Soft Computing (2022) 26:8987–8996
https://doi.org/10.1007/s00500-022-07306-2

FOUNDATIONS

Elitable GE-filters of bordered GE-algebras

Jeong-Gon Lee[1] · Manzoor Kaleem Shaik[2] · Ravikumar Bandaru[3] · Young Bae Jun[4]

Accepted: 6 June 2022 / Published online: 6 July 2022

Abstract

In this paper, the notions of maximal GE-filter and prime GE-filter of a GE-algebra are introduced and the relation between them is given. Some characterizations of prime GE-filters of a transitive GE-algebra are given in terms of the GE-filter generated by a subset of a transitive GE-algebra. We generalized Stone's theorem to transitive GE-algebras. The notion of elitable GE-filter of a bordered GE-algebra is introduced and investigated its properties. We observed that the class of all elitable GE-filters of a transitive bordered GE-algebra is a complete distributive lattice. Equivalent conditions for a GE-filter of a transitive bordered GE-algebra to be elitable GE-filter are given. We provided conditions for a subset of a transitive bordered GE-algebra to be elitable GE-filter.

Keywords Prime GE-filter · Maximal GE-filter · Elitable GE-filter

1 Introduction

Imai and Iséki introduced BCK-algebras (see Imai and Iséki 1966; Iséki 1966) in 1966 as the algebraic semantics for a non-classical logic with only implication. Various researchers have examined the generalised concepts of BCK-algebras since then. Henkin and Skolem introduced Hilbert algebras in the 1950 s to investigate intuitionistic and other non-classical logics. A. Diego established that Hilbert algebras form a locally finite variety (see Diego 1966). Later several researchers extended the theory on Hilbert algebras.

✉ Ravikumar Bandaru
ravimaths83@gmail.com

Jeong-Gon Lee
jukolee@wku.ac.kr

Manzoor Kaleem Shaik
sm30113akaleem@gmail.com

Young Bae Jun
skywine@gmail.com

1 Division of Applied Mathematics, Wonkwang University, 460, Iksan-daero, Iksan, Jeonbuk 54538, Korea

2 Department of Mathematics, St. Joseph's Degree College, Kurnool, Andhra Pradesh 518004, India

3 Department of Mathematics, GITAM (Deemed to be University), Hyderabad Campus, Hyderabad, Telangana 502329, India

4 Department of Mathematics Education, Gyeongsang National University, Jinju 52828, Korea

S. Celani given a representation theorem for Hilbert algebras by means of ordered sets and characterized the homomorphisms of Hilbert algebras in terms of applications defined between the sets of all irreducible deductive systems of the associated algebras (see Celani 2002). Chajda et al. considered the properties of deductive systems in Hilbert algebras which are upper semi-lattices as posets and shown that every maximal deductive system is prime. They have given a condition for a deductive system to be prime and shown that the annihilator of any non-empty subset of a Hilbert algebra is a deductive system which is an annihilator of the induced upper semilattice(see Chajda et al. 2002). Hong et al. introduced the concept of maximal deductive systems and shown that every bounded Hilbert algebra with at least two elements contains at least one maximal deductive system (see Hong and Jun 1996). Jun et al. introduced the concept of Hilbert filter in Hilbert algebras and studied how to generate a Hilbert filter by a set (see Jun and Kim 2005). The notion of BE-algebra was introduced by H.S. Kim and Y.H. Kim as a generalization of a dual BCK-algebra (see Kim and Kim 2006). Rezaei et al. discussed relations between Hilbert algebras and BE-algebras (see Rezaei et al. 2013). Borumand Saeid et al. introduced the notions of implicative filter, positive implicative filter, normal filter, fantastic filter, obstinate filter and maximal filter in a BE-algebra and obtained the related properties (see Borumand Saeid et al. 2013). Dudek et al. introduced the notion of poor and crazy filters in BCK-algebras and studied their properties in different types of BCK-algebras (see Dudek and Jun 2007).

The generalisation process is another important topic in the study of algebraic structures. Bandaru et al. introduced the concept of GE-algebras as a generalisation of Hilbert algebras and studied several properties (see Bandaru et al. 2021). Rezaei and colleagues introduced and discussed the concept of prominent GE-filters in GE-algebras (see Rezaei et al. 2021). Bandaru et al. introduced and investigated the concept of bordered GE-algebra (see Bandaru et al. 2021). Later, Ozturk et al. introduced and investigated the concept of Strong GE-filters, GE-ideals of bordered GE-algebras (see Ozturk et al. 2021). Song et al. introduced and discussed the concept of Imploring GE-filters of GE-algebras (see Song et al. 2021).

In this paper, we introduce and investigate the concepts of maximal GE-filter and prime GE-filter of a GE-algebra. We define prime GE-filter as the GE-filter produced by a subset of a transitive GE-algebra. We define and investigate the properties of an elitable GE-filter of a bordered GE-algebra. The class of all elitable GE-filters of a transitive bordered GE-algebra is a complete distributive lattice, as we observe. We describe an elitable GE-filter in a transitive bordered GE-algebra. In addition, we define the conditions under which a subset of a transitive bordered GE-algebra is an elitable GE-filter.

2 Preliminaries

Definition 2.1 (Bandaru et al. 2021) A *GE-algebra* is a non-empty set R with a constant 1 and a binary operation $*$ satisfying the following axioms:

(GE1) $\mu * \mu = 1$,
(GE2) $1 * \mu = \mu$,
(GE3) $\mu * (\nu * \tau) = \mu * (\nu * (\mu * \tau))$

for all $\mu, \nu, \tau \in R$.

In a GE-algebra R, a binary relation "$\leq$" is defined by

$$(\forall \beta, \gamma \in R)\,(\beta \leq \gamma \;\Leftrightarrow\; \beta * \gamma = 1)\,. \tag{2.1}$$

Definition 2.2 (Bandaru et al. 2021) A GE-algebra R is said to be

- *transitive* if it satisfies:

$$(\forall \beta, \gamma, \alpha \in R)\,(\beta * \gamma \leq (\alpha * \beta) * (\alpha * \gamma))\,. \tag{2.2}$$

- *antisymmetric* if the binary relation "$\leq$" is antisymmetric.
- *commutative* if it satisfies:

$$(\forall \beta, \gamma \in R)\,((\beta * \gamma) * \gamma = (\gamma * \beta) * \beta)\,. \tag{2.3}$$

Theorem 2.3 (Bandaru et al. 2021) *Every self-distributive BE-algebra is a GE-algebra.*

The following proposition gives the equivalent conditions for a GE-algebra to be implication algebra, dual implicative BCK-algebra and commutative Hilbert algebra.

Proposition 2.4 (Bandaru et al. 2021) *Let $(R, *, 1)$ be a GE-algebra. Then, the following are equivalent.*

(i) R is commutative,
(ii) R is implication algebra,
(iii) R is dual implicative BCK-algebra,
(iv) R is commutative Hilbert algebra.

Definition 2.5 (Bandaru et al. 2021) If a GE-algebra R has a special element, say 0, that satisfies $0 \leq \beta$ for all $\beta \in R$, we call R the *bordered GE-algebra*.

For every element β of a bordered GE-algebra R, we denote $\beta * 0$ by $\beta^{\text{ß}}$, and $(\beta^{\text{ß}})^{\text{ß}}$ is denoted by $\beta^{\text{ßß}}$.

Definition 2.6 (Bandaru et al. 2021) If a bordered GE-algebra R satisfies the condition (2.2), we say that R is a *transitive bordered GE-algebra*.

Definition 2.7 (Bandaru et al. 2021) A bordered GE-algebra R is said to be *antisymmetric* if the binary operation "$\leq$" is antisymmetric.

Proposition 2.8 (Bandaru et al. 2021) *Every GE-algebra R satisfies the following items.*

$$(\forall \beta \in R)\,(\beta * 1 = 1)\,. \tag{2.4}$$
$$(\forall \beta, \gamma \in R)\,(\beta * (\beta * \gamma) = \beta * \gamma)\,. \tag{2.5}$$
$$(\forall \beta, \gamma \in R)\,(\beta \leq \gamma * \beta)\,. \tag{2.6}$$
$$(\forall \beta, \gamma, \alpha \in R)\,(\beta * (\gamma * \alpha) \leq \gamma * (\beta * \alpha))\,. \tag{2.7}$$
$$(\forall \beta \in R)\,(1 \leq \beta \;\Rightarrow\; \beta = 1)\,. \tag{2.8}$$
$$(\forall \beta, \gamma \in R)\,(\beta \leq (\gamma * \beta) * \beta)\,. \tag{2.9}$$
$$(\forall \beta, \gamma \in R)\,(\beta \leq (\beta * \gamma) * \gamma)\,. \tag{2.10}$$
$$(\forall \beta, \gamma, \alpha \in R)\,(\beta \leq \gamma * \alpha \;\Leftrightarrow\; \gamma \leq \beta * \alpha)\,. \tag{2.11}$$

If R is transitive, then

$$(\forall \beta, \gamma, \alpha \in R)$$
$$(\beta \leq \gamma \;\Rightarrow\; \alpha * \beta \leq \alpha * \gamma,\; \gamma * \alpha \leq \beta * \alpha)\,. \tag{2.12}$$
$$(\forall \beta, \gamma, \alpha \in R)\,(\beta * \gamma \leq (\gamma * \alpha) * (\beta * \alpha))\,. \tag{2.13}$$

Lemma 2.9 (Bandaru et al. 2021) *The following are equivalent to each other in a GE-algebra R.*

$$(\forall \beta, \gamma, \alpha \in R)\,(\beta * \gamma \leq (\alpha * \beta) * (\alpha * \gamma))\,. \tag{2.14}$$

$$(\forall \beta, \gamma, \alpha \in R)\,(\beta * \gamma \leq (\gamma * \alpha) * (\beta * \alpha))\,. \tag{2.15}$$

Definition 2.10 (Bandaru et al. 2021) A subset K of a GE-algebra R is called a *GE-filter* of R if it satisfies:

$$1 \in K, \tag{2.16}$$

$$(\forall \beta, \gamma \in R)(\beta * \gamma \in K,\ \beta \in K\ \Rightarrow\ \gamma \in K). \tag{2.17}$$

Lemma 2.11 (Bandaru et al. 2021) *In a GE-algebra R, every GE-filter K of R satisfies:*

$$(\forall \beta, \gamma \in R)\,(\beta \leq \gamma,\ \beta \in K\ \Rightarrow\ \gamma \in K)\,. \tag{2.18}$$

Proposition 2.12 *(Bandaru et al. 2021) The following assertions are true in a bordered GE-algebra R.*

$$1^{\text{ß}} = 0,\ 0^{\text{ß}} = 1. \tag{2.19}$$

$$(\forall \beta \in R)\left(\beta \leq \beta^{\text{ßß}},\ 0 \leq \beta^{\text{ßß}}\right). \tag{2.20}$$

$$(\forall \beta, \gamma \in R)\left(\beta * \gamma^{\text{ß}} \leq \gamma * \beta^{\text{ß}}\right). \tag{2.21}$$

$$(\forall \beta, \gamma \in R)\left(\beta \leq \gamma^{\text{ß}}\ \Leftrightarrow\ \gamma \leq \beta^{\text{ß}}\right). \tag{2.22}$$

$$(\forall \beta, \gamma \in R)\left(\beta * \gamma^{\text{ß}} = \beta * (\gamma * \beta^{\text{ß}})\right). \tag{2.23}$$

If R is a transitive bordered GE-algebra, then

$$(\forall \beta, \gamma \in R)\left(\beta \leq \gamma\ \Rightarrow \gamma^{\text{ß}} \leq \beta^{\text{ß}}\right). \tag{2.24}$$

$$(\forall \beta, \gamma \in R)\left(\beta * \gamma \leq \gamma^{\text{ß}} * \beta^{\text{ß}}\right). \tag{2.25}$$

If R is an antisymmetric bordered GE-algebra, then

$$(\forall \beta, \gamma \in R)\left(\beta * \gamma^{\text{ß}} = \gamma * \beta^{\text{ß}}\right). \tag{2.26}$$

If R is a transitive and antisymmetric bordered GE-algebra, then

$$(\forall \beta \in R)\left(\beta^{\text{ßßß}} = \beta^{\text{ß}}\right). \tag{2.27}$$

Definition 2.13 (Bandaru et al. 2021) A *duplex bordered element* in a bordered GE-algebra R is defined as an element β of R that satisfies $\beta^{\text{ßß}} = \beta$.

The set of all duplex bordered elements of a bordered GE-algebra R is denoted by $0^2(R)$ and is referred to as the R duplex bordered set. It is clear that $0, 1 \in 0^2(R)$.

Definition 2.14 (Bandaru et al. 2021) A bordered GE-algebra R is said to be *duplex* if every element of R is a duplex bordered element, that is, $R = 0^2(R)$.

Definition 2.15 (Ozturk et al. 2021) Let R be a bordered GE-algebra. If a subset G of R meets the following conditions for all $\beta, \gamma \in R$, it is termed a *GE-ideal* of R:

(1) $0 \in G$,
(2) $\beta \in G$ and $(\beta^{\text{ß}} * \gamma^{\text{ß}})^{\text{ß}} \in G$ imply that $\gamma \in G$.

Clearly, $\{0\}$ is a GE-ideal of R.

Proposition 2.16 (Ozturk et al. 2021) *Let G be a GE-ideal of R. Then, for any $\eta, \zeta \in R$, we have*

(1) *$\eta \in G$ and $\zeta \leq \eta$ imply $\zeta \in G$.*
(2) *$(\eta * \zeta)^{\text{ß}} \in G,\ \zeta \in G\ \Rightarrow\ \eta \in G$.*

3 Maximal and prime GE-filters

Definition 3.1 (Ozturk et al. 2021) Let K be a subset of a GE-algebra R. The *GE-filter* of R generated by K is denoted by $\langle K \rangle$ and is defined to be the intersection of all GE-filters of R containing K.

Example 3.2 Let $R = \{1, e, f, g, h, a, b\}$ be a set with the binary operation "$*$" in the following Cayley Table.

$*$	1	e	f	g	h	a	b
1	1	e	f	g	h	a	b
e	1	1	1	g	a	a	1
f	1	e	1	h	h	h	b
g	1	1	f	1	1	1	1
h	1	e	1	1	1	1	b
a	1	e	f	1	1	1	1
b	1	e	f	a	h	a	1

Then, $(R, *, 1)$ is a GE-algebra. If we take a subset $G = \{1, e\}$ of R, then the GE-filter of R generated by G is $\langle G \rangle = \{1, e, f, b\}$.

The next theorem shows how the elemental structure of $\langle K \rangle$ is constructed.

Theorem 3.3 *Let K be a non-empty subset of a transitive GE-algebra R. Then, $\langle K \rangle$ consists of β's that satisfies the following condition:*

$$\begin{aligned}&(\exists \beta_1, \beta_2, \cdots, \beta_n \in K)\\ &\quad (\beta_n * (\cdots * (\beta_2 * (\beta_1 * \beta)) \cdots) = 1)\,,\end{aligned} \tag{3.1}$$

that is, $\langle K \rangle = \{\beta \in R \mid \beta_n * (\cdots * (\beta_2 * (\beta_1 * \beta)) \cdots) = 1$ for some $\beta_1, \beta_2, \cdots, \beta_n \in K\}$.

Proof Let

$$G := \{\beta \in R \mid \beta_n * (\cdots * (\beta_2 * (\beta_1 * \beta)) \cdots) = 1 \text{for some } \beta_1, \beta_2, \cdots, \beta_n \in K\}.$$

Obviously, $1 \in G$. Let $\mu, \nu \in R$ be such that $\mu * \nu \in G$ and $\mu \in G$. Then, there is $\beta_1, \beta_2, \cdots, \beta_m, \gamma_1, \gamma_2, \cdots, \gamma_n \in K$ such that

$$\beta_m * (\cdots * (\beta_2 * (\beta_1 * (\mu * \nu))) \cdots) = 1, \tag{3.2}$$

$$\gamma_n * (\cdots * (\gamma_2 * (\gamma_1 * \mu)) \cdots) = 1. \tag{3.3}$$

By (GE3), (3.2) and (GE1), we can observe that,

$$\mu * (\beta_m * (\cdots * (\beta_2 * (\beta_1 * \nu)) \cdots)) = 1,$$

that is,

$$\mu \le \beta_m * (\cdots * (\beta_2 * (\beta_1 * \nu)) \cdots). \tag{3.4}$$

By (2.12), we get

$$\gamma_1 * \mu \le \gamma_1 * (\beta_m * (\cdots * (\beta_2 * (\beta_1 * \nu)) \cdots)). \tag{3.5}$$

If we repeat this process n times, we have

$$1 = \gamma_n * (\cdots * (\gamma_2 * (\gamma_1 * \mu)) \cdots) \le \gamma_n * (\cdots * (\gamma_1 * (\beta_m * (\cdots * (\beta_2 * (\beta_1 * \nu)) \cdots))) \cdots).$$

By (2.8), it follows that

$$\gamma_n * (\cdots * (\gamma_1 * (\beta_m * (\cdots * (\beta_2 * (\beta_1 * \nu)) \cdots))) \cdots) = 1.$$

Hence, $\nu \in G$. Thus, G is a GE-filter of R. It is obvious that $K \subseteq G$. Let M be a GE-filter for R that includes K. If $\beta \in G$, then $\alpha_n * (\cdots * (\alpha_2 * (\alpha_1 * \beta)) \cdots) = 1 \in M$ for some $\alpha_1, \alpha_2, \cdots, \alpha_n \in K \subseteq M$. It follows that $\beta \in M$. Therefore, $G \subseteq M$. This shows that $G = \langle K \rangle$. □

Corollary 3.4 *For every element γ in a transitive GE-algebra R, we have*

$$\langle \gamma \rangle = \{\beta \in R \mid \gamma * \beta = 1\},$$

that is, $\langle \gamma \rangle = \{\beta \in R \mid \gamma \le \beta\}$ which is called principal GE-filter generated by γ.

We construct the smallest GE-filter containing K and η, given a GE-filter K and an element η in a transitive GE-algebra R.

Theorem 3.5 *Let K be a GE-filter of a transitive GE-algebra R and η be any element of R. Then,*

$$\langle K \cup \{\eta\} \rangle = \{\zeta \in R \mid \eta * \zeta \in K\}. \tag{3.6}$$

Proof Let $K_\eta := \{\zeta \in R \mid \eta * \zeta \in K\}$. Since $\eta * \eta = 1 \in K$, $\eta \in K_\eta$. For any $\zeta \in K$, we have $\zeta \le \eta * \zeta$ by (2.6). Hence, $\eta * \zeta \in K$ by Lemma 2.11. Thus, $\zeta \in K_\eta$, which shows that $K \subseteq K_\eta$. Hence, $K \cup \{\eta\} \subseteq K_\eta$. Since $\eta * 1 = 1 \in K$, we get $1 \in K_\eta$. Let $\zeta, \alpha \in R$ be such that $\zeta \in K_\eta$ and $\zeta * \alpha \in K_\eta$. Then, $\eta * (\zeta * \alpha) \in K$ and $\eta * \zeta \in K$. Using (2.7), (2.12) and (2.5), we have

$$\eta * (\zeta * \alpha) \le \zeta * (\eta * \alpha) \le (\eta * \zeta) * (\eta * (\eta * \alpha)) = (\eta * \zeta) * (\eta * \alpha),$$

and so $\eta * \alpha \in K$ by Lemma 2.11 and (2.17). Hence, $\alpha \in K_\eta$, and thus, K_η is a GE-filter of R. Let G be a GE-filter of R containing $K \cup \{\eta\}$ and $\zeta \in K_\eta$. Then, $\eta \in G$ and $\eta * \zeta \in K \subseteq G$. Hence, $\zeta \in G$ since G is GE-filter of R. Therefore, $K_\eta \subseteq G$. Hence, $K_\eta = \langle K \cup \{\eta\} \rangle$. □

Proposition 3.6 *Let K and G be two GE-filters of a transitive GE-algebra R. Then,*

$$\langle K \cup G \rangle = \{\beta \in R \mid \eta * (\zeta * \beta) = 1 \textit{ for some } \eta \in K \textit{ and } \zeta \in G\}$$

Proof Let $J = \{\beta \in R \mid \eta * (\zeta * \beta) = 1$ for some $\eta \in K$ and $\zeta \in G\}$ and $\beta \in J$. Then, $\eta * (\zeta * \beta) = 1$ for some $\eta \in K$ and $\zeta \in G$. Hence, $\beta \in \langle K \cup G \rangle$ by Theorem 3.3. Therefore, $J \subseteq \langle K \cup G \rangle$. Conversely, assume that $\beta \in \langle K \cup G \rangle$. Then, there are $\gamma_1, \gamma_2, ..., \gamma_i, ..., \gamma_n \in K \cup G$ such that $\gamma_n * (... * (\gamma_1 * \beta)...) = 1$. By (GE3), we can get that

$$\gamma_n * (... * (\gamma_{i+1} * (\gamma_i .... * (\gamma_1 * \beta)...))...) = 1 \in G$$

such that $\gamma_1, \gamma_2, ..., \gamma_i \in K$ and$\gamma_{i+1}, ..., \gamma_n \in G$. Since $\gamma_n \in G$ and G is a GE-filter, we have

$$\gamma_{n-1} * (... * (\gamma_{i+1} * (\gamma_i ... * (\gamma_1 * \beta)...))...) \in G.$$

By repeating this, we get $\gamma_i * (... * (\gamma_1 * \beta)....) \in G$. Take $c = \gamma_i * (... * (\gamma_1 * \beta)....)$. By repeating (GE3), we get that

$$\gamma_i * (... * (\gamma_1 * (c * \beta))....) = 1 \in K,$$

Since $\gamma_i \in K$ and K is a GE-filter of R, we get $c * \beta \in K$. Put $\eta = \zeta * \beta$. Then, $\eta * (\zeta * \beta) = (\zeta * \beta) * (\zeta * \beta) = 1$ and hence $\beta \in J$. Therefore, $J = \langle K \cup G \rangle$. □

The intersection of two GE-filters K and G of a GE-algebra is a GE-filter of R, as can be seen. Also, $K \cap G$ is the infimum of both K and G, as can be shown. The class of all GE-filters in a GE-algebra R is designated by $\mathfrak{O}(R)$. The following theorem is now established.

Theorem 3.7 *Let R be a transitive GE-algebra. Then, $\mathfrak{O}(R)$ forms a complete distributive lattice.*

Proof For any two GE-filters K_1 and K_2 of a transitive GE-algebra, define

$$K_1 \vee K_2 = \langle K_1 \cup K_2 \rangle = \{\beta \in R \mid t * (r * \beta) = 1 \text{ for some } t \in K_1 \text{ and } r \in K_2\}.$$

Then, it is obvious that $(\mathfrak{O}(R), \cap, \vee)$ is a complete lattice with respect to set inclusion. Let $K_1, K_2, K_3 \in \mathfrak{O}(R)$. Then, obviously, $K_1 \cap (K_2 \vee K_3) \supseteq (K_1 \cap K_2) \vee (K_2 \cap K_3)$. Conversely, assume that $\beta \in K_1 \cap (K_2 \vee K_3)$. Then, $\beta \in K_1$ and $\beta \in K_2 \vee K_3$. Then, there are $c \in K_2$ and $d \in K_3$ such that $c * (d * \beta) = 1$. Let $\alpha_1 = d * \beta$ and $\alpha_2 = \alpha_1 * \beta$. It can be observed that $\alpha_1 \in K_1$ and $\alpha_2 \in K_1$. Now, $c * \alpha_1 = c * (d * \beta) = 1 \in K_2$ which implies that $\alpha_1 = d * \beta \in K_2$. Hence, $\alpha_1 \in K_1 \cap K_2$. Also, $d * \alpha_2 = d * (\alpha_1 * \beta) = d * ((d * \beta) * \beta) = d * ((d * \beta) * (d * \beta)) = d * 1 = 1 \in K_3$. Then, $\alpha_2 \in K_3$ and hence $\alpha_2 \in K_1 \cap K_3$. Now, by (GE3), $\alpha_1 * (\alpha_2 * \beta) = (d * \beta) * ((\alpha_1 * \beta) * \beta) = (d * \beta) * (((d * \beta) * \beta) * \beta) = 1$. Hence, $\beta \in (K_1 \cap K_2) \vee (K_1 \cap K_3)$. Hence, $K_1 \cap (K_2 \vee K_3) \subseteq (K_1 \cap K_2) \vee (K_1 \cap K_3)$. Thus, $(\mathfrak{O}(R), \cap, \vee)$ is a complete distributive lattice. □

Corollary 3.8 *Let R be a GE-algebra with transitivity. Then, with regard to the inclusion ordering $\subseteq$, the class $\mathfrak{O}(R)$ of all GE-filters of R is a complete lattice in which for any class $\{K_\zeta\}_{\zeta \in \Delta}$ of GE-filters of R, $inf\{K_\zeta\}_{\zeta \in \Delta} = \cap_{\zeta \in \Delta} K_\alpha$ and $sup\{K_\zeta\}_{\zeta \in \Delta} = \langle \cup_{\zeta \in \Delta} K_\alpha \rangle$.*

A GE-filter K of a GE-algebra R is said to be *proper* if $K \neq R$.

Definition 3.9 A proper GE-filter N of a GE-algebra R is said to be *maximal* if $\langle N \cup \{\beta\} \rangle = R$ for any $\beta \in R \setminus N$, where $\langle N \cup \{\beta\} \rangle$ is the GE-filter generated by $N \cup \{\beta\}$.

Example 3.10 Let $R = \{1, e, f, g, h, a, b\}$ be a set with the binary operation "$*$" in the following Cayley Table.

$*$	1	e	f	g	h	a	b
1	1	e	f	g	h	a	b
e	1	1	1	g	g	a	b
f	1	e	1	h	h	a	b
g	1	e	1	1	1	a	1
h	1	e	1	1	1	a	b
a	1	e	1	g	g	1	b
b	1	1	f	h	h	a	1

Then, $(R, *, 1)$ is a GE-algebra and $H := \{1, e, f, a, b\}$ is a proper GE-filter of R. Moreover, we have $\langle H \cup \{g\} \rangle = R = \langle H \cup \{h\} \rangle$, and so H is a maximal GE-filter of R.

We now have a necessary and sufficient condition for any proper GE-filter to be maximal.

Theorem 3.11 *Let R be a transitive GE-algebra. Then, a proper GE-filter K_1 of R is maximal if and only if $K_1 \subseteq K \subseteq R$ implies $K_1 = K$ or $K = R$ for any GE-filter K of R.*

Proof Suppose K_1 is a maximal GE-filter of R. Let K be a GE-filter of R such that $K_1 \subseteq K \subseteq R$. Suppose $K \neq R$. Then, we have to show that $K_1 = K$. Suppose $K_1 \neq K$. Then, there exists $\beta \in K$ such that $\beta \notin K_1$. Since K_1 is a maximal GE-filter of R, we have $\langle K_1 \cup \{\beta\} \rangle = R$. Let $\gamma \in R$. Then, $\gamma \in \langle K_1 \cup \{\beta\} \rangle$. Then, $\beta * \gamma \in K_1 \subseteq K$ and hence, $\gamma \in K$. Therefore, $K = K_1$. Conversely, assume that the condition holds. Let $\beta \in R \setminus K_1$. Suppose $\langle K_1 \cup \{\beta\} \rangle \neq R$. Chose $\gamma \notin \langle K_1 \cup \{\beta\} \rangle$ and $\gamma \in R$. Hence, $K_1 \subseteq \langle K_1 \cup \{\beta\} \rangle \subset R$. Then by assumption, we get $K_1 = \langle K_1 \cup \{\beta\} \rangle$. Hence, $\beta \in K_1$ which is a contradiction. Thus, K_1 is a maximal GE-filter of R. □

Definition 3.12 A proper GE-filter P of a GE-algebra R is said to be *prime* if $Q \cap H \subseteq P$ implies $Q \subseteq P$ or $H \subseteq P$ for any two GE-filters Q and H of R.

Example 3.13 Consider the GE-algebra R in Example 3.10. It is easy to verify that the set $K := R \setminus \{a\}$ is a prime GE-filter of R.

Theorem 3.14 *A proper GE-filter P of a GE-algebra R is prime if and only if $\langle \beta \rangle \cap \langle \gamma \rangle \subseteq P$ implies $\beta \in P$ or $\gamma \in P$ for all $\beta, \gamma \in R$.*

Proof Assume that P is a prime GE-filter of R. Let $\beta, \gamma \in R$ be such that $\langle \beta \rangle \cap \langle \gamma \rangle \subseteq P$. Since P is prime, it implies that $\beta \in \langle \beta \rangle \subseteq P$ or $\gamma \in \langle \gamma \rangle \subseteq P$. Conversely, assume that the condition holds. Let K and G be two GE-filters of R such that $K \cap G \subseteq P$. Let $\beta \in K$ and $\gamma \in G$. Then, $\langle \beta \rangle \subseteq K$ and $\langle \gamma \rangle \subseteq G$. Hence, $\langle \beta \rangle \cap \langle \gamma \rangle \subseteq K \cap G \subseteq P$. Then, $\beta \in P$ or $\gamma \in P$. Thus, $K \subseteq P$ or $G \subseteq P$. Therefore, P is a prime GE-filter of R. □

Theorem 3.15 *Let K be a GE-filter of a transitive GE-algebra R. Then, $P \cap L \subseteq K$ if and only if $\langle K \cup P \rangle \cap \langle K \cup L \rangle = K$, for any GE-filters P and L of R.*

Proof Let $\langle K \cup P \rangle \cap \langle K \cup L \rangle = K$. Since $P \subseteq \langle K \cup P \rangle$ and $L \subseteq \langle K \cup L \rangle$, we get that $P \cap L \subseteq \langle K \cup P \rangle \cap \langle K \cup L \rangle = K$. Therefore, $P \cap L \subseteq K$. Conversely, assume that $P \cap L \subseteq K$. Clearly, $K \subseteq \langle K \cup P \rangle \cap \langle K \cup L \rangle$. Let $t \in \langle K \cup P \rangle \cap \langle K \cup L \rangle$. Since K is a GE-filter of R, we get $(\beta_n * (\ldots * (\beta_1 * t)..)) \in K$, for some $n \in \mathbb{N}$ and $\beta_1, \beta_2, \ldots. \beta_n \in P$. It follows that, there exists $b_1 \in K$ such that $\beta_n * (\ldots * (\beta_1 * t)..) = b_1$. By the similar argument, we have $\gamma_m * (\ldots * (\gamma_1 * t)..) = b_2$, for some $m \in \mathbb{N}$, $\gamma_1, \gamma_2, \ldots, \gamma_m \in L$ and $b_2 \in K$. Hence, by repeating (GE3) and by (2.4), we get $\beta_n * (\ldots * (\beta_1 * (b_1 * t))..) = 1$. Hence, $b_1 * t \in P$. By the similar argument, we can show that $b_2 * t \in L$. Since $b_1, b_2 \in K$, $b_1 * t \leq b_2 * (b_1 * t)$ and

$b_2 * t \leq b_2 * (b_1 * t)$, we get $b_2 * (b_1 * t) \in P \cap L \subseteq K$. Hence, $t \in K$. Therefore, $\langle K \cup P\rangle \cap \langle K \cup L\rangle \subseteq K$. Thus, $\langle K \cup P\rangle \cap \langle K \cup L\rangle = K$ □

Corollary 3.16 *Let Q be a GE-filter of a transitive GE-algebra R. Then, for any $\eta, \zeta \in R$,*

$$\langle \eta\rangle \cap \langle \zeta\rangle \subseteq Q \text{ if and only if } \langle Q \cup \{\eta\}\rangle \cap \langle Q \cup \{\zeta\}\rangle = Q.$$

Theorem 3.17 *In a transitive GE-algebra, every maximal GE-filter is a prime GE-filter.*

Proof Let Q be a maximal GE-filter of a transitive GE-algebra R. Let $\langle \mu\rangle \cap \langle \nu\rangle \subseteq Q$ for some $\mu, \nu \in R$. Suppose $\mu \notin Q$ and $\nu \notin Q$. Then, $\langle Q\cup\{\mu\}\rangle = R$ and $\langle Q\cup\{\nu\}\rangle = R$. Hence, $\langle Q\cup\{\mu\}\rangle\cap\langle Q\cup\{\nu\}\rangle = R$. Hence, by Corollary 3.16, $\langle \mu\rangle \cap \langle \nu\rangle \nsubseteq Q$, which is a contradiction. Hence, $\mu \in Q$ or $\nu \in Q$. Therefore, Q is a prime GE-filter of R. □

Corollary 3.18 *Let R be a transitive GE-algebra. If $Q_1, Q_2, \ldots Q_n$ and Q are maximal GE-filters of R such that $\bigcap_{j=1}^{n} Q_j \subseteq Q$. There exists $i \in \{1, 2, \ldots, n\}$ such that $Q_i = Q$.*

The following example shows that the converse of Theorem 3.17 is not valid.

Example 3.19 Let $R = \{1, e, f, g, h, a, b\}$ be a set with the binary operation "$*$" in the following Cayley Table.

$*$	1	e	f	g	h	a	b
1	1	e	f	g	h	a	b
e	1	1	1	h	h	b	b
f	1	e	1	g	g	a	a
g	1	e	1	1	1	a	b
h	1	e	1	1	1	a	b
a	1	e	f	h	h	1	1
b	1	e	f	g	h	1	1

Then, $(R, *, 1)$ is a transitive GE-algebra. It is routine to verify that $L := \{1, e, f, g, h\}$ is a prime GE-filter of R. Note that $P := \{1, f, g, h\}$ is a GE-filter of R such that $P \subsetneq L \subsetneq R$. Hence, L is not a maximal GE-filter of R.

Theorem 3.20 *Let R be a transitive GE-algebra and G be a nonempty subset of R such that G is closed under "∂", where $\beta\partial\gamma := (\gamma * \beta) * \beta$, for any $\beta, \gamma \in G$. If K is a GE-filter of R such that $K \cap G = \emptyset$, then there exist a prime GE-filter M of R such that $K \subseteq M$ and $M \cap G = \emptyset$.*

Proof Let K be a GE-filter of R such that $K \cap G = \emptyset$. Consider

$$\mathfrak{G} = \{J \in \mathfrak{G}(R) \mid K \subseteq J \text{ and } J \cap G = \emptyset\}.$$

Clearly $K \in \mathfrak{G}$. Then, by Zorn's lemma, $\mathfrak{G}$ has a maximal element, say M. Then, $K \subseteq M$ and $M \cap G = \emptyset$. We prove that M is a prime GE-filter of R. Suppose there are GE-filter K, Q of R such that $K \cap Q \subseteq M$, $K \nsubseteq M$ and $Q \nsubseteq M$. By maximality of M, we have $\langle M \cup K\rangle \cap G \neq \emptyset$ and $\langle M\cup Q\rangle\cap G \neq \emptyset$. Let $t \in \langle M\cup K\rangle\cap G$ and $r \in \langle M\cup Q\rangle\cap G$. Since $t * (t\partial r) = t * ((r * t) * t) = 1$ and $r * (t\partial r) = r * ((r * t) * t) = 1$, we have $t\partial r \in \langle M \cup K\rangle \cap \langle M \cup Q\rangle$. Also, $t, r \in G$ and G is a GE-filter of R implies that $t\partial r \in G$. Hence, $t\partial r \in (\langle M \cup K\rangle \cap \langle M \cup Q\rangle) \cap G$. Therefore, $M \neq \langle M \cup K\rangle \cap \langle M \cup Q\rangle$. Hence, by Theorem 3.15, $K \cap Q \nsubseteq M$ which is a contradiction. Therefore, M is a prime GE-filter of R. □

Corollary 3.21 *Let R be a transitive GE-algebra. Then, the following holds:*

(1) *For any $\beta \in R \setminus \{1\}$, there exists a prime GE-filter M such that $\beta \notin M$.*
(2) $\bigcap\{M \mid M$ *is a prime GE-filter of* $R\} = \{1\}$.
(3) *Any proper GE-filter K of R can be expressed as the intersection of all prime GE-filters of R containing K.*

Theorem 3.22 *If a GE-algebra R is transitive, then $\mathfrak{G}(X)$ is a chain if and only if every proper GE-filter of R is a prime GE-filter.*

Proof Suppose that $\mathfrak{G}(X)$ is a chain. Let P be a proper GE-filter of R. Let $\mu, \nu \in R$ be such that $\langle \mu\rangle \cap \langle \nu\rangle \subseteq P$. Since $\langle \mu\rangle$ and $\langle \nu\rangle$ are GE-filters of R, we get either $\langle \mu\rangle \subseteq \langle \nu\rangle$ or $\langle \nu\rangle \subseteq \langle \mu\rangle$. Hence, $\mu \in P$ or $\nu \in P$. Therefore, P is a prime GE-filter of R.

Conversely, suppose that every proper GE-filter of R is a prime GE-filter of R. Let P and L be two proper GE-filters of R. Since $P \cap L$ is a proper GE-filter of R, we get $P \subseteq P \cap L$ or $L \subseteq P \cap L$. Hence, $P \subseteq L$ or $L \subseteq P$. Therefore, $\mathfrak{G}(X)$ is a chain. □

4 Elitable GE-filters

In this section, the concept of elitable GE-filters is introduced and characterized. Some basic properties of elitable GE-filters are observed in terms of maximal GE-filters.

Definition 4.1 Let K be a nonempty subset of a bordered GE-algebra R. Then, the *elitable* of K is denoted by $K^{\otimes}$ and is defined as $K^{\otimes} := \{\beta \in R \mid \beta^{ßß} \in K\}$.

Example 4.2 Let $R = \{0, 1, e, f, g, h, a, b\}$ be a set with the binary operation "$*$" in the following Cayley Table.

$*$	0	1	e	f	g	h	a
0	1	1	1	1	1	1	1
1	0	1	e	f	g	h	a
e	0	1	1	1	0	1	1
f	g	1	e	1	g	e	e
g	1	1	e	1	1	e	e
h	1	1	1	f	1	1	1
a	0	1	1	f	g	1	1

Then, $(R, *, 1)$ is a bordered GE-algebra. Given a nonempty subset K of R, we have:

$$K^{\otimes} = \begin{cases} \{0, g, h\} & \text{if } 0 \in K, 1 \notin K, \\ \{1, e, f, a\} & \text{if } 0 \notin K, 1 \in K, \\ \emptyset & \text{if } 0 \notin K, 1 \notin K, \\ R & \text{if } 0 \in K, 1 \in K, \end{cases}$$

Given a nonempty subset K in a bordered GE-algebra R, the elitable of K may not be a GE-filter of R as seen in the following example.

Example 4.3 In Example 4.2, if we take $K := \{1, g, h\}$, then $K^{\otimes} = \{1, e, f, a\}$ and it is not a GE-filter of R since $e*h = 1$ and $e \in K^{\otimes}$ but $h \notin K^{\otimes}$.

Lemma 4.4 *Let R be a bordered GE-algebra and consider two elitable subsets K and G of R. Then, the following holds:*

$$K \subseteq G \Rightarrow K^{\otimes} \subseteq G^{\otimes}. \tag{4.1}$$

$$(K \cap G)^{\otimes} = K^{\otimes} \cap G^{\otimes}. \tag{4.2}$$

Proof Suppose $K \subseteq G$ and $\beta \in K^{\otimes}$. Then, $\beta^{\text{ßß}} \in K \subseteq G$ and hence, $\beta \in G^{\otimes}$. Thus, (4.1) holds. Let $\beta \in (K \cap G)^{\otimes}$. Then, $\beta^{\text{ßß}} \in K \cap G$. Hence, $\beta^{\text{ßß}} \in K$ and $\beta^{\text{ßß}} \in G$. Therefore, $\beta \in K^{\otimes} \cap G^{\otimes}$. Hence, $(K \cap G)^{\otimes} \subseteq K \cap G$. Suppose $\beta \in K^{\otimes} \cap G^{\otimes}$. Then, $\beta \in K^{\otimes}$ and $\beta \in G^{\otimes}$. Hence, $\beta^{\text{ßß}} \in K$ and $\beta^{\text{ßß}} \in G$. Therefore, $\beta^{\text{ßß}} \in K \cap G$ and hence, $\beta \in (K \cap G)^{\otimes}$. Therefore, $K^{\otimes} \cap G^{\otimes} \subseteq (K \cap G)^{\otimes}$. Thus, (4.2) holds. □

Lemma 4.5 *Let R be a bordered GE-algebra which is transitive. Then, for any $\beta, \gamma \in R$, we have*

(1) $\beta^{\text{ßßß}} \le \beta^{\text{ß}}$,
(2) $\beta * \gamma^{\text{ß}} \le \beta^{\text{ßß}} * y^{\text{ß}}$,
(3) $(\beta * \gamma^{\text{ßß}})^{\text{ßß}} \le \beta * \gamma^{\text{ßß}}$,
(4) $(\beta^{\text{ß}} * \gamma^{\text{ß}})^{\text{ßß}} \le \beta^{\text{ß}} * \gamma^{\text{ß}}$,
(5) $(\beta * \gamma)^{\text{ßß}} \le \beta^{\text{ßß}} * \gamma^{\text{ßß}}$.

Proof (1). Let $\beta \in R$. Then, by (GE1), (2.7) and (2.25),

$$\begin{aligned} 1 &= (\beta * 0) * (\beta * 0) \le \beta * ((\beta * 0) * 0) \\ &= \beta * \beta^{\text{ßß}} \le \beta^{\text{ßßß}} * \beta^{\text{ß}}. \end{aligned}$$

Hence, $\beta^{\text{ßßß}} * \beta^{\text{ß}} = 1$, which gives $\beta^{\text{ßßß}} \le \beta^{\text{ß}}$.

(2). Let $\beta, \gamma \in R$. Then, by (2.21) and (2.25), $\beta * \gamma^{\text{ß}} \le \gamma * \beta^{\text{ß}} \le \beta^{\text{ßß}} * \gamma^{\text{ß}}$.

(3). Let $\beta, \gamma \in R$. We can observe that $(\beta * \gamma^{\text{ßß}})^{\text{ß}} \le (\beta * \gamma^{\text{ßß}})^{\text{ßßß}}$. By (2.12), we get $\gamma^{\text{ß}} * (\beta * \gamma^{\text{ßß}})^{\text{ß}} \le \gamma^{\text{ß}} * (\beta * \gamma^{\text{ßß}})^{\text{ßßß}}$ and so $\beta * (\gamma^{\text{ß}} * (\beta * \gamma^{\text{ßß}})^{\text{ß}}) \le \beta * (\gamma^{\text{ß}} * (\beta * \gamma^{\text{ßß}})^{\text{ßßß}})$. Hence, by (GE1),(2.7), (2.20) and (2.21), we get

$$\begin{aligned} 1 &= (\beta * \gamma^{\text{ßß}}) * (\beta * \gamma^{\text{ßß}}) \\ &\le \beta * ((\beta * \gamma^{\text{ßß}}) * \gamma^{\text{ßß}}) \\ &\le \beta * (\gamma^{\text{ß}} * (\beta * \gamma^{\text{ßß}})^{\text{ß}}) \\ &\le \beta * (\gamma^{\text{ß}} * (\beta * \gamma^{\text{ßß}})^{\text{ßßß}}) \\ &\le \beta * ((\beta * \gamma^{\text{ßß}})^{\text{ßß}} * \gamma^{\text{ßß}}) \\ &\le (\beta * \gamma^{\text{ßß}})^{\text{ßß}} * (\beta * \gamma^{\text{ßß}}). \end{aligned}$$

Thus, $(\beta * \gamma^{\text{ßß}})^{\text{ßß}} * (\beta * \gamma^{\text{ßß}}) = 1$. Therefore, $(\beta * \gamma^{\text{ßß}})^{\text{ßß}} \le (\beta * \gamma^{\text{ßß}})$.

(4). By (2.21), we have $\beta^{\text{ß}} * \gamma^{\text{ß}} \le \gamma * \beta^{\text{ßß}}$. Hence, by (2.25), (3) and (2.21), we get

$$(\beta^{\text{ß}} * \gamma^{\text{ß}})^{\text{ßß}} \le (\gamma * \beta^{\text{ßß}})^{\text{ßß}} \le \gamma * \beta^{\text{ßß}} \le \beta^{\text{ß}} * \gamma^{\text{ß}}.$$

(5). By (2.25), we get $\beta * \gamma \le \beta^{\text{ßß}} * \gamma^{\text{ßß}}$. Hence, $(\beta * \gamma)^{\text{ßß}} \le (\beta^{\text{ßß}} * \gamma^{\text{ßß}})^{\text{ßß}}$. Also, by (4), we can observe that $(\beta^{\text{ßß}} * \gamma^{\text{ßß}})^{\text{ßß}} \le \beta^{\text{ßß}} * \gamma^{\text{ßß}}$. Hence, (5) follows, since R is transitive. □

Theorem 4.6 *Let R be a bordered GE-algebra which is transitive. Then, for any GE-filter K of R, we have the following:*

(1) *$K^{\otimes}$ is a GE-filter of R.*
(2) *$K \subseteq K^{\otimes}$.*
(3) *$(K^{\otimes})^{\otimes} = K^{\otimes}$.*

Proof (1). Since $1^{\text{ßß}} = 1 \in K$, we have $1 \in K^{\otimes}$. Let $\eta \in K^{\otimes}$ and $\eta * \zeta \in K^{\otimes}$. Then, $\eta^{\text{ßß}} \in K$ and $(\eta * \zeta)^{\text{ßß}} \in K$. Since $(\eta * \zeta)^{\text{ßß}} \le \eta^{\text{ßß}} * \zeta^{\text{ßß}}$ by Lemma 4.5(5) and $(\eta * \zeta)^{\text{ßß}} \in K$, we have $\eta^{\text{ßß}} * \zeta^{\text{ßß}} \in K$. Therefore, $\zeta^{\text{ßß}} \in K$ and hence, $\zeta \in K^{\otimes}$. Thus, $K^{\otimes}$ is a GE-filter of R.

(2). Let $\eta \in K$. Since $\eta \le \eta^{\text{ßß}}$ and $\eta \in K$, we have $\eta^{\text{ßß}} \in K$. Hence, $\eta \in K^{\otimes}$. Thus, $K \subseteq K^{\otimes}$.

(3). Clearly $K^{\otimes} \subseteq (K^{\otimes})^{\otimes}$ by (2). Let $\eta \in (K^{\otimes})^{\otimes}$. Then, $\eta^{\text{ßß}} \in K^{\otimes}$ and hence, $\eta^{\text{ßßßß}} \in K$. Since $\eta^{\text{ß}} \le \eta^{\text{ßßß}}$ implies that $\eta^{\text{ßßßß}} \le \eta^{\text{ßß}}$ by (2.24), and K is a GE-filter of R, we have $\eta^{\text{ßß}} \in K$. Therefore, $\eta \in K^{\otimes}$. Thus, $K^{\otimes} = (K^{\otimes})^{\otimes}$. □

The following example shows that if R is a bordered GE-algebra which is not transitive, then the elitable of a GE-filter K of R may not be a GE-filter of R.

Example 4.7 Let $R = \{0, 1, e, f, g, h\}$ be a set with a binary operation $*$ given in the following table:

$*$	0	1	e	f	g	h
0	1	1	1	1	1	1
1	0	1	e	f	g	h
e	0	1	1	1	0	1
f	g	1	e	1	g	1
g	1	1	e	1	1	1
h	1	1	e	1	1	1

Then, R is a bordered GE-algebra which is not transitive. Let $K = \{1\}$. Then, K is a GE-filter of R and its elitable is $K^{\otimes} = \{1, e, f\}$. But $K^{\otimes}$ is not a GE-filter of R since $e * h = 1 \in K^{\otimes}$ and $e \in K^{\otimes}$ but $h \notin K^{\otimes}$.

Definition 4.8 Let R be a bordered GE-algebra. Then, a nonempty subset K of R is called an *elitable GE-filter* of R if it is a GE-filter of R and its elitable is K itself.

Example 4.9 Consider a 4−element Boolean algebra $R := \{0, 1, e, e'\}$ with the partial order $\leq$. If we define

$$x * y = \begin{cases} 1 \text{ if } x \leq y \\ y \text{ otherwise} \end{cases}$$

then $(R, *, 1)$ is a bordered GE-algebra. Let $K = \{1, e, e'\}$. Then, it can be easily verified that K is an *elitable GE-filter* of R.

Example 4.10 In Example 4.2, we can observe that $G = \{1, e, f, a\}$ is an elitable GE-filter of R.

Theorem 4.11 *Let R be a bordered GE-algebra. Then, the intersection of two elitable GE-filters of R is also an elitable GE-filter of R.*

Proof Let K_1 and K_2 be two elitable GE-filters of R. Then, $K_1^{\otimes} = K_1$ and $K_2^{\otimes} = K_2$ which imply from (4.2) that

$$K_1 \cap K_2 = K_1^{\otimes} \cap K_2^{\otimes} = (K_1 \cap K_2)^{\otimes}.$$

Hence, $K_1 \cap K_2$ is an elitable GE-filter of R. □

In the following example, we can find two elitable GE-filters of R whose union is not an elitable GE-filter of R.

Example 4.12 Let $R = \{0, 1, e, f, g, h\}$ be a set with a binary operation $*$ given in the following table:

$*$	0	1	e	f	g	h
0	1	1	1	1	1	1
1	0	1	e	f	g	h
e	h	1	1	1	h	h
f	g	1	e	1	g	g
g	f	1	e	f	1	1
h	f	1	e	f	1	1

Then, R is a bordered GE-algebra. Let $K_1 = \{1, e, f\}$ and $K_2 = \{1, g, h\}$. Then, it is routine to verify that K_1 and K_2 are elitable GE-filters of R. But $K_1 \cup K_2 = \{1, e, f, g, h\}$ is not an elitable GE-filter of R since $f * 0 = g \in K_1 \cup K_2$ and $f \in K_1 \cup K_2$ but $0 \notin K_1 \cup K_2$.

Theorem 4.13 *For any transitive bordered GE-algebra R, the class $\mathfrak{O}^{\otimes}(R)$ of all elitable GE-filters of R is a complete distributive lattice.*

Proof For any two elitable GE-filters K and G of R, define a relation $\leq$ on $\mathfrak{O}^{\otimes}(X)$ by $K \leq G \Leftrightarrow K \subseteq G$. Then, it can be observed that $(\mathfrak{O}^{\otimes}(X), \leq)$ is a partially ordered set. Consider $K \cap G = (K \cap G)^{\otimes}$ and $K \sqcup G = (K \vee G)^{\otimes}$, where

$$\begin{aligned} &K \vee G \\ &\quad = \{\beta \in R \mid t * (r * \beta) = 1 \text{ for some } t \in K \text{ and } r \in G\}. \end{aligned}$$

Obviously $(K \cap G)^{\otimes}$ is infimum of K and G in $\mathfrak{O}^{\otimes}(X)$. Also, $(K \vee G)^{\otimes}$ is the upper bound of $K^{\otimes}$ and $G^{\otimes}$. Let M be any elitable GE-filter of R such that $K \subseteq M$ and $G \subseteq M$. Let $\beta \in (K \vee G)^{\otimes}$. Then, $\beta^{ßß} \in K \vee G$, and hence, there exist $t \in K$ and $r \in G$ such that $t * (r * \beta^{ßß}) = 1$. Since $K \subseteq M$ and $G \subseteq M$, we get $\beta^{ßß} \in M = M^{\otimes}$. Hence, $\beta \in M$. Therefore, $(K \vee G)^{\otimes}$ is supremum of K and G in $\mathfrak{O}^{\otimes}(X)$. Hence, $(\mathfrak{O}^{\otimes}(X), \cap, \sqcup)$ is a lattice. We can observe that the set theoretical intersection of an arbitrary set of elitable GE-filters of R is an elitable GE-filter of R again by (4.2). Hence, the set $\mathfrak{O}^{\otimes}(X)$ forms a complete lattice with respect to set inclusion. The least and greatest element of $\mathfrak{O}^{\otimes}(X)$ are $\{1\}$ and R, respectively. Now, for any $K, G, M \in \mathfrak{O}^{\otimes}(X)$, we obtain

$$\begin{aligned} (K \sqcup G) \cap (K \sqcup M) &= (K \vee G)^{\otimes} \cap (K \vee M)^{\otimes} \\ &= ((K \vee G) \cap (K \vee M))^{\otimes} \\ &= (K \vee (G \cap M))^{\otimes} \\ &= K \sqcup (G \cap M). \end{aligned}$$

Therefore, $(\mathfrak{O}^{\otimes}(R), \sqcup, \cap, \{1\}, R)$ is a distributive lattice. □

Theorem 4.14 *Given a GE-filter K of a transitive bordered GE-algebra R, the following are equivalent:*

(1) *K is an elitable GE-filter.*
(2) *$\eta^{ßß} \in K$ implies $\eta \in K$, for all $\eta \in R$.*
(3) *$(\eta * \zeta^{ß})^{ß} \in K$ implies $\zeta \in K$, for all $\eta, \zeta \in R$.*

Proof (1) ⇒ (2) Assume (1) and $\eta^{ßß} \in K$. Then, $\eta \in K^{\otimes} = K$. Thus, (2) follows.

(2) ⇒ (3) Assume (2) and $\eta, \zeta \in R$ such that $(\eta * \zeta^{ß})^{ß} \in K$. We can observe that $\zeta^{ß} \leq \eta * \zeta^{ß}$ and hence, $(\eta * \zeta^{ß})^{ß} \leq$

$\zeta^{ßß}$. Since K is an GE-filter of R and $(\eta * \zeta^{ß})^{ß} \in K$, we get $\zeta^{ßß}$. Therefore, $\zeta \in K$ by (2). Thus, (3) follows.

(3) $\Rightarrow$ (1) Assume (3). Let $\eta \in K^{\otimes}$. Then, $\eta^{ßß} \in K$. Therefore, $(1 * \eta^{ß})^{ß} = \eta^{ßß} \in K$ and hence, $\eta \in K$ by (3). Thus, $K^{\otimes} \subseteq K$, and hence, K is an elitable GE-filter of R. $\square$

Theorem 4.15 *If K is an elitable GE-filter of a transitive bordered GE-algebra R, then for $\beta, \gamma \in R$, $\beta^{ß} = \gamma^{ß}$ and $\beta \in K$ imply that $\gamma \in K$.*

Proof Let K be an elitable GE-filter of R and $\beta, \gamma \in R$ be such that $\beta^{ß} = \gamma^{ß}$ and $\beta \in K$. Then, $\beta \leq \beta^{ßß} = \gamma^{ßß}$ and hence, $\gamma^{ßß} \in K$ since K is GE-filter of R. Therefore, $\gamma \in K$ by Theorem 4.14. $\square$

Theorem 4.16 *Every maximal GE-filter of a transitive bordered GE-algebra R is an elitable GE-filter of R.*

Proof Let H be a maximal GE-filter of R. Clearly, $H \subseteq H^{\otimes}$ by Theorem 4.6(2). Now, we prove that $H^{\otimes} \subseteq H$. Suppose $H^{\otimes} \nsubseteq H$. Then, there exists $\beta \in H^{\otimes}$ such that $\beta \notin H$. Hence, $\beta^{ßß} \in H$ and $\langle H \cup \{\beta\}\rangle = R$. Since $0 \in R$, we have $\beta^{ß} = \beta * 0 \in H$. Since $0 \leq \beta$, we have $(\beta * 0) * 0 \leq (\beta * 0) * \beta$ by (2.12). That is $\beta^{ßß} \leq \beta^{ß} * \beta$. Since $\beta^{ß}, \beta^{ßß} \in H$ and H is a GE-filter of R, we get $\beta \in H$ which is a contradiction. Hence, $H^{\otimes} \subseteq H$. Thus, H is an elitable GE-filter of R. $\square$

The following example shows that the converse of Theorem 4.16 is not valid.

Example 4.17 Let $R = \{0, 1, e, f, g, h, a\}$ be a set with the binary operation "$*$" in the following Cayley Table.

$*$	0	1	e	f	g	h	a
0	1	1	1	1	1	1	1
1	0	1	e	f	g	h	a
e	0	1	1	f	a	h	a
f	0	1	1	1	g	h	g
g	h	1	1	1	1	h	1
h	a	1	1	f	a	1	a
a	h	1	1	1	1	h	1

Then, $(R, *, 1)$ is a transitive bordered GE-algebra. Consider two GE-filters $K = \{1, e, f\}$ and $G = \{1, e, f, g, a\}$ of R. Then, we can observe that K is an elitable GE-filter of R, but K is not maximal GE-filter of R since $K \subseteq G \subseteq R$ but $K \neq G$ and $G \neq R$.

We provide conditions for a subset of a transitive bordered GE-algebra to be an elitable GE-filter.

Theorem 4.18 *Let K be a nonempty subset of a transitive bordered GE-algebra R such that its elitable is K itself. If K satisfies (2.18) and*

$$(\forall \eta, \zeta \in R)(\eta, \zeta \in K \ \Rightarrow \ (\eta * \zeta^{ß})^{ß} \in K), \tag{4.3}$$

then K is an elitable GE-filter of R.

Proof Suppose $K^{\otimes} = K$ and satisfies (2.18) and (4.3). Since $K \neq \emptyset$, there exists $\eta \in K$. As $\eta \leq 1$, we have $1 \in K$ by (2.18). Let $\eta, \zeta \in R$ be such that $\eta \in K$ and $\eta * \zeta \in K$. Then, $\eta^{ßß} \in K$ and $(\eta * \zeta)^{ßß} \in K$ by (2.10) and (2.18), which induces $(\eta^{ßß} * (\eta * \zeta)^{ßßß})^{ß} \in K$ by (4.3). Since $(\eta^{ßß} * (\eta * \zeta)^{ßßß})^{ß} \leq (\eta^{ßß} * (\eta * \zeta)^{ß})^{ß}$ and $(\eta^{ßß} * (\eta * \zeta)^{ßßß})^{ß} \in K$, we have $(\eta^{ßß} * (\eta * \zeta)^{ß})^{ß} \in K$ by (2.18). Now, $(\eta * \zeta) * \eta^{ß} \leq (\eta * \zeta) * \eta^{ßßß} \leq \eta^{ßß} * (\eta * \zeta)^{ß}$ by (2.21). Hence, it follows from (2.24) and (2.18) that $((\eta * \zeta) * x^{ß})^{ß} \in K$. Since $\zeta^{ß} = \zeta * 0 \leq (\eta * \zeta) * (\eta * 0)$ by (2.2), we have $((\eta * \zeta) * (\eta * 0))^{ß} \leq \zeta^{ßß}$ by (2.24) and hence $\zeta^{ßß} \in K$ by (2.18). Thus, $\zeta \in K$, and so K is a GE-filter of R. Therefore, K is an elitable GE-filter of R. $\square$

Corollary 4.19 *If R is antisymmetric, then, for all $\eta, \zeta \in R$, the following are equivalent.*

(1) *R is duplex.*
(2) *$\eta^{ß} = \zeta^{ß}$ implies $\eta = \zeta$.*
(3) *$\eta^{ß} * \zeta^{ß} = \zeta * \eta$.*

Corollary 4.20 *If R is antisymmetric, then the conditions below are equivalent.*

(1) *R is duplex.*
(2) *Every GE-filter is an elitable GE-filter.*
(3) *Every principal GE-filter is an elitable GE-filter.*

Open Problem. We were unable to identify an example of an elitable GE-filter for an infinite GE-algebra. Is it possible to construct an elitable GE-filter for an infinite GE-algebra?

Conclusion

In this paper, we have introduced the notions of maximal GE-filter and prime GE-filter in a GE-algebra and studied the relation between them. We have characterized prime GE-filter in terms of the GE-filter generated by a subset of a transitive GE-algebra. We generalized Stone's theorem to transitive GE-algebras. We have introduced the notion of elitable GE-filter of a bordered GE-algebra and investigated its properties. We have observed that the class of all elitable GE-filters of a transitive bordered GE-algebra is a complete distributive lattice. We have given equivalent conditions for a GE-filter of a transitive bordered GE-algebra to be elitable GE-filter. We have provided conditions for a subset of a transitive bordered GE-algebra to be elitable GE-filter.

Acknowledgements This paper was supported by Wonkwang University in 2022.

Funding Funding was provided by Wonkwang University.

Data Availability No data were used to support this study.

Declarations

Conflicts of interest The authors declare that they have no conflicts of interest.

References

Bandaru RK, Borumand Saeid A, Jun YB (2021) On GE-algebras. Bull Sect Logic 50(1):81–96. https://doi.org/10.18778/0138-0680.2020.20

Bandaru RK, Öztürk MA, Jun YB (2021) Bordered GE-algebras, J Algeraic Syst (submitted)

Celani S (2002) A note on homomorphisms of Hilbert algebras. Int J Math Math Sci 29(1):55–61. https://doi.org/10.1155/S0161171202011134

Chajda I, Halas R, Jun YB (2002) Annihilators and deductive systems in commutative Hilbert algebras. Comment Math Univ Carol 43(3):407–417

Diego A (1966) Sur les algebres de Hilbert. Collection de Logique Mathematique, Edition Hermann, Serie A, XXI

Dudek WA, Jun YB (2007) Poor and crazy filters of BCK-algebras. Int J Pure Appl Math Sci 4:25–38

Hong SM, Jun YB (1996) On deductive systems of Hilbert algebras. Commun Korean Math Soc 11(3):595–600

Imai Y, Iséki K (1966) On axiom system of propositional calculi, XIV. Proc Japan Acad Ser A Math Sci 42(1):19–22

Iséki K (1966) An algebra related with a propositional calculus. Proc Japan Acad Ser A Math Sci 42(1):26–29

Jun YB, Kim KH (2005) H-filters of Hilbert algebras. Sci Math Japonicae e–2005:231–236

Kim HS, Kim YH (2006) On BE-algebras. Sci Math Japonicae Online e–2006:1299–1302

Ozturk MA, Lee JG, Bandaru RK, Jun YB (2021) Strong GE-filters and GE-ideals of bordered GE-algebras, J Math, Volume 2021, Article ID 5520023, 9 p. https://doi.org/10.1155/2021/5520023

Rezaei A, Borumand Saeid A, Borzooei RA (2013) Relation between Hilbert algebras and BE-algebras. Appl Appl Math Int J 8(2):573–584

Borumand Saeid A, Rezaei A, Borzooei RA (2013) Some types of filters in BE-algebras. Math Comput Sci 7:341–352. https://doi.org/10.1007/s11786-013-0157-6

Rezaei A, Bandaru RK, Borumand Saeid A, Jun YB (2021) Prominent GE-filters and GE-morphisms in GE-algebras. Afrika Matematika 32:1121–1136. https://doi.org/10.1007/s13370-021-00886-6

Song SZ, Bandaru RK, Jun YB (2021) Imploring GE-filters of GE-algebras, J Math, Volume 2021, Article ID 6651531, 7 pages. https://doi.org/10.1155/2021/6651531