Similarity measures of Pythagorean fuzzy soft sets and clustering analysis

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Abstract Pythagorean fuzzy set (PFS) is a broadening of intuitionistic fuzzy set that can represent the situations where the sum of membership and the non-membership values exceeds one. Adding parameterization to PFS we obtain a structure named as Pythagorean fuzzy soft set (PFSS). It has a higher capacity to deal with vagueness as it captures both the structures of a PFS and a soft set. Several practical situations demand the measure of similarity between two structures, whose sum of membership value and non-membership value exceeds one. There are no existing tools to measure the similarity between PFSS and this paper put forward similarity measures for PFSS. An axiomatic definition for similarity measure is proposed for PFSS and certain expressions for similarity measure are introduced. Further, some theorems which express the properties of similarity measures are proved. A comparative study between proposed expressions for similarity measure is carried out. Also, a clustering algorithm based on PFSS is introduced by utilizing the proposed similarity measure.
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1 Introduction

A measure of analogy between the objects is necessary for many real-life problems like clustering, pattern recognition, sequence alignment, medical diagnosis, etc. This motivates to introduce similarity measures and caste its useful expressions. Since all the problems mentioned above have to cope with uncertainties, the notion of similarity measure is explored in many generalized set-theoretical concepts that can handle vagueness.

In 1965, Zadeh's fuzzy set (FS) theory [35] was a paradigm shift in this regard, as it decreases the space between real life and mathematics. The FS points out the membership value of objects in $[0, 1]$ and which surpasses the crisp set (whose range of membership value is 0 and 1). For this reason, there are a lot of research works on FSs in the literature and many extensions are attained. We mention some of them here. Intuitionistic fuzzy set (IFS), introduced by Atanassov [2], is capable to explain the objects' membership value together with nonmembership value, μ and ν respectively with the property, $0 \leq \mu + \nu \leq 1$. Torra figured out the Hesitant fuzzy set (HFS) [30], which can overcome the difficulty that hesitancy of giving membership value by establishing more than one membership value of an object. Yager's Pythagorean fuzzy set (PFS) [34] is a broadening of IFS, which relaxes the dependency of membership and nonmembership value by $0 \leq \mu^2 + \nu^2 \leq 1$. Similarly, several generalizations of fuzzy sets are introduced for representing vagueness happening in different situations.

In 1999, Molodtsov [22] introduced soft sets and it is an adequate tool to handle vagueness in a parametric manner. A soft set is able to view as a bag with an approximate representation of the objects and involves two components that are predicate and approximate value set. As the initial description has an approximate nature, the machinery of traditional mathematics fails whereas the soft set can be used to manage many problems in this aspect. By considering this importance, several studies based on soft sets are done, and some of them can be seen in [18,1,6,15]. Later on, hybrid structures like fuzzy soft sets (FSS) [8], intuitionistic fuzzy soft set (IFSS) [19], Pythagorean fuzzy soft set (PFSS) [28] are introduced which can represent vagueness as well.

PFS is an extension of IFS. In contrast to IFS, PFS allows one to consign a membership value and a non-membership value resulting sum can exceed one. Studies on aggregation operations, decision-making problems, multi-criteria decision-making problems using PFS are appeared in papers of Garg et al. [9, 10]. Also, Peng et al. [27] give an extensive review of PFS. PFSS is a hybrid structure of the soft set and PFS, that can effectively deal with vagueness. Impressively PFSS grasp the properties of PFS and soft sets together. The structure, PFSS was established by Peng et al. [28]. After which Guleria et al. [12] gave the matrix representation, operation, and decision-making problems of PFSS. Later on, Athira et al. [4,5] studied the entropy and distance measure of PFSS. Recently, topological and group structures on PFSS are introduced and that can be seen in [25,13] also Athira and Sunil proposed the incomplete PFSS [3].

As mentioned earlier, the similarity measure is a significant implement to estimate the degrees of association among two or more items. The idea of similarity measure for FS was initially introduced by Wang [31]. Meanwhile several studies were done on similarity measure of FS [14,33] and after which the similarity measure was extended to IFS [17], PFS [26] etc. Also, Majumdar and Samanta defined certain similarity measure for soft sets [21] and FSS [20]. Later on, it has been extended to IFSS [7]. Among various structures discussed so far, PFSS being the most generalized one. Vagueness can be represented more effectively by PFSS, and in this case, comparing their affinity is not readily available in the existing literature. This paves the way to the necessity of introducing similarity measures for PFSS.

This paper contains three sections besides the introduction. Section 2 includes basic definitions that are required for future study. In section 3, definitions of similarity measures and certain properties of the proposed measures are inserted. The final section consists of the applications of the new measures to the clustering algorithm as well as the benefits of proposed similarity measures with the already established measures.

2 Preliminaries

This section addresses elementary definitions beneficial for entire discussions. That is, the main definitions connected with PFS, soft sets, and PFSS. Unless otherwise specified, $\mho$ be the universal set, $\mathfrak{L}$ be parameter set, and $P(\mho)$ be the power set of $\mho$.

Definition 1 [34] A Pythagorean fuzzy set $\mathcal{P}$ on $\mho$ is the set $\{(u, \mu_p(u), \nu_p(u)) : u \in \mho\}$ where $\mu_p : \mho \to [0,1]$ and $\nu_p : \mho \to [0,1]$ with $0 \leq \mu_p^2 + \nu_p^2 \leq 1$.

Definition 2 [34] Let $p_1 = (\mu_1, \nu_1)$ and $p_2 = (\mu_2, \nu_2)$ be two Pythagorean fuzzy numbers.The set operations are given by;

1. $p_1 \subseteq p_2$ if $\mu_1 \leq \mu_2$ & $\nu_1 \geq \nu_2$
2. $p_1 \cup p_2 = (\max\{\mu_1, \mu_2\}, \min\{\nu_1, \nu_2\})$
3. $p_1 \cap p_2 = (\min\{\mu_1, \mu_2\}, \max\{\nu_1, \nu_2\})$
4. $p^c = (\nu, \mu)$

Definition 3 [36] Let $PFS(\mho)$ be the set of all PFSs over $\mho$ and ψ be a mapping $\psi : PFS(\mho) \times PFS(\mho) \to [0,1]$. Then ψ is a similarity measure of PFSs if it holds:

1. $(\varphi_1, \varphi_2) = 1 \Leftrightarrow \varphi_1 = \varphi_2$
2. $(\varphi_1, \varphi_2) = \psi\,(\varphi_2, \varphi_1)$
3. $(\varphi_1, P_3) \leq \psi(\varphi_1, P_2) \wedge \psi(\varphi_2, \varphi_3)$ if $\varphi_1 \subseteq \varphi_2 \subseteq \varphi_3$ $\forall \varphi_1, \varphi_2, \varphi_3 \in PFS(\mho)$.

Definition 4 [22] The pair $(\mathcal{S}, \mathfrak{L})$ is soft set over $\mho$, if $\mathcal{S}$ is a mapping from $\mathfrak{L}$ to $P(\mho)$.

Definition 5 [18] Let $(\mathcal{S}_1, \mathfrak{L}_1)$ and $(\mathcal{S}_2, \mathfrak{L}_2)$ are soft sets over $\mho$. We can define the set operations as follows;

1. $(\mathcal{S}_1, \mathfrak{L}_1)\widetilde{\subseteq}(\mathcal{S}_2, \mathfrak{L}_2)$ if $\mathcal{S}_1(\mathfrak{h}) \subseteq \mathcal{S}_2(\mathfrak{h})$ and $\mathfrak{L}_1 \subseteq \mathfrak{L}_2$, $\forall \mathfrak{h} \in \mathfrak{L}_1$. We say $(\mathcal{S}_1, \mathfrak{L}_1)$ and $(\mathcal{S}_2, \mathfrak{L}_2)$ are soft equal if $(\mathcal{S}_1, \mathfrak{L}_1)\widetilde{\subseteq}(\mathcal{S}_2, \mathfrak{L}_2)$ and $(\mathcal{S}_2, \mathfrak{L}_2)\widetilde{\subseteq}(\mathcal{S}_1, \mathfrak{L}_1)$.
2. The complement $((\mathcal{S}_1, \mathfrak{L}_1)^c)$ of $(\mathcal{S}_1, \mathfrak{L}_1)$ is the soft set $({\mathcal{S}_1}^c, \mathfrak{L}_1)$, where $\mathcal{S}_1^c : \mathfrak{L}_1 \to P(\mathcal{U})$ such that $\mathcal{S}_1^c(\mathfrak{h}) = \{\mathcal{S}_1(\mathfrak{h})\}^c = \mho - \mathcal{S}_1(\mathfrak{h})$, $\forall \mathfrak{h} \in \mathfrak{L}_1$.
3. The union of $(\mathcal{S}_1, \mathfrak{L}_1)$ and $(\mathcal{S}_2, \mathfrak{L}_2)$ is denoted by $(\mathcal{S}_1, \mathfrak{L}_1)\widetilde{\cup}(\mathcal{S}_2, \mathfrak{L}_2)$ and is a soft set $(\mathcal{S}_3, \mathfrak{L}_1 \cup \mathfrak{L}_2)$ where $\mathcal{S}_3 : \mathfrak{L}_1 \cup \mathfrak{L}_2 \to P(\mho)$ is given by,
$$\mathcal{S}_3(\mathfrak{h}) = \begin{cases} \mathcal{S}_1(\mathfrak{h}), \text{ for } \mathfrak{h} \in \mathfrak{L}_1 - \mathfrak{L}_2 \\ \mathcal{S}_2(\mathfrak{h}), \text{ for } \mathfrak{h} \in \mathfrak{L}_2 - \mathfrak{L}_1 \\ \mathcal{S}_1(\mathfrak{h}) \cup \mathcal{S}_2(\mathfrak{h}), \text{ for } \mathfrak{h} \in \mathfrak{L}_1 \cap \mathfrak{L}_2 \end{cases}.$$
4. The intersection of $(\mathcal{S}_1, \mathfrak{L}_1)$ and $(\mathcal{S}_2, \mathfrak{L}_2)$ is denoted by $(\mathcal{S}_1, \mathfrak{L}_1)\widetilde{\cap}(\mathcal{S}_2, \mathfrak{L}_2)$ and is a soft set $(\mathcal{S}_4, \mathfrak{L}_1 \cap \mathfrak{L}_2)$ where $\mathcal{S}_4 : \mathfrak{L}_1 \cap \mathfrak{L}_2 \to P(\mho)$ is given by $\mathcal{S}_4(\mathfrak{h}) = \mathcal{S}_1(\mathfrak{h}) \cap \mathcal{S}_2(\mathfrak{h}), \quad \forall \mathfrak{h} \in \mathfrak{L}_1 \cap \mathfrak{L}_2$.

Definition 6 [16] Let $\mathtt{S}(\mho)$ denotes the collection of all soft sets over $\mho$. A similarity measure is a map $\psi : \mathtt{S}(\mho) \times \mathtt{S}(\mho) \to [0, 1]$ with following properties; for $(\mathcal{S}_1, \mathfrak{L})$, $(\mathcal{S}_2, \mathfrak{L})$, $(\mathcal{S}_3, \mathfrak{L}) \in \mathtt{S}(\mho)$,

1. if $(\mathcal{S}_1, \mathfrak{L}) = (\mathcal{S}_2, \mathfrak{L})$, then $\psi((\mathcal{S}_1, \mathfrak{L}), (\mathcal{S}_2, \mathfrak{L})) = 1$
2. $(\mathcal{S}_1, \mathfrak{L}), (\mathcal{S}_2, \mathfrak{L})) = \psi((\mathcal{S}_2, \mathfrak{L}), (\mathcal{S}_1, \mathfrak{L}))$
3. $(\mathcal{S}_1, \mathfrak{L}), (\mathcal{S}_3, \mathfrak{L})) \leq \psi(\mathcal{S}_1, \mathfrak{L}), (\mathcal{S}_2, \mathfrak{L}))$ and
$(\mathcal{S}_1, \mathfrak{L}), (\mathcal{S}_3, \mathfrak{L})) \leq \psi(\mathcal{S}_2, \mathfrak{L}), (\mathcal{S}_3, \mathfrak{L}))$ if $(\mathcal{S}_1, \mathfrak{L})\widetilde{\subseteq}(\mathcal{S}_2, \mathfrak{L})\widetilde{\subseteq}(\mathcal{S}_3, \mathfrak{L})$.

Definition 7 [28] A PFSS over $\mho$ is a pair $(\mathcal{P}, \mathfrak{L})$ where $\mathfrak{L}$ is a collection of parameters and $\mathcal{P}$ is a mapping from $\mathfrak{L}$ into $PFS(\mho)$ and $PFS(\mho)$ is the set of all PFS subsets of $\mho$.

From the definition 7 it is clear that PFSS is a hybrid structure of PFS and soft set.

Definition 8 [28] Suppose $\mathfrak{L}_1, \mathfrak{L}_2 \subseteq \mathfrak{L}$ and $(\mathcal{P}, \mathfrak{L}_1), (\mathcal{G}, \mathfrak{L}_2)$ are two PFSSs over $\mho$. $(\mathcal{P}, \mathfrak{L}_1)$ is said to be PFSS subset of $(\mathcal{P}, \mathfrak{L}_2)$ denoted as $(\mathcal{P}, \mathfrak{L}_1)\sqsubseteq(\mathcal{G}, \mathfrak{L}_2)$ if,

1. $\mathfrak{L}_1 \subseteq \mathfrak{L}_2$
2. $\mathcal{P}(\mathfrak{h}) \subseteq \mathcal{G}(\mathfrak{h})$, $\forall \mathfrak{h} \in \mathfrak{L}_1$
i.e., $\forall x \in \mho$ and $\mathfrak{h} \in \mathfrak{L}_1$, $\mu_{\mathcal{P}(\mathfrak{h})}(x) \leq \mu_{\mathcal{G}(\mathfrak{h})}(x)$ and $\nu_{\mathcal{P}(\mathfrak{h})}(x) \geq \nu_{\mathcal{G}(\mathfrak{h})}(x)$.

If $(\mathcal{P}, \mathfrak{L}_1)\sqsubseteq(\mathcal{G}, \mathfrak{L}_2)$ and $(\mathcal{G}, \mathfrak{L}_2)\sqsubseteq(\mathcal{P}, \mathfrak{L}_1)$ then $(\mathcal{P}, \mathfrak{L}_1)$ and $(\mathcal{G}, \mathfrak{L}_2)$ are said to be equal.

Definition 9 [28] The intersection of two PFSSs $(\mathcal{P}, \mathfrak{L}_1)$ and $(\mathcal{G}, \mathfrak{L}_2)$ over $\mho$, denoted by, $(\mathcal{P}, \mathfrak{L}_1)\sqcap(\mathcal{G}, \mathfrak{L}_2)$ and is defined to be the PFSS $(\mathcal{R}, \mathfrak{L}_1 \cap \mathfrak{L}_2)$ where $\mathcal{R}$ is a map from $\mathfrak{L}_1 \cap \mathfrak{L}_2$ into $PFS(\mho)$ given by $\mathcal{R}(\mathfrak{h}) = \mathcal{P}(\mathfrak{h}) \cap \mathcal{G}(\mathfrak{h})$, $\forall \mathfrak{h} \in \mathfrak{L}_1 \cap \mathfrak{L}_2$, while the union of two PFSSs over $\mho$, denoted as $(\mathcal{P}, \mathfrak{L}_1)\sqcup(\mathcal{G}, \mathfrak{L}_2)$ and is defined to be $(\mathcal{T}, \mathfrak{L}_1 \cup \mathfrak{L}_2)$ where $\mathcal{T}$ is a map from $\mathfrak{L}_1 \cup \mathfrak{L}_2$ into $PFS(\mho)$ given by,

$$\mathcal{T}(\mathfrak{h}) = \begin{cases} \mathcal{P}(\mathfrak{h}) & \text{for } \mathfrak{h} \in \mathfrak{L}_1 \setminus \mathfrak{L}_2 \\ \mathcal{G}(\mathfrak{h}) & \text{for } \mathfrak{h} \in \mathfrak{L}_2 \setminus \mathfrak{L}_1 \\ \mathcal{P}(\mathfrak{h}) \cup \mathcal{G}(\mathfrak{h}) & \text{for } \ \mathfrak{h} \in \mathfrak{L}_1 \cap \mathfrak{L}_2 \end{cases}$$

3 Similarity Measures of PFSS

A real-valued function that measures out the similarity among two objects is called the similarity measure. This section explains the similarity measure for PFSSs, moreover different expressions to calculate similarity measures. Also, we proved a couple of theorems that describe certain interesting properties of the similarity measures.

The definition for similarity measure should emphasize the properties essential for a measure that quantifies the similarity between PFSSs. That is, the similarity measure between two PFSSs is maximum if and only if both are equal, and it is minimum if the sets are entirely different. To avoid unnecessary complexity we assume the values of similarity measures lie between 0 and 1.

Definition 10 For the PFSS $(\mathcal{P}, \mathfrak{L})$, $(\mathcal{G}, \mathfrak{L})$, $(\mathcal{R}, \mathfrak{L})$ over the universal set $\mho$ with parameter set $\mathfrak{L}$. A mapping $\Psi : PFSS(\mho) \times PFSS(\mho) \rightarrow \mathbb{R}^+$ defines a similarity measure if it satisfies the four axioms given below,

A1. $0 \leq \Psi((\mathcal{P}, \mathfrak{L}), (\mathcal{G}, \mathfrak{L})) \leq 1$
A2. $\Psi((\mathcal{P}, \mathfrak{L}), (\mathcal{G}, \mathfrak{L})) = \Psi((\mathcal{G}, \mathfrak{L}), (\mathcal{P}, \mathfrak{L}))$
A3. $\Psi((\mathcal{P}, \mathfrak{L}), (\mathcal{G}, \mathfrak{L})) = 1$ iff $(\mathcal{P}, \mathfrak{L}) = (\mathcal{G}, \mathfrak{L})$
A4. $\Psi((\mathcal{P}, \mathfrak{L}), (\mathcal{R}, \mathfrak{L})) \leq \Psi((\mathcal{P}, \mathfrak{L}), (\mathcal{G}, \mathfrak{L}))$ and
$\Psi((\mathcal{P}, \mathfrak{L}), (\mathcal{R}, \mathfrak{L})) \leq \Psi((\mathcal{G}, \mathfrak{L}), (\mathcal{R}, \mathfrak{L}))$ if $(\mathcal{P}, \mathfrak{L}) \sqsubseteq (\mathcal{G}, \mathfrak{L}) \sqsubseteq (\mathcal{R}, \mathfrak{L})$.

Theorem 1 *Let* $\mho = \{\mathfrak{u}_1, \mathfrak{u}_2, \cdots, \mathfrak{u}_n\}$ *be universal set,* $\mathfrak{L} = \{\mathfrak{h}_1, \mathfrak{h}_2, \cdots, \mathfrak{h}_m\}$ *be the parameter set and* $(\mathcal{P}, \mathfrak{L})$, $(\mathcal{G}, \mathfrak{L})$ *are any two PFSSs over* $\mho$. *Then* $\Psi_r((\mathcal{P}, \mathfrak{L}), (\mathcal{G}, \mathfrak{L}))$, $r = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$ *are similarity measures.*

1.

$$\Psi_1((\mathcal{P}, \mathfrak{L}), (\mathcal{G}, \mathfrak{L})) = \frac{\sum_{i=1}^{n}\sum_{j=1}^{m}\begin{pmatrix}\min\{\mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i), \mu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)\}+\\ \min\{\nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i), \nu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)\}\end{pmatrix}}{\sum_{i=1}^{n}\sum_{j=1}^{m}\begin{pmatrix}\max\{\mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i), \mu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)\}+\\ \max\{\nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i), \nu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)\}\end{pmatrix}} \tag{1}$$

2.

$$\Psi_2((\mathcal{P}, \mathfrak{L}), (\mathcal{G}, \mathfrak{L})) = \frac{1}{m}\sum_{j=1}^{m}\frac{\sum_{i=1}^{n} 1 - \min\begin{Bmatrix}|\mu^2_{\mathcal{P}(\mathfrak{h}_j)}(u_i) - \mu^2_{\mathcal{G}(\mathfrak{h}_j)}(u_i)|,\\ |\nu^2_{\mathcal{P}(\mathfrak{h}_j)}(u_i) - \nu^2_{\mathcal{G}(\mathfrak{h}_j)}(u_i)|\end{Bmatrix}}{\sum_{i=1}^{n} 1 + \max\begin{Bmatrix}|\mu^2_{\mathcal{P}(\mathfrak{h}_j)}(u_i) - \mu^2_{\mathcal{G}(\mathfrak{h}_j)}(u_i)|,\\ |\nu^2_{\mathcal{P}(\mathfrak{h}_j)}(u_i) - \nu^2_{\mathcal{G}(\mathfrak{h}_j)}(u_i)|\end{Bmatrix}} \tag{2}$$

3.

$$\Psi_3((\mathcal{P}, \mathfrak{L}), (\mathcal{G}, \mathfrak{L})) = 1 - \frac{1}{2mn}\sum_{j=1}^{m}\sum_{i=1}^{n}\begin{pmatrix}|\mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i) - \mu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)|+\\ |\nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i) - \nu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)|+\\ |\pi^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i) - \pi^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)|\end{pmatrix} \tag{3}$$

4.

$$\Psi_4((\mathcal{P}, \mathfrak{L}), (\mathcal{G}, \mathfrak{L})) = \frac{1}{mn}\sum_{j=1}^{m}\sum_{i=1}^{n}\cos\left[\frac{\pi}{2}\max\{\begin{matrix}|\mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i) - \mu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)|,\\ |\nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i) - \nu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)|\}\end{matrix}\right] \tag{4}$$

5.

$$\Psi_5((\mathcal{P},\mathfrak{L}),(\mathcal{G},\mathfrak{L})) = \frac{1}{mn}\sum_{j=1}^{m}\sum_{i=1}^{n}\cos\left[\frac{\pi}{4}\begin{pmatrix}|\mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)-\mu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)| \\ +|\nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)-\nu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)|\end{pmatrix}\right] \tag{5}$$

6.

$$\Psi_6((\mathcal{P},\mathfrak{L}),(\mathcal{G},\mathfrak{L})) = 1-\frac{1}{m}\sum_{j=1}^{m}\left(\frac{\begin{array}{c}\sum_{i=1}^{n}|\mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)-\mu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)| \\ +|\nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)-\nu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)|\end{array}}{\begin{array}{c}\sum_{i=1}^{n}|\mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)+\mu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)| \\ +|\nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)+\nu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)|\end{array}}\right) \tag{6}$$

7.

$$\Psi_7((\mathcal{P},\mathfrak{L}),(\mathcal{G},\mathfrak{L})) = \frac{1}{m}\sum_{j=1}^{m}\left(\frac{\begin{array}{c}\sum_{i=1}^{n}\min\{\mu^2_{\mathcal{P}(\mathfrak{h}_j)},\mu^2_{\mathcal{G}(\mathfrak{h}_j)}\}+ \\ \min\{1-\nu^2_{\mathcal{P}(\mathfrak{h}_j)},1-\nu^2_{\mathcal{G}(\mathfrak{h}_j)}\}\end{array}}{\begin{array}{c}\sum_{i=1}^{n}\max\{\mu^2_{\mathcal{P}(\mathfrak{h}_j)},\mu^2_{\mathcal{G}(\mathfrak{h}_j)}\}+ \\ \max\{1-\nu^2_{\mathcal{P}(\mathfrak{h}_j)},1-\nu^2_{\mathcal{G}(\mathfrak{h}_j)}\}\end{array}}\right) \tag{7}$$

8.

$$\Psi_8((\mathcal{P},\mathfrak{L}),(\mathcal{G},\mathfrak{L})) = \left(\begin{array}{c}\frac{\alpha}{m}\sum_{j=1}^{m}\frac{\sum_{i=1}^{n}\min\{\mu^2_{\mathcal{P}(\mathfrak{h}_j)},\mu^2_{\mathcal{G}(\mathfrak{h}_j)}\}}{\sum_{i=1}^{n}\max\{\mu^2_{\mathcal{P}(\mathfrak{h}_j)},\mu^2_{\mathcal{G}(\mathfrak{h}_j)}\}}+ \\ \frac{1-\alpha}{m}\sum_{j=1}^{m}\frac{\sum_{i=1}^{n}\min\{\nu^2_{\mathcal{P}(\mathfrak{h}_j)},\nu^2_{\mathcal{G}(\mathfrak{h}_j)}\}}{\sum_{i=1}^{n}\max\{\nu^2_{\mathcal{P}(\mathfrak{h}_j)},\nu^2_{\mathcal{G}(\mathfrak{h}_j)}\}}\end{array}\right), \alpha\in[0,1] \tag{8}$$

9.

$$\Psi_9((\mathcal{P},\mathfrak{L}),(\mathcal{G},\mathfrak{L})) = \frac{1}{m}\sum_{j=1}^{m}\frac{\sum_{i=1}^{n}\begin{pmatrix}\mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i).\mu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)+\nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i).\nu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i) \\ +\pi^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i).\pi^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)\end{pmatrix}}{\max\left\{\begin{array}{c}\sum_{i=1}^{n}(\mu^4_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)+\nu^4_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)+\pi^4_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)), \\ \sum_{i=1}^{n}(\mu^4_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)+\nu^4_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)+\pi^4_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i))\end{array}\right\}} \tag{9}$$

10.

$$\Psi_{10}((\mathcal{P},\mathfrak{L}),(\mathcal{G},\mathfrak{L})) = 1-\frac{1}{m}\sum_{j=1}^{m}\left(\frac{\begin{array}{c}\sum_{i=1}^{n}(|\mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)-\mu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)|^2+ \\ |\nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)-\nu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)|^2+ \\ |\pi^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)-\pi^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)|^2)\end{array}}{\begin{array}{c}\sum_{i=1}^{n}(|\mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)+\mu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)|^2+ \\ |\nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)+\nu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)|^2+ \\ |\pi^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)+\pi^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)|^2)\end{array}}\right)^{\frac{1}{2}} \tag{10}$$

Proof For proving $\Psi_r((\mathcal{P},\mathfrak{L}),(\mathcal{G},\mathfrak{L}))$, $r=1,2,\cdots,10$ are similarity measure, it must satisfy the four axioms A_1, A_2, A_3 and A_4. We will prove axiom A_3 for $\Psi_9((\mathcal{P},\mathfrak{L}),(\mathcal{G},\mathfrak{L}))$, axiom A_4 for $\Psi_r((\mathcal{P},\mathfrak{L}),(\mathcal{G},\mathfrak{L}))$, $r=3,4,6$ and rest of the proof is straight forward.

For proving Axiom A_3 for $\Psi_9((\mathcal{P},\mathfrak{L}),(\mathcal{G},\mathfrak{L}))$, let $\Psi_9((\mathcal{P},\mathfrak{L}),(\mathcal{G},\mathfrak{L}))=1$ implies that

$$\frac{1}{m}\sum_{j=1}^{m}\frac{\sum_{i=1}^{n}\begin{pmatrix}\mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i).\mu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)+\nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i).\nu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)\\+\pi^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i).\pi^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)\end{pmatrix}}{\max\left\{\begin{array}{l}\sum_{i=1}^{n}(\mu^4_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)+\nu^4_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)+\pi^4_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)),\\ \sum_{i=1}^{n}(\mu^4_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)+\nu^4_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)+\pi^4_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i))\end{array}\right\}}=1$$

Then for each $j=1,2,\cdots,m$,

$$\begin{aligned}&\sum_{i=1}^{n}\begin{pmatrix}\mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i).\mu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)+\nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i).\nu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)\\+\pi^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i).\pi^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)\end{pmatrix}\\ &=\max\left\{\begin{array}{l}\sum_{i=1}^{n}(\mu^4_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)+\nu^4_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)+\pi^4_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)),\\ \sum_{i=1}^{n}(\mu^4_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)+\nu^4_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)+\pi^4_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i))\end{array}\right\}\\ &\implies \vec{V_1}\cdot\vec{V_2} \;=\; \max\{(\vec{V_1}\cdot\vec{V_1})^2,\;\;(\vec{V_2}\cdot\vec{V_2})^2\}\end{aligned}\tag{11}$$

where,
$\vec{V_1}=\left(\mu^2_{\mathcal{P}(\mathfrak{h}_j)}(u_1),...,\mu^2_{\mathcal{P}(\mathfrak{h}_j)}(u_n),\nu^2_{\mathcal{P}(\mathfrak{h}_j)}(u_1),...,\nu^2_{\mathcal{P}(\mathfrak{h}_j)}(u_n),\pi^2_{\mathcal{P}(\mathfrak{h}_j)}(u_1),...,\pi^2_{\mathcal{P}(\mathfrak{h}_j)}(u_n)\right)$
$\vec{V_2}=\left(\mu^2_{\mathcal{G}(\mathfrak{h}_j)}(u_1),...,\mu^2_{\mathcal{G}(\mathfrak{h}_j)}(u_n),\nu^2_{\mathcal{G}(\mathfrak{h}_j)}(u_1),...,\nu^2_{\mathcal{G}(\mathfrak{h}_j)}(u_n),\pi^2_{\mathcal{G}(\mathfrak{h}_j)}(u_1),...,\pi^2_{\mathcal{G}(\mathfrak{h}_j)}(u_n)\right)$.
The Cauchy Schwarz inequality for positive real numbers gives,
$\vec{V_1}\cdot\vec{V_2}\le(\vec{V_1}\cdot\vec{V_1}).(\vec{V_2}\cdot\vec{V_2})\le\max\{(\vec{V_1}\cdot\vec{V_1})^2,\;\;(\vec{V_2}\cdot\vec{V_2})^2\}$ and the equality retains iff $\vec{V_1}=c\vec{V_2}$ for a constant c.
Thus from equation 11 we get, $\left(\sum_{i=1}^{n}\mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i),\sum_{i=1}^{n}\nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i),\sum_{i=1}^{n}\pi^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)\right)=$ $c\left(\sum_{i=1}^{n}\mu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i),\sum_{i=1}^{n}\nu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i),\sum_{i=1}^{n}\pi^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)\right)$ and that directly implies that $c=1$. Thus $\vec{V_1}=\vec{V_2}$.

Now we are going to prove axiom A_4. So consider $(\mathcal{P},\mathfrak{L})\sqsubseteq(\mathcal{G},\mathfrak{L}\sqsubseteq(\mathcal{R},\mathfrak{L})$. It implies that $\mu_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)\le\mu_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)\le\mu_{\mathcal{R}(\mathfrak{h}_j)}(\mathfrak{u}_i)$ and $\nu_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)\ge\nu_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)\ge\nu_{\mathcal{R}(\mathfrak{h}_j)}(\mathfrak{u}_i)\;\;\forall i,j$.

1.

$$\Psi_3((\mathcal{P},\mathfrak{L}),(\mathcal{G},\mathfrak{L}))=1-\frac{1}{2mn}\sum_{j=1}^{m}\sum_{i=1}^{n}\begin{pmatrix}|\mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)-\mu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)|+\\|\nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)-\nu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)|+\\|\pi^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)-\pi^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)|\end{pmatrix}$$

$$\Psi_3((\mathcal{P},\mathfrak{L}),(\mathcal{R},\mathfrak{L})) = 1 - \frac{1}{2mn}\sum_{j=1}^{m}\sum_{i=1}^{n}\begin{pmatrix}|\mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i) - \mu^2_{\mathcal{R}(\mathfrak{h}_j)}(\mathfrak{u}_i)|+\\ |\nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i) - \nu^2_{\mathcal{R}(\mathfrak{h}_j)}(\mathfrak{u}_i)|+\\ |\pi^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i) - \pi^2_{\mathcal{R}(\mathfrak{h}_j)}(\mathfrak{u}_i)|\end{pmatrix}$$

Let $\Psi_3((\mathcal{P},\mathfrak{L}),(\mathcal{G},\mathfrak{L})) - \Psi_3((\mathcal{P},\mathfrak{L}),(\mathcal{R},\mathfrak{L})) = \Delta\Psi_3$.

$$\Delta\Psi_3 = \frac{1}{2mn}\sum_{j=1}^{m}\sum_{i=1}^{n}\begin{pmatrix}|\mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i) - \mu^2_{\mathcal{R}(\mathfrak{h}_j)}(\mathfrak{u}_i)| + |\nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i) - \nu^2_{\mathcal{R}(\mathfrak{h}_j)}(\mathfrak{u}_i)|+\\ |\pi^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i) - \pi^2_{\mathcal{R}(\mathfrak{h}_j)}(\mathfrak{u}_i)| - |\mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i) - \mu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)|-\\ |\nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i) - \nu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)| - |\pi^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i) - \pi^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)|\end{pmatrix}$$

$$= \frac{1}{2mn}\sum_{j=1}^{m}\sum_{i=1}^{n}\begin{pmatrix}\mu^2_{\mathcal{R}(\mathfrak{h}_j)}(\mathfrak{u}_i) - \mu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i) + \nu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i) - \nu^2_{\mathcal{R}(\mathfrak{h}_j)}(\mathfrak{u}_i)+\\ |\mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i) + \nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i) - (\mu^2_{\mathcal{R}(\mathfrak{h}_j)}(\mathfrak{u}_i) + \nu^2_{\mathcal{R}(\mathfrak{h}_j)}(\mathfrak{u}_i)|-\\ |\mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i) + \nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i) - (\mu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i) + \nu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)|\end{pmatrix}$$

Consider four cases;

Case 1: $\mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i) + \nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i) \geq \max\{\mu^2_{\mathcal{R}(\mathfrak{h}_j)}(\mathfrak{u}_i) + \nu^2_{\mathcal{R}(\mathfrak{h}_j)}(\mathfrak{u}_i), \mu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i) + \nu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)\}$
then $\Delta\Psi_3 = \frac{1}{2mn}\sum\limits_{j=1}^{m}\sum\limits_{i=1}^{n}\left(\nu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i) - \nu^2_{\mathcal{R}(\mathfrak{h}_j)}(\mathfrak{u}_i)\right) \geq 0$.
Case 2: $\mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i) + \nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i) \leq \min\{\mu^2_{\mathcal{R}(\mathfrak{h}_j)}(\mathfrak{u}_i) + \nu^2_{\mathcal{R}(\mathfrak{h}_j)}(\mathfrak{u}_i), \mu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i) + \nu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)\}$
then $\Delta\Psi_3 = \frac{1}{2mn}\sum\limits_{j=1}^{m}\sum\limits_{i=1}^{n}2\left(\mu^2_{\mathcal{R}(\mathfrak{h}_j)}(\mathfrak{u}_i) - \mu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)\right) \geq 0$.
Case 3: $\mu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i) + \nu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i) \leq \mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i) + \nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i) \leq \mu^2_{\mathcal{R}(\mathfrak{h}_j)}(\mathfrak{u}_i) + \nu^2_{\mathcal{R}(\mathfrak{h}_j)}(\mathfrak{u}_i)$ then,
$\Delta\Psi_3 = \frac{1}{2mn}\sum\limits_{j=1}^{m}\sum\limits_{i=1}^{n}2\left(\mu^2_{\mathcal{R}(\mathfrak{h}_j)}(\mathfrak{u}_i) + \nu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i) - \mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i) - \nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)\right)$
$\geq \frac{1}{2mn}\sum\limits_{j=1}^{m}\sum\limits_{i=1}^{n}2\begin{pmatrix}[\mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i) + \nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i) - \nu^2_{\mathcal{R}(\mathfrak{h}_j)}(\mathfrak{u}_i)]+\\ \nu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i) - \mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i) - \nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)\end{pmatrix}$
$= \frac{1}{2mn}\sum\limits_{j=1}^{m}\sum\limits_{i=1}^{n}2\left(\nu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i) - \nu^2_{\mathcal{R}(\mathfrak{h}_j)}(\mathfrak{u}_i)\right) \geq 0$.
Case 4: $\mu^2_{\mathcal{R}(\mathfrak{h}_j)}(\mathfrak{u}_i) + \nu^2_{\mathcal{R}(\mathfrak{h}_j)}(\mathfrak{u}_i) \leq \mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i) + \nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i) \leq \mu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i) + \nu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)$ then
$\Delta\Psi_3 = \frac{1}{2mn}\sum\limits_{j=1}^{m}\sum\limits_{i=1}^{n}2\left(\mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i) + \nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i) - \nu^2_{\mathcal{R}(\mathfrak{h}_j)}(\mathfrak{u}_i) - \mu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)\right) \geq$
$\frac{1}{2mn}\sum\limits_{j=1}^{m}\sum\limits_{i=1}^{n}2\begin{pmatrix}\mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i) + \nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i) - [\mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)\\ + \nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i) - \mu^2_{\mathcal{R}(\mathfrak{h}_j)}(\mathfrak{u}_i)] - \mu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)\end{pmatrix}$
$= \frac{1}{2mn}\sum\limits_{j=1}^{m}\sum\limits_{i=1}^{n}2\left(\mu^2_{\mathcal{R}(\mathfrak{h}_j)}(\mathfrak{u}_i) - \mu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)\right) \geq 0$.
By considering all the above cases it is obtained that $\Psi_3((\mathcal{P},\mathfrak{L}),(\mathcal{G},\mathfrak{L})) \geq$

$\Psi_3((\mathcal{P},\mathfrak{L}),(\mathcal{R},\mathfrak{L}))$ and same way it can be proved
$\Psi_3((\mathcal{G},\mathfrak{L}),(\mathcal{R},\mathfrak{L})) \geq \Psi_3((\mathcal{P},\mathfrak{L}),(\mathcal{R},\mathfrak{L}))$.

2.

$$\Psi_4((\mathcal{P},\mathfrak{L}),(\mathcal{G},\mathfrak{L})) = \frac{1}{mn}\sum_{j=1}^{m}\sum_{i=1}^{n}\cos\left[\frac{\pi}{2}\max\{\begin{matrix}|\mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)-\mu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)|,\\ |\nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)-\nu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)|\}\end{matrix}\right]$$

$$\Psi_4((\mathcal{P},\mathfrak{L}),(\mathcal{R},\mathfrak{L})) = \frac{1}{mn}\sum_{j=1}^{m}\sum_{i=1}^{n}\cos\left[\frac{\pi}{2}\max\{\begin{matrix}|\mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)-\mu^2_{\mathcal{R}(\mathfrak{h}_j)}(\mathfrak{u}_i)|,\\ |\nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)-\nu^2_{\mathcal{R}(\mathfrak{h}_j)}(\mathfrak{u}_i)|\}\end{matrix}\right]$$

Since $|\mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)-\mu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)| \geq |\mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)-\mu^2_{\mathcal{R}(\mathfrak{h}_j)}(\mathfrak{u}_i)|$, $|\nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)-\nu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)| \geq |\nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)-\nu^2_{\mathcal{R}(\mathfrak{h}_j)}(\mathfrak{u}_i)|$ and cosine is increasing on $[0,\pi/2]$, it is directly obtained that $\Psi_4((\mathcal{G},\mathfrak{L}),(\mathcal{R},\mathfrak{L})) \geq \Psi_4((\mathcal{P},\mathfrak{L}),(\mathcal{R},\mathfrak{L}))$.
Similarly we get $\Psi_4((\mathcal{G},\mathfrak{L}),(\mathcal{R},\mathfrak{L})) \geq \Psi_4((\mathcal{P},\mathfrak{L}),(\mathcal{R},\mathfrak{L}))$.

3.

$$\Psi_6((\mathcal{P},\mathfrak{L}),(\mathcal{G},\mathfrak{L})) = 1-\frac{1}{m}\sum_{j=1}^{m}\left(\frac{\begin{matrix}\sum_{i=1}^{n}|\mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)-\mu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)|\\ +|\nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)-\nu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)|\end{matrix}}{\begin{matrix}\sum_{i=1}^{n}|\mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)+\mu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)|\\ +|\nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)+\nu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)|\end{matrix}}\right)$$

$$\Psi_6((\mathcal{P},\mathfrak{L}),(\mathcal{R},\mathfrak{L})) = 1-\frac{1}{m}\sum_{j=1}^{m}\left(\frac{\begin{matrix}\sum_{i=1}^{n}|\mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)-\mu^2_{\mathcal{R}(\mathfrak{h}_j)}(\mathfrak{u}_i)|\\ +|\nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)-\nu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)|\end{matrix}}{\begin{matrix}\sum_{i=1}^{n}|\mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)+\mu^2_{\mathcal{R}(\mathfrak{h}_j)}(\mathfrak{u}_i)|\\ +|\nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)+\nu^2_{\mathcal{R}(\mathfrak{h}_j)}(\mathfrak{u}_i)|\end{matrix}}\right)$$

We have, $\begin{pmatrix}|\mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)-\mu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)|+\\ |\nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)-\nu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)|\end{pmatrix} \leq \begin{pmatrix}|\mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)-\mu^2_{\mathcal{R}(\mathfrak{h}_j)}(\mathfrak{u}_i)|+\\ |\nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)-\nu^2_{\mathcal{R}(\mathfrak{h}_j)}(\mathfrak{u}_i)|\end{pmatrix}$

$$\implies \frac{1}{\begin{pmatrix}|\mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)-\mu^2_{\mathcal{R}(\mathfrak{h}_j)}(\mathfrak{u}_i)|+\\ |\nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)-\nu^2_{\mathcal{R}(\mathfrak{h}_j)}(\mathfrak{u}_i)|\end{pmatrix}} \leq \frac{1}{\begin{pmatrix}|\mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)-\mu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)|+\\ |\nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)-\nu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)|\end{pmatrix}}$$

$$\implies 1+\frac{2\left(\mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)+\nu^2_{\mathcal{R}(\mathfrak{h}_j)}(\mathfrak{u}_i)\right)}{\begin{pmatrix}|\mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)-\mu^2_{\mathcal{R}(\mathfrak{h}_j)}(\mathfrak{u}_i)|+\\ |\nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)-\nu^2_{\mathcal{R}(\mathfrak{h}_j)}(\mathfrak{u}_i)|\end{pmatrix}} \leq 1+\frac{2\left(\mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)+\nu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)\right)}{\begin{pmatrix}|\mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)-\mu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)|+\\ |\nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)-\nu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)|\end{pmatrix}}$$

$$\implies \frac{\begin{pmatrix}|\mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)+\mu^2_{\mathcal{R}(\mathfrak{h}_j)}(\mathfrak{u}_i)|+\\ |\nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)+\nu^2_{\mathcal{R}(\mathfrak{h}_j)}(\mathfrak{u}_i)|\end{pmatrix}}{\begin{pmatrix}|\mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)-\mu^2_{\mathcal{R}(\mathfrak{h}_j)}(\mathfrak{u}_i)|+\\ |\nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)-\nu^2_{\mathcal{R}(\mathfrak{h}_j)}(\mathfrak{u}_i)|\end{pmatrix}} \leq \frac{\begin{pmatrix}|\mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)+\mu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)|+\\ |\nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)+\nu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)|\end{pmatrix}}{\begin{pmatrix}|\mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)-\mu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)|+\\ |\nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)-\nu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)|\end{pmatrix}}$$

$$\implies 1-\frac{1}{m}\sum_{j=1}^{m}\frac{\sum_{i=1}^{n}\begin{pmatrix}|\mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)-\mu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)|+\\ |\nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)-\nu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)|\end{pmatrix}}{\sum_{i=1}^{n}\begin{pmatrix}|\mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)+\mu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)|+\\ |\nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)+\nu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)|\end{pmatrix}} \geq$$

$$1-\frac{1}{m}\sum_{j=1}^{m}\frac{\sum_{i=1}^{n}\begin{pmatrix}|\mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)-\mu^2_{\mathcal{R}(\mathfrak{h}_j)}(\mathfrak{u}_i)|+\\ |\nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)-\nu^2_{\mathcal{R}(\mathfrak{h}_j)}(\mathfrak{u}_i)|\end{pmatrix}}{\sum_{i=1}^{n}\begin{pmatrix}|\mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)+\mu^2_{\mathcal{R}(\mathfrak{h}_j)}(\mathfrak{u}_i)|+\\ |\nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)+\nu^2_{\mathcal{R}(\mathfrak{h}_j)}(\mathfrak{u}_i)|\end{pmatrix}}.$$

Thus $\Psi_6((\mathcal{P},\mathfrak{L}),(\mathcal{G},\mathfrak{L})) \geq \Psi_6((\mathcal{P},\mathfrak{L}),(\mathcal{R},\mathfrak{L}))$ and similarly it can be proven that $\Psi_4((\mathcal{G},\mathfrak{L}),(\mathcal{R},\mathfrak{L})) \geq \Psi_4((\mathcal{P},\mathfrak{L}),(\mathcal{R},\mathfrak{L}))$.

Theorem 2 *For $r = 1,2,3,4,5,6,8,9,10$ and $\alpha=\beta=1/2$,*

1. $\Psi_r((\mathcal{P},\mathfrak{L}),(\mathcal{G},\mathfrak{L})^c) = \Psi_r((\mathcal{P},\mathfrak{L})^c,(\mathcal{G},\mathfrak{L}))$
2. $\Psi_r((\mathcal{P},\mathfrak{L})^c,(\mathcal{G},\mathfrak{L})^c) = \Psi_r((\mathcal{P},\mathfrak{L}),(\mathcal{G},\mathfrak{L}))$
3. $\Psi_r((\mathcal{P},\mathfrak{L})\sqcap(\mathcal{G},\mathfrak{L}),(\mathcal{P},\mathfrak{L})\sqcup(\mathcal{G},\mathfrak{L})) = \Psi_r((\mathcal{P},\mathfrak{L}),(\mathcal{G},\mathfrak{L}))$, $r \neq 3,9,10$

Proof

$$\Psi_1((\mathcal{P},\mathfrak{L}),(\mathcal{G},\mathfrak{L})^c) = \frac{\sum_{i=1}^{n}\sum_{j=1}^{m}\begin{pmatrix}\min\{\mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i),\nu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)\}+\\ \min\{\nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i),\mu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)\}\end{pmatrix}}{\sum_{i=1}^{n}\sum_{j=1}^{m}\begin{pmatrix}\max\{\mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i),\nu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)\}+\\ \max\{\nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i),\mu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)\}\end{pmatrix}}$$

$$= \frac{\sum_{i=1}^{n}\sum_{j=1}^{m}\begin{pmatrix}\min\{\nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i),\mu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)\}+\\ \min\{\mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i),\nu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)\}\end{pmatrix}}{\sum_{i=1}^{n}\sum_{j=1}^{m}\begin{pmatrix}\max\{\nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i),\mu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)\}+\\ \max\{\mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i),\nu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)\}\end{pmatrix}}$$

$$= \Psi_1((\mathcal{P},\mathfrak{L})^c,(\mathcal{G},\mathfrak{L}))$$

Similarly the rest of the proof can be done.

Theorem 3 *For $r = 2,3,4,5$ we get,*

1. $\Psi_r((\mathcal{P},\mathfrak{L}),(\mathcal{P},\mathfrak{L})\sqcap(\mathcal{G},\mathfrak{L})) = \Psi_r((\mathcal{G},\mathfrak{L}),(\mathcal{P},\mathfrak{L})\sqcup(\mathcal{G},\mathfrak{L}))$
2. $\Psi_r((\mathcal{P},\mathfrak{L}),(\mathcal{P},\mathfrak{L})\sqcup(\mathcal{G},\mathfrak{L})) = \Psi_r((\mathcal{G},\mathfrak{L}),(\mathcal{P},\mathfrak{L})\sqcap(\mathcal{G},\mathfrak{L}))$
3. $\Psi_r((\mathcal{P},\mathfrak{L}),(\mathcal{P},\mathfrak{L})+(\mathcal{G},\mathfrak{L})) = \Psi_r((\mathcal{G},\mathfrak{L}),(\mathcal{P},\mathfrak{L}).(\mathcal{G},\mathfrak{L}))$

4. $\Psi_r((\mathcal{P},\mathfrak{L}),(\mathcal{P},\mathfrak{L}).(\mathcal{G},\mathfrak{L})) = \Psi_r((\mathcal{G},\mathfrak{L}),(\mathcal{P},\mathfrak{L})+(\mathcal{G},\mathfrak{L}))$

Proof Here we give proof for $r = 3$ and the proof for remaining values of r follows similarly.

$$\Psi_3((\mathcal{P},\mathfrak{L}),(\mathcal{P},\mathfrak{L})\sqcap(\mathcal{G},\mathfrak{L})) = 1 - \frac{1}{2mn}\sum_{j=1}^{m}\sum_{i=1}^{n}\begin{pmatrix}|\mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i) - \mu^2_{\mathcal{P}\sqcap\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)|+\\ |\nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i) - \nu^2_{\mathcal{P}\sqcap\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)|+\\ |\pi^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i) - \pi^2_{\mathcal{P}\sqcap\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)|\end{pmatrix}$$

$$= 1 - \frac{1}{2mn}\sum_{j=1}^{m}\sum_{i=1}^{n}\begin{pmatrix}|\mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i) - \min\{\mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i),\mu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)\}|+\\ |\nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i) - \min\{\nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i),\nu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)\}|+\\ |\pi^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i) - \min\{\pi^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i),\pi^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)\}|\end{pmatrix} \quad (12)$$

$$\Psi_3((\mathcal{G},\mathfrak{L}),(\mathcal{P},\mathfrak{L})\sqcup(\mathcal{G},\mathfrak{L})) = 1 - \frac{1}{2mn}\sum_{j=1}^{m}\sum_{i=1}^{n}\begin{pmatrix}|\mu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i) - \mu^2_{\mathcal{P}\sqcup\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)|+\\ |\nu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i) - \nu^2_{\mathcal{P}\sqcup\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)|\\ +|\pi^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i) - \pi^2_{\mathcal{P}\sqcup\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)|\end{pmatrix}$$

$$= 1 - \frac{1}{2mn}\sum_{j=1}^{m}\sum_{i=1}^{n}\begin{pmatrix}|\mu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i) - \max\{\mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i),\mu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)\}|+\\ |\nu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i) - \max\{\nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i),\nu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)\}|+\\ |\pi^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i) - \max\{\pi^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i),\pi^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)\}|\end{pmatrix} \quad (13)$$

We have $|\pi^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i) - \pi^2_{\mathcal{P}\sqcap\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)| =$
$|-\mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i) - \nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i) + \min\{\mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i),\mu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)\} + \min\{\nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i),\nu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)\}|$
and $|\pi^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i) - \pi^2_{\mathcal{P}\sqcup\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)| =$
$|-\mu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i) - \nu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i) + \max\{\mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i),\mu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)\} + \min\{\nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i),\nu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)\}|$.

Consider all the four cases;
Case 1: If $\mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i) \le \mu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)$ and $\nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i) \le \nu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)$, then
$|\pi^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i) - \pi^2_{\mathcal{P}\sqcap\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)|$

$$\begin{aligned}&= |-\mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i) - \nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i) + \mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i) + \nu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)|\\ &= |\nu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i) - \nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)|\end{aligned}$$

$|\pi^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i) - \pi^2_{\mathcal{P}\sqcup\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)|$

$$\begin{aligned}&= |-\mu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i) - \nu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i) + \mu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i) + \nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)|\\ &= |\nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i) - \nu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)|.\end{aligned}$$

Case 2: If $\mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i) \le \mu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)$ and $\nu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i) \le \nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)$, then
$|\pi^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i) - \pi^2_{\mathcal{P}\sqcap\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)|$

$$\begin{aligned}&= |-\mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i) - \nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i) + \mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i) + \nu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)|\\ &= |\nu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i) - \nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)|\end{aligned}$$

$$|\pi^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i) - \pi^2_{\mathcal{P}\sqcup\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)|$$
$$= |-\mu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i) - \nu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i) + \nu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i) + \nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)|$$
$$= |\nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i) - \nu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)|.$$

Case 3: If $\mu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i) \leq \mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)$ and $\nu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i) \leq \nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)$, then

$$|\pi^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i) - \pi^2_{\mathcal{P}\sqcap\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)|$$
$$= |-\mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i) - \nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i) + \mu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i) + \nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)|$$
$$= |\mu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i) - \mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)|$$

$$|\pi^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i) - \pi^2_{\mathcal{P}\sqcup\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)|$$
$$= |-\mu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i) - \nu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i) + \mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i) + \nu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)|$$
$$= |\mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i) - \mu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)|.$$

Case 4: If $\mu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i) \leq \mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)$ and $\nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i) \leq \nu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)$, then

$$|\pi^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i) - \pi^2_{\mathcal{P}\sqcap\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)|$$
$$= |-\mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i) - \nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i) + \mu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i) + \nu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)|$$
$$= |\mu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i) - \mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i) + \nu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i) - \nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)|$$

$$|\pi^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i) - \pi^2_{\mathcal{P}\sqcup\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)|$$
$$= |-\mu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i) - \nu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i) + \mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i) + \nu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)|$$
$$= |\mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i) - \mu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i) + \nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i) - \nu^2_{\mathcal{G}(\mathfrak{h}_j)(\mathfrak{u}_i)}|.$$

In all the four cases we have

$$|\pi^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i) - \pi^2_{\mathcal{P}\sqcap\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)| = |\pi^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i) - \pi^2_{\mathcal{P}\sqcup\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)|.$$

Thus,

$$\Psi_3((\mathcal{P},\mathfrak{L}),(\mathcal{P},\mathfrak{L})\sqcap(\mathcal{G},\mathfrak{L})) = \Psi_3((\mathcal{G},\mathfrak{L}),(\mathcal{P},\mathfrak{L})\sqcup(\mathcal{G},\mathfrak{L})).$$

Similarly, the remaining properties can be done.

4 Applications of Similarity Measures of PFSS

Here, a comparison study is carried out and that shows the superiority of the measure that we introduced. The applications of the obtained results are explained through the method of cluster analysis. A suitable clustering algorithm is proposed and displayed an example.

4.1 Comparative Study

This section marks out the advantages of similarity measures of PFSS over the existing measures. Table 1 contains already established similarity measures for IFSS.

Table 1 Table showing already established similarity measures

Authors	**Similarity measures** $\Psi((\mathcal{P},\mathfrak{L}),(\mathcal{G},\mathfrak{L}))$
Çağman et al. [7]	$\Psi_C(\mathcal{P},\mathcal{G}) = \dfrac{\sum_{i=1}^{n}\sum_{j=1}^{m}\begin{pmatrix}\lvert(\mu_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)-\nu_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)). \\ (\mu_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)-\nu_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i))\rvert\end{pmatrix}}{\sum_{i=1}^{n}\sum_{j=1}^{m}\max\begin{Bmatrix}\lvert\lvert\mu_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)-\nu_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)\rvert\rvert^2, \\ \lvert\lvert\mu_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)-\nu_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)\rvert\rvert^2\end{Bmatrix}}$
Muthukumar et al. [24]	$\Psi_M(\mathcal{P},\mathcal{G}) = \dfrac{\sum_{j=1}^{m}\begin{pmatrix}\mu_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}).\mu_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u})+ \\ \nu_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}).\nu_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u})\end{pmatrix}}{\sum_{j=1}^{m}\begin{pmatrix}(\mu_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}))^2 \vee \mu_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}))^2+ \\ (\nu_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}))^2 \vee \nu_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}))^2\end{pmatrix}}$
Mukherjee et al. [23]	$\Psi_{Mu}(\mathcal{P},\mathcal{G}) = \dfrac{1}{1+\frac{1}{2mn}\sum_{i=1}^{n}\sum_{j=1}^{m}\begin{pmatrix}\lvert\mu_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)-\mu_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)\rvert^2+ \\ \lvert\nu_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)-\nu_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)\rvert^2\end{pmatrix}^{\frac{1}{2}}}$
Sarala et al. [29]	$\Psi_S(\mathcal{P},\mathcal{G}) = 1-\frac{1}{2mn}\sum_{i=1}^{n}\sum_{j=1}^{m}\begin{pmatrix}\lvert\mu_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)-\mu_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)\rvert+ \\ \lvert\nu_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i)-\nu_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)\rvert\end{pmatrix}$

A similarity measure for IFSS cannot be directly adopted to a similarity measure for PFSS, even though PFSS is a generalization of IFSS. For example, consider the similarity measure $S_C(\mathcal{P},\mathcal{G})$ given in the Table 1. The following example shows that $\Psi_C(\mathcal{P},\mathcal{G}) = 0$ for entirely different PFSS $\mathcal{P}$ and $\mathcal{G}$.

Example 1 Consider the universal set $\mho = \{u_1, u_2\}$ and the parameter set $\mathfrak{L} = \{e_1, e_2\}$. The PFSSs $(\mathcal{P}_1,\mathfrak{L})$ and $(\mathcal{P}_2,\mathfrak{L})$ is given as,
$(\mathcal{P}_1,\mathfrak{L}) = \begin{pmatrix}(0.11, 0.89) & (0.31, 0.31) \\ (0.56, 0.56) & (0.72, 0.00)\end{pmatrix}$ and $(\mathcal{P}_2,\mathfrak{L}) = \begin{pmatrix}(0.42, 0.42) & (0.61, 0.42) \\ (0.91, 0.11) & (0.33, 0.33)\end{pmatrix}$.
Then $S_C(\mathcal{P}_1,\mathcal{P}_2) = 0$ where $(\mathcal{P}_1,\mathfrak{L})$ and $(\mathcal{P}_2,\mathfrak{L})$ are two different PFSSs. Therefore $\Psi_C(\mathcal{P},\mathcal{G})$ is not a similarity measure for PFSSs. Note that the proposed similarity measure, for example $\Psi_3(\mathcal{P}_1,\mathcal{P}_2)$ of $(\mathcal{P}_1,\mathfrak{L})$ and $(\mathcal{P}_2,\mathfrak{L})$ is 0.53.

The next example describes a comparative analysis of the new notion of similarity measure with the remaining notions of similarity measures within Table 1 in the context of a pattern recognition problem.

Example 2 Take three patterns $\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3$ and an unknown pattern $\mathfrak{U}$ into account. It is to be identified that the pattern $\mathfrak{Q}$ belongs to which class i.e., either $\mathfrak{T}_1$, $\mathfrak{T}_2$ or $\mathfrak{T}_3$. The description of each patterns in terms of PFSS is given below.

Consider the universal set $\mho = \{u_1, u_2\}$ and the parameter set $\mathfrak{L} = \{e_1, e_2\}$. Then the patterns are given as,

$$\mathfrak{T}_1 = \begin{pmatrix} (0.19, 0.31) & (0.60, 0.12) \\ (0.58, 0.22) & (0.79, 0.11) \end{pmatrix}, \mathfrak{T}_2 = \begin{pmatrix} (0.25, 0.25) & (0.26, 0.30) \\ (0.61, 0.22) & (0.61, 0.31) \end{pmatrix},$$
$$\mathfrak{T}_3 = \begin{pmatrix} (0.34, 0.64) & (0.59, 0.15) \\ (0.29, 0.12) & (0.49, 0.18) \end{pmatrix}, \mathfrak{U} = \begin{pmatrix} (0.28, 0.48) & (0.41, 0.19) \\ (0.44, 0.16) & (0.66, 0.23) \end{pmatrix}.$$

The table 2 explains a situation in which pattern recognition using existing similarity measures fail.

Table 2 Pattern recognition

	$\Psi(\mathfrak{T}_1, \mathfrak{U})$	$\Psi(\mathfrak{T}_2, \mathfrak{U})$	$\Psi(\mathfrak{T}_3, \mathfrak{U})$	Decision
Ψ_M	**0.74**	**0.74**	**0.74**	Can not be classified
Ψ_{Mu}	**0.96**	**0.96**	**0.96**	Can not be classified
Ψ_S	0.88	**0.89**	**0.89**	Can not be classified
Ψ_3(Proposed)	0.82	0.86	0.82	$\mathfrak{P}_2$

4.2 **PFSSs Clustering Algorithm**

Clustering analysis has been extensively studied and applied in heterogeneous fields. A straight forward and practical algorithm is proposed in the PFSSs environment. Before doing this some useful definitions are incorporated.

Definition 11 Let $(\mathcal{P}_k, \mathfrak{L})$ be k number of PFSSs over $\mho$. $\mathcal{M} = [M_{uv}]_{k \times k}$ is called similarity matrix if $\mathcal{M}_{uv} = \Psi((\mathcal{P}_u, \mathfrak{L}), (\mathcal{P}_v, \mathfrak{L}))$ and $\Psi((\mathcal{P}_u, \mathfrak{L}), (\mathcal{P}_v, \mathfrak{L}))$ denotes the similarity measure between $(\mathcal{P}_u, \mathfrak{L})$ and $(\mathcal{P}_v, \mathfrak{L})$ which satisfying,

i $0 \leq \mathcal{M}_{uv} \leq 1$, $u, v = 1, 2, \cdots, k$
ii $\mathcal{M}_{uu} = 1$, $u = 1, 2, \cdots, k$
iii $\mathcal{M}_{uv} = \mathcal{M}_{vu}$, $u, v = 1, 2, \cdots, k$

Definition 12 [32] Let $\mathcal{M} = [\mathcal{M}_{uv}]_{k \times k}$ be a similarity matrix, if $\mathcal{M}^2 = \mathcal{M} \circ \mathcal{M} = [\mathcal{M}_{uv}]_k \times k$; then $\mathcal{M}^2$ is defined as composition matrix of $\mathcal{M}$, where $\mathcal{M}_{uv} = \max_p \{\min\{\mathcal{M}_{up}, \mathcal{M}_{pv}\}\}$, $u, v = 1, 2, \cdots, m$.

Definition 13 [32] Let $\mathcal{M} = (\mathcal{M}_{uv})_{k \times k}$ be a similarity matrix. Then in accordance with a finite compositions $\mathcal{M} \to \mathcal{M}^2 \to \mathcal{M}^4 \to, \cdots, \mathcal{M}^{2p} \to \cdots$, there must exist an integer $p > 0$ such that $\mathcal{M}^{2p} = \mathcal{M}^{2(p+1)}$, and $\mathcal{M}^{2p}$ is known as equivalent similarity matrix.

Definition 14 [32] For a given equivalent similarity matrix $\mathcal{M} = (\mathcal{M}_{uv})_{k\times k}$, Γ-cutting matrix of $\mathcal{M}$ is $\mathcal{M}^{\Gamma} = [\mathcal{M}^{\Gamma}_{uv}]_{k\times k}$ where,

$$\mathcal{M}^{\Gamma}_{uv} = \begin{cases} 1 & \text{if } \mathcal{M}_{uv} \geq \Gamma \\ 0 & \text{if } \mathcal{M}_{uv} < \Gamma \end{cases} .$$

and $\Gamma \in [0, 1]$ is the confidence interval.

Suppose we have k alternatives $P_1, P_2, \cdots, P_k$ characterised by n attributes, $a_1, a_2, \cdots, a_n$. Further, consider there are r experts $\mathfrak{h}_1, \mathfrak{h}_2, \cdots, \mathfrak{h}_r$ where each of them provide their preferences in terms of PFSSs in accordance with attributes given. Our goal is to group the given alternatives $P_u(u = 1, 2, \cdots, k)$ by considering the opinions of each experts with equal importance. The step by step algorithm for fulfilling our goal is given below.

Step 1: Represent experts' view points about each alternatives with respect the attributes in terms of PFSSs. Here, $\{a_1, a_2, \cdots a_n\}$ is the universal set and $\{\mathfrak{h}_1, \mathfrak{h}_2, \cdots, \mathfrak{h}_r\}$ is parameter set.
Step 2: Construct the similarity matrix $\mathcal{M} = (m_{uv})_{k\times k}$, where $m_{uv} = S_3(P_u, P_v)$ and it can be rewritten as,

$$\Psi_3((\mathcal{P}, \mathfrak{L}), (\mathcal{G}, \mathfrak{L})) = 1 - \frac{1}{2mn} \sum_{j=1}^{m} \sum_{i=1}^{n} \begin{pmatrix} |\mu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i) - \mu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)| + \\ |\nu^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i) - \nu^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)| + \\ |\pi^2_{\mathcal{P}(\mathfrak{h}_j)}(\mathfrak{u}_i) - \pi^2_{\mathcal{G}(\mathfrak{h}_j)}(\mathfrak{u}_i)| \end{pmatrix}$$

Step 3: Construct the corresponding similarity matrix $\mathcal{M}^{2k}$, where $k = 1, 2, 3, \ldots$ such as $\mathcal{M} \to \mathcal{M}^2 \to \mathcal{M}^4 \to, \cdots, \mathcal{M}^{2k} \to \cdots$, until $\mathcal{M}^{2k} = \mathcal{M}^{2k+1}$.
Step 4: Using definition 14, formulate Γ-cutting matrix $\mathcal{M}^{\Gamma} = [\mathcal{M}^{\Gamma}_{uv}]_{k\times k}$ for a confidence interval Γ.
Step 5: Classify the PFSSs according to the rule; If every elements in the u^{th} row (column) of $\mathcal{M}^{\Gamma}$ matrix are identical as the corresponding elements in v^{th} row (column) of $\mathcal{M}^{\Gamma}$ matrix then respective P_us belong to the same class.

4.2.1 ***Illustrative Example***

Consider the situation explained by Garg et al. in the paper [11] with the following additional aspect. Instead of considering the opinion of a single expert, consider five experts and represent each of their explanations separately. The PFSSs are represented for the ten software evaluated by five experts concerning the four criteria skill in the image processing (IP), measurement equipment for coordinate/distance/area/volume (ME), producing contour lines by applying digital elevation models (DEM)/digital surface models (DEM/DSM) and, production of 3D modeling/texturing abilities (PM/TA) which are given in tables 3, 4, $\cdots$, 12.

Table 3 Table showing explanations of software Δ_1

	IP	ME	DEM/DSM	PM/TA
Expert_1	(0.481,0.392)	(0.690,0.211)	(0.772,0.233)	(0.781,0.240)
Expert_2	(0.890,0.200)	(0.672,0.312)	(0.691,0.481)	(0.611,0.352)
Expert_3	(0.783,0.152)	(0.662,0.282)	(0.672,0.411)	(0.920,0.290)
Expert_4	(0.872,0.272)	(0.422,0.591)	(0.823,0.513)	(0.740,0.060)
Expert_5	(0.513,0.581)	(0.661,0.041)	(0.691,0.231)	(0.760,0.380)

Table 4 Table showing explanations of software Δ_2

	IP	ME	DEM/DSM	PM/TA
Expert_1	(0.586,0.318)	(0.662,0.332)	(0.471,0.481)	(0.371,0.542)
Expert_2	(0.890,0.140)	(0.910,0.290)	(0.760,0.550)	(0.640,0.530)
Expert_3	(0.881,0.172)	(0.702,0.261)	(0.661,0.321)	(0.873,0.143)
Expert_4	(0.770,0.450)	(0.271,0.682)	(0.721,0.561)	(0.822,0.292)
Expert_5	(0.572,0.573)	(0.773,0.133)	(0.741,0.234)	(0.774,0.264)

Table 5 Table showing explanations of software Δ_3

	IP	ME	DEM/DSM	PM/TA
Expert_1	(0.251,0.451)	(0.512,0.321)	(0.322,0.543)	(0.450,0.331)
Expert_2	(0.890,0.011)	(0.641,0.442)	(0.724,0.342)	(0.452,0.442)
Expert_3	(0.754,0.254)	(0.781,0.222)	(0.782,0.561)	(0.889,0.332)
Expert_4	(0.778,0.144)	(0.343,0.853)	(0.872,0.591)	(0.521,0.311)
Expert_5	(0.471,0.672)	(0.442,0.031)	(0.713,0.343)	(0.783,0.371)

Table 6 Table showing explanations of software Δ_4

	IP	ME	DEM/DSM	PM/TA
Expert_1	(0.342,0.272)	(0.551,0.241)	(0.563,0.233)	(0.713,0.291)
Expert_2	(0.881,0.000)	(0.570,0.360)	(0.730,0.450)	(0.880,0.441)
Expert_3	(0.811, 0.156)	(0.783,0.223)	(0.891,0.452)	(0.885,0.465)
Expert_4	(0.571,0.572)	(0.633,0.173)	(0.771,0.441)	(0.662,0.233)
Expert_5	(0.554,0.234)	(0.714,0.312)	(0.892,0.342)	(0.662,0.142)

Table 7 Table showing explanations of software Δ_5

	IP	ME	DEM/DSM	PM/TA
Expert_1	(0.761,0.331)	(0.322,0.682)	(0.551,0.651)	(0.273,0.673)
Expert_2	(0.783,0.191)	(0.681,0.352)	(0.666,0.546)	(0.462,0.600)
Expert_3	(0.680, 0.272)	(0.750,0.262)	(0.752,0.431)	(0.891,0.289)
Expert_4	(0.560,0.150)	(0.570,0.271)	(0.781,0.654)	(0.554,0.252)
Expert_5	(0.514,0.592)	(0.622,0.562)	(0.721,0.322)	(0.773,0.363)

Table 8 Table showing explanations of software Δ_6

	IP	ME	DEM/DSM	PM/TA
Expert_1	(0.721,0.321)	(0.445,0.274)	(0.672,0.322)	(0.401,0.301)
Expert_2	(0.721,0.244)	(0.381,0.624)	(0.574,0.567)	(0.213,0.733)
Expert_3	(0.520,0.330)	(0.460,0.770)	(0.670,0.540)	(0.440,0.620)
Expert_4	(0.780,0.372)	(0.332,0.672)	(0.822,0.472)	(0.722,0.132)
Expert_5	(0.882,0.121)	(0.881,0.214)	(0.784,0.234)	(0.652,0.652)

Table 9 Table showing explanations of software Δ_7

	IP	ME	DEM/DSM	PM/TA
Expert_1	(0.621,0.261)	(0.571,0.292)	(0.522,0.362)	(0.722,0.131)
Expert_2	(0.651,0.251)	(0.732,0.362)	(0.772,0.361)	(0.731,0.231)
Expert_3	(0.681,0.162)	(0.892,0.272)	(0.661,0.321)	(0.751,0.142)
Expert_4	(0.670,0.240)	(0.860,0.321)	(0.780,0.330)	(0.660,0.130)
Expert_5	(0.520,0.120)	(0.610,0.240)	(0.640,0.460)	(0.870,0.230)

Table 10 Table showing explanations of software Δ_8

	IP	ME	DEM/DSM	PM/TA
Expert_1	(0.241,0.481)	(0.641,0.352)	(0.561,0.122)	(0.591,0.311)
Expert_2	(0.881,0.242)	(0.262,0.573)	(0.567,0.611)	(0.212,0.661)
Expert_3	(0.871,0.132)	(0.342,0.783)	(0.573,0.573)	(0.442,0.722)
Expert_4	(0.661,0.471)	(0.232,0.712)	(0.342,0.621)	(0.321,0.711)
Expert_5	(0.552,0.732)	(0.561,0.461)	(0.531,0.622)	(0.212,0.782)

Table 11 Table showing explanations of software Δ_9

	IP	ME	DEM/DSM	PM/TA
Expert_1	(0.574,0.174)	(0.554,0.060)	(0.550,0.190)	(0.420,0.420)
Expert_2	(0.821,0.121)	(0.212,0.782)	(0.661,0.231)	(0.232,0.872)
Expert_3	(0.820,0.001)	(0.770,0.120)	(0.770,0.140)	(0.890,0.230)
Expert_4	(0.780,0.330)	(0.330,0.780)	(0.540,0.720)	(0.110,0.780)
Expert_5	(0.131,0.671)	(0.330,0.780)	(0.560,0.680)	(0.321,0.771)

Table 12 Table showing explanations of software Δ_{10}

	IP	ME	DEM/DSM	PM/TA
Expert_1	(0.45,0.23)	(0.51,0.11)	(0.56,0.26)	(0.88,0.13)
Expert_2	(0.83,0.12)	(0.33,0.82)	(0.76,0.32)	(0.45,0.87)
Expert_3	(0.89,0.23)	(0.73,0.34)	(0.89,0.30)	(0.17,0.82)
Expert_4	(0.56,0.17)	(0.78,0.12)	(0.88,0.42)	(0.77,0.12)
Expert_5	(0.47,0.66)	(0.67,0.21)	(0.66,0.22)	(0.78,0.11)

Step 1 From the tables 3, 4, $\cdots$, 12, PFSS representation of software $\Delta_x (x = 1, 2, \cdots 10)$ can be done easily where, universal set is $\{\text{IP}, \text{ME}, \text{DEM/DSM}, \text{PM/TA}\}$ and parameter set is $\{\text{Expert}_1, \text{Expert}_2, \text{Expert}_3, \text{Expert}_4, \text{Expert}_5\}$.

Step 2 The similarity matrix is framed using $\Psi_3(\Delta_x, \Delta_y)$.

$$\mathcal{M} = \begin{pmatrix}
1.0000 & 0.8362 & 0.8100 & 0.7850 & 0.7860 & 0.7210 & 0.7796 & 0.69657 & 0.6911 & 0.7423 \\
0.8362 & 1.0000 & 0.7792 & 0.7527 & 0.7741 & 0.7094 & 0.7537 & 0.6962 & 0.7070 & 0.7121 \\
0.8100 & 0.7792 & 1.0000 & 0.7593 & 0.7857 & 0.6947 & 0.7455 & 0.6986 & 0.7332 & 0.7201 \\
0.7850 & 0.7527 & 0.7593 & 1.0000 & 0.7430 & 0.6668 & 0.7723 & 0.6599 & 0.6744 & 0.7613 \\
0.7860 & 0.7741 & 0.7857 & 0.7430 & 1.0000 & 0.7113 & 0.7562 & 0.6423 & 0.6854 & 0.7033 \\
0.7210 & 0.7094 & 0.6947 & 0.6668 & 0.7113 & 1.0000 & 0.6968 & 0.7249 & 0.6632 & 0.6778 \\
0.7796 & 0.7537 & 0.7455 & 0.7723 & 0.7562 & 0.6968 & 1.0000 & 0.6275 & 0.6530 & 0.7269 \\
0.6965 & 0.6962 & 0.6986 & 0.6599 & 0.6423 & 0.7249 & 0.6275 & 1.0000 & 0.7474 & 0.6661 \\
0.6911 & 0.7070 & 0.7332 & 0.6744 & 0.6854 & 0.6632 & 0.6530 & 0.7474 & 1.0000 & 0.6787 \\
0.7423 & 0.7121 & 0.7201 & 0.7613 & 0.7033 & 0.6778 & 0.7269 & 0.6661 & 0.6787 & 1.0000
\end{pmatrix}$$

Step 3 Calculate $\mathcal{M}^2 = \mathcal{M} \circ \mathcal{M}$

$$\mathcal{M}^2 = \begin{pmatrix}
1.0000 & 0.8362 & 0.8100 & 0.7850 & 0.7860 & 0.7210 & 0.7796 & 0.7210 & 0.7332 & 0.7613 \\
0.8362 & 1.0000 & 0.8100 & 0.7850 & 0.7860 & 0.7210 & 0.7796 & 0.7094 & 0.7332 & 0.7527 \\
0.8100 & 0.8100 & 1.0000 & 0.7850 & 0.7860 & 0.7210 & 0.7796 & 0.7332 & 0.7332 & 0.7593 \\
0.7850 & 0.7850 & 0.7850 & 1.0000 & 0.7850 & 0.7210 & 0.7796 & 0.6986 & 0.7332 & 0.7613 \\
0.7860 & 0.7860 & 0.7860 & 0.7850 & 1.0000 & 0.7210 & 0.7796 & 0.7113 & 0.7332 & 0.7430 \\
0.7210 & 0.7210 & 0.7210 & 0.7210 & 0.7210 & 1.0000 & 0.7210 & 0.7249 & 0.7249 & 0.7210 \\
0.7796 & 0.7796 & 0.7796 & 0.7796 & 0.7796 & 0.7210 & 1.0000 & 0.6986 & 0.7332 & 0.7613 \\
0.7210 & 0.7094 & 0.7332 & 0.6986 & 0.7113 & 0.7249 & 0.6986 & 1.0000 & 0.7474 & 0.6986 \\
0.7332 & 0.7332 & 0.7332 & 0.7332 & 0.7332 & 0.7249 & 0.7332 & 0.7474 & 1.0000 & 0.7201 \\
0.7613 & 0.7527 & 0.7593 & 0.7613 & 0.7430 & 0.7210 & 0.7613 & 0.6986 & 0.7201 & 1.0000
\end{pmatrix}$$

Calculate $\mathcal{M}^4 = \mathcal{M}^2 \circ \mathcal{M}^2$

$$\mathcal{M}^4 = \begin{pmatrix}
1.0000 & 0.8362 & 0.8100 & 0.7850 & 0.7860 & 0.7249 & 0.7796 & 0.7332 & 0.7332 & 0.7613 \\
0.8362 & 1.0000 & 0.8100 & 0.7850 & 0.7860 & 0.7249 & 0.7796 & 0.7332 & 0.7332 & 0.7613 \\
0.8100 & 0.8100 & 1.0000 & 0.7850 & 0.7860 & 0.7249 & 0.7796 & 0.7332 & 0.7332 & 0.7613 \\
0.7850 & 0.7850 & 0.7850 & 1.0000 & 0.7850 & 0.7249 & 0.7796 & 0.7332 & 0.7332 & 0.7613 \\
0.7860 & 0.7860 & 0.7860 & 0.7850 & 1.0000 & 0.7249 & 0.7796 & 0.7332 & 0.7332 & 0.7613 \\
0.7249 & 0.7249 & 0.7249 & 0.7249 & 0.7249 & 1.0000 & 0.7249 & 0.7249 & 0.7249 & 0.7210 \\
0.7796 & 0.7796 & 0.7796 & 0.7796 & 0.7796 & 0.7249 & 1.0000 & 0.7332 & 0.7332 & 0.7613 \\
0.7332 & 0.7332 & 0.7332 & 0.7332 & 0.7332 & 0.7249 & 0.7332 & 1.0000 & 0.7474 & 0.7332 \\
0.7332 & 0.7332 & 0.7332 & 0.7332 & 0.7332 & 0.7249 & 0.7332 & 0.7474 & 1.0000 & 0.7332 \\
0.7613 & 0.7613 & 0.7613 & 0.7613 & 0.7613 & 0.7210 & 0.7613 & 0.7332 & 0.7332 & 1.0000
\end{pmatrix}$$

Calculate $\mathcal{M}^8 = \mathcal{M}^4 \circ \mathcal{M}^4$

$$\mathcal{M}^8 = \begin{pmatrix}
1.0000 & 0.8362 & 0.8100 & 0.7850 & 0.7860 & 0.7249 & 0.7796 & 0.7332 & 0.7332 & 0.7613 \\
0.8362 & 1.0000 & 0.8100 & 0.7850 & 0.7860 & 0.7249 & 0.7796 & 0.7332 & 0.7332 & 0.7613 \\
0.8100 & 0.8100 & 1.0000 & 0.7850 & 0.7860 & 0.7249 & 0.7796 & 0.7332 & 0.7332 & 0.7613 \\
0.7850 & 0.7850 & 0.7850 & 1.0000 & 0.7850 & 0.7249 & 0.7796 & 0.7332 & 0.7332 & 0.7613 \\
0.7860 & 0.7860 & 0.7860 & 0.7850 & 1.0000 & 0.7249 & 0.7796 & 0.7332 & 0.7332 & 0.7613 \\
0.7249 & 0.7249 & 0.7249 & 0.7249 & 0.7249 & 1.0000 & 0.7249 & 0.7249 & 0.7249 & 0.7249 \\
0.7796 & 0.7796 & 0.7796 & 0.7796 & 0.7796 & 0.7249 & 1.0000 & 0.7332 & 0.7332 & 0.7613 \\
0.7332 & 0.7332 & 0.7332 & 0.7332 & 0.7332 & 0.7249 & 0.7332 & 1.0000 & 0.7474 & 0.7332 \\
0.7332 & 0.7332 & 0.7332 & 0.7332 & 0.7332 & 0.7249 & 0.7332 & 0.7474 & 1.0000 & 0.7332 \\
0.7613 & 0.7613 & 0.7613 & 0.7613 & 0.7613 & 0.7249 & 0.7613 & 0.7332 & 0.7332 & 1.0000
\end{pmatrix}$$

Calculate $\mathcal{M}^{16} = \mathcal{M}^8 \circ \mathcal{M}^8$

$$\mathcal{M}^{16} = \begin{pmatrix}
1.0000 & 0.8362 & 0.8100 & 0.7850 & 0.7860 & 0.7249 & 0.7796 & 0.7332 & 0.7332 & 0.7613 \\
0.8362 & 1.0000 & 0.8100 & 0.7850 & 0.7860 & 0.7249 & 0.7796 & 0.7332 & 0.7332 & 0.7613 \\
0.8100 & 0.8100 & 1.0000 & 0.7850 & 0.7860 & 0.7249 & 0.7796 & 0.7332 & 0.7332 & 0.7613 \\
0.7850 & 0.7850 & 0.7850 & 1.0000 & 0.7850 & 0.7249 & 0.7796 & 0.7332 & 0.7332 & 0.7613 \\
0.7860 & 0.7860 & 0.7860 & 0.7850 & 1.0000 & 0.7249 & 0.7796 & 0.7332 & 0.7332 & 0.7613 \\
0.7249 & 0.7249 & 0.7249 & 0.7249 & 0.7249 & 1.0000 & 0.7249 & 0.7249 & 0.7249 & 0.7249 \\
0.7796 & 0.7796 & 0.7796 & 0.7796 & 0.7796 & 0.7249 & 1.0000 & 0.7332 & 0.7332 & 0.7613 \\
0.7332 & 0.7332 & 0.7332 & 0.7332 & 0.7332 & 0.7249 & 0.7332 & 1.0000 & 0.7474 & 0.7332 \\
0.7332 & 0.7332 & 0.7332 & 0.7332 & 0.7332 & 0.7249 & 0.7332 & 0.7474 & 1.0000 & 0.7332 \\
0.7613 & 0.7613 & 0.7613 & 0.7613 & 0.7613 & 0.7249 & 0.7613 & 0.7332 & 0.7332 & 1.0000
\end{pmatrix}$$

It is obtained that $\mathcal{M}^8 = \mathcal{M}^{16}$. Thus the equivalent similarity matrix is identified as $\mathcal{M}^8$.

Step 4 Let the confidence level $\Gamma = 0.7860$. The Γ-cutting matrix $\mathcal{M}^\Gamma$ is obtained as,

$$\mathcal{M}^\Gamma = \begin{pmatrix}
1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

Step 5 The classification of $\Delta_x(x = 1, 2, \cdots 10)$s by using $\mathcal{M}^{\Gamma}$ matrix is obtained as, $\{\Delta_1, \Delta_2, \Delta_3, \Delta_5\}$, $\{\Delta_4\}$,$\{\Delta_6\}$,$\{\Delta_7\}$,$\{\Delta_8\}$, $\{\Delta_9\}$, $\{\Delta_{10}\}$.

Thus, under the given criteria with confidence level $\Gamma = 0.7860$, we are able to conclude that the data processing and analysis of software $\Delta_1, \Delta_2, \Delta_3$ and Δ_5 come under the same group, and each of the remaining ones form single groups.

Table 13 Table showing clustering of Δ_xs for different confidence level

Confidence level	Clusters	Number of Classes
$0.0000 \leq \Gamma \leq 0.7249$	$\{\Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_5, \Delta_6, \Delta_7, \Delta_8, \Delta_9, \Delta_{10}\}$	1
$0.7249 < \Gamma \leq 0.7332$	$\{\Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_5, \Delta_7, \Delta_8, \Delta_9, \Delta_{10}\}, \{\Delta_6\}$	2
$0.7332 < \Gamma \leq 0.7613$	$\{\Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_5, \Delta_7, \Delta_{10}\}, \{\Delta_6\}, \{\Delta_8, \Delta_9\}$	3
$0.7613 < \Gamma \leq 0.7796$	$\{\Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_5, \Delta_7\}, \{\Delta_6\}, \{\Delta_8\}, \{\Delta_9\}, \{\Delta_{10}\}$	5
$0.7796 < \Gamma \leq 0.7850$	$\{\Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_5\}, \{\Delta_6\}, \{\Delta_7\}, \{\Delta_8\}, \{\Delta_9\}, \{\Delta_{10}\}$	6
$0.7850 < \Gamma \leq 0.7860$	$\{\Delta_1, \Delta_2, \Delta_3, \Delta_5\}, \{\Delta_4\},\{\Delta_6\}, \{\Delta_7\}, \{\Delta_8\},\{\Delta_9\},\{\Delta_{10}\}$	7
$0.7860 < \Gamma \leq 0.8100$	$\{\Delta_1, \Delta_2, \Delta_3\}, \{\Delta_4\}, \{\Delta_5\}, \{\Delta_6\}, \{\Delta_7\}, \{\Delta_8\}, \{\Delta_9\}, \{\Delta_{10}\}$	8
$0.8100 < \Gamma \leq 0.8362$	$\{\Delta_1, \Delta_2\}, \{\Delta_3\}\{\Delta_4\}, \{\Delta_5\}, \{\Delta_6\}, \{\Delta_7\}, \{\Delta_8\}, \{\Delta_9\}, \{\Delta_{10}\}$	9
$0.8362 < \Gamma \leq 1.0000$	$\{\Delta_1\}, \{\Delta_2\}, \{\Delta_3\}, \{\Delta_4\}, \{\Delta_5\}, \{\Delta_6\}, \{\Delta_7\}, \{\Delta_8\}, \{\Delta_9\}, \{\Delta_{10}\}$	10

Since the $\mathcal{M}^{\Gamma}$ matrix varies according to different confidence levels, we obtain different clustering results. From Table 13, it is obtained that as the confidence level increases, the number of patterns obtained also increases. The clustering effect diagram (Figure 1) obtained by analysing the table 13 and we can conclude that software are separated mainly into the following two directions, $\{\Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_5, \Delta_7, \Delta_8, \Delta_9, \Delta_{10}\}$ and $\{\Delta_6\}$. Thus, one can use the clustering effect diagram to take decisions in the above problem with a prescribed confidence level. Since in the proposed clustering algorithm we considered the opinions of five experts for choosing the software, it will be more accurate than the present algorithms.

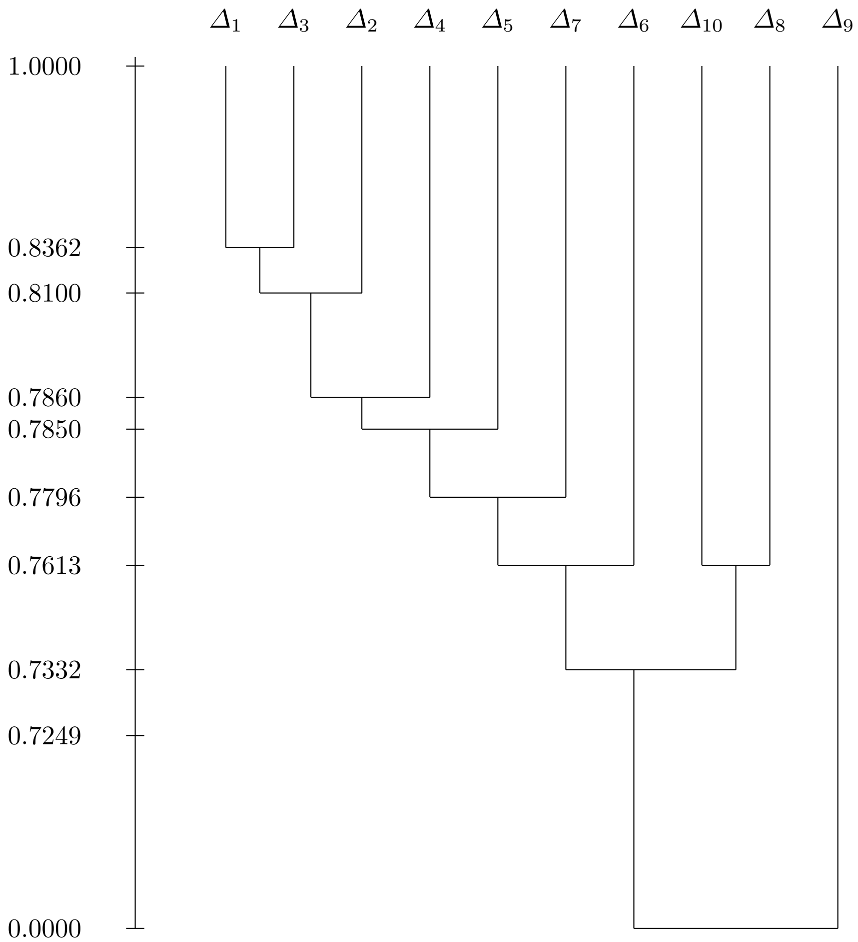

Fig.1 Clustering effect diagram corresponding to Ten software

5 Conclusion

This paper put forward the notion of similarity measure for PFSSs. Different expressions to compute similarity measures are obtained and a comparative study is executed. Finally, a real-life application, viz. cluster analysis is explained. This paper can fill up the deficiency of a measuring tool for finding similarities of objects when the vagueness is represented as PFSSs. We plan to study the concept of information measures of PFSS which are useful in decision-making problems in future.

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7 Declarations

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Consent to participate Informed consent was obtained from all individual participants included in the study.
Consent for publication Not applicable.

8 Data availability statement

Data sharing not applicable to this article as no data sets were generated or analysed during the current study.

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