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Derivations of equality algebras

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Abstract

In this paper, we introduced the concept of derivation on equality algebra E by using the notions of inner and outer derivations. Then we investigated some properties of (inner, outer) derivation and we introduced some suitable conditions that they help us to define a derivation on E. We introduced kernel and fixed point sets of derivation on E and prove that under which condition they are filters of E. Finally we prove that the equivalence relations on $(E, \rightsquigarrow, 1)$ coincide with the equivalence relations on E with derivation d.

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1 Introduction

Derivations are important topic to the theory of near ring and it is studied in [28, 19]. In [17] the authors applied the notions of near ring to BCI-algebras. Then other researchers started a new field of studying on the generalizations of derivations and application to many logical algebras. For instance, they introduced the concept of left-right (right-left) f-derivations, left derivations of BCI/BCK/BCC-algebras, (regular) (α, β)-derivations and (f, g)-derivations of BCI-algebras, then they investigated some fundamental properties of them and studied the relation among them (See [34, 3, 4, 13, 26, 25, 29, 6]).

An equality algebra, is introduced by Jenei in [15] and it continued in [10, 11, 14, 16, 32]. In the last years many mathematician studied on equality algebra and investigate this algebra in different topics. For example Zebardast et al. [33] studied the relation among equality algebra with other logical algebras, and in [11] they studied different kinds of filters in equality algebra and investigated the quotient structure that is made by them. Kologani and et al. in [1] introduced &-equality algebra and proved that it can be an EQ-algebra and in [2] they introduced right left mapping on equality and in this paper by using some properties of these mapping we introduce the notions of inner and outer derivations on equality algebra and some properties are investigated and we introduce the conditions that they help us to define a derivation on equality algebras. We introduce kernel and fixed point sets of derivation on E and prove that under what condition they are filters of E. Finally, we prove that the equivalence relations on $(E, \rightsquigarrow, 1)$ coincide with the equivalence relations on E with derivation d.

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2 Preliminaries

In this section, we recollect some definitions and results which will be used in the next sections.

Definition 2.1. [15] An algebraic structure $(E, \wedge, \sim, 1)$ is called an *equality algebra* if for any $r, s, t \in E$, it satisfies the following conditions.
$(E1)$ $(E, \wedge, 1)$ is a commutative idempotent integral monoid,
$(E2)$ The operation "$\sim$" is commutative,
$(E3)$ $r \sim r = 1$,
$(E4)$ $r \sim 1 = r$,
$(E5)$ If $r \leq s \leq t$, then $r \sim t \leq s \sim t$ and $r \sim t \leq r \sim s$,
$(E6)$ $r \sim s \leq (r \wedge t) \sim (s \wedge t)$,
$(E7)$ $r \sim s \leq (r \sim t) \sim (s \sim t)$,
where $r \leq s$ if and only if $r \wedge s = r$. The equality algebra $(E, \wedge, \sim, 1)$ is simply denoted by E for short.

In an equality algebra $(E, \wedge, \sim, 1)$, we define two operations "$\rightsquigarrow$" and "$\leftrightarrow$" on E as follows:

$$r \rightsquigarrow s := r \sim (r \wedge s) \text{ and } r \leftrightarrow s := (r \rightsquigarrow s) \wedge (s \rightsquigarrow r).$$

Proposition 2.2. [15] *Let $(E, \wedge, \sim, 1)$ be an equality algebra. Then, for all $r, s, t \in E$, the following assertions are valid:*
(i) $r \rightsquigarrow s = 1$ *if and only if* $r \leq s$,
(ii) $r \rightsquigarrow (s \rightsquigarrow t) = s \rightsquigarrow (r \rightsquigarrow t)$,
(iii) $1 \rightsquigarrow r = r,\ r \rightsquigarrow 1 = 1,\ r \rightsquigarrow r = 1$,
(iv) $r \leq s \rightsquigarrow t$ *if and only if* $s \leq r \rightsquigarrow t$,
(v) $r \leq s \rightsquigarrow r$,
(vi) $r \leq (r \rightsquigarrow s) \rightsquigarrow s$,
(vii) $r \rightsquigarrow s \leq (s \rightsquigarrow t) \rightsquigarrow (r \rightsquigarrow t)$,
$(viii)$ *If* $s \leq r$, *then* $r \leftrightarrow s = r \rightsquigarrow s = r \sim s$,
(ix) $r \sim s \leq r \leftrightarrow s \leq r \rightsquigarrow s$,
(x) *If* $r \leq s$, *then* $s \rightsquigarrow t \leq r \rightsquigarrow t$ *and* $t \rightsquigarrow r \leq t \rightsquigarrow s$,
(xi) $((r \rightsquigarrow s) \rightsquigarrow s) \rightsquigarrow s = r \rightsquigarrow s$.

An equality algebra E is said to be *bounded* if there exists an element $0 \in E$ such that $0 \leq r$ for all $r \in E$. In a bounded equality algebra E, we define the negation "$\neg$" on E by $\neg r = r \rightsquigarrow 0 = r \sim 0$ for all $r \in E$.

A subset F of E is called a *filter* of E (see [16]) if for any $r, s \in E$, it satisfies the following conditions:
(F_1) If $r \in F$ such that $r \leq s$, then $s \in F$.
(F_2) If $r \in F$ and $r \sim s \in F$, then $s \in F$.
Denote by $\mathcal{F}(E)$ the set of all filters of E.

Lemma 2.3. [14] *Let E be an equality algebra. A subset F of E is a filter of E if and only if $1 \in F$ and for any $r, s \in F$, if $r \in F$ and $r \rightsquigarrow s \in F$, then $s \in F$.*

Definition 2.4. [33] An equality algebra $(E, \wedge, \sim, 1)$ is said to be *commutative* if for any $r, s \in E$, $(r \rightsquigarrow s) \rightsquigarrow s = (s \rightsquigarrow r) \rightsquigarrow r$.

Definition 2.5. [2] An equality algebra E is called a *positive implicative equality algebra* if for any $r, s, t \in E$, $r \rightsquigarrow (s \rightsquigarrow t) = (r \rightsquigarrow s) \rightsquigarrow (r \rightsquigarrow t)$.

Let E and Y be equality algebras. A mapping $f : E \to Y$ is called a $\rightsquigarrow$*-homomorphism* if for any $r, s \in E$, $f(r \rightsquigarrow s) = f(r) \rightsquigarrow f(s)$.

Definition 2.6. [1] Given an equality algebra $(E, \wedge, \sim, 1)$ and $r, s \in E$, we define

$$E(r, s) := \{m \in E \mid r \leq s \rightsquigarrow m\}. \tag{2.1}$$

Clearly, 1, r and s are contained in $E(r,s)$.

Definition 2.7. [1] An equality algebra $(E,\wedge,\sim,1)$ is called an $\odot$-*equality algebra* if for all $r,s\in E$, the set $E(r,s)$ has the least element which is denoted by $r\odot s$.

Proposition 2.8. [1] *If $E=(E,\wedge,\sim,1)$ is an $\odot$-equality algebra, then for any $r,s\in E$, the following statements hold:*
(i) $r\odot s=s\odot r$,
(ii) $(r\odot s)\odot t=r\odot(s\odot t)$,
(iii) If $r\leq s$, then $r\odot t\leq s\odot t$.

Lemma 2.9. [1] *Let $E=(E,\wedge,\sim,1)$ be an equality algebra in which there exists a binary operation "$\odot$" such that for any $r,s,t\in E$, $r\rightsquigarrow(s\rightsquigarrow t)=(r\odot s)\rightsquigarrow t$. Then $\mathcal{E}=(E,\wedge,\sim,1)$ is an $\odot$-equality algebra.*

Note. From now on, we denote E as an equality algebra, unless otherwise specified.

3 Derivations of equality algebra

In this section, we define inner (outer) derivation on equality algebra and some related properties are investigated. We introduce kernel and fixed point sets of derivation on E and prove that under which condition they are filters of E. Also, we prove that any derivation is a modal on an equality algebra as a BE-algebra.

Notation. For any $r,s\in X$, we consider $[r;s^2]=(r\rightsquigarrow s)\rightsquigarrow s$.

Definition 3.1. A self mapping $d:E\to E$ is called an

(1) *inner derivation* on X if for any $r,s\in E$;

$$d(r\rightsquigarrow s)=[r\rightsquigarrow d(s);(d(r)\rightsquigarrow s)^2]. \tag{3.1}$$

(2) *outer derivation* on X if for $r,s\in E$;

$$d(r\rightsquigarrow s)=[d(r)\rightsquigarrow s;(r\rightsquigarrow d(s))^2]. \tag{3.2}$$

(3) *derivation* on E if d is an inner and outer derivation on E.

The sets of all outer derivations, inner derivations and derivations on E is denoted by $\mathcal{OD}(E)$, $\mathcal{ID}(E)$ and $\mathcal{D}(E)$, respectively.

Example 3.2. (i) Obviously, $d:=id_E\in\mathcal{D}(E)$.
(ii) Let $E=\{0,m,n,1\}$ be a chain such that $0\leq m\leq n\leq 1$. Define the operation $\sim$ on E as follows:

$\sim$	0	m	n	1
0	1	0	0	0
m	0	1	m	m
n	0	m	1	n
1	0	m	n	1

Then $\mathcal{E}=(E,\wedge,\sim,1)$ is a bounded equality algebra, and the implication $(\rightsquigarrow)$ is given by,

$\rightsquigarrow$	0	m	n	1
0	1	1	1	1
m	0	1	1	1
n	0	m	1	1
1	0	m	n	1

− Define the map $d : E \to E$ such that $d(0) = 0$ and $d(m) = d(n) = d(1) = 1$. Then $d \in \mathcal{D}(E)$.
− Define $d(0) = 0$, $d(n) = n$ and $d(m) = d(1) = 1$, then $d \in \mathcal{OD}(E)$ which is not an inner derivation on E because

$$n = d(n) = d(1 \rightsquigarrow n) \neq [(1 \rightsquigarrow d(n)) \rightsquigarrow (d(n) \rightsquigarrow 1)] \rightsquigarrow (d(n) \rightsquigarrow 1) = 1.$$

− Define $d(0) = 0$, $d(m) = n$ and $d(n) = d(1) = 1$, then $d \in \mathcal{ID}(E)$ which is not an outer derivation on E because

$$n = d(m) = d(n \rightsquigarrow m) \neq [(d(n) \rightsquigarrow m) \rightsquigarrow (n \rightsquigarrow d(m))] \rightsquigarrow (n \rightsquigarrow d(m)) = 1.$$

Remark 3.3. Let E be a commutative equality algebra. Then every outer derivation on E is an inner derivation on E and vice versa. Suppose d is an outer derivation on E. Then for any $r, s \in E$, we have

$$\begin{aligned} d(r \rightsquigarrow s) &= [d(r) \rightsquigarrow s; (r \rightsquigarrow d(s))^2] \\ &= [(d(r) \rightsquigarrow s) \rightsquigarrow (r \rightsquigarrow d(s))] \rightsquigarrow (r \rightsquigarrow d(s)) \\ &= [(r \rightsquigarrow d(s)) \rightsquigarrow (d(r) \rightsquigarrow s)] \rightsquigarrow (d(r) \rightsquigarrow s) \\ &= [r \rightsquigarrow d(s); (d(r) \rightsquigarrow s)^2]. \end{aligned}$$

Proposition 3.4. *Let $d \in \mathcal{OD}(E)$, then for any $r, s \in E$, the following statements hold:*

(i) $d(1) = 1$,

(ii) $d(r) = (r \rightsquigarrow d(r)) \rightsquigarrow d(r)$,

(iii) $r \leq d(r)$,

(iv) $d(r \rightsquigarrow d(r)) = 1$,

(v) *If E is a bounded equality algebra and $d(0) = 0$, then $d(\neg r) = \neg r$, where $\neg r = r \rightsquigarrow 0$.*

Proof. (i) For any $r \in E$, since $d \in \mathcal{OD}(E)$, then

$$\begin{aligned} d(1) = d(r \rightsquigarrow 1) = [d(r) \rightsquigarrow 1; (r \rightsquigarrow d(1))^2] &= [(d(r) \rightsquigarrow 1) \rightsquigarrow (r \rightsquigarrow d(1))] \rightsquigarrow (r \rightsquigarrow d(1)) \\ &= (r \rightsquigarrow d(1)) \rightsquigarrow (r \rightsquigarrow d(1)) = 1. \end{aligned}$$

(ii) For any $r \in E$, since $d \in \mathcal{OD}(E)$, by (i) we have

$$d(r) = d(1 \rightsquigarrow r) = [d(1) \rightsquigarrow r; (1 \rightsquigarrow d(r))^2] = [r; d(r)^2] = (r \rightsquigarrow d(r)) \rightsquigarrow d(r).$$

(iii) For any $r \in E$, we show that $r \rightsquigarrow d(r) = 1$. By Proposition 2.2(ii) and (ii),

$$r \rightsquigarrow d(r) = r \rightsquigarrow ((r \rightsquigarrow d(r)) \rightsquigarrow d(r)) = (r \rightsquigarrow d(r)) \rightsquigarrow (r \rightsquigarrow d(r)) = 1,$$

and so $r \leq d(r)$.

(iv) Let $r \in E$. Since $d \in \mathcal{OD}(E)$, we have

$$d(r \rightsquigarrow d(r)) = ((d(r) \rightsquigarrow d(r)) \rightsquigarrow (r \rightsquigarrow d(d(r)))) \rightsquigarrow (r \rightsquigarrow d(d(r))) = (r \rightsquigarrow d(d(r))) \rightsquigarrow (r \rightsquigarrow d(d(r))) = 1.$$

By (iii) and Proposition 2.2(xi), the proof of (v) is clear. ∎

Proposition 3.5. *Let $d \in \mathcal{ID}(E)$, then for any $r, s \in E$, the following statements hold:*

(i) $d(1) = 1$,

(ii) $d(r) = (d(r) \rightsquigarrow r) \rightsquigarrow r$,

(iii) $r \leq d(r)$,

(iv) *If* $r \leq s$, *then* $r \leq d(s)$,

(v) $d(r \rightsquigarrow d(r)) = 1$,

(vi) *If* E *is a bounded equality algebra, then* $d(0) = \neg\neg(d(0))$.

Proof. (i) For any $r \in E$, since $d \in \mathcal{ID}(E)$, then

$$d(1) = d(r \rightsquigarrow 1) = [r \rightsquigarrow d(1); (d(r) \rightsquigarrow 1)^2] = [(r \rightsquigarrow d(1)) \rightsquigarrow (d(r) \rightsquigarrow 1)] \rightsquigarrow (d(r) \rightsquigarrow 1) = 1.$$

(ii) For any $r \in E$, since $d \in \mathcal{ID}(E)$, then by (i) we have

$$d(r) = d(1 \rightsquigarrow r) = [1 \rightsquigarrow d(r); (d(1) \rightsquigarrow r)^2] = [d(r); r^2] = (d(r) \rightsquigarrow r) \rightsquigarrow r.$$

(iii) For any $r \in E$, we show that $r \rightsquigarrow d(r) = 1$. By Proposition 2.2(ii) and (ii),

$$r \rightsquigarrow d(r) = r \rightsquigarrow ((d(r) \rightsquigarrow r) \rightsquigarrow r) = (d(r) \rightsquigarrow r) \rightsquigarrow (r \rightsquigarrow r) = 1,$$

and so $r \leq d(r)$.

(iv) Let $r, s \in E$ such that $r \leq s$. Since $d \in \mathcal{ID}(E)$, we get

$$r \rightsquigarrow d(s) = r \rightsquigarrow ((d(s) \rightsquigarrow s) \rightsquigarrow s) = (d(s) \rightsquigarrow s) \rightsquigarrow (r \rightsquigarrow s),$$

since $r \rightsquigarrow s = 1$, we have $r \leq d(s)$.

(v) Since $d \in \mathcal{ID}(E)$, by (iii) and (i), the proof is clear.

By (ii) the proof of (v) is clear. □

Remark 3.6. *(i)* Let $d \in \mathcal{D}(E)$. Then $d(1) = 1$ if and only if $r \leq d(r)$, for any $r \in E$.
(ii) Let $d \in \mathcal{D}(E)$. Then for any $r, s \in E$, clearly $s \leq d(r \rightsquigarrow s)$.

Proposition 3.7. *Let* $d \in \mathcal{OD}(E)$. *Then for any* $r, s \in E$, $d(r \rightsquigarrow s) = r \rightsquigarrow d(s)$.

Proof. Let $d \in \mathcal{OD}(E)$ and $r, s \in E$. Then by definition, we have $d(r \rightsquigarrow s) = [(d(r) \rightsquigarrow s) \rightsquigarrow (r \rightsquigarrow d(s))] \rightsquigarrow (r \rightsquigarrow d(s))$. By Proposition 3.4(iii), we have $r \leq d(r)$ and $s \leq d(s)$. By Proposition 2.2(x), we have $d(r) \rightsquigarrow s \leq r \rightsquigarrow s$ and $r \rightsquigarrow s \leq r \rightsquigarrow d(s)$, then $d(r) \rightsquigarrow s \leq r \rightsquigarrow d(s)$. Thus $(d(r) \rightsquigarrow s) \rightsquigarrow (r \rightsquigarrow d(s)) = 1$, and so

$$d(r \rightsquigarrow s) = [(d(r) \rightsquigarrow s) \rightsquigarrow (r \rightsquigarrow d(s))] \rightsquigarrow (r \rightsquigarrow d(s)) = 1 \rightsquigarrow (r \rightsquigarrow d(s)) = r \rightsquigarrow d(s).$$

Hence, $d(r \rightsquigarrow s) = r \rightsquigarrow d(s)$. □

In the next example we show that the converse of Proposition 3.7 does not hold, in general.

Example 3.8. Let E be the equality algebra as in Example 3.2(ii). Define a self mapping $d : E \to E$ such that $d(0) = 0$, $d(1) = 1$ and $d(m) = d(n) = n$. Then for any $r, s \in E$, $d(r \rightsquigarrow s) = r \rightsquigarrow d(s)$, but $d \notin \mathcal{OD}(E)$. Because

$$n = d(m) = d(n \rightsquigarrow m) \neq [(d(n) \rightsquigarrow m) \rightsquigarrow (n \rightsquigarrow d(m))] \rightsquigarrow (n \rightsquigarrow d(m)) = [(n \rightsquigarrow m) \rightsquigarrow (n \rightsquigarrow n)] \rightsquigarrow (n \rightsquigarrow n) = 1.$$

Proposition 3.9. *Let* d *be a self mapping on* E. *Then* $d \in \mathcal{OD}(E)$ *if and only if for any* $r, s \in E$, $d(r \rightsquigarrow s) = r \rightsquigarrow d(s)$ *and* $r \leq d(r)$.

Proof. Let d be a self mapping on E and let d be in $\mathcal{OD}(E)$. By Propositions 3.7 and 3.4(iii), the proof is clear.
Conversely, since for any $r \in E$, $r \leq d(r)$, by Proposition 2.2(x), we have $d(r) \rightsquigarrow s \leq d(r) \rightsquigarrow d(s)$ and $d(r) \rightsquigarrow d(s) \leq r \rightsquigarrow d(s)$. Thus $d(r) \rightsquigarrow s \leq r \rightsquigarrow d(s)$. Hence

$$d(r \rightsquigarrow s) = r \rightsquigarrow d(s) = 1 \rightsquigarrow (r \rightsquigarrow d(s)) = [(d(r) \rightsquigarrow s) \rightsquigarrow (r \rightsquigarrow d(s))] \rightsquigarrow (r \rightsquigarrow d(s)).$$

Therefore, $d \in \mathcal{OD}(E)$. □

Remark 3.10. By Proposition 3.7, if $d \in \mathcal{OD}(E)$ such that for any $r, s \in E$, $r \leq s$, then $1 = d(1) = d(r \rightsquigarrow s) = r \rightsquigarrow d(s)$, and so $r \leq d(s)$.

Proposition 3.11. *Let $d \in \mathcal{OD}(E)$. Then for any $r, s \in E$, the following statements hold:*
(i) $d(\neg r) = r \rightsquigarrow d(0)$,
(ii) $d(0) \leq d(\neg r)$,
(iii) If $d(0) = 0$, *then* $d(\neg r) = \neg r$,
(iv) $d^n(r \rightsquigarrow s) = r \rightsquigarrow d^n(s)$, *for any* $n \in \mathbb{N}$,
(v) $d(r) \rightsquigarrow d(s) \leq d(r \rightsquigarrow s)$,
(vi) $d(r) \rightsquigarrow s \leq r \rightsquigarrow d(s)$.

Proof. The proof is straightforward. □

Proposition 3.12. *Let $d_1, d_2 \in \mathcal{OD}(E)$. Then $d_1 \circ d_2 \in \mathcal{OD}(E)$.*

Proof. Since $d_1, d_2 \in \mathcal{OD}(E)$, for any $r \in E$, $r \leq d_1(r)$ and $r \leq d_2(r)$. Then by Remark 3.10, since $r \leq d_2(r)$ and $d_1 \in \mathcal{OD}(E)$, $r \leq d_1(d_2(r)) = (d_1 \circ d_2)(r)$. By Proposition 3.7, for any $r, s \in E$, we have $d_1(r \rightsquigarrow s) = r \rightsquigarrow d_1(s)$ and $d_2(r \rightsquigarrow s) = r \rightsquigarrow d_2(s)$. Thus

$$(d_1 \circ d_2)(r \rightsquigarrow s) = d_1(d_2(r \rightsquigarrow s)) = d_1(r \rightsquigarrow d_2(s)) = r \rightsquigarrow d_1(d_2(s)) = r \rightsquigarrow (d_1 \circ d_2)(s).$$

Therefore, by Proposition 3.9, $d_1 \circ d_2 \in \mathcal{OD}(E)$. □

Proposition 3.13. *Let E be a commutative equality algebra and $d_1, d_2 \in \mathcal{OD}(E)$ such that $d_1(0) = d_2(0) = 0$ and $d_1 \circ d_2 = 0$. Then $d_1 = 0$ or $d_2 = 0$.*

Proof. By Proposition 3.12, $d_1 \circ d_2 \in \mathcal{OD}(E)$. Suppose $d_2(r) \neq 0$ for any $r \in E - \{0\}$. Since E is commutative, $d_1(0) = d_2(0) = 0$ and $d_1 \circ d_2 = 0$, by Remark 3.10, we have

$$\begin{aligned}
d_1(r) &= d_1(1 \rightsquigarrow r) \\
&= d_1((0 \rightsquigarrow r) \rightsquigarrow r) \\
&= d_1([(d_1(d_2(r))) \rightsquigarrow r] \rightsquigarrow r) \\
&= d_1(\underbrace{[r \rightsquigarrow (d_1(d_2(r)))]}_{1} \rightsquigarrow (d_1(d_2(r)))) \quad \text{by commutativity} \\
&= d_1(d_1(d_2(r))) \\
&= d_1(0) \\
&= 0.
\end{aligned}$$

Then $d_1 = 0$. Similarly, we can prove that $d_2 = 0$. □

Remark 3.14. Clearly, the set of all outer derivations on E with $\circ$ operation, i.e., $(OD(E), \circ, id_E)$ is a semigroup.

Proposition 3.15. *Let $m \in E$. Define $d_m : E \to E$ is a self mapping on E such that for any $r \in E$, $d_m(r) = m \rightsquigarrow r$, then $d_m \in \mathcal{OD}(E)$.*

Proof. Let $r, s, m \in E$. By Proposition 2.2(v), $r \leq m \rightsquigarrow r$, and by Proposition 2.2(x),

$$(m \rightsquigarrow r) \rightsquigarrow s \leq r \rightsquigarrow s \leq m \rightsquigarrow (r \rightsquigarrow s) = r \rightsquigarrow (m \rightsquigarrow s).$$

Then by Propositon 3.7, we have

$$\begin{aligned}
d_m(r \rightsquigarrow s) = m \rightsquigarrow (r \rightsquigarrow s) = r \rightsquigarrow (m \rightsquigarrow s) &= 1 \rightsquigarrow [r \rightsquigarrow (m \rightsquigarrow s)] \\
&= [((m \rightsquigarrow r) \rightsquigarrow s) \rightsquigarrow (r \rightsquigarrow (m \rightsquigarrow s))] \rightsquigarrow (r \rightsquigarrow (m \rightsquigarrow s)) \\
&= [(d_m(r) \rightsquigarrow s) \rightsquigarrow (r \rightsquigarrow d_m(s))] \rightsquigarrow (r \rightsquigarrow d_m(s)).
\end{aligned}$$

Therefore, $d_m \in \mathcal{OD}(E)$. □

Proposition 3.16. *Let E be a positive implicative equality algebra and $m \in E$. Define $d_m : E \to E$ is a self mapping on E such that for any $r \in E$, $d_m(r) = m \rightsquigarrow r$. Then the following statements hold:*
(i) $d_m \in \mathcal{OD}(E)$.
(ii) d_m is an order preserving map.
(iii) d_m is an $\rightsquigarrow$-endomorphism on E.
(iv) d_m is an idempotent map.
(v) $d_0(r) = 1$ and $d_1(r) = r$.

Proof. (i) By Proposition 3.15, the proof is clear.
(ii) Let $r, s \in E$ such that $r \leq s$. Then by Proposition 2.2(x), $m \rightsquigarrow r \leq m \rightsquigarrow s$, and so $d_m(r) \leq d_m(s)$. Hence, d_m is an order preserving map.
(iii) Let $r, s \in E$. Since E is a positive implicative equality algebra, we have

$$d_m(r \rightsquigarrow s) = m \rightsquigarrow (r \rightsquigarrow s) = (m \rightsquigarrow r) \rightsquigarrow (m \rightsquigarrow s) = d_m(r) \rightsquigarrow d_m(s).$$

Hence, $d_m(r \rightsquigarrow s) = d_m(r) \rightsquigarrow d_m(s)$.
(iv) Let $r \in E$. Then

$$d_m(d_m(r)) = d_m(m \rightsquigarrow r) = m \rightsquigarrow (m \rightsquigarrow r) = (m \rightsquigarrow m) \rightsquigarrow (m \rightsquigarrow r) = m \rightsquigarrow r = d_m(r).$$

(v) The proof is clear. □

Theorem 3.17. *Let E be a positive implicative $\odot$-equality algebra. Then $(d_E, \circ, d_1)$ is a commutative semigroup, where $d_E = \{d_m \mid d_m \in \mathcal{OD}(E) \text{ and } m \in E\}$.*

Proof. Since E is a positive implicative equality algebra, by Proposition 3.16(i), for any $m \in E$, $d_m \in \mathcal{OD}(E)$. Let $m, n \in E$ such that $d_m, d_n \in d_E$. Since E is an $\odot$-equality algebra, by Lemma 2.9, for any $r \in E$,

$$d_m \circ d_n(r) = d_m(n \rightsquigarrow r) = m \rightsquigarrow (n \rightsquigarrow r) = (m \odot n) \rightsquigarrow r = d_{m \odot n}(r).$$

Since $m \odot n \in E$, by Proposition 3.16(i), obviously, $d_{m \odot n} \in d_E$, and so $(d_E, \circ)$ is closed. Moreover, by Proposition 2.8(i) and (ii), it is easy to see that $(d_E, \circ)$ is commutative and associative. Also, $d_m \circ d_1(r) = m \rightsquigarrow (1 \rightsquigarrow r) = m \rightsquigarrow r = d_m(r)$, for any $r, m \in E$. Hence, d_1 is an identity element of $(d_E, \circ)$. Therefore, $(d_E, \circ, d_1)$ is a commutative semigroup. □

Proposition 3.18. *Mapping $\xi : (E, \rightsquigarrow) \to (d_E, \circ)$ which is defined by $\xi(r) = d_r$, for any $r \in E$, is one-to-one.*

Proof. Let $r, s \in E$ such that $\xi(r) = \xi(s)$ if and only if $d_r = d_s$ if and only if for any $m \in E$, $d_r(m) = d_s(m)$, and so $r \rightsquigarrow m = s \rightsquigarrow m$. If $r = m$, then $1 = r \rightsquigarrow r = s \rightsquigarrow r$. Thus $s \leq r$. If $s = m$, then $r \rightsquigarrow s = s \rightsquigarrow s$ and so $r \rightsquigarrow s = 1$. Thus $r \leq s$. Hence $r = s$. Therefore, ξ is one-to-one. □

By routine calculation, clearly, ξ is not an $\rightsquigarrow$-endomorphism. In the next proposition we investigate that under what condition ξ is an $\rightsquigarrow$-endomorphism.

Proposition 3.19. *Let E be an equality algebra such that for any $r, s, t \in E$,*

$$(r \rightsquigarrow s) \rightsquigarrow t = (r \rightsquigarrow t) \rightsquigarrow (s \rightsquigarrow t).$$

Then the mapping $\xi : (E, \rightsquigarrow) \to (d_E, \circ)$ which is defined by $\xi(r) = d_r$, for any $r \in E$, is a one-to-one $\rightsquigarrow$-endomorphism.

Proof. Let $r, s, t \in E$. Then by assumption we have

$$\xi(r \rightsquigarrow s)(t) = d_{r \rightsquigarrow s}(t) = (r \rightsquigarrow s) \rightsquigarrow t = (r \rightsquigarrow t) \rightsquigarrow (s \rightsquigarrow t) = d_r(t) \rightsquigarrow d_s(t) = (\xi(r) \rightsquigarrow \xi(s))(t).$$

Thus ξ is an $\rightsquigarrow$-endomorphism and by Proposition 3.18 , ξ is one-to-one. □

Definition 3.20. [9] A *BE-algebra* is an algebra $(A, \rightsquigarrow, 1)$ of the type $(2, 0)$ such that for all $r, s, t \in A$ the following axioms are fulfilled:
$(BE1)$ $r \rightsquigarrow r = 1$,
$(BE2)$ $r \rightsquigarrow 1 = 1$,
$(BE3)$ $1 \rightsquigarrow r = r$,
$(BE4)$ $r \rightsquigarrow (s \rightsquigarrow t) = s \rightsquigarrow (r \rightsquigarrow t)$.

Definition 3.21. [12] Let A be a BE-algebra. A mapping $f : A \to A$ is called a *modal operator on* A if for all $r, s \in A$, it satisfies the following conditions:
$(M1)$ $r \leq f(r)$,
$(M2)$ $f(f(r)) = f(r)$,
$(M3)$ $f(r \rightsquigarrow s) \leq f(r) \rightsquigarrow f(s)$.
The pair (A, f) is called a *modal BE-algebra*.

Corollary 3.22. *Let E be a positive implicative equality algebra and $m \in E$. If $d_m : E \to E$ is a self mapping defined in Proposition 3.16, then d_m is a modal operator on E as a BE-algebra.*

Proof. By Proposition 2.2(ii) and (iii), it is clear that $(E, \rightsquigarrow, 1)$ is a BE-algebra. Then by Proposition 3.4(iii), we have $r \leq d(r)$ and so (M1) holds. Also, by Proposition 3.16(iv) and (iii), since E is a positive implicative equality algebra, (M2) and (M3) hold. Therefore, by Definition 3.21, d_m is a modal operator on E as a BE-algebra. □

Let $d : E \to E$ be a self mapping on E. Define $\ker(d) = \{r \in E \mid d(r) = 1\}$. If $d \in \mathcal{D}(E)$, then by Propositions 3.4(i), $d(1) = 1$, and so $\ker(d) \neq \emptyset$.

Example 3.23. Let E be the equality algebra as Example 3.2(ii). Define a self mapping d on E by $d(0) = 0$ and $d(m) = d(n) = d(1) = 1$. Obviously, $d \in \mathcal{D}(E)$ such that $\ker(d) = \{m, n, 1\}$.

Proposition 3.24. *Let $d \in \mathcal{D}(E)$. Then the following statements hold:*
(i) $\ker(d)$ *is closed under the operation* $\rightsquigarrow$.
(ii) d *is an idempotent map on* $\ker(d)$.
(iii) d *is an order preserving map on* $\ker(d)$.
(iv) *If $r \in \ker(d)$, then for any $s \in E$, $s \rightsquigarrow r \in \ker(d)$.*

Proof. The proof is straightforward. □

Theorem 3.25. *Let E be an equality algebra such that for any $r, s, t \in E$, $(r \rightsquigarrow s) \rightsquigarrow t = (s \rightsquigarrow t) \rightsquigarrow (r \rightsquigarrow t)$. Suppose $d_m : E \to E$, for $m \in E$ is a self mapping on E such that $d_m(r) = (r \rightsquigarrow m) \rightsquigarrow m$. Then the following statemants hold:*
(i) d_m *is an order preserving map.*
(ii) $d_m \in \mathcal{OD}(E)$.
(iii) d_m *is a* $\rightsquigarrow$*-endomorphism on* E.
(iv) d *is idempotent.*
(v) $\ker(d_m) \in \mathcal{F}(E)$.

Proof. *(i)* Let $r, s \in E$ such that $r \leq s$. By twice using of Proposition 2.2(x), we have $d(r) = (r \rightsquigarrow m) \rightsquigarrow m \leq (s \rightsquigarrow m) \rightsquigarrow m = d(s)$. Hence, d_m is an order preserving map.
(ii) Let $r, s \in E$. Then by Proposition 2.2(vi), $r \leq (r \rightsquigarrow m) \rightsquigarrow m$ and $s \leq (s \rightsquigarrow m) \rightsquigarrow m$. Thus by Proposition 2.2(x), $((r \rightsquigarrow m) \rightsquigarrow m) \rightsquigarrow s \leq ((r \rightsquigarrow m) \rightsquigarrow m) \rightsquigarrow ((s \rightsquigarrow m) \rightsquigarrow m)$ and $((r \rightsquigarrow m) \rightsquigarrow m) \rightsquigarrow ((s \rightsquigarrow m) \rightsquigarrow m) \leq r \rightsquigarrow ((s \rightsquigarrow m) \rightsquigarrow m)$ and so $((r \rightsquigarrow m) \rightsquigarrow m) \rightsquigarrow s \leq r \rightsquigarrow ((s \rightsquigarrow m) \rightsquigarrow m)$. Then by

assumption and Proposition 2.2(xi), we get

$$\begin{aligned}
d(r \rightsquigarrow s) &= ((r \rightsquigarrow s) \rightsquigarrow m) \rightsquigarrow m = [(s \rightsquigarrow m) \rightsquigarrow (r \rightsquigarrow m)] \rightsquigarrow m \\
&= ((r \rightsquigarrow m) \rightsquigarrow m) \rightsquigarrow ((s \rightsquigarrow m) \rightsquigarrow m) = (s \rightsquigarrow m) \rightsquigarrow (((r \rightsquigarrow m) \rightsquigarrow m) \rightsquigarrow m) \\
&= (s \rightsquigarrow m) \rightsquigarrow (r \rightsquigarrow m) = r \rightsquigarrow ((s \rightsquigarrow m) \rightsquigarrow m) \\
&= 1 \rightsquigarrow (r \rightsquigarrow ((s \rightsquigarrow m) \rightsquigarrow m)) \\
&= [(((r \rightsquigarrow m) \rightsquigarrow m) \rightsquigarrow s) \rightsquigarrow (r \rightsquigarrow ((s \rightsquigarrow m) \rightsquigarrow m))] \rightsquigarrow (r \rightsquigarrow ((s \rightsquigarrow m) \rightsquigarrow m)) \\
&= [(d(r) \rightsquigarrow s) \rightsquigarrow (r \rightsquigarrow d(s))] \rightsquigarrow (r \rightsquigarrow d(s)).
\end{aligned}$$

Therefore, $d \in \mathcal{OD}(E)$.
(iii) Let $r, s \in E$. Since by (ii), $d \in \mathcal{OD}(E)$, by Proposition 3.7, $d(r \rightsquigarrow s) = r \rightsquigarrow d(s)$. Then by Proposition 2.2(xi) we have

$$\begin{aligned}
d(r) \rightsquigarrow d(s) &= ((r \rightsquigarrow m) \rightsquigarrow m) \rightsquigarrow ((s \rightsquigarrow m) \rightsquigarrow m) = (s \rightsquigarrow m) \rightsquigarrow (((r \rightsquigarrow m) \rightsquigarrow m) \rightsquigarrow m) = (s \rightsquigarrow m) \rightsquigarrow (r \rightsquigarrow m) \\
&= r \rightsquigarrow ((s \rightsquigarrow m) \rightsquigarrow m) = r \rightsquigarrow d(s) = d(r \rightsquigarrow s).
\end{aligned}$$

Hence d is an $\rightsquigarrow$-endomorphism on E.
(iv) Let $r \in E$. Then by Proposition 2.2(xi),

$$d(r) = (r \rightsquigarrow m) \rightsquigarrow m = [((r \rightsquigarrow m) \rightsquigarrow m) \rightsquigarrow m] \rightsquigarrow m = (d(r) \rightsquigarrow m) \rightsquigarrow m = d(d(r)).$$

Therefore, d is idempotent.
(v) Let $r, s \in E$ such that $r, r \rightsquigarrow s \in \ker(d_m)$. Then by Proposition 2.2(ii), (xi), (vi) and (x), we have

$$\begin{aligned}
(s \rightsquigarrow m) \rightsquigarrow m &= 1 \rightsquigarrow [(s \rightsquigarrow m) \rightsquigarrow m] \\
&= [((r \rightsquigarrow s) \rightsquigarrow m) \rightsquigarrow m] \rightsquigarrow [(s \rightsquigarrow m) \rightsquigarrow m] \\
&= (s \rightsquigarrow m) \rightsquigarrow [(((r \rightsquigarrow s) \rightsquigarrow m) \rightsquigarrow m) \rightsquigarrow m] \\
&= (s \rightsquigarrow m) \rightsquigarrow [(r \rightsquigarrow s) \rightsquigarrow m] \\
&= (r \rightsquigarrow s) \rightsquigarrow [(s \rightsquigarrow m) \rightsquigarrow m] \\
&= (r \rightsquigarrow s) \rightsquigarrow [1 \rightsquigarrow ((s \rightsquigarrow m) \rightsquigarrow m)] \\
&= (r \rightsquigarrow s) \rightsquigarrow [((r \rightsquigarrow m) \rightsquigarrow m) \rightsquigarrow ((s \rightsquigarrow m) \rightsquigarrow m)] \\
&= (r \rightsquigarrow s) \rightsquigarrow [(s \rightsquigarrow m) \rightsquigarrow (((r \rightsquigarrow m) \rightsquigarrow m) \rightsquigarrow m)] \\
&= (r \rightsquigarrow s) \rightsquigarrow [(s \rightsquigarrow m) \rightsquigarrow (r \rightsquigarrow m)] \\
&= (r \rightsquigarrow s) \rightsquigarrow [r \rightsquigarrow ((s \rightsquigarrow m) \rightsquigarrow m)] \\
&= 1
\end{aligned}$$

Hence $d_m(s) = (s \rightsquigarrow m) \rightsquigarrow m = 1$, and so $s \in \ker(d_m)$. Therefore, $\ker(d_m) \in \mathcal{F}(E)$. □

Corollary 3.26. *Let E be an equality algebra such that for any $r, s \in E$, $\neg(r \rightsquigarrow s) = \neg s \rightsquigarrow \neg r$ and $d : E \to E$ be a self mapping on E such that $d(r) = \neg\neg r$. Then the following statemants hold:*
(i) d is an order preserving map.
(ii) $d \in \mathcal{OD}(E)$.
(iii) d is an $\rightsquigarrow$-endomorphism on E.
(iv) d is idempotent.
(v) $\ker d \in \mathcal{F}(E)$.

Proof. It is enough to let $m = 0$ in Theorem 3.25. □

Example 3.27. Let E be the equality algebra as in Example 3.2(ii). Routine calculations show that for any $m, n \in E$, $\neg(m \rightsquigarrow n) = \neg n \rightsquigarrow \neg m$.

Theorem 3.28. *Let $d : E \to E$ be a self mapping on E such that $d(1) = 1$ and for any $r \in E - \{1\}$, $d(r) = m$ where $m \in E$. Then $d \in \mathcal{D}(E)$ if and only if $m = 1$.*

Proof. Let $m = 1$. Then for any $r \in E$, $d(r) = 1$ and so d is a fixed map equal 1. Thus $d \in \mathcal{D}(E)$.

Conversly, suppose $m \neq 1$, $d \in \mathcal{D}(E)$ and $r, s \in E - \{1\}$ such that $r \neq s$. By Definition 3.1, d is an inner and an outer derivation on E. Thus by Proposition 3.7, we have

$$d(r \rightsquigarrow s) = [(r \rightsquigarrow d(s)) \rightsquigarrow (d(r) \rightsquigarrow s)] \rightsquigarrow (d(r) \rightsquigarrow s) \text{ and } d(r \rightsquigarrow s) = r \rightsquigarrow d(s).$$

By definition of inner derivation, we have

$$m = d(r \rightsquigarrow s) = [(r \rightsquigarrow d(s)) \rightsquigarrow (d(r) \rightsquigarrow s)] \rightsquigarrow (d(r) \rightsquigarrow s) = [(r \rightsquigarrow m) \rightsquigarrow (m \rightsquigarrow s)] \rightsquigarrow (m \rightsquigarrow s).$$

If $r = m$, then

$$m = [(r \rightsquigarrow m) \rightsquigarrow (m \rightsquigarrow s)] \rightsquigarrow (m \rightsquigarrow s) = [(m \rightsquigarrow m) \rightsquigarrow (m \rightsquigarrow s)] \rightsquigarrow (m \rightsquigarrow s) = (m \rightsquigarrow s) \rightsquigarrow (m \rightsquigarrow s) = 1,$$

which is a contradiction. If $s = m$, then

$$m = [(r \rightsquigarrow m) \rightsquigarrow (m \rightsquigarrow s)] \rightsquigarrow (m \rightsquigarrow s) = [(r \rightsquigarrow m) \rightsquigarrow (m \rightsquigarrow m)] \rightsquigarrow (m \rightsquigarrow m) = (r \rightsquigarrow m) \rightsquigarrow 1 = 1,$$

which is a contradiction. Now, suppose $d \in \mathcal{OD}(E)$. Then if $r = m$ we have

$$m = d(r \rightsquigarrow s) = r \rightsquigarrow d(s) = r \rightsquigarrow m = m \rightsquigarrow m = 1,$$

which is a contradiction. Hence, in both cases we have a contradiction. Therefore, if $m \neq 1$, then $d \notin \mathcal{D}(E)$. $\square$

Corollary 3.29. *Let E be a commutative equality algebra and $d : E \rightsquigarrow E$ be a derivation on E such that $d(1) = 1$ and for any $r \in E - \{1\}$, $d(r) = m$, where $m \in E$. Then $r \leq m$ or $r \rightsquigarrow m = m$.*

Proof. Since E is a commutative equality algebra, by Remark 3.3, an outer and inner derivations on E are same. So we discuss about outer derivation on E. Then by Proposition 3.7 we have $d(r \rightsquigarrow s) = r \rightsquigarrow d(s)$. Since $d \in \mathcal{OD}(E)$, if $r \leq s$, then $1 = d(r \rightsquigarrow s) = r \rightsquigarrow d(s) = r \rightsquigarrow m$, and so $r \leq m$. If r and s are incomprable, then $m = d(r \rightsquigarrow s) = r \rightsquigarrow d(s) = r \rightsquigarrow m$. $\square$

Proposition 3.30. *Let $d \in \mathcal{D}(E)$. If $r \in E$ is a co-atom of E, then $d(r)$ is a co-atom of E.*

Proof. Let $d \in \mathcal{D}(E)$ and r be a co-atom of E. Clearly, $r < 1$ and for any $s \in E$ such that $r \leq s < 1$ implies $r = s$. Suppose $d(r) \neq 1$. By Proposition 3.4(iii), $r \leq d(r) < 1$. Since r is co-atom, we get $r = d(r)$. Hence $d(r)$ is a co-atom of E. $\square$

Proposition 3.31. *Let $d \in \mathcal{D}(E)$ and $F \in \mathcal{F}(E)$. Then $d(F) \subseteq F$.*

Proof. Let $d \in \mathcal{D}(E)$ and $s \in d(F)$. Then there exists $r \in F$ such that $d(r) = s$. By Proposition 3.4(iii), $r \leq d(r) = s$. Since $F \in \mathcal{F}(E)$ and $r \in F$, we have $s \in F$, and so $d(F) \subseteq F$. $\square$

In the next example we show that the converse of above proposition is not true.

Example 3.32. Let E be equality algebra as in Example 3.2(ii), where $d(1) = d(m) = 1$, $d(n) = n$ and $d(0) = 0$. Suppose $F = \{m, 1\}$. Obviously, $d(F) \subseteq F$ but $F \notin \mathcal{F}(E)$ because $m \leq n$ and $m \in F$ but $n \notin F$.

Let $d : E \to E$ be a self mapping on E. Define $\mathrm{Fix}_d(E) = \{r \in E \mid \ d(r) = r\}$. If $d \in \mathcal{D}(E)$, then by Propositions 3.4(i) and 3.5, $d(1) = 1$, and so $\mathrm{Fix}_d(E) \neq \emptyset$.

Example 3.33. Let E be the equality algebra as Example 3.2(ii). Define a self mapping d on E by $d(0) = 0$ and $d(m) = d(n) = d(1) = 1$. Clearly, $d \in \mathcal{D}(E)$ such that $\mathrm{Fix}_d(E) = \{0, 1\}$.

Proposition 3.34. *Let $d \in \mathcal{D}(E)$. Then the following statements hold:*
(i) $\mathrm{Fix}_d(E)$ *is closed under the operation* $\rightsquigarrow$.
(ii) d *is an idempotent map on* $\mathrm{Fix}_d(E)$.
(iii) d *is an order preserving map on* $\mathrm{Fix}_d(E)$.
(iv) $\mathrm{Fix}_d(E) \cap \ker(d) = \{1\}$.
(v) If d *is an idempotent derivation on* E*, then* $\mathrm{Im}(d) = \mathrm{Fix}_d(E)$.

Proof. (*i*) Let $r, s \in \mathrm{Fix}_d(E)$. Then $d(r) = r$ and $d(s) = s$. Since $d \in \mathcal{D}(E)$, we get d is an inner and outer derivations on E. Suppose $d \in \mathcal{OD}(E)$. Then by Proposition 3.7, $d(r \rightsquigarrow s) = r \rightsquigarrow d(s) = r \rightsquigarrow s$ and so $r \rightsquigarrow s \in \mathrm{Fix}_d(E)$. Now, let $d \in \mathcal{ID}(E)$. Since $r, s \in \mathrm{Fix}_d(E)$, we have

$$d(r \rightsquigarrow s) = [(r \rightsquigarrow d(s)) \rightsquigarrow (d(r) \rightsquigarrow s)] \rightsquigarrow (d(r) \rightsquigarrow s) = [(r \rightsquigarrow s)) \rightsquigarrow (r \rightsquigarrow s)] \rightsquigarrow (r \rightsquigarrow s) = r \rightsquigarrow s.$$

Hence, $r \rightsquigarrow s \in \mathrm{Fix}_d(E)$. Therefore, $\mathrm{Fix}_d(E)$ is closed under the operation $\rightsquigarrow$.
(*ii*) Let $r \in \mathrm{Fix}_d(E)$. Then $d(r) = r$ and so $d(d(r)) = d(r) = r$. Hence $d(d(r)) = d(r)$, and so d is an idempotent map on $\mathrm{Fix}_d(E)$.
(*iii*) The proof is clear.
(*iv*) Let $r \in \mathrm{Fix}_d(E) \cap \ker(d)$. Then $1 = d(r) = r$, and so $r = 1$.
(*v*) Let d be an idempotent derivation on E and $r \in E$. If $r \in \mathrm{Fix}_d(E)$, then $d(r) = r$, and so $r \in \mathrm{Im}(d)$. Hence, $\mathrm{Fix}_d(E) \subseteq \mathrm{Im}(d)$. Now, suppose $s \in \mathrm{Im}(d)$. Then there exists $r \in E$ such that $d(r) = s$. Since d is an idempotent derivation on E, we get $d(s) = d(d(r)) = d(r) = s$, and so $s \in \mathrm{Fix}_d(E)$. Hence $\mathrm{Im}(d) \subseteq \mathrm{Fix}_d(E)$. Therefore, $\mathrm{Fix}_d(E) = \mathrm{Im}(d)$ □

Corollary 3.35. *Let $d \in \mathcal{D}(E)$. Then $\langle(\mathrm{Fix}_d(E), \rightsquigarrow, 1), d\rangle$ is a modal BE-algebra.*

Proposition 3.36. *Let $d_1, d_2 \in \mathcal{OD}(E)$ such that $d_1 \circ d_2 = 1$. Then $\mathrm{Fix}_{d_2}(E) \subseteq \ker(d_1)$.*

Proof. Let $r \in \mathrm{Fix}_{d_2}(E)$. Then $d_2(r) = r$. Since $d_1 \in \mathcal{OD}(E)$, by Proposition 3.7 we have

$$d_1(r) = d_1(1 \rightsquigarrow r) = d_1(1 \rightsquigarrow d_2(r)) = 1 \rightsquigarrow d_1(d_2(r)) = 1.$$

Hence, $r \in \ker(d_1)$, and so $\mathrm{Fix}_{d_2}(E) \subseteq \ker(d_1)$. □

Proposition 3.37. *Let d_1 and d_2 be two order preserving derivations on E. Then $d_1 = d_2$ if and only if $\mathrm{Fix}_{d_1}(E) = \mathrm{Fix}_{d_2}(E)$.*

Proof. Let $d_1, d_2 \in \mathcal{D}(E)$ such that d_1 and d_2 be order preserving maps such that $d_1 = d_2$. Suppose $r \in \mathrm{Fix}_{d_1}(E)$. Then $d_1(r) = r$. Since $d_1 = d_2$, we get $d_2(r) = r$ and so $r \in \mathrm{Fix}_{d_2}(E)$. Hence, $\mathrm{Fix}_{d_1}(E) \subseteq \mathrm{Fix}_{d_2}(E)$. By the similar way, we can prove that $\mathrm{Fix}_{d_2}(E) \subseteq \mathrm{Fix}_{d_1}(E)$, and so $\mathrm{Fix}_{d_1}(E) = \mathrm{Fix}_{d_2}(E)$.

Conversely, by Proposition 3.34, we get d_1 and d_2 are idempotent maps on $\mathrm{Fix}_{d_1}(E)$ and $\mathrm{Fix}_{d_2}(E)$, respectively. Then $d_1(d_1(r)) = d_1(r)$, and so $d_1(r) \in \mathrm{Fix}_{d_1}(E)$. Since $\mathrm{Fix}_{d_1}(E) = \mathrm{Fix}_{d_2}(E)$, we have $d_1(r) \in \mathrm{Fix}_{d_2}(E)$. Thus, $d_2(d_1(r)) = d_1(r)$. Also, by Proposition 3.4(iii), for any $r \in E$, $r \leq d_1(r)$. Since d_1 and d_2 are order preserving derivations on E, we get $d_2(r) \leq d_2(d_1(r)) = d_1(r)$. Hence, for any $r \in E$, $d_2(r) \leq d_1(r)$. By the similar way, we can prove that $d_1(r) \leq d_2(r)$. Therefore, $d_1 = d_2$. □

Corollary 3.38. *Let d_1 and d_2 be two idempotent derivations on E. If $\mathrm{Im}(d_1) = \mathrm{Im}(d_2)$, then for any $u \in \mathrm{Im}(d_i)$, where $i = 1, 2$, $d_1 = d_2$.*

Proof. By Propositions 3.34(v) and 3.37, the proof is clear. □

Proposition 3.39. *Let $d \in \mathcal{D}(E)$. Then d is one to one if and only if d is an identity map if and only if $\ker(d) = \{1\}$.*

Proof. Let d be one to one. Since $d \in \mathcal{D}(E)$, we get d is an outer and inner derivations on E. From $d \in \mathcal{OD}(E)$, for any $r \in E$ we have $d(d(r) \rightsquigarrow r) = d(r) \rightsquigarrow d(r) = 1 = d(1)$, and so $d(r) \rightsquigarrow r = 1$. Then by Proposition 3.4(ii), $d(r) \leq r \leq d(r)$. Hence $d(r) = r$. Moreover, since $d \in \mathcal{ID}(E)$, we have

$$d(d(r) \rightsquigarrow r) = [(d(r) \rightsquigarrow d(r)) \rightsquigarrow (d^2(r) \rightsquigarrow r)] \rightsquigarrow (d^2(r) \rightsquigarrow r) = 1 = d(1).$$

Then by Proposition 3.4(ii), $d(r) \leq r \leq d(r)$, and so $d(r) = r$. Therefore, in both cases, d is an identity.

Obviously, d is an identity, then $\ker(d) = \{1\}$.

Now, suppose $\ker(d) = \{1\}$. Let $d(r) = d(s)$, for any $r, s \in E$. Then $d(r) \rightsquigarrow d(s) = 1$. Since $d \in \mathcal{OD}(E)$, by Proposition 3.11(v), $1 = d(r) \rightsquigarrow d(s) \leq d(r \rightsquigarrow s)$, we have $d(r \rightsquigarrow s) = 1$ and so $r \rightsquigarrow s \in \ker(d)$. Thus $r \rightsquigarrow s = 1$. Hence, $r \leq s$. By the similar way, we can see that $s \leq r$ and so $r = s$. Therefore, d is one to one. □

Corollary 3.40. *Let $d \in \mathcal{D}(E)$. Then the following statements hold:*
(i) If d is one to one, then d is onto derivation on E.
(ii) If d is an idempotent onto derivation on E, then d is one to one.

Proof. (i) Let $d \in \mathcal{D}(E)$ such that d be one to one. By Proposition 3.39, d is an identity derivation on E. Then d is idempotent and $\mathrm{Fix}_d(E) = E$. By Proposition 3.34(v), $\mathrm{Im}(d) = \mathrm{Fix}_d(E) = E$. Hence, d is onto derivation on E.
(ii) Let d be an idempotent onto derivation on E and $r \in \ker(d)$. Then $d(r) = 1$. Since $r \in E = \mathrm{Im}(d)$, there exists $m \in E$ such that $d(m) = r$. By idempotency of d, we have $r = d(m) = d(d(m)) = d(r) = 1$, and so $r = 1$. Hence, $\ker(d) = \{1\}$. Therefore, by Proposition 3.39, d is one to one. □

In the following example we show that for any equality algebra E, $\mathrm{Fix}_d(E)$ and $\ker(d)$ may not be a filter of E.

Example 3.41. (i) Let $d : E \to E$ be an identity map. Clearly, $\mathrm{Fix}_d(E) = E$ and $\ker(d) = \{1\}$. In this case $\mathrm{Fix}_d(E)$ and $\ker(d)$ are filters of E.
(ii) Let E be the equality algebra as Example 3.2(ii). Define a self mapping d on E by $d(0) = 0$ and $d(m) = d(n) = d(1) = 1$. Obviously, $d \in \mathcal{D}(E)$ such that $\mathrm{Fix}_d(E) = \{0, 1\} \notin \mathcal{F}(E)$ and $\ker(d) = \{m, n, 1\} \in \mathcal{F}(E)$.
(iii) Let E be the equality algebra as Example 3.2(ii). Define a self mapping d on E by $d(0) = 0$, $d(n) = n$ and $d(m) = d(1) = 1$. Then by routine calculation, $d \in \mathcal{OD}(E)$ such that $\mathrm{Fix}_d(E) = \{0, n, 1\} \notin \mathcal{F}(E)$ and $\ker(d) = \{m, 1\} \notin \mathcal{F}(E)$ because $m \leq n$ and $m \in \ker(d)$ but $n \notin \ker(d)$.

In the following proposition we investigate that under which condition, $\ker(d) \in \mathcal{F}(E)$.

Proposition 3.42. *Let $d : E \to E$ be a self mapping on E. Then the following statements hold:*
(i) If d is an $\rightsquigarrow$-endomorphism on E, then $\ker(d) \in \mathcal{F}(E)$.
(ii) If $d \in \mathcal{OD}(E)$, order preserving map and idempotent, then $\ker(d) \in \mathcal{F}(E)$.
(iii) If for any $r, s \in E$, $d(r \rightsquigarrow s) \leq d(r) \rightsquigarrow d(s)$, then $\ker(d) \in \mathcal{F}(E)$.

Proof. (i) Since d is an $\rightsquigarrow$-endomorphism on E, we get $d(1) = 1$ and so $1 \in \ker(d)$. Suppose $r, r \rightsquigarrow s \in \ker(d)$, for $r, s \in E$. Then $d(r) = d(r \rightsquigarrow s) = 1$. Thus

$$1 = d(r \rightsquigarrow s) = d(r) \rightsquigarrow d(s) = 1 \rightsquigarrow d(s) = d(s).$$

Hence $d(s) = 1$ and so $s \in \ker(d)$. Therefore, $\ker(d) \in \mathcal{F}(E)$.
(ii) Let d be an order preserving map, idempotent outer derivation on E. Then $d(1) = 1$ and so $1 \in \ker(d)$. Suppose $r, r \rightsquigarrow s \in \ker(d)$, for $r, s \in E$. Then $d(r) = d(r \rightsquigarrow s) = 1$. Thus $1 = d(r \rightsquigarrow s) = r \rightsquigarrow d(s)$, and so $r \leq d(s)$. Since d is an order preserving map and idempotent map, we get $d(r) \leq d(d(s)) = d(s)$. Since $r \in \ker(d)$ and $1 = d(r) \leq d(s)$, we have $d(s) = 1$ and so $s \in \ker(d)$. Therefore, $\ker(d) \in \mathcal{F}(E)$.
(iii) By (i) and Proposition 3.11(v), the proof is clear. □

Example 3.43. Let E be the equality algebra as in Example 3.2(ii) and $d : E \to E$ be a self mapping on E such that $d(1) = d(m) = 1$, $d(n) = m$ and $d(0) = 0$. Then $d \in \mathcal{OD}(E)$. Clearly $m \leq n$ but $1 = d(m) \nleq d(n) = m$, we get d is not an order preserving map. Also, $m = d(n) \neq d(d(n)) = d(m) = 1$, so d is not an idempotent derivation. Hence we can see that $\ker(d) \notin \mathcal{F}(E)$.

Proposition 3.44. *Let E be a commutative equality algebra. If d is an idempotent derivation on E, then $\ker(d) \in \mathcal{F}(E)$.*

Proof. Since $d \in \mathcal{D}(E)$, by Proposition 3.4(i), $d(1) = 1$ and so $1 \in \ker(d)$. Let $r, s \in E$ such that $r, r \rightsquigarrow s \in \ker(d)$. Then $d(r) = d(r \rightsquigarrow s) = 1$. Since $d \in \mathcal{D}(E)$, $1 = d(r \rightsquigarrow s) = r \rightsquigarrow d(s)$, and so $r \leq d(s)$. Moreover, from E is a commutative equality algebra, we have $d(s) = 1 \rightsquigarrow d(s) = (r \rightsquigarrow d(s)) \rightsquigarrow d(s) = (d(s) \rightsquigarrow r) \rightsquigarrow r$. Since $r \in \ker(d)$ and d is idempotent, we have

$$d(s) = d(d(s)) = d((d(s) \rightsquigarrow r) \rightsquigarrow r) = (d(s) \rightsquigarrow r) \rightsquigarrow d(r) = (d(s) \rightsquigarrow r) \rightsquigarrow 1 = 1.$$

Hence, $d(s) = 1$ and so $s \in \ker(d)$. Therefore, $\ker(d) \in \mathcal{F}(E)$. $\square$

Proposition 3.45. *Let E be an equality algebra in which every pair of elements is comparable. If $d \in \mathcal{D}(E)$, then $\ker(d)$ is a sub-algebra of E.*

Proof. Let $r, s \in ker(d)$ such that $r \leq s$. Then $r \wedge s = r$ and so $d(r \wedge s) = d(r) = 1$. Thus $r \wedge s \in ker(d)$. Moreover, we have $r \sim s = s \rightsquigarrow r$ by Proposition 2.2(viii). Then

$$d(r \sim s) = d(s \rightsquigarrow r) = s \rightsquigarrow d(r) = s \rightsquigarrow 1 = 1,$$

and

$$d(r \sim s) = d(s \rightsquigarrow r) = [(s \rightsquigarrow d(r)) \rightsquigarrow (d(s) \rightsquigarrow r)] \rightsquigarrow (d(s) \rightsquigarrow r) = r \rightsquigarrow r = 1,$$

and so $r \sim s \in ker(d)$. Therefore, $\ker(d)$ is a sub-algebra of E. $\square$

In the following proposition, we investigate that under which condition $\mathrm{Fix}_d(E) \in \mathcal{F}(E)$.

Proposition 3.46. *Let $F \in \mathcal{F}(E)$ such that for any $r, s \in E$, if $r, s \notin F$, then $r \leq s$ or $s \leq r$. Then there exists a derivation on E such as d such that $\mathrm{Fix}_d(E) = F$ and so $\mathrm{Fix}_d(E) \in \mathcal{F}(E)$.*

Proof. Let $F \in \mathcal{F}(E)$. Define a self mapping $d : E \to E$ such that for any $r \in F$, $d(r) = r$ and for any $r \notin F$, $d(r) = 1$. Clearly, $\mathrm{Fix}_d(E) = F$ and so $\mathrm{Fix}_d(E) \in \mathcal{F}(E)$. It is enough to prove that $d \in \mathcal{D}(E)$. For this we have to prove that d is an inner and outer derivation on E. At first we show that $d \in \mathcal{OD}(E)$. For this, we have four cases:

Case 1. Suppose $r, s \in F$. Then $d(r) = r$ and $d(s) = s$. Since $F \in \mathcal{F}(E)$ and $s \leq r \rightsquigarrow s$, we have $r \rightsquigarrow s \in F$. Then

$$\begin{aligned} d(r \rightsquigarrow s) = r \rightsquigarrow s = 1 \rightsquigarrow (r \rightsquigarrow s) &= [(r \rightsquigarrow s) \rightsquigarrow (r \rightsquigarrow s)] \rightsquigarrow (r \rightsquigarrow s) \\ &= [(d(r) \rightsquigarrow s) \rightsquigarrow (r \rightsquigarrow d(s))] \rightsquigarrow (r \rightsquigarrow d(s)). \end{aligned}$$

Case 2. Suppose $r \notin F$ and $s \in F$. Then $d(r) = 1$ and $d(s) = s$. Since $F \in \mathcal{F}(E)$ and $s \leq r \rightsquigarrow s$, we have $r \rightsquigarrow s \in F$. Then

$$\begin{aligned} d(r \rightsquigarrow s) = r \rightsquigarrow s = 1 \rightsquigarrow (r \rightsquigarrow s) &= [(1 \rightsquigarrow s) \rightsquigarrow (r \rightsquigarrow s)] \rightsquigarrow (r \rightsquigarrow s) \\ &= [(d(r) \rightsquigarrow s) \rightsquigarrow (r \rightsquigarrow d(s))] \rightsquigarrow (r \rightsquigarrow d(s)). \end{aligned}$$

Case 3. Suppose $r \in F$ and $s \notin F$. Then $r \rightsquigarrow s \notin F$. Because, if $r \rightsquigarrow s \in F$, since $r \in F$ and $F \in \mathcal{F}(E)$, we have $s \in F$, which is a contradiction. So $r \rightsquigarrow s \notin F$. Since $d(r) = r$ and $d(s) = 1$, we have

$$\begin{aligned} d(r \rightsquigarrow s) = 1 = (r \rightsquigarrow s) \rightsquigarrow 1 &= [(r \rightsquigarrow s) \rightsquigarrow (r \rightsquigarrow 1)] \rightsquigarrow (r \rightsquigarrow 1) \\ &= [(d(r) \rightsquigarrow s) \rightsquigarrow (r \rightsquigarrow d(s))] \rightsquigarrow (r \rightsquigarrow d(s)). \end{aligned}$$

Case 4. Suppose $r, s \notin F$. Then $d(r) = d(s) = 1$ and by assumption $r \leq s$. Thus

$$d(r \rightsquigarrow s) = 1 = 1 \rightsquigarrow 1 = [(1 \rightsquigarrow s) \rightsquigarrow (r \rightsquigarrow 1)] \rightsquigarrow (r \rightsquigarrow 1) = [(d(r) \rightsquigarrow s) \rightsquigarrow (r \rightsquigarrow d(s))] \rightsquigarrow (r \rightsquigarrow d(s)).$$

Hence $d \in \mathcal{OD}(E)$. By the similar way we can prove that $d \in \mathcal{ID}(E)$. Therefore, $d \in \mathcal{D}(E)$. □

Obviously, the condition "if $r, s \notin F$, then $r \leq s$ or $s \leq r$" in Proposition 3.46 is necessary. Because if $r \rightsquigarrow s \in F$, then $d(r \rightsquigarrow s) = r \rightsquigarrow s$ but $[(d(r) \rightsquigarrow s) \rightsquigarrow (r \rightsquigarrow d(s))] \rightsquigarrow (r \rightsquigarrow d(s)) = 1$, and so $r \rightsquigarrow s = 1$. Hence $r \leq s$.

In the following example we show that there exists a filter of an equality algebra which satisfiying to the condition "if $r, s \notin F$, then $r \leq s$ or $s \leq r$".

Example 3.47. Let E be the equality algebra as in Example 3.2(ii). Suppose $F = \{n, 1\}$. Obviously, $F \in \mathcal{F}(E)$ and for $r, s \notin F$, $r \leq s$ or $s \leq r$.

Lemma 3.48. *Let $F \in \mathcal{F}(E)$. For any $r, s \in E$, define the relation θ_F on E as follows:*

$$r\theta_F s \iff r \rightsquigarrow s \in F \text{ and } s \rightsquigarrow r \in F.$$

Then θ_F is an equivalence relation on E.

Proof. Clearly, θ_F is reflexive and symetric relation. Suppose $r, s, t \in E$ such that $r\theta_F s$ and $s\theta_F t$. Then $r \rightsquigarrow s \in F$, $s \rightsquigarrow r \in F$, $s \rightsquigarrow t \in F$ and $t \rightsquigarrow s \in F$. By Proposition 2.2(vii), we have

$$r \rightsquigarrow s \leq (s \rightsquigarrow t) \rightsquigarrow (r \rightsquigarrow t) \text{ and } t \rightsquigarrow s \leq (s \rightsquigarrow r) \rightsquigarrow (t \rightsquigarrow r).$$

Since $F \in \mathcal{F}(E)$, $r \rightsquigarrow s \in F$ and $t \rightsquigarrow s \in F$, we get $(s \rightsquigarrow t) \rightsquigarrow (r \rightsquigarrow t) \in F$ and $(s \rightsquigarrow r) \rightsquigarrow (t \rightsquigarrow r) \in F$. Moreover, from $F \in \mathcal{F}(E)$, $s \rightsquigarrow t \in F$ and $s \rightsquigarrow r \in F$, we have $r \rightsquigarrow t \in F$ and $t \rightsquigarrow r \in F$, and so $r\theta_F t$. Therefore, θ_F is an equivalence relation on E. □

Theorem 3.49. *Let $d \in \mathcal{D}(E)$ such that $d(r \rightsquigarrow s) \leq d(r) \rightsquigarrow d(s)$, for any $r, s \in E$. Then the equivalence relations on $(E, \rightsquigarrow, 1)$ coincide with the equivalence relations on E with derivation d.*

Proof. Let $F \in \mathcal{F}(E)$ and $r, s \in E$ such that $r\theta_F s$. By Lemma 3.48, we get θ_F is an equivalence relation on E. Then by definition of θ_F we have $r \rightsquigarrow s, s \rightsquigarrow r \in F$. By Proposition 3.4(ii), $r \rightsquigarrow s \leq d(r \rightsquigarrow s)$ and $s \rightsquigarrow r \leq d(s \rightsquigarrow r)$. Since $F \in \mathcal{F}(E)$, we have $d(r \rightsquigarrow s), d(s \rightsquigarrow r) \in F$. By assumption, $d(r \rightsquigarrow s) \leq d(r) \rightsquigarrow d(s)$ and $d(s \rightsquigarrow r) \leq d(s) \rightsquigarrow d(r)$. Thus $d(r) \rightsquigarrow d(s) \in F$ and $d(s) \rightsquigarrow d(r) \in F$. Hence $d(r)\theta_F d(s)$. □

4 Conclusion

In this paper, the notions of inner and outer derivations on equality algebra are introduced and some properties are investigated. Moreover, the notions of kernel and fixed point sets of derivation on E and proved that under which condition they are filters of E. Finally, coincidence of the equivalence relations on $(E, \rightsquigarrow, 1)$ with the equivalence relations on E with derivation d is studied.

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5 Compliance with ethical standards

Conflict of interest: The authors declare that there is no conflict of interest.

Human and animal rights: This article does not contain any studies with human participants or animals performed by any of the authors.Informed consent was obtained from all individual participants included in the study.

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