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ON THE GENUS OF REDUCED COZERO-DIVISOR GRAPH OF COMMUTATIVE RINGS

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Abstract Let R be a commutative ring with identity and let $\Omega(R)^*$ be the set of all nontrivial principal ideals of R. The reduced cozero-divisor graph $\Gamma_r(R)$ of R is an undirected simple graph with $\Omega(R)^*$ as the vertex set and two distinct vertices (x) and (y) in $\Omega(R)^*$ are adjacent if and only if $(x) \nsubseteq (y)$ and $(y) \nsubseteq (x)$. In this paper, we characterize all classes of commutative Artinian non-local rings for which the reduced cozero-divisor graph has genus at most one.

Keywords planar genus · reduced cozero-divisor graph · Artinian ring

Mathematics Subject Classification (2000) 05C10 · 05C25 · 05C75

1 Introduction

Algebraic graph theory is an interesting and an inspiring field for the researchers to study the properties of the graphs based on an algebraic structures during the past years. The study of assigning the graph to a commutative

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ring was initiated by Beck[8] in the name of coloring of commutative rings and subsequently it was modified as zero-divisor graph [4]. There are many authors studied various types of algebraic graphs in the literature. For the entire literature and developments on graphs of rings, one can refer [1]. Afkhami and Khashyarmanesh [2], defined the cozero-divisor graph of commutative rings. Let R be a commutative ring with identity 1 and let $W^*(R)$ be the set of all non-zero non-unit elements of R. The cozero-divisor graph $\Gamma'(R)$ of R is an undirected simple graph with $W^*(R)$ as the vertex set and two distinct vertices x and y in $W^*(R)$ are adjacent if and only if $x \notin Ry$ and $y \notin Rx$, where for $z \in R$, Rz is the ideal generated by z. In [3], Wilkens et al. defined the reduced cozero-divisor graph of commutative rings. For a given R, let $\Omega(R)^*$ be the set of all nontrivial principal ideals of R. The reduced cozero-divisor graph of R, denoted by $\Gamma_r(R)$, is the simple undirected graph with $\Omega(R)^*$ as the vertex-set and two distinct vertices (a) and (b) are adjacent in $\Gamma_r(R)$ if and only if $(a) \nsubseteq (b)$ and $(b) \nsubseteq (a)$. The motive of developing the reduced cozero-divisor graph of commutative ring is to reduce the complexity of the cozero-divisor graph by eliminating the multiple generators of the same ideal to portray the graph effective. Kala et al. [10] determined all the finite commutative nonlocal rings whose reduced cozero-divisor graph is planar. In this paper, we characterize all commutative Artinian non-local rings whose reduced cozero-divisor graph has genus one. Throughout this paper, we assume that R is a finite commutative non-local ring with identity. For basic definitions on rings, one may consult [6].

2 Preliminaries

Let $G = (V, E)$ be an undirected simple graph with vertex set V and edge set E. A graph in which each pair of distinct vertices is joined by the edge is called a complete graph. A complete graph with n vertices is denoted by K_n. An r-partite graph is the one whose vertex set can be partitioned into r subsets so that no edge has both ends in any one subset of the vertex partition. A complete r-partite graph is one in which each vertex in one subset of the partition is joined to every vertex in all other subsets of the partition. The complete bipartite graph (2-partite graph) with subsets sizes m and n is denoted by $K_{m,n}$. The girth of G is the length of a shortest cycle in G and is denoted by $\mathrm{gr}(G)$. If G has no cycles, we define the girth of G to be infinite. A graph G is said to be planar if it can be drawn in the plane so that its edges intersect only at their ends. A subdivision of a graph is a graph obtained from it by replacing edges with pairwise internally-disjoint paths. A remarkably simple characterization of planar graphs was given by Kuratowski in 1930. Kuratowski's Theorem says that a graph G is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$(see [9]).

Let n be a non-negative integer and S_n be an orientable surface of genus n. The genus of the graph G, denoted by $g(G)$, is the smallest n such that G embeds into S_n. For details on the notion of embedding of a graph in a surface, one can see [12]. Graphs of genus 0 are planar graphs and graphs of genus 1 are called toroidal graphs. If H is a subgraph of G, then $g(H) \leq g(G)$. The following results are very useful for further reference in this paper.

Lemma 1 [12, Theorem 6.37] *If $m, n \geq 2$ are integers, then* $g(K_{m,n}) = \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil$.

Theorem 1 [12, Euler Formula] *If G is a finite connected graph with n vertices, m edges and genus g, then $n - m + f = 2 - 2g$, where f is the number of faces created when G is minimally embedded on a surface of genus g.*

Lemma 2 [7, Theorem 1] *Let G be a connected graph with k blocks $B_1, \ldots, B_k$. Then $g(G) = \sum_{i=1}^{k} g(B_i)$.*

Lemma 3 [5, Lemma 2.1] *If G is a graph with n vertices, m edges, girth $gr(G)$ and genus g, then*

$$\frac{m(gr(G) - 2)}{2gr(G)} - \frac{n}{2} + 1 \leq g.$$

Theorem 2 [11, Proposition 4.4.4] *Let G be a connected graph with $n \geq 3$ vertices,q edges and genus g. Then*

$$g \geq \left\lceil \frac{q}{6} - \frac{n}{2} + 1 \right\rceil.$$

Theorem 3 [10, Theorem 3.1] *Let $R = F_1 \times \cdots \times F_n$ be a finite commutative ring with identity, where each F_j is a field and $n \geq 2$. Then $\Gamma_r(R)$ is planar if and only if R is isomorphic either $F_1 \times F_2 \times F_3$ or $F_1 \times F_2$.*

3 Planarity of $\Gamma_r(R)$

The planar characterizations of the reduced cozero-divisor graph obtained by Kala et al. [10] are given below.

Theorem 4 [10, Theorem 3.2] *Let $R = R_1 \times R_2 \times \cdots \times R_n$ be a commutative ring with identity 1, where each $(R_i, \mathfrak{m}_i)$ is a local ring with $\mathfrak{m}_i \neq (0)$ and $n \geq 2$. Then $\Gamma_r(R)$ is planar if and only if $R = R_1 \times R_2$ such that $\mathfrak{m}_i$ is the only non-zero principal ideal in R_i for $i = 1, 2$.*

Theorem 5 [10, Theorem 3.3] *Let $R = R_1 \times \cdots \times R_n \times F_1 \times \cdots \times F_m$ be a finite commutative ring with identity, where each $(R_i, \mathfrak{m}_i)$ is a local ring with $\mathfrak{m}_i \neq \{0\}$ and $m, n \geq 1$. Then $\Gamma_r(R)$ is planar if and only if R satisfies the following conditions:*

(i) *$n + m = 2$;*
(ii) *There exists only two non-zero principal ideals $\langle a_1 \rangle, \langle a_2 \rangle$ in R_1 such that $\langle a_1 \rangle \nsubseteq \langle a_2 \rangle$ and $\langle a_2 \rangle \nsubseteq \langle a_1 \rangle$;*

(iii) $\mathfrak{m}_1 = \langle a_1 \rangle$ *is a principal ideal with nilpotency at most* $k = 4$ *and*
if $k = 2$*, then* $\langle a_1 \rangle$ *is the only principal ideal in* R_1*;*
if $k = 3$*, then* $\langle a_1 \rangle$ *and* $\langle a_1^2 \rangle$ *are the only ideals in* R_1*;*
if $k = 3$*, then* $\langle a_1 \rangle$*,* $\langle a_1^2 \rangle$ *and* $\langle a_1^3 \rangle$ *are the only ideals in* R_1*.*

Let us have the following lemma in order to show that when the condition (ii) in Theorem 5 is true, then the reduced cozero-divisor graph is not planar. Note that the condition (1) is nothing but $n = 1$ and $m = 1$ and hence $R \cong R_1 \times F_1$.

Lemma 4 *Let* $(R_1, \mathfrak{m}_1)$ *be a local ring,* F_1 *be a field and let* $R = R_1 \times F_1$*. If there exist only two non-zero principal ideals* $\langle a_1 \rangle$ *and* $\langle a_2 \rangle$ *of* R_1 *such that* $\langle a_1 \rangle \nsubseteq \langle a_2 \rangle$ *and* $\langle a_2 \rangle \nsubseteq \langle a_1 \rangle$*, then* $\mathit{\Gamma}_r(R) > 0$*.*

Proof. Assume that there exist two non-zero principal ideals $\langle a_1 \rangle$ and $\langle a_2 \rangle$ of the local ring $(R_1, \mathfrak{m}_1)$ such that $\langle a_1 \rangle \nsubseteq \langle a_2 \rangle$ and $\langle a_2 \rangle \nsubseteq \langle a_1 \rangle$. Then $a_1, a_2 \in \mathfrak{m}_1$.

Consider the ideals $I_1 = \langle a_1 \rangle \times \langle 1 \rangle$, $I_2 = \langle a_2 \rangle \times \langle 1 \rangle$, $I_3 = \langle 0 \rangle \times \langle 1 \rangle$, $I_4 = \langle a_1 \rangle \times \langle 0 \rangle$, $I_5 = \langle a_2 \rangle \times \langle 0 \rangle$, $I_6 = \langle a_1 + a_2 \rangle \times \langle 0 \rangle$, $I_7 = R_1 \times \langle 0 \rangle$ of R and let $Z = \{I_1, I_2, I_3, I_4, I_5, I_6, I_7\} \subset \mathit{\Omega}(R)^*$. One can check that the subgraph induced by Z is a subdivision of $K_{3,3}$ with partition subsets $\{I_1, I_2, I_3\}$, $\{I_5, I_6, I_7\}$ and a subdivision of the edge joining I_2 and I_5 through the vertex I_4. Hence $g(\mathit{\Gamma}_r(R) > 0$. □

Having identified a flow in Theorem 5, we state below a characterization of all finite commutative non-local rings with identity whose $\mathit{\Gamma}_r(R)$ is planar. Hence we have the following modified characterization in Theorem 5 for $\mathit{\Gamma}_r(R)$ to be planar.

Theorem 6 *Let* $R = R_1 \times \cdots \times R_n \times F_1 \times \cdots \times F_m$ *be a finite commutative ring with identity, where each* $(R_i, \mathfrak{m}_i)$ *is a local ring with* $\mathfrak{m}_i \neq \{0\}$ *for* $1 \leq i \leq n$*,* F_i *is a field for* $1 \leq i \leq m$ *and* $m, n \geq 1$*. Then* $\mathit{\Gamma}_r(R)$ *is planar if and only if* R *satisfies the following conditions:*

(1) $n = m = 1$;
(2) $\mathfrak{m}_1 = \langle a_1 \rangle$ *is a principal ideal with nilpotency at most* $k = 4$ *and*
  (i) *if* $k = 2$*, then* $\mathfrak{m}_1 = \langle a_1 \rangle$ *is the only principal ideal in* R_1*;*
  (ii) *if* $k = 3$*, then* R_1 *contains at most three non-zero principal ideals.*

Proof. Assume that $\mathit{\Gamma}_r(R)$ is planar for $R = R_1 \times \cdots \times R_n \times F_1 \times \cdots \times F_m$. Suppose $n \geq 2$. Let I and J be non-trivial principal ideals in R_1 and R_2 respectively. Consider the ideals $I_1 = \langle 0 \rangle \times R_2 \times \cdots \times \langle 0 \rangle \times F_1 \times \cdots \times \langle 0 \rangle$, $I_2 = \langle 0 \rangle \times J \times \cdots \times \langle 0 \rangle \times F_1 \times \cdots \times \langle 0 \rangle$, $I_3 = I \times J \times \cdots \times \langle 0 \rangle \times F_1 \times \cdots \times \langle 0 \rangle$, $I_4 = R_1 \times J \times \cdots \times \langle 0 \rangle \times \langle 0 \rangle \times \cdots \times \langle 0 \rangle$, $I_5 = R_1 \times R_2 \times \cdots \times \langle 0 \rangle \times \langle 0 \rangle \times \cdots \times \langle 0 \rangle$, $I_6 = I \times R_2 \times \cdots \times \langle 0 \rangle \times \langle 0 \rangle \times \cdots \times \langle 0 \rangle$ and let $X = \{I_1, I_2, I_3, I_4, I_5, I_6\} \subset \mathit{\Omega}(R)^*$. Then the subgraph induced by X of $\mathit{\Gamma}_r(R)$ contains $K_{3,3}$ as a subgraph with vertex partition $\{I_1, I_2, I_3\}$ and $\{I_4, I_5, I_6\}$. From this, we get that $g(\mathit{\Gamma}_r(R)) \geq 1$, which is a contradiction to the assumption that $\mathit{\Gamma}_r(R)$ is planar. Hence $n = 1$.

Suppose $m \geq 2$. Let I be a non-trivial principal ideal in R_1. Consider the ideals $J_1 = R_1 \times \langle 0 \rangle \times \langle 0 \rangle \times \langle 0 \rangle \times \cdots \times \langle 0 \rangle$, $J_2 = R_1 \times F_1 \times \langle 0 \rangle \times \langle 0 \rangle \times \cdots \times \langle 0 \rangle$, $J_3 = I \times F_1 \times \langle 0 \rangle \times \langle 0 \rangle \times \cdots \times \langle 0 \rangle$, $J_4 = I \times \langle 0 \rangle \times F_2 \times \langle 0 \rangle \times \cdots \times \langle 0 \rangle$, $J_5 = \langle 0 \rangle \times F_1 \times F_2 \times \langle 0 \rangle \times \cdots \times \langle 0 \rangle$, $J_6 = \langle 0 \rangle \times \langle 0 \rangle \times F_2 \times \langle 0 \rangle \times \cdots \times \langle 0 \rangle$ and let $Y = \{J_1, J_2, J_3, J_4, J_5, J_6\}$. Then the subgraph of $\mathit{\Gamma}_r(R)$ induced by Y contains $K_{3,3}$ as a subgraph with vertex partition $\{J_1, J_2, J_3\}$ and $\{J_4, J_5, J_6\}$. From this $g(\mathit{\Gamma}_r(R)) \geq 1$, which is a contradiction to the assumption that $\mathit{\Gamma}_r(R)$ is planar. Hence $m = 1$.

From the above arguments $R = R_1 \times F_1$.

Since R is finite, R_1 is an Artinian ring and so every ideal of R_1 is finitely generated. If $\mathfrak{m}_1$ is not principal, then there exist $a_1, a_2 \in \mathfrak{m}_1$ such that $\langle a_1 \rangle \nsubseteq \langle a_2 \rangle$. By Lemma 4, $g(\mathit{\Gamma}_r(R)) \geq 0$ and so $\mathfrak{m}_1$ is principal. Thus $\mathfrak{m}_1 = \langle a_1 \rangle$. Since R_1 is Artinian, $\mathfrak{m}_1$ is a nil-ideal with nilpotency $k > 1$ and so $\mathfrak{m}_1^k = \langle 0 \rangle$, $\mathfrak{m}_1^{k-1} \neq \langle 0 \rangle$. Suppose $k \geq 5$. Then the subgraph induced by $\{\langle 0 \rangle \times \langle 1 \rangle, \langle a_1^4 \rangle \times \langle 1 \rangle, \langle a_1^3 \rangle \times \langle 1 \rangle, \langle a_1^2 \rangle \times \langle 0 \rangle, \langle a_1 \rangle \times \langle 0 \rangle, R_1 \times \langle 0 \rangle\}$ of $\mathit{\Gamma}_r(R)$ contains $K_{3,3}$ as a subgraph. From this, we get that $g(\mathit{\Gamma}_r(R)) \geq 1$, which is a contradiction to the assumption that $\mathit{\Gamma}_r(R)$ is planar. Hence $k \leq 4$ and so $\mathfrak{m}_1 = \langle a_1 \rangle$ is a principal ideal of nil-potence at most 4. The other parts of condition (2) in the statement are trivially true.

Converse follows from Figure 1. In fact, rings are considered through the nilpotent index of $\mathfrak{m}_1$ and planar embeddings of the corresponding reduced cozero-divisor graphs are given in Figure 1. $\square$

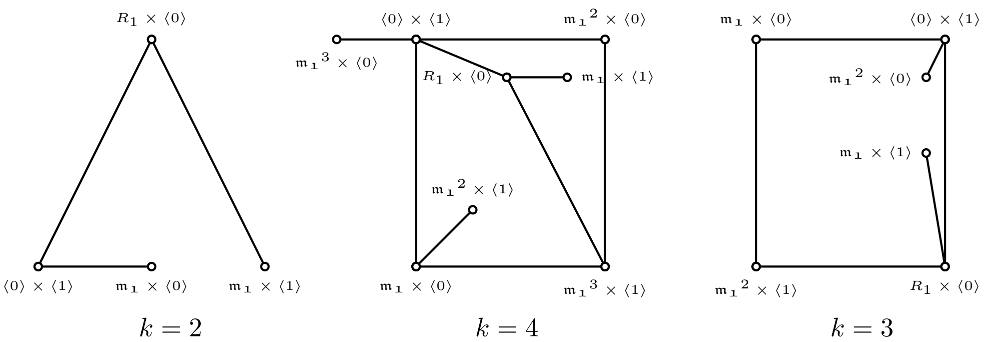


Figure 1: Planar Embeddings of $\mathit{\Gamma}_r(R_1 \times F_1)$

4 Genus of $\mathit{\Gamma}_r(R)$

The main goal of this paper is to obtain a characterization of commutative rings R for which $\mathit{\Gamma}_r(R)$ is toroidal. Towards this attempt, in this section, we classify all finite commutative non-local rings with identity whose cozero-divisor graph $\mathit{\Gamma}_r(R)$ is of genus one. The following theorem gives a tool to identify rings R for which $\mathit{\Gamma}_r(R)$ is not toroidal.

Theorem 7 *Let $R = F_1 \times F_2 \times \cdots \times F_n$ be a commutative ring with identity where each F_i is a field for $1 \leq i \leq n$ and $n \geq 4$. Then $g(\Gamma_r(R)) \geq 2$.*

Proof. Let $R = F_1 \times F_2 \times \cdots \times F_n$ and $n \geq 4$. Consider the ideals $I_1 = F_1 \times \langle 0 \rangle \times \langle 0 \rangle \times \langle 0 \rangle$, $I_2 = \langle 0 \rangle \times F_2 \times \langle 0 \rangle \times \langle 0 \rangle$, $I_3 = F_1 \times \langle 0 \rangle \times F_3 \times \langle 0 \rangle$, $I_4 = \langle 0 \rangle \times F_2 \times F_3 \times \langle 0 \rangle$, $I_5 = F_1 \times \langle 0 \rangle \times F_3 \times F_4$, $I_6 = \langle 0 \rangle \times F_2 \times \langle 0 \rangle \times F_4$, $I_7 = F_1 \times F_2 \times \langle 0 \rangle \times \langle 0 \rangle$, $I_8 = \langle 0 \rangle \times \langle 0 \rangle \times \langle 0 \rangle \times F_4$, $I_9 = \langle 0 \rangle \times \langle 0 \rangle \times F_3 \times F_4$, $I_{10} = F_1 \times F_2 \times F_3 \times \langle 0 \rangle$, $I_{11} = F_1 \times \langle 0 \rangle \times \langle 0 \rangle \times F_4$. The subgraph induced by $A = \{I_1, I_2, \ldots, I_{11}\} \subset V(\Gamma_r(R))$ of $\Gamma_r(R)$ contains graph H given in Figure 2 as a subgraph. Since, the graph H has two blocks, both isomorphic to $K_{3,3}$ and so by Lemma 2, $g(\Gamma_r(R)) \geq g(H) \geq 2$. □

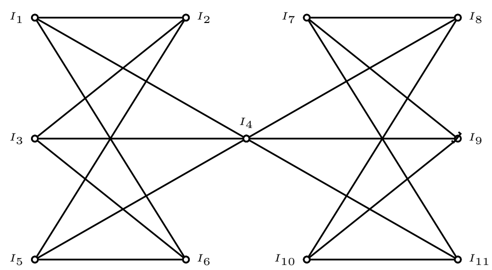


Figure 2: A graph H with $g(H) = 2$

Since an Artinian ring R is not isomorphic to product of fields, it is natural to look into the genus of $\Gamma_r(R))$ where R is an Artinian ring. The following theorems attempts to find the same. In the rest of the section, we look into the characterization for toroidal reduced cozero-divisor graph of finite commutative non-local rings with identity.

Theorem 8 *Let $R = R_1 \times R_2 \times \cdots \times R_n$ be a commutative ring with identity where each $(R_i, \mathfrak{m}_i)$ is a local ring with $\mathfrak{m}_i \neq \{0\}$ and $n \geq 2$. Let η_i be the nilpotent index of $\mathfrak{m}_i$. Then $g(\Gamma_r(R))$=1 if and only if R satisfies the following conditions:*

(1) *$n = 2$;*
(2) *$\mathfrak{m}_1 = \langle a_1 \rangle$ and $\mathfrak{m}_2 = \langle b_1 \rangle$ for some $a_1 \in R_1$, $b_1 \in R_2$ and $1 < \eta_1, \eta_2 \leq 3$;*
  (i) *if $\eta_1 = 3$ and $\eta_2 = 2$, then $\mathfrak{m}_1$ and $\mathfrak{m}_1^2$ are the only non-trivial principal ideals in R_1 and $\mathfrak{m}_2$ is the only non-trivial principal ideal in R_2.*
  (ii) *if $\eta_1 = 2$ and $\eta_2 = 3$, then $\mathfrak{m}_1$ is the only non-trivial principal ideal in R_1 and $\mathfrak{m}_2$ and $\mathfrak{m}_2^2$ are the only non-trivial principal ideals in R_2.*

Proof. Assume that $g(\Gamma_r(R)) = 1$ for $R = R_1 \times R_2 \times \cdots \times R_n$. Suppose $n \geq 3$. Let I, J and K be non-trivial principal ideals in R_1, R_2 and R_3 respectively. Consider the ideals $J_1 = \langle 0 \rangle \times R_2 \times K \times \cdots \times \langle 0 \rangle$, $J_2 = \langle 0 \rangle \times J \times R_3 \times \cdots \times \langle 0 \rangle$, $J_3 = \langle 0 \rangle \times R_2 \times R_3 \times \cdots \times \langle 0 \rangle$, $J_4 = I \times J \times R_3 \times \cdots \times \langle 0 \rangle$, $J_5 = I \times \langle 0 \rangle \times R_3 \times \cdots \times \langle 0 \rangle$, $J_6 = R_1 \times \langle 0 \rangle \times \langle 0 \rangle \times \cdots \times \langle 0 \rangle$, $J_7 = R_1 \times R_2 \times \langle 0 \rangle \times \cdots \times \langle 0 \rangle$, $J_8 = R_1 \times \langle 0 \rangle \times K \times \cdots \times \langle 0 \rangle$, $J_9 = I \times R_2 \times \langle 0 \rangle \times \cdots \times \langle 0 \rangle$. Then the subgraph induced by $A = \{J_1, J_2, J_3, J_4, J_5, J_6, J_7, J_8, J_9\}$ of $\Gamma_r(R)$ contains $K_{5,4}$ as a subgraph with vertex partition $\{J_1, J_2, J_3, J_4, J_5\}$ and $\{J_6, J_7, J_8, J_9\}$. By Lemma 1, $g(\Gamma_r(R)) > 1$ which is a contradiction. Hence $n = 2$ and so $R = R_1 \times R_2$.

Theorem 4 and $g(\Gamma_r(R)) = 1$ together imply that either R_1 or R_2 contains at least two non-trivial principal ideals. Since R is finite, every ideal in R_i is finitely generated. Let $\Phi_1 = \{a_1, a_2, \ldots, a_t : a_i \in R_1 \text{ for } 1 \leq i \leq t\}$ and $\Phi_2 = \{b_1, b_2, \ldots, b_k : b_i \in R_2 \text{ for } 1 \leq i \leq k\}$ be minimal generating sets of $\mathfrak{m}_1$ and $\mathfrak{m}_2$ respectively. Then $\langle a_i \rangle \nsubseteq \langle a_j \rangle$ for all $i \neq j$ and $\langle b_i \rangle \nsubseteq \langle b_j \rangle$ for all $i \neq j$.

Suppose $t \geq 2$ and $k \geq 2$. Let $I_1 = R_1 \times \langle b_2 \rangle$, $I_2 = R_1 \times \langle b_1 \rangle$, $I_3 = \langle a_2 \rangle \times \langle b_2 \rangle$, $I_4 = \langle a_2 \rangle \times \langle b_1 \rangle$, $I_5 = \langle 0 \rangle \times \langle b_2 \rangle$, $I_6 = \langle 0 \rangle \times \langle b_1 \rangle$, $I_7 = \langle a_2 \rangle \times R_2$, $I_8 = \langle a_1 \rangle \times \langle b_2 \rangle$, $I_9 = \langle a_1 \rangle \times \langle b_1 \rangle$, $I_{10} = \langle 0 \rangle \times R_2$, $I_{11} = R_1 \times \langle 0 \rangle$. Then $B = \{I_1, I_2, I_3, I_4, I_5, I_6, I_7, I_8, I_9, I_{10}, I_{11}\} \subset \Omega(R)^*$ and the subgraph induced by B contains two blocks, both isomorphic to $K_{3,3}$ as in Figure 2. By Lemma 2, $g(\Gamma_r(R)) > 1$, which is a contradiction. Hence $t = 1$ or $k = 1$.

Without loss of generality, let us assume that $t = 1$. Suppose $k \geq 2$. Let $I_1 = R_1 \times \langle 0 \rangle$, $I_2 = R_1 \times \langle b_1 \rangle$, $I_3 = R_1 \times \langle b_2 \rangle$, $J_1 = \langle 0 \rangle \times \langle b_1 + b_2 \rangle$, $J_2 = \langle 0 \rangle \times R_2$, $J_3 = \langle a_1 \rangle \times \langle b_1 + b_2 \rangle$, $J_4 = \langle a_1 \rangle \times R_2$, $K_1 = \langle 0 \rangle \times \langle b_1 \rangle$, $K_2 = \langle a_1 \rangle \times \langle b_1 \rangle$, $K_3 = \langle a_1 \rangle \times \langle b_2 \rangle$, $T_1 = \langle 0 \rangle \times \langle b_2 \rangle$, $T_2 = \langle a_1 \rangle \times \langle 0 \rangle$, $T_3 = R_1 \times \langle b_1 + b_2 \rangle$. Since $I_i J_j \in E(\Gamma_r(R))$, for $1 \leq i \leq 3$, $1 \leq j \leq 4$ and $i \neq j$, the subgraph H of $\Gamma_r(R)$ induced by $\{I_1, I_2, I_3, J_1, J_2, J_3, J_4, K_1, K_2, K_3, T_1, T_2, T_3\}$ contains $K_{3,4}$ as a subgraph. Using Theorem 1, $g(H) \geq 1$. Suppose $g(H)$=1. Let H' be the subgraph obtained from H by deleting the vertices T_1, T_2, T_3 and edges J_2J_3, I_2I_3 and H'' be the subgraph obtained from H' by deleting the vertices K_1, K_2, K_3. Then $H'' \cong K_{3,4}$ and so g(H'')=1. Since $g(H) = 1$, we have $1 = g(H'') \leq g(H') \leq g(H) = 1$ and so $g(H')$=1.

Note that $|V(H')| = 10$ and one can check that $|E(H')| = 24$. In fact, $V(H') = \{I_1, I_2, I_3, J_1, J_2, J_3, J_4, K_1, K_2, K_3\}$ and $E(H') = \{I_1J_1,\ I_1J_2,\ I_1J_3,\ I_1J_4,\ I_1K_1,\ I_1K_2,\ I_1K_3,\ I_2J_1,\ I_2J_2,\ I_2J_3,\ I_2J_4,\ I_2K_3,\ I_3J_1,\ I_3J_2,\ I_3J_3,\ I_3J_4,\ I_3K_1,\ I_3K_2,\ J_1K_2,\ J_1K_3,\ J_2K_2,\ J_2K_3,\ K_1K_3,\ K_2K_3\}$. By Theorem 1, the number of faces in any embedding of H' in the torus shall be 16. Let $\{F'_1, \ldots, F'_{16}\}$ be the set of all faces corresponding to an embedding H' in the torus. Since $H'' \cong K_{3,4}$, by Theorem 1, we get that there are 5 faces for any embedding of H'' in the torus. Let $\{F''_1, \ldots, F''_5\}$ be the set of faces of H'' corresponding to an embedding of H'' on the torus. Further the faces of H'' can be either one octagonal face and 4 rectangular faces, or two hexagonal faces and 3 rectangular faces. Clearly, boundaries of all faces are 4-cycles but with two 6-cycles or one 8-cycle. Next, we prove $g(H) > 1$ by a deletion and insertion argument.

Since $K_1K_3, K_2K_3 \in E(H')$, K_1, K_2, K_3 should be inserted in the same face say F''_a of H'' to avoid crossing. As $I_1K_1,\ I_1K_2,\ I_1K_3,\ I_2K_3,\ I_3K_1,\ I_3K_2,\ J_1K_2,\ J_1K_3,\ J_2K_2,\ J_2K_3 \in E(H')$, one should have I_1, I_2, I_3, J_1, J_2 in the boundary of F''_a. Consider the following edges of H'. Let $e_1 = K_1K_3,\ e_2 = K_2K_3,\ e_3 = J_1K_2,\ e_4 = J_1K_3,\ e_5 = J_2K_2,\ e_6 = J_2K_3,\ e_7 = I_1K_1,\ e_8 = I_1K_2,\ e_9 = I_1K_3,\ e_{10} = I_2K_3,\ e_{11} = I_3K_1,\ e_{12} = I_3K_2$. From this, it is clear that K_1, K_2, K_3 should be inserted into the same face.

Suppose if we try to insert K_2 first, then we obtain the following Figure 3. It is easy to observe from the figure that we cannot insert K_3 without edge crossings. Thus we get a contradiction.

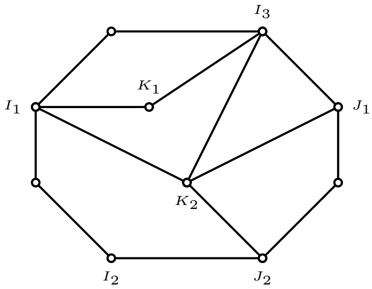


Figure 3

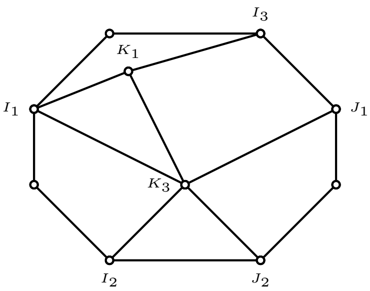


Figure 4(a): F_a''

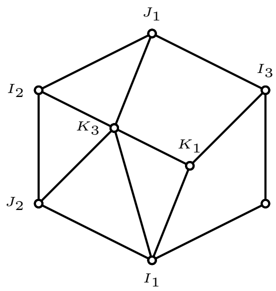


Figure 4(b): F_b''

If we insert K_1, K_2, K_3 and e_i $(1 \leq i \leq 12)$ in the octagonal face F_a'', then we obtain the Figure 4(a). However from Figure 4(a), it is clear that when we insert the vertex K_2 into the face F_a'', then we get an edge crossing. If we insert K_1, K_2, K_3 and e_i $(1 \leq i \leq 12)$ in the hexagonal face F_b'', then we obtain the Figure 4(b). However from Figure 4(b), it is clear that there is no way to insert the vertex K_2 into the face F_b'' without crossing in the embedding of H'. Therefore we get $g(H) > 1$ and hence we get that $g(\Gamma_r(R)) > 1$, which is a contradiction. Hence $k = 1$ and so $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are principal ideals generated by a_1 and b_1 respectively. Since each R_i is Artinian, $\mathfrak{m}_i^{\eta_i} = \langle 0 \rangle$, $\mathfrak{m}_i^{\eta_i - 1} \neq \langle 0 \rangle$ for $i = 1, 2$.

Suppose $\eta_i \geq 4$ for $i = 1, 2$. Let $L_1 = \mathfrak{m}_1 \times \langle 0 \rangle$, $L_2 = \langle 0 \rangle \times \mathfrak{m}_2$, $L_3 = \mathfrak{m}_1 \times \mathfrak{m}_2^2$, $L_4 = \langle 0 \rangle \times R_2$, $L_5 = \mathfrak{m}_1 \times \mathfrak{m}_2^3$, $L_6 = \mathfrak{m}_1^2 \times \mathfrak{m}_2$, $L_7 = \mathfrak{m}_1 \times \mathfrak{m}_2$, $L_8 = R_1 \times \mathfrak{m}_2^3$, $L_9 = R_1 \times \langle 0 \rangle$, $L_{10} = \mathfrak{m}_1 \times R_2$, $_{11} = R_1 \times \mathfrak{m}_2^2$ and $C = \{L_1, L_2, \ldots, L_{11}\} \subset V(\Gamma_r(R))$. Note that the subgraph induced by C contains two blocks both isomorphic to $K_{3,3}$ as in Figure 2 by taking $L_i = I_i$ for $1 \leq i \leq 11$. From this, we have $g(\Gamma_r(R)) > 1$, which is a contradiction. Hence either $\eta_1 \leq 3$ or $\eta_2 \leq 3$. Without loss of generality, let us take $\eta_1 \leq 3$.

Case 1. Assume that $\eta_1 = 3$.

Suppose $\eta_2 \geq 3$. Consider the subgraph H of $\Gamma_r(R)$ induced by the non-trivial principal ideals $Y_1 = \mathfrak{m}_1 \times \mathfrak{m}_2$, $Y_2 = \mathfrak{m}_1 \times R_2$, $Y_3 = \mathfrak{m}_1^2 \times R_2$, $Y_4 = R_1 \times \mathfrak{m}_2$, $X_1 = \langle 0 \rangle \times \mathfrak{m}_2^2$, $X_2 = \mathfrak{m}_1^2 \times \langle 0 \rangle$, $X_3 = \mathfrak{m}_1^2 \times \mathfrak{m}_2^2$, $U_1 = \langle 0 \rangle \times \mathfrak{m}_2$, $U_2 = \langle 0 \rangle \times R_2$, $U_3 = \mathfrak{m}_1^2 \times \mathfrak{m}_2$, $V_1 = \mathfrak{m}_1 \times \langle 0 \rangle$, $V_2 = \mathfrak{m}_1 \times \mathfrak{m}_2^2$, $V_3 = R_1 \times \langle 0 \rangle$, $V_4 = R_1 \times \mathfrak{m}_2^2$. Since $U_iV_j \in E(\Gamma_r(R))$, H contains $K_{3,4}$ as a subgraph. Using Theorem 1, $g(H) \geq 1$. Suppose $g(H)$=1. Let $H' = H - \{X_1, X_2, X_3\} - \{V_2V_3, U_2U_3\}$ and $H'' = H' - \{Y_1, Y_2, Y_3, Y_4\}$. Then $H'' \cong K_{3,4}$ and so g(H'')=1. Since g(H)=1 and $g(H'') \leq g(H') \leq g(H)$, we get $g(H')$=1.

Note that $|V(H')|$=11, $|E(H')|$=25. In fact, $V(H') = \{Y_1, Y_2, Y_3, Y_4, U_1, U_2, U_3, V_1, V_2, V_3\}$ and $E(H') = \{Y_1Y_3,\ Y_1U_2,\ Y_1V_3,\ Y_1V_4,\ Y_2Y_4,\ Y_2V_3,\ Y_2V_4,\ Y_3Y_4,\ Y_3V_1,\ Y_3V_2,\ Y_3V_3,\ Y_3V_4,\ Y_4U_2,\ U_1V_1,\ U_1V_2,\ U_1V_3,\ U_1V_4,\ U_2V_1,\ U_2V_2,\ U_2V_3,\ U_2V_4,\ U_3V_1,\ U_3V_2,\ U_3V_3,\ U_3V_4\}$.

Using the fact that $n - m + f = 2 - 2g$, there are 14 faces in any embedding of H' on the torus. Let $\{F'_1, \ldots, F'_{14}\}$ be the set of all faces corresponding to an embedding of H' on the torus. Since H'' is isomorphic to $K_{3,4}$, by Euler's formula, any embedding of H'' in S_1 has 5 faces, one octagonal face and 4 rectangular faces, or two hexagonal faces and 3 rectangular faces. Hence boundaries of faces are 4-cycles or two 6-cycles or one 8-cycle. Let $\{F''_1, \ldots, F''_5\}$ be the set of all faces of H'' obtained by deleting Y_1, Y_2, Y_3, Y_4 and all the edges incident with Y_1, Y_2, Y_3, Y_4 from the representation of H'. Next, we prove $g(H) > 1$ by deletion and insertion argument.

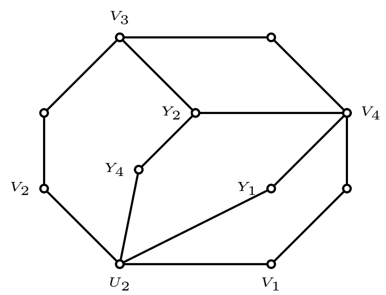


Figure 5(a): F''_m

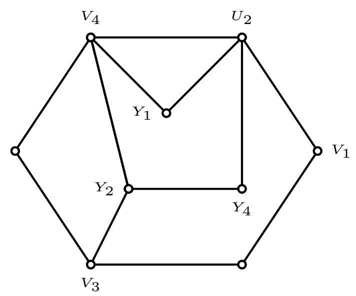


Figure 5(b): F''_n

Since $Y_2Y_4, Y_3Y_4 \in E(H')$, vertices Y_2, Y_3, Y_4 should be inserted in the same face say F''_m of H'' to avoid crossing. Note that $Y_1Y_3, Y_1V_3, Y_1V_4, Y_1U_2, Y_2V_3, Y_2V_4, Y_3V_1, Y_3V_2, Y_3V_3, Y_3V_4, Y_4V_2 \in E(H')$ and therefore V_1, V_2, V_3, V_4, U_2 are on the boundary of F''_m. The following are edges in H'. Let $e_1 = Y_2Y_4, e_2 = Y_3Y_4, e_3 = Y_1Y_3, e_4 = Y_1V_3, e_5 = Y_1V_4, e_6 = Y_1U_2, e_7 = Y_2V_3, e_8 = Y_2V_4, e_9 = Y_3V_1, e_{10} = Y_3V_2, e_{11} = Y_3V_3, e_{12} = Y_3V_4, e_{13} = Y_4U_2$. From this, it is clear that Y_1, Y_2, Y_3, Y_4 should be inserted into the same face.

If we insert Y_1, Y_2, Y_3, Y_4 and e_i $(1 \leq i \leq 13)$ in the octagonal face F''_m, then we obtain the Figure 5(a). However from Figure 5(a), it is clear that, if we insert the vertex Y_3 into the face F''_m, then we get an edge crossing, which is a contradiction. If we insert Y_1, Y_2, Y_3, Y_4 and e_i $(1 \leq i \leq 13)$ in the hexagonal face F''_n, then we obtain the Figure 5(b). However from Figure 5(b), it is clear that there is no way to insert the vertex Y_3 into the faces F''_n without crossing in the embedding of H'. Therefore we get, $g(H) > 1$ and so $g(\varGamma_r(R)) > 1$, which is a contradiction. Hence $\eta_2 = 2$ and so $\mathfrak{m}_1, \mathfrak{m}_1^2$ and $\mathfrak{m}_2$ are the only non-trivial principal ideals in R_1 and R_2 respectively.

Case 2. $\eta_1 = 2$.

Suppose $\eta_2 \geq 4$. Let $U_1 = \langle 0 \rangle \times \mathfrak{m}_2$, $U_2 = \langle 0 \rangle \times R_2$, $U_3 = \mathfrak{m}_1 \times \mathfrak{m}_2$, $U_4 = \mathfrak{m}_1 \times R_2$, $V_1 = R_1 \times \langle 0 \rangle$, $V_2 = R_1 \times \mathfrak{m}_2^2$, $V_3 = R_1 \times \mathfrak{m}_2^3$, $Y_1 = \langle 0 \rangle \times \mathfrak{m}_2^2$, $Y_2 = \langle 0 \rangle \times \mathfrak{m}_2^3$, $Y_3 = \mathfrak{m}_1 \times \langle 0 \rangle$, $Y_4 = \mathfrak{m}_1 \times \mathfrak{m}_2^3$, $X_1 = \mathfrak{m}_1 \times \mathfrak{m}_2^2$, $X_2 = R_1 \times \mathfrak{m}_2$ and $D = \{U_1, U_2, U_3, U_4, V_1, V_2, V_3, Y_1, Y_2, Y_3, Y_4, X_1,\ X_2\} \subseteq \varOmega(R)^*$. Let G be the subgraph of $\varGamma_r(R)$ induced by D, $G' = G - \{X_1, X_2\} - \{U_2U_3\}$ and $G'' = G' - \{Y_1, Y_2, Y_3, Y_4\}$. Then $G'' \cong K_{3,4}$ and so $g(K_{3,4}) = 1$. Since $V_iU_j \in E(\varGamma_r(R))$ and G contains $K_{3,4}$ as a subgraph, and so $g(G) \geq 1$. If $g(G) = 1$, then we get that $g(G')$=1. Note that $|V(G')| = 11$, $|E(G')| = 23$. By Euler's formula, there are 12 faces when embedding G' on the torus. Let $\{F'_1, \ldots, F'_{12}\}$ be the set of faces of G' through a representation of G' on the torus. Again by Theorem 1, $K_{3,4}$ has 5 faces, one octagonal face and 4 rectangular faces, or two hexagonal faces and 3 rectangular faces. Observe that $K_{3,4}$ has boundaries which are 4-cycles or two 6-cycles or one 8-cycle. Let $\{F''_1, \ldots, F''_5\}$ be the set of faces of G'' obtained by deleting Y_1, Y_2, Y_3 and Y_4 and all the edges incident with Y_1, Y_2, Y_3 and Y_4 from the representation of G'.

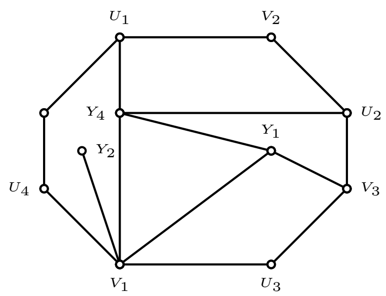


Figure 6(a): F''_ℓ

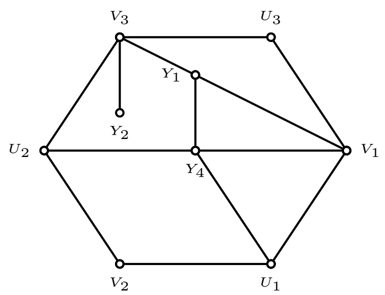


Figure 6(b): F''_s

Note that $Y_1Y_3, Y_1Y_4 \in E(G')$. Hence Y_1, Y_3, Y_4 should be inserted in the same face say F''_ℓ of G'' to avoid edge crossing. Also note that $Y_1V_1, Y_1V_3, Y_2V_1, Y_2Y_3, Y_3U_1,\ Y_3U_2, Y_4U_1, Y_4U_2, Y_4V_1 \in E(G')$ and therefore V_1, U_1, V_3, U_2 are boundary vertices of F''_ℓ. Consider the following edges of G'. Let $e_1 = Y_1V_1, e_2 = Y_1V_3, e_3 = Y_2V_1, e_4 = Y_3U_1, e_5 = Y_3U_2, e_6 = Y_4U_1, e_7 = Y_4U_2, e_8 = Y_4V_1, e_9 = Y_3Y_2, e_{10} = Y_3Y_1, e_{11} = Y_1Y_4$. From this, it is clear that Y_1, Y_2, Y_3, Y_4 should be inserted into the same face. If we insert Y_1, Y_2, Y_3, Y_4 and e_i $(1 \leq i \leq 11)$ in the octagonal face F''_ℓ then we obtain the Figure 6(a). However from Figure 6(a), it is clear that there is no way to insert the vertex Y_3 into the face F''_ℓ without crossings. If we insert Y_1, Y_2, Y_3, Y_4 and e_i $(1 \le i \le 11)$ in the hexagonal face F''_s then we obtain the Figure 6(b). However from Figure 6(b), it is clear that there is no way to insert the vertex Y_3 into the faces F''_s without crossings. Therefore we get, $g(G) > 1$ and hence $g(\Gamma_r(R)) > 1$. Hence $\eta_2 \le 3$. Since $g(\Gamma_r(R)) = 1$ and by Theorem 5, $\eta_2 \ne 2$. Thus, $\eta_2 = 3$ and so $\mathfrak{m}_1$ and $\mathfrak{m}_2$, $\mathfrak{m}_2^2$ are the only non-trivial principal ideals in R_1 and R_2 respectively.

Converse follows from Figures 7(a) and 7(b). □

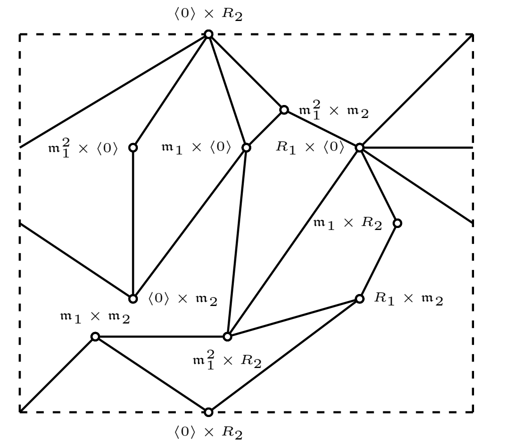


Figure 7(a): $\eta_1 = 3, \eta_2 = 2$

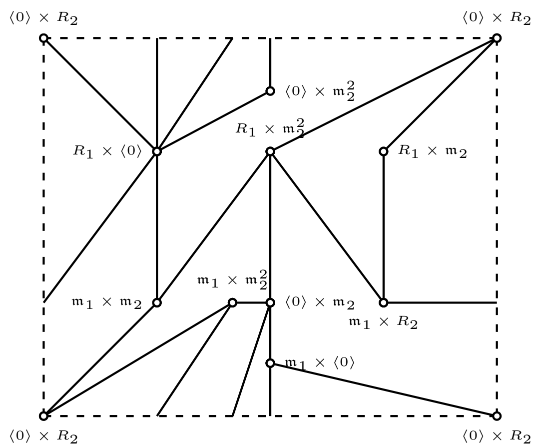


Figure 7(b): $\eta_1 = 2, \eta_2 = 3$

Embeddings of $\Gamma_r(R_1 \times R_2)$ in S_1

Theorem 9 *For integers $n, m \geq 1$, let $R = R_1 \times R_2 \times \cdots \times R_n \times F_1 \times \cdots \times F_m$ be a commutative ring with identity where each $(R_i, \mathfrak{m}_i)$ $(1 \leq i \leq n)$ is a local ring with $\mathfrak{m}_i \neq \{0\}$ and each F_j $(1 \leq j \leq m)$ is a field. Then $g(\Gamma_r(R)) = 1$ if and only if R satisfies one of the following conditions:*

(1) *$R \cong R_1 \times F_1 \times F_2$ and $\mathfrak{m}_1$ is the only non-trivial principal ideal in R_1;*
(2) *$R \cong R_1 \times F_1$ and*
  (i) *$\mathfrak{m}_1 = \langle b_1, b_2 \rangle$, and also $\langle b_1 \rangle$, $\langle b_2 \rangle$, $\langle b_1 b_2 \rangle$ and $\langle b_1 + b_2 \rangle$ are the only non-trivial principal ideals of R_1.*
  (ii) *$\mathfrak{m}_1 = \langle b_1 \rangle$ is a principal ideal in R_1 with nilpotency $\eta = 5$ or 6;*
    (a) *If $\eta = 5$, then $\mathfrak{m}$, $\mathfrak{m}^2$, $\mathfrak{m}^3$ and $\mathfrak{m}^4$ are the only non-trivial principal ideals of R_1.*
    (b) *If $\eta = 6$, then $\mathfrak{m}$, $\mathfrak{m}^2$, $\mathfrak{m}^3$, $\mathfrak{m}^4$ and $\mathfrak{m}^5$ are the only non-trivial principal ideals of R_1.*

Proof. Assume that $g(\Gamma_r(R)) = 1$ for $R_1 \times R_2 \times \cdots \times R_n \times F_1 \times \cdots \times F_m$.

Suppose $n \geq 2$. Let J_1 and J_2 are the non-trivial principal ideals in R_1 and R_2 respectively and let $X_1 = \langle 0 \rangle \times R_2 \times \cdots \times \langle 0 \rangle \times F_1 \times \cdots \times \langle 0 \rangle$, $X_2 = \langle 0 \rangle \times J_2 \times \cdots \times \langle 0 \rangle \times F_1 \times \cdots \times \langle 0 \rangle$, $X_3 = J_1 \times J_2 \times \cdots \times \langle 0 \rangle \times F_1 \times \cdots \times \langle 0 \rangle$, $X_4 = J_1 \times \langle 0 \rangle \times \cdots \times \langle 0 \rangle \times F_1 \times \cdots \times \langle 0 \rangle$, $X_5 = \langle 0 \rangle \times \langle 0 \rangle \times \cdots \times \langle 0 \rangle \times F_1 \times \cdots \times \langle 0 \rangle$, $Y_1 = R_1 \times \langle 0 \rangle \times \cdots \times \langle 0 \rangle \times \cdots \times \langle 0 \rangle$, $Y_2 = R_1 \times J_2 \times \cdots \times \langle 0 \rangle \times \cdots \times \langle 0 \rangle$, $Y_3 = R_1 \times R_2 \times \cdots \times \langle 0 \rangle \times \cdots \times \langle 0 \rangle$, $Y_4 = J_1 \times R_2 \times \cdots \times \langle 0 \rangle \times \cdots \times \langle 0 \rangle$ and $A = \{X_1, X_2, X_3, X_4, X_5, Y_1, Y_2, Y_3, Y_4\} \subseteq \Omega(R)^*$. Then the subgraph induced by A in $\Gamma_r(R)$ contains $K_{5,4}$ as subgraph with vertex partitions $\{X_1, X_2, X_3, X_4, X_5\}$ and $\{Y_1, Y_2, Y_3, Y_4\}$. By Lemma 1, $g(\Gamma_r(R)) > 1$ which is a contradiction. Hence $n = 1$.

Suppose $m \geq 3$. Let I be a non-trivial principal ideal in R_1 and let $Y_1 = R_1 \times \langle 0 \rangle \times \langle 0 \rangle \times \langle 0 \rangle \times \cdots \times \langle 0 \rangle$, $Y_2 = I \times \langle 0 \rangle \times F_2 \times \langle 0 \rangle \times \cdots \times \langle 0 \rangle$, $Y_3 = R_1 \times F_1 \times \langle 0 \rangle \times \langle 0 \rangle \times \cdots \times \langle 0 \rangle$, $Y_4 = \langle 0 \rangle \times F_1 \times F_2 \times F_3 \times \cdots \times \langle 0 \rangle$, $Y_5 = I \times F_1 \times \langle 0 \rangle \times F_3 \times \cdots \times \langle 0 \rangle$, $Y_6 = I \times F_1 \times F_2 \times \langle 0 \rangle \times \cdots \times \langle 0 \rangle$, $Y_7 = \langle 0 \rangle \times F_1 \times \langle 0 \rangle \times \langle 0 \rangle \times \cdots \times \langle 0 \rangle$, $Y_8 = R_1 \times \langle 0 \rangle \times \langle 0 \rangle \times F_3 \times \cdots \times \langle 0 \rangle$, $Y_9 = I \times \langle 0 \rangle \times \langle 0 \rangle \times \langle 0 \rangle \times \cdots \times \langle 0 \rangle$, $Y_{10} = \langle 0 \rangle \times F_1 \times F_2 \times \langle 0 \rangle \times \cdots \times \langle 0 \rangle$, $Y_{11} = I \times \langle 0 \rangle \times \langle 0 \rangle \times F_3 \times \cdots \times \langle 0 \rangle$. Then the subgraph induced by $B = \{Y_1, Y_2, \ldots Y_{11}\} \subseteq \Omega(R)^*$ of $\Gamma_r(R)$ contains H as in Figure 2 as subgraph by identifying $Y_i = I_i$ for $1 \leq i \leq 11$. This gives that $g(\Gamma_r(R)) \geq 2$ which is a contradiction. Hence $m \leq 2$.

Assume that $n = 1$ and $m = 2$ and so $R = R_1 \times F_1 \times F_2$. Since R is finite and R_1 is an Artinian, every ideal in R_1 is finitely generated. Let $\Phi = \{a_1, a_2, \ldots, a_k : a_i \in R_1 \text{ for } 1 \leq i \leq k\}$ be a minimal generating set for $\mathfrak{m}_1$ in R_1. Then $k \geq 1$ and $\langle a_i \rangle \nsubseteq \langle a_j \rangle$ for all $i \neq j$.

Suppose $k \geq 2$. Let $U_1 = \langle a_1 \rangle \times \langle 0 \rangle \times \langle 0 \rangle$, $U_2 = \langle 0 \rangle \times \langle 0 \rangle \times F_2$, $U_3 = \langle a_2 \rangle \times \langle 0 \rangle \times \langle 0 \rangle$, $U_4 = \langle 0 \rangle \times F_1 \times \langle 0 \rangle$, $U_5 = R_1 \times \langle 0 \rangle \times \langle 0 \rangle$, $U_6 = \langle 0 \rangle \times F_1 \times F_2$, $U_7 = \langle a_1 \rangle \times F_1 \times \langle 0 \rangle$, $U_8 = \langle a_1 \rangle \times \langle 0 \rangle \times F_2$, $U_9 = \langle a_2 \rangle \times \langle 0 \rangle \times F_2$, $U_{10} = \langle a_2 \rangle \times F_1 \times \langle 0 \rangle$, $U_{11} = R_1 \times \langle 0 \rangle \times F_2$. Then the subgraph induced by $C = \{U_1, U_2, U_3, U_4, U_5, U_6, U_7, U_8, U_9, U_{10}, U_{11}\} \subseteq \Omega(R)^*$ contains two blocks, both isomorphic to $K_{3,3}$ as in Figure 2 by taking $U_i = I_i$ for $1 \leq i \leq 11$. By Lemma 2, $g(\Gamma_r(R)) > 1$, a contradiction. Hence $k = 1$ and so $\mathfrak{m}_1$ is a principal ideal generated by a_1.

Since R_1 is Artinian, $\mathfrak{m}_1^\eta = \langle 0\rangle$, $\mathfrak{m}_1^{\eta-1} \neq \langle 0\rangle$ for some $\eta \in \mathbb{N}$. Suppose $\eta \geq 3$. Then the subgraph induced by the $I_1 = \{\mathfrak{m}_1 \times \langle 0\rangle \times \langle 0\rangle$, $I_2 = \langle 0\rangle \times \langle 0\rangle \times F_2$, $I_3 = \mathfrak{m}_1^2 \times \langle 0\rangle \times \langle 0\rangle$, $I_4 = \langle 0\rangle \times F_1 \times \langle 0\rangle$, $I_5 = R_1 \times \langle 0\rangle \times \langle 0\rangle$, $I_6 = \langle 0\rangle \times F_1 \times F_2$, $I_7 = \mathfrak{m}_1 \times F_1 \times \langle 0\rangle$, $I_8 = \mathfrak{m}_1 \times \langle 0\rangle \times F_2$, $I_9 = \mathfrak{m}_1^2 \times \langle 0\rangle \times F_2$, $I_{10} = \mathfrak{m}_1^2 \times F_1 \times \langle 0\rangle$, $I_{11} = R_1 \times \langle 0\rangle \times F_2\}$ in $\Gamma_r(R)$ contains H as in Figure 2 as a subgraph. By Lemma 2, $g(\Gamma_r(R)) > 1$, a contradiction. Hence $\eta = 2$ and so R_1 contains exactly one non-trivial principal ideal m_1.

Assume that $n = 1, m = 1$ and so $R = R_1 \times F_1$. Consider $\Phi = \{b_1, b_2, \ldots, b_t : b_i \in R_1$ for $1 \leq i \leq t\}$ be a minimal generating set for $\mathfrak{m}$ in R_1. Then $t \geq 1$ and $\langle b_i\rangle \nsubseteq \langle b_j\rangle$ for all $i \neq j$.

Suppose $t \geq 3$. Let $X = \{S_1, S_2, S_3, S_4, S_5, S_6, S_7, T_1, T_2, T_3, U_1, U_2, U_3\} \subset V(\Gamma_r(R))$, where $S_1 = (0) \times F_1$, $S_2 = \langle b_1\rangle \times F_1$, $S_3 = \langle b_2\rangle \times F_1$, $S_4 = \langle b_3\rangle \times F_1$, $S_5 = \langle b_1 + b_2\rangle \times F_1$, $S_6 = \langle b_1 + b_3\rangle \times F_1$, $S_7 = \langle b_2 + b_3\rangle \times F_1$, $T_1 = \langle b_1 + b_2\rangle \times (0)$, $T_2 = \langle b_1 + b_3\rangle \times (0)$, $T_3 = \langle b_2 + b_3\rangle \times (0)$, $V_1 = \langle b_3\rangle \times (0)$, $V_2 = \langle b_2\rangle \times (0)$, $V_3 = \langle b_1\rangle \times (0)$. Then the subgraph induced by X contains a subdivision of $K_{7,3}$ with vertex partitions $\{S_1, S_2, S_3, S_4, S_5, S_6, S_7\}$, $\{T_1, T_2, T_3\}$ and the edges joining T_1 and S_5, T_2 and S_6 and T_3 and S_7 through the vertices U_1, U_2, and U_3 respectively. Applying Lemma 1, $g(\Gamma_r(R)) > 1$, a contradiction. Hence $t \leq 2$.

Case 1. $t = 2$.

Assume that $\mathfrak{m}^2 \neq 0$. Then $b_i^2 \neq 0$ for some i. Without loss of generality, we assume that $b_1^2 \neq 0$. Consider the non-trivial principal ideals $U_1 = R_1 \times \langle 0\rangle$, $U_2 = \langle b_1 + b_2\rangle \times \langle 0\rangle$, $U_3 = \langle b_2\rangle \times \langle 0\rangle$, $X_1 = \langle 0\rangle \times \langle 1\rangle$, $X_2 = \langle b_1\rangle \times \langle 1\rangle$, $X_3 = \left\langle b_1^2\right\rangle \times \langle 1\rangle$, $V_1 = \langle b_1 + b_2\rangle \times \langle 1\rangle$, $Y_1 = \langle b_2\rangle \times \langle 1\rangle$,, $Y_2 = \langle b_1\rangle \times \langle 0\rangle$, $Y_3 = \left\langle b_1^2\right\rangle \times \langle 0\rangle$ of R and let $X = \{U_1, U_2, U_3, X_1, X_2, X_3, V_1, Y_1, Y_2, Y_3\} \subseteq \Omega(R)^*$. Let H be the subgraph induced by X in $\Gamma_r(R)$, $H' = H - \{v_1\}$ and $H'' = H' - \{Y_1, Y_2, Y_3\}$. Then $H'' \cong K_{3,3}$ and so $g(H'') = 1$. Since $u_i x_i \in E(\Gamma_r(R))$ and H contains $K_{3,3}$ as a subgraph, $g(H) \geq 1$. Suppose that $g(H)$=1. Then we get $g(H')$=1. Note that $|V(G')| = 9$ and $|E(G')| = 20$. By Euler's formula, there are 11 faces for any embedding of H' on the torus. Fix a representation of H' and let $\{F'_1, \ldots, F'_{11}\}$ be the set of faces of H'. Again by Theorem 1, $K_{3,3}$ has 3 faces. Let $\{F''_1, \ldots, F''_3\}$ be the set of faces of H'' obtained by deleting Y_1, Y_2, Y_3 and all the edges incident with Y_1, Y_2, Y_3 from the representation of H'.

Note that $Y_1Y_2, Y_1Y_3 \in E(H')$. Hence Y_1, Y_2, Y_3 should be inserted in the same face say F''_q of H'' to avoid crossing. Also note that $Y_1U_1,\ Y_1U_2,\ Y_1X_2,\ Y_1X_3,\ Y_2U_3,\ Y_2X_1,\ Y_2X_3,\ Y_3U_3,\ Y_3X_1 \in E(G')$ and therefore X_1, X_3, U_2, U_3 are in the boundary of F''_q. Consider the edges $e_1 = Y_1Y_2, e_2 = Y_1Y_3, e_3 = Y_1U_1, e_4 = Y_1U_2, e_5 = Y_1X_2, e_6 = Y_1X_3, e_7 = Y_2U_3, e_8 = Y_2X_1, e_9 = Y_2X_3, e_{10} = Y_3U_3, e_{11} = Y_3X_1$ of H'. If we insert Y_1, Y_2, Y_3 and e_i $(1 \leq i \leq 11)$ in the face F''_q, then we obtain the Figure 8. However from Figure 8, it is clear that there is no way to insert the vertex Y_1 into the face F''_q without crossing in the embedding of H'. Therefore we get, $g(H) > 1$. Since H is a subgraph of $g(\Gamma_r(R))$, $g(\Gamma_r(R)) > 1$, a contradiction. Therefore, $b_i^2 = 0$ for all i. Thus, the all non-trivial principal ideals of R_1 are of the form: $\langle b_1\rangle$, $\langle b_2\rangle$, $\langle b_1 b_2\rangle$ and $\langle b_1 + \alpha b_2\rangle$, where $\alpha \in U(R_1)$.

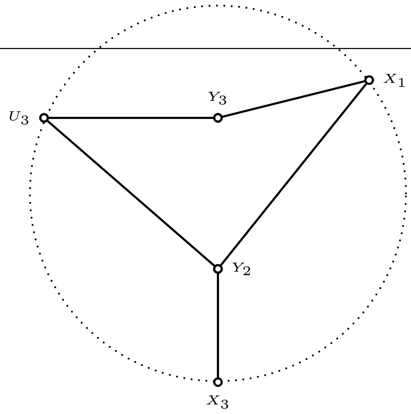


Figure 8: F_q''

Note that $|U(R_1)| \geq 2$. Let $\alpha, \beta \in U(R_1)$ with $\alpha \neq \beta$. Suppose $\langle b_1 + \alpha b_2\rangle \neq \langle b_1 + \beta b_2\rangle$. Then $|V(\Gamma_r(R))| > 10$ and $|E(\Gamma_r(R))| > 33$. By Theorem 2, $g(\Gamma_r(R)) > 1$, a contradiction. Hence, $\langle b_1 + \alpha b_2\rangle = \langle b_1 + \beta b_2\rangle$ for all $\alpha \neq \beta \in U(R_1)$ and so $\langle b_1\rangle$, $\langle b_2\rangle$, $\langle b_1 b_2\rangle$ and $\langle b_1 + b_2\rangle$ are only non-trivial principal ideals of R_1.

Case 2. $t = 1$.

Suppose $\eta \geq 7$. Let $A = \{U_1, U_2, U_3, U_4, V_1, V_2, V_3, V_4, V_5\} \subset \Omega(R)^*$, where $U_1 = \langle 0\rangle \times \langle 1\rangle$,, $U_2 = \mathfrak{m}_1^5 \times \langle 1\rangle$,, $U_3 = \mathfrak{m}_1^6 \times \langle 1\rangle$,, $U_4 = \mathfrak{m}_1^4 \times \langle 1\rangle$,, $V_1 = \mathfrak{m}_1 \times \langle 0\rangle$, $V_2 = \mathfrak{m}_1^2 \times \langle 0\rangle$, $V_3 = \mathfrak{m}_1^3 \times \langle 0\rangle$, $V_4 = R_1 \times \langle 0\rangle$, $V_5 = \mathfrak{m}_1^4 \times \langle 0\rangle$. Then the subgraph induced by A in $\Gamma_r(R)$ contains a subgraph which is isomorphic to the graph given in Figure 9. By Lemma 3, $g(\Gamma_r(R)) > 1$, a contradiction. Since $g(\Gamma_r(R)) = 1$ and by Theorem 6, $\eta > 4$. Hence $\eta = 5$, or 6. If $\eta = 5$, then R_1 contains exactly four non-trivial principal ideals $\mathfrak{m}_1$, $\mathfrak{m}_1^2$, $\mathfrak{m}_1^3$ and $\mathfrak{m}_1^4$.

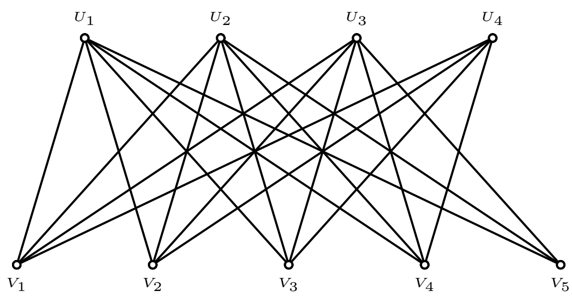


Figure 9

If $\eta = 6$, then R_1 contains exactly five non-trivial principal ideals $\mathfrak{m}_1$, $\mathfrak{m}_1^2$, $\mathfrak{m}_1^3$, $\mathfrak{m}_1^4$ and $\mathfrak{m}_1^5$.

Converse follows from Figure 10, 11, 12 and 13.

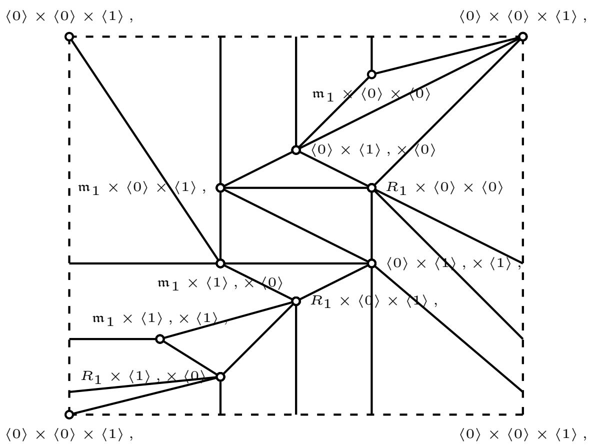


Figure 10: Embedding of $\Gamma_r(R_1 \times F_1 \times F_2)$ in S_1

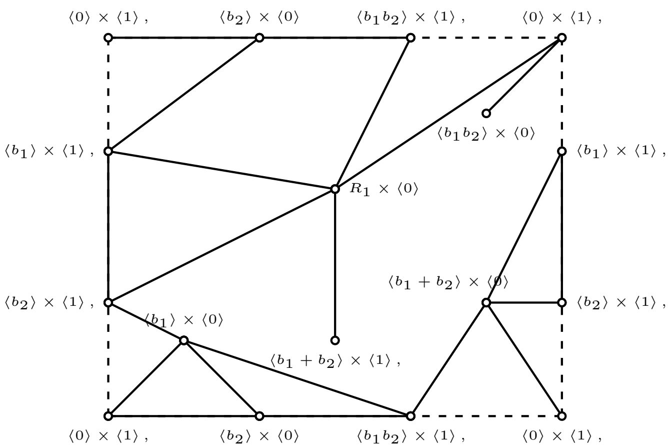


Figure 11: Embedding of $\Gamma_r(R_1 \times F_1)$ in S_1 and $t = 2$

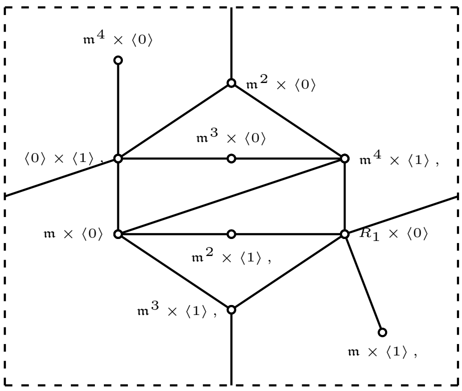
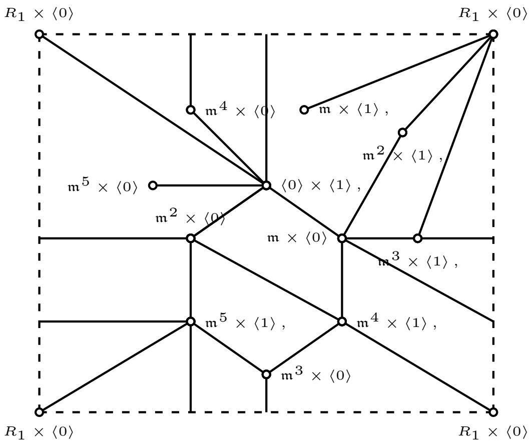


Figure 12: $\eta = 5$

Embedding of $\Gamma_r(R_1 \times F_1)$ in S_1 and $t = 1$



□

Figure 13: $\eta = 6$

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Ethical Approval

Conflict of interest

The authors declare that they have no conflict of interest.

Data Availability Statement

The authors have not used any data for the preparation of this manuscript.

Competing Interests

The authors have no relevant financial or non-financial interests to disclose.

Author Contributions

All authors contributed to the study conception and design. The first draft of the manuscript was written by E. Jesili and all authors commented on previous versions of the manuscript. All authors read and approved the final manuscript.

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