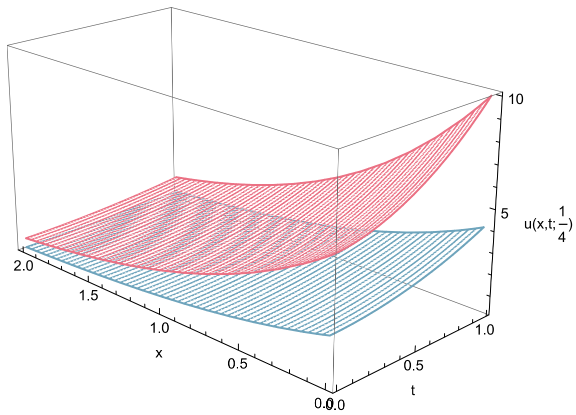

Fig. 3 Graph of $\upsilon(x,t)$ of Example 53

6 Fuzzy Traveling Wave Solution of The Fuzzy Fractional Diffusion Equation

Consider the following fuzzy fractional linear diffusion equation

$$\mathscr{D}^{\alpha}_{t_{gH}}\upsilon = \mathscr{K} \odot \upsilon_{xx_{gH}} \tag{21}$$

with the initial condition

$$\upsilon(x,0) = f(x), \tag{22}$$

where $f(x) \in \mathbb{E}$.

Step 1. To find a traveling wave solution for equation (21), consider

$$\upsilon(x,t) = \mathscr{U}(\xi), \quad \text{where} \quad \xi = x - \mathscr{K}\frac{t^{\alpha}}{\alpha},$$

where $\mathscr{U}$ is a continuous function and gH-differentiable in ξ.

Step 2. We have

$$\frac{\partial \xi}{\partial t} = -\mathscr{K} t^{\alpha-1} < 0, \qquad \frac{\partial \xi}{\partial x} = 1,$$

Let $\upsilon(x,t)$ is $\alpha_{i.gH}$-differentiable with respect to t, and $\upsilon(x,t)$ and $\upsilon_{x_{gH}}$ are $[(i)-p]$-differentiable with respect to x. Then $\mathscr{U}(\xi)$ is a $[(i)-gH]$-differentiable fuzzy function and

$$\mathscr{D}^{\alpha}_{t_{i.gH}}\upsilon = t^{1-\alpha} \odot \frac{d_{i.gH}\mathscr{U}}{d\xi} \odot \frac{\partial \xi}{\partial t} = \ominus\mathscr{K} \odot \frac{d_{i.gH}\mathscr{U}}{d\xi}.$$

$$\upsilon_{x_{i.gH}} = \frac{d_{i.gH}\mathscr{U}}{d\xi} \odot \frac{\partial \xi}{\partial x} = \frac{d_{i.gH}\mathscr{U}}{d\xi}, \quad \Rightarrow \quad \upsilon_{xx_{i.gH}} = \frac{d^2_{i.gH}\mathscr{U}}{d\xi^2}.$$

Hence the equation (21) is reduced to the following fuzzy ordinary differential equation of ξ

$$\frac{d^2_{i.gH}\mathscr{U}}{d\xi^2} \oplus \frac{d_{i.gH}\mathscr{U}}{d\xi} = 0. \tag{23}$$

Step 3. To find fuzzy solutions for ordinary differential equation (23), we need some auxiliary boundary conditions. Which in this article, we consider the following auxiliary boundary conditions

$$\lim_{\xi\to\pm\infty} \mathscr{U}(\xi) = 0, \quad \lim_{\xi\to\pm\infty} \frac{d\mathscr{U}}{d\xi} = 0, \quad \lim_{\xi\to\pm\infty} \frac{d^2\mathscr{U}}{d\xi^2} = 0. \tag{24}$$

We integrate both sides of Eq(23). According to the auxiliary boundary conditions expressed in Eq.(24), the integration constants are zero and

$$\frac{d_{i.gH}\mathscr{U}}{d\xi} \oplus \mathscr{U} = 0. \tag{25}$$

This equation has the following fuzzy solution [4]

$$U(\xi) = \mathscr{C}e^{-\xi},$$

which satisfies the condition $U(\xi) = 0$ when $\xi \to \infty$. Therefore

$$\upsilon(x,t) = \mathscr{C}e^{-(x-\mathscr{K}\frac{t^\alpha}{\alpha})}.$$

Using the initial condition (22), we can write

$$\mathscr{C} = f(x)e^x,$$

and finally the fuzzy solution for the fuzzy linear diffusion equation is equal

$$\upsilon(x,t) = f(x)e^{\mathscr{K}\frac{t^\alpha}{\alpha}}. \tag{26}$$

The other case of differentiability can be examined in a similar way.

Example 61 *Consider the following fuzzy fractional Diffusion equation*

$$\begin{cases} \mathscr{D}^{\frac{1}{3}}_{t_{gH}}\upsilon = \upsilon_{xx_{gH}}, \\ \upsilon(x,0) = \left(3e^x, 6.2e^x, 9.9e^x\right). \end{cases}$$

So using equation (26) *we have*

$$\upsilon(x,t) = \left(3e^{-x+3t^{\frac{1}{3}}}, 6.2e^{-x+3t^{\frac{1}{3}}}, 9.9e^{-x+3t^{\frac{1}{3}}}\right).$$

We plot this solution in Figure 4.

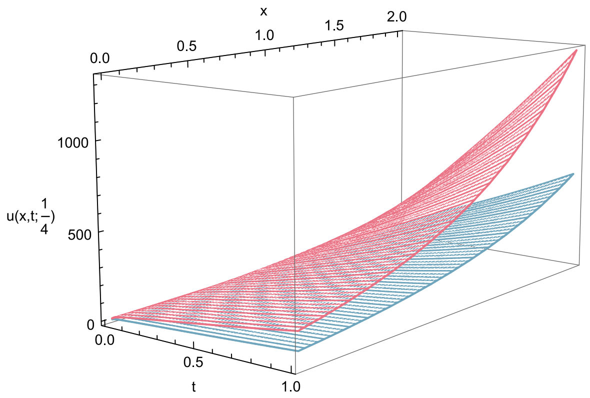

Fig. 4 Graph of $\upsilon(x,t)$ for $r = \frac{1}{4}$ of Example 61

7 Conclusion

In this paper, we have defined the generalized Hukuhara conformable fractional derivative and the type of differentiability of this derivative is studied in detail, and we have proved some novel properties for it. The fuzzy traveling wave solution of the fractional Wave equation and fractional Diffusion equation was obtained by considering the type of α_{gH}-differentiability. To demonstrate the efficiency of the method, the fuzzy traveling wave solutions of some examples were obtained. All results show that this method is a compelling and efficient method for obtaining an analytical solution for the fuzzy linear fractional partial differential equation.

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Compliance with Ethical Standards

Conflict of interest

The authors declare that they have no conflict of interest.

Ethical approval

This article does not contain any studies with human participants or animals performed by any of the authors.

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