New methods for computing fuzzy eigenvalues and fuzzy eigenvectors of fuzzy matrices using nonlinear programming approach

Somaye khosravi
Islamic Azad University of Lahijan
Ali Asghar Hosseinzadeh (hosseinzadeh_ali@yahoo.com)
Islamic Azad University of Lahijan https://orcid.org/0000-0002-0592-1795
armin Ghane Kanafi
Islamic Azad University of Lahijan
amir hosein Refahi Sheikhani
Islamic Azad University of Lahijan

Research Article

Keywords: Fuzzy number, Fuzzy eigenvalues, Fuzzy eigenvector, fuzzy triangular matrix

Posted Date: August 26th, 2022

DOI: https://doi.org/10.21203/rs.3.rs-1426064/v1

New methods for computing fuzzy eigenvalues and fuzzy eigenvectors of fuzzy matrices using nonlinear programming approach

Somaye khosravi, Ali Asghar Hosseinzadeh*, Armin Ghane Kanafi, Amir Hossein Refahi Sheikhani

Department of Mathematics, Lahijan Branch, Islamic Azad University, Lahijan, Iran

Abstract

In this paper, we propose a new method to obtain eigenvalues and fuzzy triangular eigenvectors of a fuzzy triangular matrix $(\widetilde{\mathrm{A}})$, which are the elements of the given fuzzy triangular matrix . To this purpose, we solve the $1-$cut of a fuzzy triangular matrix $(\widetilde{\mathrm{A}})$ to obtain the $1-$cut of eigenvalues and eigenvectors. Then, based on the results obtained in a $1-$cut mode, we use three new models to determine the left and right widths for those eigenvalues and eigenvectors. So, after some manipulation, in each of the models, the fully fuzzy linear systems (FFLSs) transformed to $2\mathrm{n}$ crisp linear equations and some crisp linear non-equations (that, the first model includes $2(\mathrm{n}+1)$, the second model includes $2(\mathrm{n}+3)$ and the third model includes $6\mathrm{n}+2$ crisp linear non-equation). Then, we suggest a nonlinear programming problem (NLP) to calculation simultaneous equations and non-equations. Furthermore, we define three other new eigenvalues (namely, fuzzy escribed eigenvalue, fuzzy peripheral eigenvalue, and fuzzy approximate eigenvalue) for a fuzzy triangular matrix $(\widetilde{\mathrm{A}})$ that does not have any suitable solution. the fuzzy escribed eigenvalue which is placed in a tolerable fuzzy triangular eigenvalue set (TTFES), the fuzzy peripheral eigenvalue placed in a controllable fuzzy triangular eigenvalue set (CTFES), and the fuzzy approximate eigenvalue placed in an approximate fuzzy triangular eigenvalue set (ATFES). Finally, numerical examples are presented to illustrate the proposed method.

Keywords: Fuzzy number , Fuzzy eigenvalues , Fuzzy eigenvector, fuzzy triangular matrix

1. Introduction

* **Correspondence Author, Email: : hosseinzadeh_ali@yahoo.com**

The system of linear equations plays a crucial role in various areas such as physics, statistics, operational research, engineering, and social sciences. One of the major applications of fuzzy number arithmetic is in solving linear systems whose parameters are all or partially represented by fuzzy numbers. Therefore, it is immensely important to develop a numerical procedure that would appropriately treat general fuzzy linear systems and solve them.

The system of linear equations $A\widetilde{X} = \tilde{b}$ where the coefficient matrix $A = (a_{ij}), 1 \leq i, j \leq n$ is a crisp matrix and the elements, $\tilde{x}_i, \tilde{b}_i, 1 \leq i \leq n$ of the vectors $\widetilde{X}, \tilde{b}$ are fuzzy numbers, called a fuzzy linear system (FLS). A general model for solving a FLS first was proposed by Friedman et al.[14]. Friedman et al.[15] investigated a dual fuzzy linear system using nonnegative matrix theory. To continue Friedman et al.'s work, Allahviranloo $[2-4]$ proposed various numerical methods to solve fuzzy linear systems. In addition, Abbasbandy et al. [1] used the LU decomposition method for solving a fuzzy system of the linear equation when the coefficient matrix is symmetric positive definite. In 2011 Allahviranloo and Salahshour [6] proposed a practical method to solve a FLS. In that method, they solved 1-cut of a fuzzy linear system. Then, by some fuzzification, they obtained the fuzzy vector solutions by symmetric spreads of each element of a fuzzy vector solution. Also for more references see $[19-21]$.

The system of linear equations $\widetilde{A}\widetilde{X} = \tilde{b}$ where, $\widetilde{A} = (\tilde{a}_{ij})_{n\times n}, \widetilde{X}^T = (\tilde{x}_1, \cdots, \tilde{x}_n)_{1\times n}, \tilde{b}^T = (\tilde{b}_1, \cdots, \tilde{b}_n)_{1\times n}$ such that $\tilde{a}_{ij}, \tilde{b}_i, \tilde{x}_j$ are fuzzy numbers, for all $i, j = 1, \cdots, n$, called a fully fuzzy linear system(FFLS).

FFLSs have been studied by many authors. Buckley and Qu in their consecutive works $[8-10]$ suggested different solutions for FFLS. Based on their works, Muzzioli and Reynaerts [17] introduced an algorithm to find vector solutions by transforming the system $A_1x + b_1 = A_2x + b_2$ into the FFLS $Ax = b$ where $A = A_1 - A_2$ and $b = b_1 - b_2$. their approach contains solving of $2^{n(n+1)}$ crisp systems for all $\alpha \in [0,1]$. obviously, for a large n, obtaining such a solution is not easy work and such an approach has a big propagation error.

Dehghan et al. $[11, 12]$proposed some methods to solve FFLS such as the Cramer's rule, Gaussian elimination, LU decomposition $(\text{Doolittle algorithm})$, and linear programming (LP) for solving square and non-square fully fuzzy systems. However, their methods are not available for the nonnegative solution. Vroman et al. [18] suggested a practical algorithm using parametric functions in which the variables were given by the fuzzy coefficients of the system. In addition, they showed that their algorithm is better than the method of Buckley and Qu. Allahviranloo et al. [7] suggested a method to solve FFLS. To this end, they solved $1-\text{cut}$ of an FFLS, then allocated some unknown symmetric spreads to each row of a $1-$ cut and then the symmetric spreads of the solution are computed by solving a $2\mathrm{n}$ linear equations. In 2014, Allahviranloo and Hosseinzade. $[5]$ proposed a novel method to obtain the fuzzy trapezoidal solution for an FFLS. Their method is constructed based on solving two FILSs. In addition, they introduced two different models for those FFLSs that do not have a feasible solution.

in this paper, we propose a method for obtaining eigenvalues and fuzzy triangular eigenvectors for a fuzzy triangular matrix $(\widetilde{\mathrm{A}})$ using a nonlinear programming problem (NLP). Moreover, we define three other new eigenvalues namely, fuzzy escribed eigenvalue, fuzzy peripheral eigenvalue, and fuzzy approximate eigenvalue for a fuzzy triangular matrix.

The structure of this paper organized as follows:

In section 2, we introduce the notation, the definitions, and preliminary results, which will be used throughout. In section 3, we design our new method to obtain eigenvalues and fuzzy triangular eigenvectors of a fuzzy triangular matrix. In section 4, we define three new eigenvalues namely, fuzzy escribed eigenvalue, fuzzy peripheral eigenvalue, and fuzzy approximate eigenvalue for a fuzzy triangular matrix. Numerical examples are given in section 5 to examine our method and conclusions drawn in section 6.

2. preliminaries

The basic definitions of a fuzzy number given in [13, 16, 23] as follows:

Definition 2.1. An interval number $[x]$ is defined as the set of real numbers such that $[x]=[\underline{\mathrm{x}},\overline{\mathrm{x}}]=\{\acute{x}\in R: \underline{x}\leq \acute{x}\leq \overline{x}\}$ where $\underline{x}\leq \overline{x}$. We denote the set of all interval numbers by$\mathbb{I}$.

Definition 2.2. A vector $[X]=([x_1],[x_2],\dots,[x_n])^T$, where $[x_i]$=$[\underline{x_i},\overline{x_i}]$, $1\leq i\leq n$ are interval numbers, is called an interval number vector. In this case, we denote$[X]\in\mathbb{I}^n$.

Definition 2.3. Let $[x]=[\underline{x},\overline{x}]$ *and* $[y]=[\underline{y},\overline{y}]$ be to interval numbers, then

$$[\underline{x},\overline{x}]\oplus\left[\underline{y},\overline{y}\right]=\left[\underline{x}+\underline{y},\overline{x}+\overline{y}\right],\ [\underline{x},\overline{x}]\ominus\left[\underline{y},\overline{y}\right]=\left[\underline{x}-\overline{y},\overline{x}-\underline{y}\right],$$

$$[\underline{x},\overline{x}]\otimes\left[\underline{y},\overline{y}\right]=[\underline{c},\overline{c}]\qquad \begin{cases}\underline{c}=min\{\underline{x}.\underline{y},\overline{x}.\underline{y},\underline{x}.\overline{y},\ \overline{x}.\overline{y}\}\\ \overline{c}=max\left\{\underline{x}.\underline{y},\overline{x}.\underline{y},\underline{x}.\overline{y},\ \overline{x}.\overline{y}\right\}\end{cases},$$

$$[\underline{x},\overline{x}]\oslash\left[\underline{y},\overline{y}\right]=[\underline{x},\overline{x}]\otimes\left[1/\overline{y},1/\underline{y}\right]\qquad,\ \overline{y},\underline{y}\neq 0,\qquad(2.1)$$

$$k.\left[\underline{x}, \overline{x}\right] = \begin{cases} \left[k\underline{x}, k\overline{x}\right] & k \geq 0 \\ \left[k\underline{x}, k\overline{x}\right] & k < 0 \end{cases} \qquad k \in R$$

Definition 2.4. The width of an interval number $[x]$ is defined as follows:[5]

$W([x]) = \underline{x} - \overline{x}$.

Definition 2.5. For arbitrary interval numbers $[x] = [\underline{x}, \overline{x}]$ and $[y] = [\underline{y}, \overline{y}]$, and arbitrary interval number vectors $[X] = ([x_1], [x_2], \dots, [x_n])^T$ and $[Y] = ([y_1], [y_2], \dots, [y_n])^T$ we defined:

1) $[y] \subseteq [x] \iff \underline{x} \leq \underline{y} \leq \overline{y} \leq \overline{x}$
2) *[Y]*$\subseteq [X] \quad \iff \quad \sum_{j=1}^{n} [\underline{y_j}, \overline{y_j}] \subseteq \sum_{j=1}^{n} [\underline{x_j}, \overline{x_j}]$

Definition 2.6. $[20]$ For any two arbitrary interval number vectors $X, Y \in \mathbb{I}^n$ the metric, $d: \mathbb{I}^n \to \mathbb{R}$, in the space $\mathbb{I}^n$ defined as:

$$d(X, Y) = Max\left\{\max_{1 \leq j \leq n} \left|\underline{x_j} - \underline{y_j}\right|, \max_{1 \leq j \leq n} \left|\overline{x_j} - \overline{y_j}\right|\right\} \tag{2.2}$$

Obviously, $\mathbb{I}^n$ is a complete metric space with the metric d.

3. Eigenvalues and fuzzy eigenvectors

The basic definition of fuzzy numbers given in [16].

Definition 3.1. A fuzzy number is a function $\widetilde{u}: \mathbb{R} \longrightarrow [0,1]$ satisfying the following

Properties:

i. $\widetilde{u}$ *is normal, i.e.* $\exists x_0 \in \mathbb{R}$ *with* $\widetilde{u}(x_0) = 1$,
ii. $\widetilde{u}$ *is a convex fuzzy set,*
iii. $\widetilde{u}$ *is upper semi-continuous on* $\mathbb{R}$,
iv. $\overline{\{x \in \mathbb{R}: \widetilde{u}(x) > 0\}}$ *is compact, where* $\overline{A}$ *denotes the closure of* A.

The set of all these fuzzy numbers is denoted by F. Obviously, $\mathbb{R} \subset \mathrm{F}$. Here $\mathbb{R} \subset \mathrm{F}$ is understood as $\mathbb{R} = \{\chi_{\{x\}}: x \text{ is usual real number}\}$. For $0 < r \leq 1$, we define r-cuts of fuzzy number $\widetilde{u}$ as $[\widetilde{u}]_r = \{x \in \mathbb{R}: \widetilde{u}(x) \geq r\}$ and the support and core of $\tilde{A}$ are defined by the sets $S(\widetilde{u}) = \{x \in \mathbb{R}: \widetilde{u}(x) > 0\}$ and $C(\widetilde{u}) = \{x \in \mathbb{R}: \widetilde{u}(x) = 1\}$, respectively. Then, from $(i) - (iv)$ it follows that $[\widetilde{u}]_r$ is a bounded closed interval for each $r \in [0,1]$ [22]. In this paper, we denote the r-cuts of fuzzy number $\tilde{u}$ as $[\tilde{u}]_r = [\underline{u}(r), \overline{u}(r)]$, for each $r \in [0,1]$.

Remark 3.1. For arbitrary fuzzy numbers $\tilde{u} = [\underline{u}(r), \overline{u}(r)]$ and $\tilde{v} = [\underline{v}(r), \overline{v}(r)]$, and $\lambda \in \mathbb{R}$, $r - cuts$ of the sum $\tilde{u} + \tilde{v}$ and the product $\lambda.\tilde{u}$ are defined based on interval arithmetic as

$$\begin{cases} [\tilde{u}+\tilde{v}]_r = [\tilde{u}]_r + [\tilde{v}]_r = [\underline{u}(r)+\underline{v}(r), \overline{u}(r)+\overline{v}(r)], \\ [\tilde{u}-\tilde{v}]_r = [\tilde{u}]_r - [\tilde{v}]_r = [\underline{u}(r)-\overline{v}(r), \overline{u}(r)-\underline{v}(r)], \\ [\lambda.\tilde{u}]_r = \lambda.[\tilde{u}]_r = \begin{cases} [\lambda\underline{u}(r), \lambda\overline{u}(r)], & \lambda \geq 0, \\ [\lambda\overline{u}(r), \lambda\underline{u}(r)], & \lambda < 0. \end{cases} \end{cases} \tag{3.3}$$

Definition 3.2. Two fuzzy numbers $\tilde{u}$ and $\tilde{v}$ are said to be equal, if and only if $[\tilde{u}]_r = [\tilde{v}]_r$, i.e., $\underline{u}(r) = \underline{v}(r)\ and\ \overline{u}(r) = \overline{v}(r)$, for each $r \in [0,1]$.

Definition 3.3. A fuzzy triangular number $\tilde{u} = (a_1, a_2, a_3)$, defined as follows:

$$\tilde{u}(x) = \begin{cases} 0 & a_1 > x \\ \frac{x-a_1}{a_2-a_1} & a_1 \leq x \leq a_2 \\ \frac{a_3-x}{a_3-a_2} & a_2 \leq x \leq a_3 \\ 0 & a_3 < x \end{cases}$$

Where a_2 is the core, a_1 and a_3 are the left and right points of support.

We denote the set of fuzzy triangular numbers by F_T. Clearly, for the fuzzy triangular number $\tilde{a} = (a_1, a_2, a_3)$ we have $S(\tilde{a}) = [a_1, a_3]$ and $C(\tilde{a}) = a_2$.

Definition 3.4. Two fuzzy triangular numbers $\tilde{A} = (a_1, a_2, a_3)$ and $\tilde{B} = (b_1, b_2, b_3)$ are said to be equal, if and only if $a_1 = b_1, a_2 = b_2\ and\ a_3 = b_3$.

Definition 3.5. For arbitrary fuzzy triangular numbers $\tilde{A}$ and $\tilde{B}$, addition, subtraction, and scalar multiplication are defined as follows:

$$\tilde{A} + \tilde{B} = (a_1, a_2, a_3) + (b_1, b_2, b_3) = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$$

$$\tilde{A} - \tilde{B} = (a_1, a_2, a_3) - (b_1, b_2, b_3) = (a_1 - b_3, a_2 - b_2, a_3 - b_1)$$

$$\lambda.\tilde{A} = \lambda.(a_1, a_2, a_3) = \begin{cases} (\lambda a_1, \lambda a_2, \lambda a_3), & \lambda \geq 0, \\ (\lambda a_3, \lambda a_2, \lambda a_1), & \lambda < 0. \end{cases}$$

Definition 3.6. A vector $\tilde{X} = (\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n)^T$, where $\tilde{x}_i \in F_T \quad 1 \leq i \leq n$, is called a fuzzy triangular number vector. In this case, we denote $\tilde{X} \in F_T^n$.

Consider the fuzzy square matrix of $\tilde{A} = [\tilde{a}_{ij}]_{i,j=1}^{n}$ that $\tilde{a}_{ij} \in F_T$.

Definition 3.7. The fuzzy triangular number $\tilde{\lambda} \neq \tilde{0}$ is the fuzzy eigenvalue of the fuzzy triangular matrix $\tilde{A}$, if there is a fuzzy eigenvector $\tilde{X} \neq \tilde{0}$ such that $\tilde{A}\tilde{X} = \tilde{\lambda}\tilde{X}$.

Otherwise, the $n \times n$ linear system of equations

$$\begin{cases} \tilde{a}_{11}\tilde{x}_1 + \tilde{a}_{12}\tilde{x}_2 + \cdots + \tilde{a}_{1n}\tilde{x}_n = \tilde{\lambda}\tilde{x}_1, \\ \tilde{a}_{21}\tilde{x}_1 + \tilde{a}_{22}\tilde{x}_2 + \cdots + \tilde{a}_{2n}\tilde{x}_n = \tilde{\lambda}\tilde{x}_2, \\ \qquad\qquad \vdots \\ \tilde{a}_{n1}\tilde{x}_1 + \tilde{a}_{n2}\tilde{x}_2 + \cdots + \tilde{a}_{nn}\tilde{x}_n = \tilde{\lambda}\tilde{x}_n, \end{cases} \qquad (3.4)$$

is called a fully fuzzy nonlinear system (FFNLS), where $\tilde{a}_{ij}, \tilde{\lambda}$ and $\tilde{x}_i, 1 \leq i, j \leq n$, are fuzzy numbers. The matrix form of the system (3.4) is as follows:

$\tilde{A}\tilde{X} = \tilde{\lambda}\tilde{X}$.

Definition 3.8. The $n \times n$ linear system

$$\begin{cases} [\tilde{a}_{11}]_\alpha[\tilde{x}_1]_\alpha + [\tilde{a}_{12}]_\alpha[\tilde{x}_2]_\alpha + \cdots + [\tilde{a}_{1n}]_\alpha[\tilde{x}_n]_\alpha = [\tilde{\lambda}]_\alpha[\tilde{x}_1]_\alpha, \\ [\tilde{a}_{21}]_\alpha[\tilde{x}_1]_\alpha + [\tilde{a}_{22}]_\alpha[\tilde{x}_2]_\alpha + \cdots + [\tilde{a}_{2n}]_\alpha[\tilde{x}_n]_\alpha = [\tilde{\lambda}]_\alpha[\tilde{x}_2]_\alpha, \\ \qquad\qquad \vdots \\ [\tilde{a}_{n1}]_\alpha[\tilde{x}_1]_\alpha + [\tilde{a}_{n2}]_\alpha[\tilde{x}_2]_\alpha + \cdots + [\tilde{a}_{nn}]_\alpha[\tilde{x}_n]_\alpha = [\tilde{\lambda}]_\alpha[\tilde{x}_n]_\alpha, \end{cases} \qquad (3.5)$$

Is called the $\alpha - cut$ system of FFLS (3.4), where $[\tilde{a}_{ij}]_\alpha, [\tilde{x}_j]_\alpha$ and $[\tilde{\lambda}]_\alpha, 1 \leq i, j \leq n$ and $\alpha \in [0,1]$, are $\alpha - cut$ of fuzzy numbers $\tilde{a}_{ij}, \tilde{x}_j$and $\tilde{\lambda}$, respectively. In addition, its matrix form represented by $[\tilde{A}]_\alpha \otimes [\tilde{X}]_\alpha = [\tilde{\lambda}]_\alpha \otimes [\tilde{X}]_\alpha$.

Definition 3.9. We define the following eigenvalue sets for FFLS (3.4)

- **United fuzzy triangular eigenvalue set:**

$$UTFES = \{\lambda \in F_T \,\big|\, [\tilde{A}]_1 \otimes [\tilde{X}]_1 = [\tilde{\lambda}]_1 \otimes [\tilde{X}]_1,\ [\tilde{A}]_0 \otimes [\tilde{X}]_0 = [\tilde{\lambda}]_0 \otimes [\tilde{X}]_0\},$$

- **Tolerable fuzzy triangular eigenvalue set:**

$$TTFES = \{\lambda \in F_T \,\big|\, [\tilde{A}]_1 \otimes [\tilde{X}]_1 = [\tilde{\lambda}]_1 \otimes [\tilde{X}]_1,\ [\tilde{A}]_0 \otimes [\tilde{X}]_0 \subseteq [\tilde{\lambda}]_0 \otimes [\tilde{X}]_0\},$$

- **Controllable fuzzy triangular eigenvalue set:**

$$CTFES = \{\lambda \in F_T \,\big|\, [\tilde{A}]_1 \otimes [\tilde{X}]_1 = [\tilde{\lambda}]_1 \otimes [\tilde{X}]_1, [\tilde{A}]_0 \otimes [\tilde{X}]_0 \supseteq [\tilde{\lambda}]_0 \otimes [\tilde{X}]_0\},$$

- **Approximate fuzzy triangular eigenvalue set:**

$$ATFES = \{\lambda \in F_T \left| \left[\tilde{A}\right]_1 \otimes [\tilde{X}]_1 = \left[\tilde{\lambda}\right]_1 \otimes [\tilde{X}]_1, \varepsilon > d\ \left(\left[\tilde{A}\right]_0 \otimes [\tilde{X}]_0, \left[\tilde{\lambda}\right]_0 \otimes [\tilde{X}]_0\right)\right.\},$$

Where $[\tilde{A}]_1 = ([\tilde{a}_{ij}]_1)_{i,j=1}^{n}$ and $[\tilde{A}]_0 = ([\tilde{a}_{ij}]_0)_{i,j=1}^{n}$ are interval number-value matrices and $[\tilde{X}]_1 = ([\tilde{x}_1]_1, [\tilde{x}_2]_1, \dots, [\tilde{x}_n]_1)^T, [\tilde{X}]_0 = ([\tilde{x}_1]_0, [\tilde{x}_2]_0, \dots, [\tilde{x}_n]_0)^T, [\tilde{\lambda}]_1 = ([\tilde{\lambda}_1]_1, [\tilde{\lambda}_2]_1, \dots, [\tilde{\lambda}_n]_1)^T$ and $[\tilde{\lambda}]_0 = ([\tilde{\lambda}_1]_0, [\tilde{\lambda}_2]_0, \dots, [\tilde{\lambda}_n]_0)^T$ are interval number valued eigenvectors and interval number valued eigenvalues and $[\tilde{x}_i]_0, [\tilde{\lambda}_i]_0, [\tilde{x}_i]_1, [\tilde{\lambda}_i]_1, 1 \le i \le n$ are $0-$ cuts and $1-$ cuts of fuzzy numbers $\tilde{x}_i$ and $\tilde{\lambda}_i$, respectively.

3.1. Find eigenvalues and fuzzy eigenvectors for a fuzzy matrix

In this section, we shall describe a new practical method to solve eigenvalues and fuzzy triangular eigenvectors of a fuzzy triangular matrix $\tilde{A}$.

Hence, we consider the following system:

$$\tilde{A}\tilde{X} = \tilde{\lambda}\tilde{X}, \tag{3.6}$$

Such that either $\underline{a_{ij}(0)} \ge 0$ or $\overline{a_{ij}}(0) \le 0, 1 \le i, j \le n$. Now, to get a suitable solution of system (3.6), it is sufficient to solve the follows FILSs:

$$\begin{cases} \left[\tilde{A}\right]_1 \otimes [\tilde{X}]_1 = \left[\tilde{\lambda}\right]_1 \otimes [\tilde{X}]_1 & (3.7) \\ \left[\tilde{A}\right]_0 \otimes [\tilde{X}]_0 = \left[\tilde{\lambda}\right]_0 \otimes [\tilde{X}]_0 & (3.8) \end{cases}$$

$$\sum_{j=1}^{n}\left(\left[\underline{a_{ij}}, \overline{a_{ij}}\right]_0 \otimes \left[\underline{x_j}, \overline{x_j}\right]_0\right) = \left[\underline{\lambda}, \overline{\lambda}\right] \otimes \left[\underline{x_i}, \overline{x_i}\right], \quad i = 1,2, \dots, n. \tag{3.9}$$

$$J_i^- =: \left\{j \middle| \overline{a_{ij}} \le 0\ \&\ \underline{a_{ij}} \ne 0\right\} \qquad J_i^+ =: \left\{j \middle| \underline{a_{ij}} \ge 0\right\} \quad i = 1,2, \dots, n. \tag{3.10}$$

Such that

$$\begin{array}{l} \left[\tilde{x}_j\right]_1 \subseteq \left[\tilde{x}_j\right]_0 \quad i.e., \quad x_j^1 \le x_j^2 \le x_j^3, \quad j = 1, \dots, n. \\ \left[\tilde{\lambda}_j\right]_1 \subseteq \left[\tilde{\lambda}_j\right]_0 \quad i.e., \quad \lambda_j^1 \le \lambda_j^2 \le \lambda_j^3, \quad j = 1, \dots, n. \end{array} \tag{3.11}$$

First, by solving the system (3.7), we obtain the eigenvalues $\lambda_1, \dots, \lambda_n$ and the corresponding eigenvectors $X_1, \dots, X_n$ for the crisp matrix $\left[\tilde{A}\right]_1$. Consequently, after obtaining the solution of the crisp system(3.7), we are going to determine the left and right widths for the eigenvalues $\lambda_j,\ 1 \le j \le n$ and the eigenvectors $X_j,\ 1 \le j \le n$, which are obtained from system (3.7).

Consider the eigenvalue $\lambda_k, k \in \{1, \cdots, n\}$ and corresponding eigenvector X_k.

We assume that simultaneous zero cannot belong to $S(\tilde{\lambda}_k)$ and $S(\tilde{X}_j)$.

Therefore, we consider the following three case:

Case A:

$0 \notin S(\tilde{X}_j), 1 \leq j \leq n$, Consider the follows partition.

$$P^+ = \{j | \underline{X}_j \geq 0 \,\& X_j \neq 0\} \quad P^- = \{j | \overline{X}_j \leq 0\} \tag{3.12}$$

Subcase I:

$0 \notin S(\tilde{\lambda}_k)$ Then $\lambda_k, \underline{\lambda}_k, \overline{\lambda}_k > 0$ or $\lambda_k, \underline{\lambda}_k, \overline{\lambda}_k < 0$. Therefore, according to the sign of the values obtained $\boldsymbol{x_j}$, λ_j from equation (3.7), equation (3.8) written as follows:

$$\sum_{J_i^+ P^+} \left[\underline{a_{ij}}, \overline{a_{ij}}\right] \otimes \left[\underline{x_j}, \overline{x_j}\right] + \sum_{J_i^+ P^-} \left[\underline{a_{ij}}, \overline{a_{ij}}\right] \otimes \left[\underline{x_j}, \overline{x_j}\right] + \sum_{J_i^- P^+} \left[\underline{a_{ij}}, \overline{a_{ij}}\right] \otimes \left[\underline{x_j}, \overline{x_j}\right] +$$
$$\sum_{J_i^- P^-} \left[\underline{a_{ij}}, \overline{a_{ij}}\right] \otimes \left[\underline{x_j}, \overline{x_j}\right] = \left[\underline{\lambda_k}, \overline{\lambda_k}\right] \otimes \left[\underline{x_i}, \overline{x_i}\right] \qquad 1 \leq i \leq n \tag{3.13}$$

That here:

$$\begin{aligned}
\sum_{J_i^+ P^+} \left[\underline{a_{ij}}, \overline{a_{ij}}\right] \otimes \left[\underline{x_j}, \overline{x_j}\right] &= \sum_{J_i^+ P^+} \left[\underline{a_{ij}}\ \underline{x_j}, \overline{a_{ij}}\ \overline{x_j}\right] \\
\sum_{J_i^+ P^-} \left[\underline{a_{ij}}, \overline{a_{ij}}\right] \otimes \left[\underline{x_j}, \overline{x_j}\right] &= \sum_{J_i^+ P^-} \left[\overline{a_{ij}}\underline{x_j}, \underline{a_{ij}}\overline{x_j}\right] \\
\sum_{J_i^- P^+} \left[\underline{a_{ij}}, \overline{a_{ij}}\right] \otimes \left[\underline{x_j}, \overline{x_j}\right] &= \sum_{J_i^- P^+} \left[\underline{a_{ij}}\overline{x_j}, \overline{a_{ij}}\underline{x_j}\right] \\
\sum_{J_i^- P^-} \left[\underline{a_{ij}}, \overline{a_{ij}}\right] \otimes \left[\underline{x_j}, \overline{x_j}\right] &= \sum_{J_i^- P^-} \left[\overline{a_{ij}}\ \overline{x_j}, \underline{a_{ij}}\ \underline{x_j}\right]
\end{aligned} \tag{3.14}$$

And

$$\left[\underline{\lambda_k}, \overline{\lambda_k}\right] \otimes \left[\underline{x_i}, \overline{x_i}\right] = \begin{cases} \left[\underline{\lambda_k}\underline{x_j}, \overline{\lambda_k}\ \overline{x_j}\right] & \lambda_k > 0,\ j \in P^+ \\ \left[\overline{\lambda_k}\underline{x_j}, \underline{\lambda_k}\ \overline{x_j}\right] & \lambda_k > 0,\ j \in P^- \\ \left[\underline{\lambda_k}\ \overline{x_j}, \overline{\lambda_k}\underline{x_j}\right] & \lambda_k < 0,\ j \in P^+ \\ \left[\overline{\lambda_k}\ \overline{x_j}, \underline{\lambda_k}\ \underline{x_j}\right] & \lambda_k < 0,\ j \in P^- \end{cases} \tag{3.15}$$

In summary, due to be clear the sign of the intervals and using the interval multiplication definition, model (3.13) written as follows.

$$\hat{S} = \begin{cases} \left(\underline{AX}\right)_i = \left(\underline{\lambda_k X_i}\right) \\ \left(\overline{AX}\right)_i = \left(\overline{\lambda_k X_i}\right) \\ 0 < \underline{\lambda_k} \le \lambda_k \le \overline{\lambda_k} \quad or\ \underline{\lambda_k} \le \lambda_k \le \overline{\lambda_k} < 0 \\ \underline{x_j} \le x_j \le \overline{x_j} \\ 1 \le j \le n \end{cases} \tag{3.16}$$

Subcase II:

$0 \in S\left(\widetilde{\lambda_k}\right)$ then $\lambda_k, \overline{\lambda_k} > 0,\ \underline{\lambda_k} < 0\ \ or\ \lambda_k, \underline{\lambda_k} < 0,\ \overline{\lambda_k} > 0$, therefore, according to the sign of the values obtained x_j , λ_j from equation (3.7), equation (3.8) written as follows:

$$\sum_{J_i^+ P^+} \left[\underline{a_{ij}}, \overline{a_{ij}}\right] \otimes \left[\underline{x_j}, \overline{x_j}\right] + \sum_{J_i^+ P^-} \left[\underline{a_{ij}}, \overline{a_{ij}}\right] \otimes \left[\underline{x_j}, \overline{x_j}\right] + \sum_{J_i^- P^+} \left[\underline{a_{ij}}, \overline{a_{ij}}\right] \otimes \left[\underline{x_j}, \overline{x_j}\right] +$$
$$\sum_{J_i^- P^-} \left[\underline{a_{ij}}, \overline{a_{ij}}\right] \otimes \left[\underline{x_j}, \overline{x_j}\right] = \left[\underline{\lambda_k}, \overline{\lambda_k}\right] \otimes \left[\underline{x_i}, \overline{x_i}\right] \qquad 1 \le i \le n \tag{3.17}$$

That here:

$$\sum_{J_i^+ P^+} \left[\underline{a_{ij}}, \overline{a_{ij}}\right] \otimes \left[\underline{x_j}, \overline{x_j}\right] = \sum_{J_i^+ P^+} \left[\underline{a_{ij}}\,\underline{x_j}, \overline{a_{ij}}\,\overline{x_j}\right], \sum_{J_i^+ P^-} \left[\underline{a_{ij}}, \overline{a_{ij}}\right] \otimes \left[\underline{x_j}, \overline{x_j}\right] = \sum_{J_i^+ P^-} \left[\overline{a_{ij}}\underline{x_j}, \underline{a_{ij}}\overline{x_j}\right]$$
$$\sum_{J_i^- P^+} \left[\underline{a_{ij}}, \overline{a_{ij}}\right] \otimes \left[\underline{x_j}, \overline{x_j}\right] = \sum_{J_i^- P^+} \left[\underline{a_{ij}}\overline{x_j}, \overline{a_{ij}}\underline{x_j}\right], \sum_{J_i^- P^-} \left[\underline{a_{ij}}, \overline{a_{ij}}\right] \otimes \left[\underline{x_j}, \overline{x_j}\right] = \sum_{J_i^- P^-} \left[\overline{a_{ij}}\,\overline{x_j}, \underline{a_{ij}}\,\underline{x_j}\right]$$
$$1 \le j \le n \tag{3.18}$$

And

$$\left[\underline{\lambda_k}, \overline{\lambda_k}\right] \otimes \left[\underline{x_i}, \overline{x_i}\right] = \begin{cases} \left[\underline{\lambda_k}\,\overline{x_j}, \overline{\lambda_k}\,\overline{x_j}\right] & \lambda_k, \overline{\lambda_k} > 0\ , j \in P^+ \\ \left[\overline{\lambda_k}\underline{x}_j, \underline{\lambda_k}\,\underline{x}_j\right] & \lambda_k, \overline{\lambda_k} > 0\ , j \in P^- \\ \left[\underline{\lambda_k}\overline{x}_j, \overline{\lambda_k}\overline{x}_j\right] & \lambda_k, \underline{\lambda_k} < 0\ , j \in P^+ \\ \left[\overline{\lambda_k}\,\underline{x}_j, \underline{\lambda_k}\,\underline{x}_j\right] & \lambda_k, \underline{\lambda_k} < 0\ , j \in P^- \\ & 1 \le j \le n \end{cases} \tag{3.19}$$

Consequently, according to the definition (interval multiplication), model (3.17) written as follows:

$$\hat{S}=\begin{cases}\left(\underline{AX}\right)_i=\left(\underline{\lambda_k X_i}\right)\\ \left(\overline{AX}\right)_i=\left(\overline{\lambda_k X_i}\right)\\ \underline{\lambda_k}<0<\lambda_k\leq\overline{\lambda_k}\quad or\ \underline{\lambda_k}\leq\lambda_k\ <0<\ \overline{\lambda_k}\\ \underline{x_j}\leq x_j\leq\overline{x_j}\\ 1\leq j\leq n\end{cases}\tag{3.20}$$

Case B:

$0\notin S\left(\widetilde{\lambda_k}\right)$ then $\underline{\lambda_k},\lambda_k,\overline{\lambda_k}>0\ or\ \underline{\lambda_k},\lambda_k,\overline{\lambda_k}<0$, Consider the following partition.

$$q^+=\left\{j|\overline{X}_j\geq 0\ \&X_j\neq 0\right\}\ q^-=\left\{j|\underline{X}_j\leq 0\ \right\}\tag{3.21}$$

Therefore, the equation (3.8) is rewritten as follows:

$$\sum_{J_i^+q^+}\left[\underline{a_{ij}},\overline{a_{ij}}\right]\otimes\left[\underline{x_j},\overline{x_j}\right]+\sum_{J_i^+q^-}\left[\underline{a_{ij}},\overline{a_{ij}}\right]\otimes\left[\underline{x_j},\overline{x_j}\right]+\sum_{J_i^-q^+}\left[\underline{a_{ij}},\overline{a_{ij}}\right]\otimes\left[\underline{x_j},\overline{x_j}\right]+$$
$$\sum_{J_i^-q^-}\left[\underline{a_{ij}},\overline{a_{ij}}\right]\otimes\left[\underline{x_j},\overline{x_j}\right]=\left[\underline{\lambda_k},\overline{\lambda_k}\right]\otimes\left[\underline{x_i},\overline{x_i}\right]\qquad 1\leq i\leq n\tag{3.22}$$

That here:

$$\begin{aligned}&\sum_{J_i^+q^+}\left[\underline{a_{ij}},\overline{a_{ij}}\right]\otimes\left[\underline{x_j},\overline{x_j}\right]=\sum_{J_i^+q^+}\left[\underline{a_{ij}}x_j^{'}+\overline{a_{ij}}x_j^{''},\overline{a_{ij}}\ \overline{x_j}\right]\\&\sum_{J_i^+q^-}\left[\underline{a_{ij}},\overline{a_{ij}}\right]\otimes\left[\underline{x_j},\overline{x_j}\right]=\sum_{J_i^+q^-}\left[\overline{a_{ij}}\underline{x_j},\overline{a_{ij}}\ x_j^{'}+\underline{a_{ij}}x_j^{''}\right]\\&\sum_{J_i^-q^+}\left[\underline{a_{ij}},\overline{a_{ij}}\right]\otimes\left[\underline{x_j},\overline{x_j}\right]=\sum_{J_i^-q^+}\left[\underline{a_{ij}}\overline{x_j},\underline{a_{ij}}\ x_j^{''}+\overline{a_{ij}}x_j^{'}\right]\\&\sum_{J_i^-q^-}\left[\underline{a_{ij}},\overline{a_{ij}}\right]\otimes\left[\underline{x_j},\overline{x_j}\right]=\sum_{J_i^-q^-}\left[\underline{a_{ij}}x_j^{'}+\overline{a_{ij}}x_j^{''},\underline{a_{ij}}\ \underline{x_j}\right]\\&\qquad x_j^{'}\geq 0,\ x_j^{''}\leq 0,\ 1\leq j\leq n\end{aligned}\tag{3.23}$$

And

$$\left[\underline{\lambda_k},\overline{\lambda_k}\right]\otimes\left[\underline{x_i},\overline{x_i}\right]=\begin{cases}\left[\underline{\lambda_k}x_j^{'}+\overline{\lambda_k}x_j^{''},\overline{\lambda_k}\ \overline{x_j}\right] & \lambda_k>0\ \ j\in q^+\\ \left[\overline{\lambda_k}\underline{x}_j,\overline{\lambda_k}x_j^{'}+\ \underline{\lambda_k}\ x_j^{''}\right] & \lambda_k>0\ \ j\in q^-\\ \left[\underline{\lambda_k}\overline{x}_j,\underline{\lambda_k}x_j^{''}+\ \overline{\lambda_k}\ x_j^{'}\right] & \lambda_k<0\ \ j\in q^+\\ \left[\underline{\lambda_k}x_j^{'}+\overline{\lambda_k}x_j^{''},\underline{\lambda_k}\ \underline{x}_j\right] & \lambda_k<0\ \ j\in q^-\\ x_j^{'}\geq 0,\ x_j^{''}\leq 0,\ 1\leq j\leq n\end{cases}\tag{3.24}$$

Finally, the model (3.22) obtained as follows:

$$\hat{S}=\begin{cases}\left(\underline{AX}\right)_i=\left(\underline{\lambda_k X_i}\right)\\ \left(\overline{AX}\right)_i=\left(\overline{\lambda_k X_i}\right)\\ \underline{\lambda_k},\lambda_k,\overline{\lambda_k}>0\ or\ \underline{\lambda_k},\lambda_k,\overline{\lambda_k}<0\\ \underline{x_j}\leq x_j\leq\overline{x_j}\\ 1\leq j\leq n\\ 0\leq x_j^{'}\leq M\delta_j\\ -M\left(1-\delta_j\right)\leq x_j^{''}\leq 0\quad \delta_j\in\{0,1\}\\ x_j^{'}+x_j^{''}=\underline{x_j}\ or\ x_j^{'}+x_j^{''}=\overline{x_j}\end{cases}\tag{3.25}$$

Solving models $(3.16), (3.20)\ and (3.25)$ we obtain the values of $\underline{\lambda_k}, \overline{\lambda_k}$ and $\underline{x_j}, \overline{x_j}, 1 \leq j \leq n,$ that is the suitable solution of the system (3.8). the basis of our proposed method is using the following nonlinear programming problem (NLP):

$$\begin{cases}Max\ \overline{\lambda_k}-\underline{\lambda_k}\\ s.t\ \ (X,\lambda)\in\hat{S}\end{cases}\tag{3.26}$$

$$\begin{cases}Max\ \sum_{j=1}^{n}\overline{x_j}-\underline{x_j}\\ s.t\ \ (X,\lambda)\in\hat{S}\end{cases}\tag{3.27}$$

Due to the nonlinearity of models $(3.26)\ and\ (3.27)$, it is possible to obtain different solutions for them. That these models have the most width for $\tilde{\lambda}\ and\ \tilde{x}$, respectively.

Where $\hat{S}$ is one of the regions $(3.16), (3.20)\ or\ (3.25)$.

The above $NLPS$ (3.26) and (3.27) can be easily solved by using LINGO 18 software package.

Theorem 3.1. Suppose that the $NLPS$ (3.26) and (3.27) are feasible and

$X=\left(\underline{x}_{11},x_{11},\overline{x}_{11},\underline{x}_{21},x_{21},\overline{x}_{21},\cdots\underline{x}_{n1},x_{n1},\overline{x}_{n1}\right),$

Is its optimum solution. Also, the fuzzy triangular number vector $\tilde{X}=(\tilde{x}_1,\tilde{x}_2,\dots,\tilde{x}_n)^T$, be constructed as follows:

$\left[\tilde{x}_j\right]_0=\left[x_j^1,x_j^3\right]$

Then, $\tilde{X}$ is a suitable solution for the system (3.6). Otherwise, if the $NLPS$ (3.26) and (3.27) are unfeasible, the system (3.6) does not have any suitable solution.

Proof. First, let us assume that the $NLPS$ (3.26) and (3.27) are feasible and

$X = (\underline{x}_{11}, x_{11}, \overline{x}_{11}, \underline{x}_{21}, x_{21}, \overline{x}_{21}, \cdots \underline{x}_{n1}, x_{n1}, \overline{x}_{n1},)$

Is its optimum solution, then with considering Eqs. (3.6)- (3.25) we can conclude that $\tilde{X} = (\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n)^T$, where

$$[\tilde{x}_j]_1 \subseteq [\tilde{x}_j]_0 \quad i.e., \quad x_j^1 \le x_j^2 \le x_j^3,$$

$$\underline{x}_j \le x_j \le \overline{x}_j$$

$$\tilde{x}_j = (x_j^1, x_j^2, x_j^3), \quad j = 1, \dots, n.$$

Is a suitable solution for system (3.6).

Now, let us assume that the $NLPS$ (3.26) and $(3.27)$are unfeasible. By contradiction assume that the system (3.6) has a suitable solution $\tilde{X} = (\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n)^T$, therefore we have:

$[\tilde{A}]_1 \otimes [\tilde{X}]_1 = [\tilde{\lambda}]_1 \otimes [\tilde{X}]_1$ and $[\tilde{A}]_0 \otimes [\tilde{X}]_0 = [\tilde{\lambda}]_0 \otimes [\tilde{X}]_0$. By considering the proposed method, for solving fuzzy triangular matrix $\tilde{A}$, in section 3, we can rewrite:

$$\begin{cases} Max\ \overline{\lambda_k} - \underline{\lambda_k} \\ s.t\ \ (X, \lambda) \in \hat{S} \end{cases}$$

$$\begin{cases} Max \displaystyle\sum_{j=1}^{n} \overline{x_j} - \underline{x_j} \\ s.t\ \ (X, \lambda) \in \hat{S} \end{cases}$$

Hence

$$X = (\underline{x}_{11}, x_{11}, \overline{x}_{11}, \underline{x}_{21}, x_{21}, \overline{x}_{21}, \cdots \underline{x}_{n1}, x_{n1}, \overline{x}_{n1})$$

are feasible solutions of the $NLPS$ (3.26) and (3.27). However, this is a contradiction. Thus, system (3.6) does not have any suitable solution.

4. Finding the fuzzy escribed, peripheral and approximate eigenvalue

Fuzzy nonlinear system (3.4) does not necessarily have an exact solution. Therefore, in this section, we are going to introduce three eigenvalues for $\tilde{A}$ that does not have any suitable solution. These eigenvalues are **fuzzy escribed eigenvalue**, **fuzzy peripheral eigenvalue, and fuzzy approximate eigenvalue** that defined as following.

Definition 4.1. The fuzzy triangular number $\tilde{\lambda}$ called a **fuzzy escribed eigenvalue** of $\tilde{A}$, if:

1) $\tilde{\lambda} \in F_T, \tilde{X} \in F_T^n$
2) $\forall\, \widetilde{\lambda'} \in F_T, \forall\, \tilde{Y} \in F_T^n: AX = \lambda X \;\&\; AY = \lambda Y \;\&$

$$d\left(\left[\tilde{A}\right]_0 \otimes [\tilde{X}]_0, \left[\tilde{\lambda}\right]_0 \otimes [\tilde{X}]_0 \right) \le d\left(\left[\tilde{A}\right]_0 \otimes \left[\tilde{Y}\right]_0, \left[\tilde{\lambda}\right]_0 \otimes \left[\tilde{Y}\right]_0 \right) \qquad (4.28)$$

We denote fuzzy escribed eigenvalue by $\tilde{\lambda}_{\subseteq}$.

Definition 4.2. The fuzzy triangular number $\tilde{\lambda}$ called a **fuzzy peripheral eigenvalue** of $\tilde{A}$, if:

1) $\tilde{\lambda} \in F_T, \tilde{X} \in F_T^n$
2) $\forall\, \widetilde{\lambda'} \in F_T, \forall\, \tilde{Y} \in F_T^n: AX = \lambda X \;\&\; AY = \lambda Y \;\&$

$$d\left(\left[\tilde{A}\right]_0 \otimes [\tilde{X}]_0, \left[\tilde{\lambda}\right]_0 \otimes [\tilde{X}]_0 \right) \ge d\left(\left[\tilde{A}\right]_0 \otimes \left[\tilde{Y}\right]_0, \left[\tilde{\lambda}\right]_0 \otimes \left[\tilde{Y}\right]_0 \right) \qquad (4.29)$$

We denote fuzzy escribed eigenvalue by $\tilde{\lambda}_{\supseteq}$.

In the following, we explain a method to obtaining $\left(\tilde{\lambda}_{\subseteq}, \tilde{X}_{\subseteq}\right)$ and $\left(\tilde{\lambda}_{\supseteq}, \tilde{X}_{\supseteq}\right)$ of the system $\tilde{A}\tilde{x} = \tilde{\lambda}\tilde{x}$.

We first solve the 1-cut system of FFLS, i.e., $\left[\tilde{A}\right]_1 \otimes [\tilde{X}]_1 = \left[\tilde{\lambda}\right]_1 \otimes [\tilde{X}]_1$.

We can construct the algebraic solution of the 1-cut system like the previous part technique as following:

$$[\tilde{X}]_1 = ([\tilde{x}_1]_1, [\tilde{x}_2]_1, \dots, [\tilde{x}_n]_1).$$

Then for finding 0-cut of $\left(\tilde{\lambda}_{\subseteq}, \tilde{X}_{\subseteq}\right)$ it is sufficient to solve the following NLP problem

$$\begin{aligned} Min \quad z = Max &\left\{ max_{1 \le i \le n} \left| \underline{(AX_0)_i} - \underline{(\lambda X_0)_i} \right|, max_{1 \le i \le n} \left| \overline{(AX_0)_i} - \overline{(\lambda X_0)_i} \right| \right\} \\ &\underline{(AX_0)_i} \ge \underline{(\lambda X_0)_i}\,, \overline{(AX_0)_i} \le \overline{(\lambda X_0)_i} \end{aligned} \qquad (4.30)$$

Now, we define:

$$w_1 = max_{1\le i\le n}\left|\underline{(AX_0)_i} - \underline{(\lambda X_0)_i}\right|, w_2 = max_{1\le i\le n}\left|\overline{(AX_0)_i} - \overline{(\lambda X_0)_i}\right| \qquad (4.31)$$
$$z = Max\{w_1, w_2\}.$$

With consider, Eq. $\underline{(AX_0)_i} \ge \underline{(\lambda X_0)_i}$, $\overline{(AX_0)_i} \le \overline{(\lambda X_0)_i}$ $\quad i = 1, \dots, n,$ we have:

$$\underline{(AX_0)_i} - \underline{(\lambda X_0)_i} \le z,\ \overline{(\lambda X_{0_i})} - \overline{(AX_{0_i})} \le z \quad i = 1, \dots, n. \qquad (4.32)$$

Then by using (4.32) we can rewrite system (4.30) as follows:

$$\begin{aligned} & Min\ \ z_{\subseteq} \\ & s.t. \\ & S_{\subseteq} = \begin{cases} \underline{(AX_0)_i} - \underline{(\lambda X_0)_i} \le z,\ \overline{(\lambda X_{0_i})} - \overline{(AX_{0_i})} \le z \\ \underline{(AX_0)_i} \ge \underline{(\lambda X_0)_i}, \overline{(AX_0)_i} \le \overline{(\lambda X_0)_i}, \\ (X, \lambda) \in \hat{S} \qquad\qquad (4.33) \end{cases} \end{aligned}$$

where $\hat{S}$ is one of the regions $(3.16), (3.20), or\ (3.25)$.

Similarly, to obtain$(\tilde{\lambda}_{\supseteq}, \tilde{X}_{\supseteq})$, it is sufficient to solve the following NLP problem

$$\begin{aligned} & Min\ \ z_{\supseteq} \\ & s.t. \\ & S_{\supseteq} = \begin{cases} \underline{(\lambda X_0)_i} - \underline{(AX_0)_i} \le z, \overline{(AX_{0_i})} - \overline{(\lambda X_{0_i})} \le z \\ \underline{(AX_0)_i} \le \underline{(\lambda X_0)_i}, \overline{(AX_0)_i} \ge \overline{(\lambda X_0)_i}, \\ (X, \lambda) \in \hat{S} \qquad\qquad (4.34) \end{cases} \end{aligned}$$

where $\hat{S}$ is one of the regions $(3.16), (3.20), or\ (3.25)$.

By solving the NLPS (4.33) and (4.34), we can obtain fuzzy escribed and peripheral eigenvalue.

Theorem 4.1. for a fuzzy matrix $\tilde{A}$, $\tilde{\lambda}$ is a suitable eigenvalue if there exists a fuzzy peripheral eigenvalue $\tilde{\lambda}_{\supseteq}$ and a fuzzy escribed eigenvalue $\tilde{\lambda}_{\subseteq}$ such that $\tilde{\lambda}_{\supseteq} = \tilde{\lambda}_{\subseteq} = \tilde{\lambda}$ and in models $(4.33)\ and\ (4.34)$, $z^* = 0$.

Proof: The proof is obvious. □

Given that, for the system (3.4), there may not be an algebraic, escribed, and peripheral solution, so here we define the approximate algebraic solution. In fact, we are looking for the value of $\tilde{\lambda}$ and the vector $\tilde{X}$ in the approximate solution, which the upper and lower boundary values $[\tilde{A}]_0 \otimes [\tilde{X}]_0$ and $[\tilde{\lambda}]_0 \otimes [\tilde{X}]_0$ have the least distance with each other.

Definition 4.3. The fuzzy triangular number $\tilde{\lambda}$ called a **fuzzy approximate eigenvalue** of $\tilde{A}$, if:

1) $\tilde{\lambda} \in F_T, \tilde{X} \in F_T^n$, $AX = \lambda X$
2) $\exists\, \varepsilon > 0, d\left([\tilde{A}]_0 \otimes [\tilde{X}]_0, [\tilde{\lambda}]_0 \otimes [\tilde{X}]_0 \right) < \varepsilon$

We denote fuzzy approximate eigenvalue by $\tilde{\lambda}_{\simeq}$.

We first solve the 1-cut system of FFLS, i.e., $[\tilde{A}]_1 \otimes [\tilde{X}]_1 = [\tilde{\lambda}]_1 \otimes [\tilde{X}]_1$.

We can construct the algebraic solution of the 1-cut system like the previous part technique as following:

$$[\tilde{X}]_1 = ([\tilde{x}_1]_1, [\tilde{x}_2]_1, \dots, [\tilde{x}_n]_1).$$

Then for finding 0-cut of $(\tilde{\lambda}_{\simeq}, \tilde{X}_{\simeq})$ it is sufficient to solve the following NLP problem

$$\begin{aligned} &Min\ \ \varepsilon \\ &s.t. \\ &\quad \begin{cases} -\varepsilon \leq \underline{(AX_0)_i} - \underline{(\lambda X_0)_i} \leq \varepsilon \\ -\varepsilon \leq \overline{(AX_0)_i} - \overline{(\lambda X_0)_i} \leq \varepsilon, \\ (X, \lambda) \in \hat{S} \end{cases} \end{aligned} \tag{4.35}$$

where $\hat{S}$ is one of the regions $(3.16), (3.20), or\ (3.25)$.

Theorem 4.2. model (4.35) is always feasible.

Proof: The proof is obvious. □

Remark: it is clear that the value of ε in model (4.35) is the same as the approximate solution error $(\tilde{\lambda}_{\simeq}, \tilde{X}_{\simeq})$, and if the algebraic solution (definition(3.9)) exists for system (3.4) then the solution error of model (4.35) is zero.

5. numerical example

In this section, we use the following examples to illustrate our proposed method, computed by lingo 18. In each example, we are looking for an algebraic solution(UTFES). If this solution is not available, we try to calculate the escribed, peripheral and approximate solutions.

Example 5.1. Consider a fuzzy triangular matrix $\tilde{A}$, assuming that the fuzzy triangular number $\tilde{\lambda} = (\underline{\lambda}, \lambda, \overline{\lambda})$ is a fuzzy eigenvalue and the fuzzy triangular vector $\tilde{X} = (\tilde{x}_1, \tilde{x}_2)$ is a fuzzy eigenvector of matrix $\tilde{A}$, we have:

$$\begin{cases}(6.8,12,15.25)\tilde{x}_1 + (15,16,28)\tilde{x}_2 = (\tilde{\lambda})\begin{pmatrix}\tilde{x}_1\\ \tilde{x}_2\end{pmatrix}\\ (7.2,8,10.2)\tilde{x}_1 + (19.5,20,22)\tilde{x}_2 = (\tilde{\lambda})\begin{pmatrix}\tilde{x}_1\\ \tilde{x}_2\end{pmatrix}\end{cases} \qquad (5.36)$$

Where $\tilde{a}_{ij,}$ $1 \le i,j \le n$, are fuzzy triangular numbers. First, we solve 1-cut system (5.36) which is crisp:

$$\begin{cases}12x_1 + 16x_2 = \lambda\begin{pmatrix}x_1\\ x_2\end{pmatrix}\\ 8x_1 + 20x_2 = \lambda\begin{pmatrix}x_1\\ x_2\end{pmatrix}\end{cases} \qquad (5.37)$$

So, the crisp solution $(1 - cut)$ obtained as follows:

$$\lambda = \begin{bmatrix}4 & 0\\ 0 & 28\end{bmatrix} \qquad X = \begin{bmatrix}16 & -8\\ -8 & -8\end{bmatrix}$$
$$\lambda_1 = 4,\ \lambda_2 = 28\ \ and\ \ x^1 = \begin{bmatrix}16\\ -8\end{bmatrix}, x^2 = \begin{bmatrix}-8\\ -8\end{bmatrix}$$

Now, we are going to get the three models mentioned in section 3 for system (5.36).

By using models (3.16), (3.20), and (3.25) for $\lambda_1 = 4$, $x^1 = \begin{bmatrix}16\\ -8\end{bmatrix}$, problem (5.35), written as follows:

Model(3.16):

$$\text{MAX} \quad \text{z} = \lambda_2 - \lambda_1$$
$$s.t.\begin{cases}6.8 * x_{11} + 28 * x_{21} = \lambda_1 * x_{11}\\ 7.2 * x_{11} + 22 * x_{21} = \lambda_2 * x_{21}\\ 15.25 * x_{12} + 15 * x_{22} = \lambda_2 * x_{12}\\ 10.2 * x_{12} + 19.5 * x_{22} = \lambda_1 * x_{22}\\ x_{11} \le 16, x_{12} \ge 16, x_{21} \le -8, x_{22} \ge -8\\ \lambda_1 > 0,\ \lambda_2 \ge 4, \lambda_1 \le 4\end{cases} \qquad (5.38)$$

That here:

$$\tilde{\lambda} = (\lambda_1, \lambda, \lambda_2)\ \ , \tilde{X} = \begin{bmatrix}(x_{11}, x_1, x_{12})\\ (x_{21}, x_2, x_{22})\end{bmatrix},$$

Model (3.20):

$$
\text{MAX} \quad z = \lambda_2 - \lambda_3 - \lambda_4
$$
$$
s.t.\begin{cases} 6.8 * x_{11} + 28 * x_{21} = \lambda_3 * x_{11} + \lambda_4 * x_{12} \\ 15.25 * x_{12} + 15 * x_{22} = \lambda_2 * x_{12} \\ 7.2 * x_{11} + 22 * x_{21} = \lambda_2 * x_{21} \\ 10.2 * x_{12} + 19.5 * x_{22} = \lambda_3 * x_{22} + \lambda_4 * x_{21} \\ x_{11} \le 16, x_{12} \ge 16, x_{21} \le -8, x_{22} \ge -8 \\ \lambda_2 \ge 4,\ \lambda_1 = \lambda_3 + \lambda_4 \le 4,\ \lambda_3 \ge 0, \lambda_4 \le 0, \\ \lambda_3 \le 1000 * s_1, \lambda_4 \ge 1000 - 1000 * s_1 \end{cases} \tag{5.39}
$$

That here:

$$
\tilde{\lambda} = (\lambda_3 + \lambda_4, \lambda, \lambda_2) \ , \tilde{X} = \begin{bmatrix} (x_{11}, x_1, x_{12}) \\ (x_{21}, x_2, x_{22}) \end{bmatrix},
$$

Model (3.25):

$$
\text{MAX} \quad z = \lambda_2 - \lambda_1
$$
$$
s.t.\begin{cases} 6.8 * x_{111} + 15.25 * x_{112} + 28 * x_{21} = \lambda_1 * x_{111} + \lambda_2 * x_{112} \\ 15.25 * x_{12} + 28 * x_{221} + 15 * x_{222} = \lambda_2 * x_{12} \\ 7.2 * x_{111} + 10.2 * x_{112} + 22 * x_{21} = \lambda_2 * x_{21} \\ 10.2 * x_{12} + 22 * x_{221} + 19.5 * x_{222} = \lambda_2 * x_{221} + \lambda_1 * x_{222} \\ x_{11} = x_{111} + x_{112} \le 16, x_{12} \ge 16, x_{21} \le -8, x_{22} = x_{221} + x_{222} \ge -8, \\ x_{111} \ge 0, x_{111} \le 1000 * s_{1,} x_{112} \le 0, x_{112} \ge -1000 + 1000 * s_1 \\ x_{221} \ge 0, x_{221} \le 1000 * s_{2,} x_{222} \le 0, x_{222} \ge -1000 + 1000 * s_2 \\ \lambda_1 \le 4, \lambda_1 > 0, \lambda_2 \ge 4, \end{cases} \tag{5.40}
$$

That here:

$$
\tilde{\lambda} = (\lambda_1, \lambda, \lambda_2) \ , \tilde{X} = \begin{bmatrix} (x_{111} + x_{112}, x_1, x_{12}) \\ (x_{21}, x_2, x_{221} + x_{222}) \end{bmatrix},
$$

Then, using the LINGO 18 software package, we will see that model (5.38) is infeasible, but models (5.39) and (5.40) have the following answers, respectively:

$$
\begin{aligned} &\lambda_2 = 22.00000,\ \ \lambda_4 = -23.04995,\ \ \lambda_3 = 0.000000, \\ &\mathrm{x}_{11} = 0.000000, \quad \mathrm{x}_{12} = 273.3317, \\ &\mathrm{x}_{21} = -225.0100, \quad \mathrm{x}_{22} = 122.9993, \end{aligned} \tag{5.41}
$$

And for case(5.40):

$$
\begin{aligned} &\lambda_2 = 35.85842 \qquad , \quad \lambda_1 = 0.1000000, \\ &\mathrm{x}_{11} = 0.000000,\ \ \mathrm{x}_{12} = 16.00000,\ \ x_{111} = 0.000000, x_{112} = -1000.000, \\ &\mathrm{x}_{21} = -736.0149,\ \ \mathrm{x}_{22} = 0.000000\ , x_{221} = 11.77624, x_{222} = 0.000000, \end{aligned} \tag{5.42}
$$

besides, the eigenvector and the fuzzy triangular eigenvalue for matrix $\tilde{A}$ obtained as follows:

Case(5.39):

$$\begin{aligned}&\tilde{x}_1 = (0.000000,16\,,273.3317\,),\quad \tilde{x}_2 = (-225.0100,-8\,,122.9993),\\ &\tilde{\lambda} = (\underline{\lambda},\lambda\,,\overline{\lambda}) = (-23.04995,4,22.00000\,)\end{aligned} \tag{5.43}$$

Case(5.40):

$$\begin{aligned}&\tilde{x}_1 = (-1000.000,16\,,16.00000\,),\quad \tilde{x}_2 = (-736.0149,-8\,,11.77624),\\ &\tilde{\lambda} = (\underline{\lambda},\lambda\,,\overline{\lambda}) = (0.1000000,4,35.85842\,)\end{aligned} \tag{5.44}$$

And for $\lambda_2 = 28$, $x^2 = \begin{bmatrix}-8\\-8\end{bmatrix}$, as above, using the lingo 18 software package, we will see that models (3.16) and (3.20) are infeasible, but model (3.25) has the following answers:

$$\begin{aligned}&\lambda_2 = 35.85842\quad,\quad \lambda_1 = 0.1000000,\\ &\mathrm{x}_{11} = -10.86935,\quad \mathrm{x}_{12} = 0.00000,\ x_{121} = 0.000000, x_{122} = 0.000000,\\ &\mathrm{x}_{21} = -8.000000,\quad \mathrm{x}_{22} = 0.000000, x_{221} = 0.000000, x_{222} = 0.000000,\end{aligned} \tag{5.45}$$

Therefore, the suitable solution obtained as follows:

$$\begin{aligned}&\tilde{x}_1 = (-10.86935,-8\,,0.00000\,),\quad \tilde{x}_2 = (-8.000000,-8\,,0.000000),\\ &\tilde{\lambda} = (\underline{\lambda},\lambda\,,\overline{\lambda}) = (0.1000000,28,35.85842\,)\end{aligned} \tag{5.46}$$

Example 5.2. Consider a fuzzy triangular matrix $\tilde{A}$, assuming that the fuzzy triangular number $\tilde{\lambda} = (\underline{\lambda},\lambda,\overline{\lambda})$ is a fuzzy eigenvalue and the fuzzy triangular vector $\tilde{X} = (\tilde{x}_1,\tilde{x}_2,\tilde{x}_3\,)$ is a fuzzy eigenvector of matrix $\tilde{A}$, we have:

$$\begin{cases}(0.5,1,1.8)x_1 + (-3.4,-3,-0.9)x_2 + (6,7,8)x_3 = (\underline{\lambda},\overline{\lambda})\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}\\ (-6,-4,-2)x_1 + (9.5,12,15.5)x_2 + (1,2,3)x_3 = (\underline{\lambda},\overline{\lambda})\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}\\ (2,6,10)x_1 + (3.75,4,11)x_2 + (-7,-5,-1)x_3 = (\underline{\lambda},\overline{\lambda})\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}\end{cases} \tag{5.47}$$

So, the crisp solution $(1-cut)$ obtained as follows:

$$\lambda = \begin{bmatrix}-10.0289 & 0 & 0\\ 0 & 5.0837 & 0\\ 0 & 0 & 12.9451\end{bmatrix}\quad X = \begin{bmatrix}-0.6937 & 1.3521 & -0.8675\\ -0.2167 & 0.4928 & 5.7876\\ 1.0000 & 1.0000 & 1.0000\end{bmatrix},$$

$\lambda_1 = -10.0289,\ \lambda_2 = 5.0837,\ \lambda_3 = 12.9451\ \ and\ \ x^1 = \begin{bmatrix}-0.6937\\-0.2167\\1.0000\end{bmatrix}, x^2 = \begin{bmatrix}1.3521\\0.4928\\1.0000\end{bmatrix}, x^3 = \begin{bmatrix}-0.8675\\5.7876\\1.0000\end{bmatrix}$

Now, we are going to get the three models mentioned in section 3 for system (5.47).

By using models (3.16), (3.20) and (3.25) for $\lambda_3 = 12.9451$, $x^3 = \begin{bmatrix} -0.8675 \\ 5.7876 \\ 1.0000 \end{bmatrix}$, problem (5.47), written as follows:

Model(3.16):

$$\begin{aligned} &\text{MAX} \quad \mathrm{z} = \lambda_2 - \lambda_1 \\ &s.t. \begin{cases} 1.8 * x_{11} - 3.4 * x_{22} + 6 * x_{31} = \lambda_2 * x_{11} \\ 0.5 * x_{12} - 0.9 * x_{21} + 8 * x_{32} = \lambda_1 * x_{12} \\ -2 * x_{12} + 9.5 * x_{21} + 1 * x_{31} = \lambda_1 * x_{21} \\ -6 * x_{11} + 15.5 * x_{22} + 3 * x_{32} = \lambda_2 * x_{22} \\ 10 * x_{11} + 3.75 * x_{21} - 7 * x_{32} = \lambda_1 * x_{31} \\ 2 * x_{12} + 11 * x_{22} - 1 * x_{31} = \lambda_2 * x_{32} \\ x_{11} \le -0.8675, x_{12} \ge -0.8675 \\ x_{21} \le 5.7876, x_{22} \ge 5.7876 \\ x_{31} \le 1.0000, x_{32} \ge 1.0000 \\ \lambda_2 \ge 12.9451, \lambda_1 \le 12.9451 \end{cases} \end{aligned} \tag{5.48}$$

That here:

$$\tilde{\lambda} = (\lambda_1, \lambda, \lambda_2) \ , \tilde{X} = \begin{bmatrix} (x_{11}, x_1, x_{12}) \\ (x_{21}, x_2, x_{22}) \\ (x_{31}, x_2, x_{32}) \end{bmatrix},$$

Model (3.20):

$$\begin{aligned} &MAX \quad z = \lambda_2 - \lambda_3 - \lambda_4 \\ &s.t. \begin{cases} 1.8 * x_{11} - 3.4 * x_{22} + 6 * x_{31} = \lambda_2 * x_{11} \\ 0.5 * x_{12} - 0.9 * x_{21} + 8 * x_{32} = \lambda_3 * x_{12} + \lambda_4 * x_{11} \\ -2 * x_{12} + 9.5 * x_{21} + 1 * x_{31} = \lambda_3 * x_{21} + \lambda_4 * x_{22} \\ -6 * x_{11} + 15.5 * x_{22} + 3 * x_{32} = \lambda_2 * x_{22} \\ 10 * x_{11} + 3.75 * x_{21} - 7 * x_{32} = \lambda_3 * x_{31} + \lambda_4 * x_{32} \\ 2 * x_{12} + 11 * x_{22} - 1 * x_{31} = \lambda_2 * x_{32} \\ x_{11} \le -0.8675, x_{12} \ge -0.8675 \\ x_{21} \le 5.7876, x_{22} \ge 5.7876 \\ x_{31} \le 1.0000, x_{32} \ge 1.0000 \\ \lambda_2 \ge 12.9451, \lambda_1 = \lambda_3 + \lambda_4 \le 12.9451, \\ \lambda_3 \ge 0, \lambda_3 \le 1000 * s_1, \lambda_4 \le 0, \lambda_4 \ge -1000 + 1000 * s_1 \end{cases} \end{aligned} \tag{5.49}$$

That here:

$$\tilde{\lambda} = (\lambda_3 + \lambda_4, \lambda, \lambda_2) \ , \tilde{X} = \begin{bmatrix} (x_{11}, x_1, x_{12}) \\ (x_{21}, x_2, x_{22}) \\ (x_{31}, x_2, x_{32}) \end{bmatrix},$$

Model (3.25):

$$
\begin{aligned}
&\text{MAX}\quad z=\lambda_2-\lambda_1\\
&s.t.\begin{cases}
1.8*x_{11}-3.4*x_{22}+6*x_{311}+8*x_{312}=\lambda_2*x_{11}\\
1.8*x_{121}+0.5*x_{122}-0.9*x_{211}-3.4*x_{212}+8*x_{32}=\lambda_2*x_{121}+\lambda_1*x_{122}\\
-6*x_{121}-2*x_{122}+9.5*x_{211}+15.5*x_{212}+1*x_{311}+3*x_{312}=\lambda_1*x_{211}+\lambda_2*x_{212}\\
-6*x_{11}+15.5*x_{22}+3*x_{32}=\lambda_2*x_{22}\\
10*x_{11}+3.75*x_{211}+11*x_{212}-7*x_{32}=\lambda_1*x_{311}+\lambda_2*x_{312}\\
10*x_{121}+2*x_{122}+11*x_{22}-1*x_{311}-7*x_{312}=\lambda_2*x_{32}\\
x_{11}\le -0.8675, x_{12}=x_{121}+x_{122}\ge -0.8675,\\
x_{21}=x_{211}+x_{212}\le 5.7876, x_{22}\ge 5.7876,\\
x_{31}=x_{311}+x_{312}\le 1.0000, x_{32}\ge 1.0000,\\
x_{121}\ge 0, x_{121}\le 1000*s_{1,}x_{122}\le 0, x_{122}\ge -1000+1000*s_1\\
x_{211}\ge 0, x_{211}\le 1000*s_{2,}x_{212}\le 0, x_{212}\ge -1000+1000*s_2\\
x_{311}\ge 0, x_{311}\le 1000*s_{3,}x_{312}\le 0, x_{312}\ge -1000+1000*s_3 \quad (5.50)\\
\lambda_1\le 12.9451, \lambda_2\ge 12.9451,
\end{cases}
\end{aligned}
$$

That here:

$$\tilde{\lambda}=(\lambda_1,\lambda,\lambda_2)\ ,\tilde{X}=\begin{bmatrix}(x_{11},x_1,x_{121}+x_{122})\\(x_{211}+x_{212},x_2,x_{22})\\(x_{311}+x_{312},x_3,x_{32})\end{bmatrix},$$

Then, using the LINGO 18 software package, we will see that models (5.48) and (5.49) are infeasible, but model (5.50) has the following answers:

$$
\begin{gathered}
\lambda_2=22.36984,\quad \lambda_1=0.100000,\\
\mathrm{x}_{11}=-5.529008\ ,\ \mathrm{x}_{12}=0.1325000\ ,\ x_{121}=5.529008, x_{122}=0.000000\ ,\\
\mathrm{x}_{21}=0.7876000,\quad \mathrm{x}_{22}=9.309340, x_{211}=0.000000, x_{212}=-9.309340\ ,\\
\mathrm{x}_{31}=0.000000\ ,\quad \mathrm{x}_{32}=10.25988\ , x_{311}=0.000000, x_{312}=-10.25988,
\end{gathered}
\qquad (5.51)
$$

Besides, corresponding suitable solution of FFLS based on the compute above given by:

$$
\begin{aligned}
&\tilde{x}_1=(-5.529008\ ,-0.8675, 0.1325000),\quad \tilde{x}_2=(0.7876000, 5.7876, 9.309340),\\
&\tilde{x}_3=(0.000000,1.0000, 10.25988)\quad \tilde{\lambda}=\left(\underline{\lambda},\lambda\ ,\overline{\lambda}\right)=(0.100000,12.9451, 22.36984\ )
\end{aligned}
\qquad (5.52)
$$

Similarly, to obtain widths $\lambda_1=-10.0289$, $x^1=\begin{bmatrix}-0.6937\\-0.2167\\1.0000\end{bmatrix}$, and $\lambda_2=5.0837, x^2=\begin{bmatrix}1.3521\\0.4928\\1.0000\end{bmatrix}$, we behavior as above. That we will see for $\lambda_1=-10.0289$, $x^1=\begin{bmatrix}-0.6937\\-0.2167\\1.0000\end{bmatrix}$, all three the models $(3.16), (3.20), and$ (3.25) are infeasible, and for $\lambda_2=5.0837, x^2=\begin{bmatrix}1.3521\\0.4928\\1.0000\end{bmatrix}$ models (3.16) and (3.20) are infeasible, but model (3.25) has the following answers:

$$\lambda_2 = 22.36984, \quad \lambda_1 = 0.100000,$$
$$\begin{aligned} &\mathrm{x}_{11} = 0.3521000, \quad \mathrm{x}_{12} = 1.352100, \quad x_{111} = 0.000000, x_{112} = -1.352100, \\ &\mathrm{x}_{21} = 0.4928000, \quad \mathrm{x}_{22} = 2.276567, x_{211} = 0.000000, x_{212} = -2.276567, \\ &\mathrm{x}_{31} = 0.000000, \quad \mathrm{x}_{32} = 2.509020, x_{311} = 0.000000, x_{312} = -2.509020, \end{aligned} \tag{5.53}$$

Therefore, we have:

$$\begin{aligned} &\tilde{x}_1 = (0.3521000\ , 1.3521, 1.352100), \quad \tilde{x}_2 = (0.4928000, 0.4928, 2.276567), \\ &\tilde{x}_3 = (0.000000, 1.0000, 2.509020) \quad \tilde{\lambda} = (\underline{\lambda}, \lambda, \overline{\lambda}) = (0.100000, 12.9451, 22.36984\ ) \end{aligned} \tag{5.54}$$

Now, we consider three $\alpha - cut$ levels ($i.e.\ \alpha = 0.3, 0.5, 0.7$) in model (5.50) that $\lambda_3 = 12.9451$, $x^3 = \begin{bmatrix} -0.8675 \\ 5.7876 \\ 1.0000 \end{bmatrix}$, and based on the interval solutions under different $\alpha - cut$ levels, the membership function can then be generated through statistical regression analysis methods. Therefore, we have:

Table 1

Solution process by model(5.50).

$\alpha = 0.3$		
Eigenvalues	NLP models	solutions
x_1	$MAX = \lambda_2 - \lambda_1$ $1.56 * x_{11} - 3.52 * x_{22} + 6.3 * x_{311} + 7.7 * x_{312} = \lambda_2 * x_{11}$ $1.56 * x_{121} + 0.65 * x_{122} - 1.53 * x_{211} - 3.52 * x_{212} + 7.7 * x_{32} = \lambda_2 * x_{121} + \lambda_1 * x_{122}$ $-5.4 * x_{121} - 2.6 * x_{122} + 10.25 * x_{211} + 14.45 * x_{212} + 1.3 * x_{311} + 2.7 * x_{312} = \lambda_1 * x_{211} + \lambda_2 * x_{212}$ $-5.4 * x_{11} + 14.45 * x_{22} + 2.7 * x_{32} = \lambda_2 * x_{22}$ $8.8 * x_{11} + 3.825 * x_{211} + 8.9 * x_{212} - 6.4 * x_{32} = \lambda_1 * x_{311} + \lambda_2 * x_{312}$ $8.8 * x_{121} + 3.2 * x_{122} + 8.9 * x_{22} - 2.2 * x_{311} - 6.4 * x_{312} = \lambda_2 * x_{32}$ $0.2 \leq \lambda_1 \leq 12.9, 13 \leq \lambda_2 \leq 22$ $X_{11} >= -5.529008, X_{121} + X_{122} <= 5.529008, X_{121} >= 0.1, X_{121} <= 1000 * S_1, X_{122} <= 0, X_{122} >= -1000 + 1000 * S_1$ $X_{211} + X_{212} >= -9.309340, X_{22} <= 9.309340, X_{211} >= 0, X_{211} <= 1000 * S_2, X_{212} <= 0, X_{212} >= -1000 + 1000 * S_2$ $X_{311} + X_{312} >= -10.25988, X_{32} <= 10.25988, X_{311} >= 0, X_{311} <= 1000 * S_3, X_{312} <= 0, X_{312} >= -1000 + 1000 * S_3$ $X_{11} <= -0.87, X_{121} + X_{122} >= -0.87, X_{211} + X_{212} <= 5.8, X_{22} >= 5.8, X_{311} + X_{312} <= 1, X_{32} >= 1$	$\lambda_1 = 0.2000000$ $\lambda_2 = 20.41815$ $X_{11} = -3.477525$ $X_{121} = 3.477525$ $X_{122} = 0.000000$ $X_{22} = 5.800000$ $X_{211} = 0.000000$ $X_{212} = -5.800000$ $X_{32} = 5.865413$ $X_{311} = 0.000000$ $X_{312} = -5.865413$
x_2	$MAX = \lambda_2 - \lambda_1$ $1.56 * x_{11} - 3.52 * x_{22} + 6.3 * x_{311} + 7.7 * x_{312} = \lambda_2 * x_{11}$ $1.56 * x_{121} + 0.65 * x_{122} - 1.53 * x_{211} - 3.52 * x_{212} + 7.7 * x_{32} = \lambda_2 * x_{121} + \lambda_1 * x_{122}$ $-5.4 * x_{121} - 2.6 * x_{122} + 10.25 * x_{211} + 14.45 * x_{212} + 1.3 * x_{311} + 2.7 * x_{312} = \lambda_1 * x_{211} + \lambda_2 * x_{212}$ $-5.4 * x_{11} + 14.45 * x_{22} + 2.7 * x_{32} = \lambda_2 * x_{22}$ $8.8 * x_{11} + 3.825 * x_{211} + 8.9 * x_{212} - 6.4 * x_{32} = \lambda_1 * x_{311} + \lambda_2 * x_{312}$ $8.8 * x_{121} + 3.2 * x_{122} + 8.9 * x_{22} - 2.2 * x_{311} - 6.4 * x_{312} = \lambda_2 * x_{32}$ $0.2 \leq \lambda_1 \leq 12.9, 13 \leq \lambda_2 \leq 22$ $X_{11} >= -5.529008, X_{121} + X_{122} <= 5.529008, X_{121} >= 0, X_{121} <= 1000 * S_1, X_{122} <= 0, X_{122} >= -1000 + 1000 * S_1$ $X_{211} + X_{212} >= -9.309340, X_{22} <= 9.309340, X_{211} >= 0.1, X_{211} <= 1000 * S_2, X_{212} <= 0, X_{212} >= -1000 + 1000 * S_2$ $X_{311} + X_{312} >= -10.25988, X_{32} <= 10.25988, X_{311} >= 0, X_{311} <= 1000 * S_3, X_{312} <= 0, X_{312} >= -1000 + 1000 * S_3$ $X_{11} <= -0.87, X_{121} + X_{122} >= -0.87, X_{211} + X_{212} <= 5.8, X_{22} >= 5.8, X_{311} + X_{312} <= 1, X_{32} >= 1$	$\lambda_1 = 0.2000000$ $\lambda_2 = 20.41815$ $X_{11} = -2.216488$ $X_{121} = 1.862351$ $X_{122} = 0.000000$ $X_{22} = 5.800000$ $X_{211} = 1.604428$ $X_{212} = 0.000000$ $X_{32} = 4.434871$ $X_{311} = 0.000000$ $X_{312} = -2.247337$
x_3		

	$MAX = \lambda_2 - \lambda_1$ $1.56 * x_{11} - 3.52 * x_{22} + 6.3 * x_{311} + 7.7 * x_{312} = \lambda_2 * x_{11}$ $1.56 * x_{121} + 0.65 * x_{122} - 1.53 * x_{211} - 3.52 * x_{212} + 7.7 * x_{32} = \lambda_2 * x_{121} + \lambda_1 * x_{122}$ $-5.4 * x_{121} - 2.6 * x_{122} + 10.25 * x_{211} + 14.45 * x_{212} + 1.3 * x_{311} + 2.7 * x_{312} = \lambda_1 * x_{211} + \lambda_2 * x_{212}$ $-5.4 * x_{11} + 14.45 * x_{22} + 2.7 * x_{32} = \lambda_2 * x_{22}$ $8.8 * x_{11} + 3.825 * x_{211} + 8.9 * x_{212} - 6.4 * x_{32} = \lambda_1 * x_{311} + \lambda_2 * x_{312}$ $8.8 * x_{121} + 3.2 * x_{122} + 8.9 * x_{22} - 2.2 * x_{311} - 6.4 * x_{312} = \lambda_2 * x_{32}$ $0.2 \le \lambda_1 \le 12.9, 13 \le \lambda_2 \le 22$ $X_{11} >= -5.529008, X_{121} + X_{122} <= 5.529008, X_{121} >= 0, X_{121} <= 1000 * S_1, X_{122} <= 0, X_{122} >= -1000 + 1000 * S_1$ $X_{211} + X_{212} >= -9.309340, X_{22} <= 9.309340, X_{211} >= 0, X_{211} <= 1000 * S_2, X_{212} <= 0, X_{212} >= -1000 + 1000 * S_2$ $X_{311} + X_{312} >= -10.25988, X_{32} <= 10.25988, X_{311} >= 0, X_{311} <= 1000 * S_3, X_{312} <= -0.1, X_{312} >= -1000 + 1000 * S_3$ $X_{11} <= -0.87, X_{121} + X_{122} >= -0.87, X_{211} + X_{212} <= 5.8, X_{22} >= 5.8, X_{311} + X_{312} <= 1, X_{32} >= 1$	$\lambda_1 = 0.2000000$ $\lambda_2 = 20.41815$ $X_{11} = -3.477526$ $X_{121} = 3.477527$ $X_{122} = 0.000000$ $X_{22} = 5.800000$ $X_{211} = 0.000000$ $X_{212} = -5.800004$ $X_{32} = 5.865414$ $X_{311} = 0.000000$ $X_{312} = -5.865416$
$\alpha = 0.5$		
Eigenvalues	NLP models	solutions
x_1	$MAX = \lambda_2 - \lambda_1$ $1.28 * x_{11} - 3.26 * x_{22} + 6.65 * x_{311} + 7.35 * x_{312} = \lambda_2 * x_{11}$ $1.28 * x_{121} + 0.825 * x_{122} - 2.265 * x_{211} - 3.26 * x_{212} + 7.35 * x_{32} = \lambda_2 * x_{121} + \lambda_1 * x_{122}$ $-4.7 * x_{121} - 3.3 * x_{122} + 11.125 * x_{211} + 13.225 * x_{212} + 1.65 * x_{311} + 2.35 * x_{312} = \lambda_1 * x_{211} + \lambda_2 * x_{212}$ $-4.7 * x_{11} + 13.225 * x_{22} + 2.35 * x_{32} = \lambda_2 * x_{22}$ $7.4 * x_{11} + 3.9125 * x_{211} + 6.45 * x_{212} - 5.7 * x_{32} = \lambda_1 * x_{311} + \lambda_2 * x_{312}$ $7.4 * x_{121} + 4.6 * x_{122} + 6.45 * x_{22} - 3.6 * x_{311} - 5.7 * x_{312} = \lambda_2 * x_{32}$ $0.5 \le \lambda_1 \le 12.9, 13 \le \lambda_2 \le 20.41815$ $X_{11} >= -3.477525, X_{121} + X_{122} <= 3.477525, X_{121} >= 0.1, X_{121} <= 1000 * S_1, X_{122} <= 0, X_{122} >= -1000 + 1000 * S_1$ $X_{211} + X_{212} >= -5.8, X_{22} <= 5.8, X_{211} >= 0, X_{211} <= 1000 * S_2, X_{212} <= 0, X_{212} >= -1000 + 1000 * S_2$ $X_{311} + X_{312} >= -5.865413, X_{32} <= 5.865413, X_{311} >= 0, X_{311} <= 1000 * S_3, X_{312} <= 0, X_{312} >= -1000 + 1000 * S_3$ $X_{11} <= -0.87, X_{121} + X_{122} >= -0.87, X_{211} + X_{212} <= 5.8, X_{22} >= 5.8, X_{311} + X_{312} <= 1, X_{32} >= 1$	$\lambda_1 = 0.5000000$ $\lambda_2 = 17.99655$ $X_{11} = -3.357019$ $X_{121} = 3.357019$ $X_{122} = 0.000000$ $X_{22} = 5.800000$ $X_{211} = 0.4266358E-07$ $X_{212} = -5.800000$ $X_{32} = 5.062553$ $X_{311} = 0.000000$ $X_{312} = -5.062553$
x_2	$MAX = \lambda_2 - \lambda_1$ $1.28 * x_{11} - 3.26 * x_{22} + 6.65 * x_{311} + 7.35 * x_{312} = \lambda_2 * x_{11}$ $1.28 * x_{121} + 0.825 * x_{122} - 2.265 * x_{211} - 3.26 * x_{212} + 7.35 * x_{32} = \lambda_2 * x_{121} + \lambda_1 * x_{122}$ $-4.7 * x_{121} - 3.3 * x_{122} + 11.125 * x_{211} + 13.225 * x_{212} + 1.65 * x_{311} + 2.35 * x_{312} = \lambda_1 * x_{211} + \lambda_2 * x_{212}$ $-4.7 * x_{11} + 13.225 * x_{22} + 2.35 * x_{32} = \lambda_2 * x_{22}$ $7.4 * x_{11} + 3.9125 * x_{211} + 6.45 * x_{212} - 5.7 * x_{32} = \lambda_1 * x_{311} + \lambda_2 * x_{312}$ $7.4 * x_{121} + 4.6 * x_{122} + 6.45 * x_{22} - 3.6 * x_{311} - 5.7 * x_{312} = \lambda_2 * x_{32}$ $0.5 \le \lambda_1 \le 12.9, 13 \le \lambda_2 \le 18.57814$ $X_{11} >= -2.216488, X_{121} + X_{122} <= 1.862351, X_{121} >= 0, X_{121} <= 1000 * S_1, X_{122} <= 0, X_{122} >= -1000 + 1000 * S_1$ $X_{211} + X_{212} >= 1.604428, X_{22} <= 5.8, X_{211} >= 0.1, X_{211} <= 1000 * S_2, X_{212} <= 0, X_{212} >= -1000 + 1000 * S_2$ $X_{311} + X_{312} >= -2.247337, X_{32} <= 4.434871, X_{311} >= 0, X_{311} <= 1000 * S_3, X_{312} <= 0, X_{312} >= -1000 + 1000 * S_3$ $X_{11} <= -0.87, X_{121} + X_{122} >= -0.87, X_{211} + X_{212} <= 5.8, X_{22} >= 5.8, X_{311} + X_{312} <= 1, X_{32} >= 1$	$\lambda_1 = 4.037175$ $\lambda_2 = 16.41320$ $X_{11} = -2.137860$ $X_{121} = 1.504952$ $X_{122} = 0.000000$ $X_{22} = 5.800000$ $X_{211} = 1.604428$ $X_{212} = 0.000000$ $X_{32} = 3.593029$ $X_{311} = 0.000000$ $X_{312} = -1.829205$
x_3	$MAX = \lambda_2 - \lambda_1$ $1.28 * x_{11} - 3.26 * x_{22} + 6.65 * x_{311} + 7.35 * x_{312} = \lambda_2 * x_{11}$ $1.28 * x_{121} + 0.825 * x_{122} - 2.265 * x_{211} - 3.26 * x_{212} + 7.35 * x_{32} = \lambda_2 * x_{121} + \lambda_1 * x_{122}$ $-4.7 * x_{121} - 3.3 * x_{122} + 11.125 * x_{211} + 13.225 * x_{212} + 1.65 * x_{311} + 2.35 * x_{312} = \lambda_1 * x_{211} + \lambda_2 * x_{212}$ $-4.7 * x_{11} + 13.225 * x_{22} + 2.35 * x_{32} = \lambda_2 * x_{22}$ $7.4 * x_{11} + 3.9125 * x_{211} + 6.45 * x_{212} - 5.7 * x_{32} = \lambda_1 * x_{311} + \lambda_2 * x_{312}$ $7.4 * x_{121} + 4.6 * x_{122} + 6.45 * x_{22} - 3.6 * x_{311} - 5.7 * x_{312} = \lambda_2 * x_{32}$ $0.5 \le \lambda_1 \le 12.9, 13 \le \lambda_2 \le 20.41815$ $X_{11} >= -3.477526, X_{121} + X_{122} <= 3.477527, X_{121} >= 0, X_{121} <= 1000 * S_1, X_{122} <= 0, X_{122} >= -1000 + 1000 * S_1$ $X_{211} + X_{212} >= -5.800004, X_{22} <= 5.8, X_{211} >= 0, X_{211} <= 1000 * S_2, X_{212} <= 0, X_{212} >= -1000 + 1000 * S_2$ $X_{311} + X_{312} >= -5.865416, X_{32} <= 5.865414, X_{311} >= 0, X_{311} <= 1000 * S_3, X_{312} <= -0.1, X_{312} >= -1000 + 1000 * S_3$ $X_{11} <= -0.87, X_{121} + X_{122} >= -0.87, X_{211} + X_{212} <= 5.8, X_{22} >= 5.8, X_{311} + X_{312} <= 1, X_{32} >= 1$	$\lambda_1 = 0.5000000$ $\lambda_2 = 17.99655$ $X_{11} = -3.357019$ $X_{121} = 3.357019$ $X_{122} = 0.000000$ $X_{22} = 5.800000$ $X_{211} = 0.000000$ $X_{212} = -5.800000$ $X_{32} = 5.062553$ $X_{311} = 0.000000$ $X_{312} = -5.062553$
$\alpha = 0.7$		
Eigenvalues	NLP models	solutions
x_1	$MAX = \lambda_2 - \lambda_1$ $1.084 * x_{11} - 3.078 * x_{22} + 6.895 * x_{311} + 7.105 * x_{312} = \lambda_2 * x_{11}$ $1.084 * x_{121} + 0.9475 * x_{122} - 2.7795 * x_{211} - 3.078 * x_{212} + 7.105 * x_{32} = \lambda_2 * x_{121} + \lambda_1 * x_{122}$ $-4.21 * x_{121} - 3.79 * x_{122} + 11.7375 * x_{211} + 12.3675 * x_{212} + 1.895 * x_{311} + 2.105 * x_{312} = \lambda_1 * x_{211} + \lambda_2 * x_{212}$ $-4.21 * x_{11} + 12.3675 * x_{22} + 2.105 * x_{32} = \lambda_2 * x_{22}$ $6.42 * x_{11} + 3.97375 * x_{211} + 4.735 * x_{212} - 5.21 * x_{32} = \lambda_1 * x_{311} + \lambda_2 * x_{312}$ $6.42 * x_{121} + 5.58 * x_{122} + 4.735 * x_{22} - 4.58 * x_{311} - 5.21 * x_{312} = \lambda_2 * x_{32}$ $0.7 \le \lambda_1 \le 12.9, 13 \le \lambda_2 \le 17.99655$ $X_{11} >= -3.357019, x_{121} + X_{122} <= 3.357019, X_{121} >= 0.1, X_{121} <= 1000 * S_1, X_{122} <= 0, X_{122} >= -1000 + 1000 * S_1$ $X_{211} + X_{212} >= -5.8, X_{22} <= 5.8, X_{211} >= 0, X_{211} <= 1000 * S_2, X_{212} <= 0, X_{212} >= -1000 + 1000 * S_2$ $X_{311} + X_{312} >= -5.062553, X_{32} <= 5.062553, X_{311} >= 0, X_{311} <= 1000 * S_3, X_{312} <= -0.1, X_{312} >= -1000 + 1000 * S_3$ $X_{11} <= -0.87, X_{121} + X_{122} >= -0.87, X_{211} + X_{212} <= 5.8, X_{22} >= 5.8, X_{311} + X_{312} <= 1, X_{32} >= 1$	$\lambda_1 = 0.7000000$ $\lambda_2 = 16.27231$ $X_{11} = -3.207526$ $X_{121} = 3.207526$ $X_{122} = 0.000000$ $X_{22} = 5.800000$ $X_{211} = 0.4886187E-07$ $X_{212} = -5.800000$ $X_{32} = 4.344056$ $X_{311} = 0.000000$ $X_{312} = -4.344056$

x_2	$MAX = \lambda_2 - \lambda_1$ $1.084 * x_{11} - 3.078 * x_{22} + 6.895 * x_{311} + 7.105 * x_{312} = \lambda_2 * x_{11}$ $1.084 * x_{121} + 0.9475 * x_{122} - 2.7795 * x_{211} - 3.078 * x_{212} + 7.105 * x_{32} = \lambda_2 * x_{121} + \lambda_1 * x_{122}$ $-4.21 * x_{121} - 3.79 * x_{122} + 11.7375 * x_{211} + 12.3675 * x_{212} + 1.895 * x_{311} + 2.105 * x_{312} = \lambda_1 * x_{211} + \lambda_2 * x_{212}$ $-4.21 * x_{11} + 12.3675 * x_{22} + 2.105 * x_{32} = \lambda_2 * x_{22}$ $6.42 * x_{11} + 3.97375 * x_{211} + 4.735 * x_{212} - 5.21 * x_{32} = \lambda_1 * x_{311} + \lambda_2 * x_{312}$ $6.42 * x_{121} + 5.58 * x_{122} + 4.735 * x_{22} - 4.58 * x_{311} - 5.21 * x_{312} = \lambda_2 * x_{32}$ $4.037175 \leq \lambda_1 \leq 12.9, 13 \leq \lambda_2 \leq 16.41320$ $X_{11} >= -2.137860, x_{121} + X_{122} <= 1.504952, X_{121} >= 0, X_{121} <= 1000 * S_1, X_{122} <= 0, X_{122} >= -1000 + 1000 * S_1$ $X_{211} + X_{212} >= 1.604428, X_{22} <= 5.8, X_{211} >= 0.1, X_{211} <= 1000 * S_2, X_{212} <= 0, X_{212} >= -1000 + 1000 * S_2$ $X_{311} + X_{312} >= -1.829205, X_{32} <= 3.593029, X_{311} >= 0, X_{311} <= 1000 * S_3, X_{312} <= 0, X_{312} >= -1000 + 1000 * S_3$ $X_{11} <= -0.87, X_{121} + X_{122} >= -0.87, X_{211} + X_{212} <= 5.8, X_{22} >= 5.8, X_{311} + X_{312} <= 1, X_{32} >= 1$	$\lambda_1 = 6.841805$ $\lambda_2 = 14.88045$ $X_{11} = -2.039475$ $X_{121} = 1.141952$ $X_{122} = 0.000000$ $X_{22} = 5.800000$ $X_{211} = 1.604428$ $X_{212} = 0.000000$ $X_{32} = 2.845093$ $X_{311} = 0.000000$ $X_{312} = -1.447589$
x_3	$MAX = \lambda_2 - \lambda_1$ $1.084 * x_{11} - 3.078 * x_{22} + 6.895 * x_{311} + 7.105 * x_{312} = \lambda_2 * x_{11}$ $1.084 * x_{121} + 0.9475 * x_{122} - 2.7795 * x_{211} - 3.078 * x_{212} + 7.105 * x_{32} = \lambda_2 * x_{121} + \lambda_1 * x_{122}$ $-4.21 * x_{121} - 3.79 * x_{122} + 11.7375 * x_{211} + 12.3675 * x_{212} + 1.895 * x_{311} + 2.105 * x_{312} = \lambda_1 * x_{211} + \lambda_2 * x_{212}$ $-4.21 * x_{11} + 12.3675 * x_{22} + 2.105 * x_{32} = \lambda_2 * x_{22}$ $6.42 * x_{11} + 3.97375 * x_{211} + 4.735 * x_{212} - 5.21 * x_{32} = \lambda_1 * x_{311} + \lambda_2 * x_{312}$ $6.42 * x_{121} + 5.58 * x_{122} + 4.735 * x_{22} - 4.58 * x_{311} - 5.21 * x_{312} = \lambda_2 * x_{32}$ $0.7 \leq \lambda_1 \leq 12.9, 13 \leq \lambda_2 \leq 17.99655$ $X_{11} >= -3.357019, x_{121} + X_{122} <= 3.357019, X_{121} >= 0, X_{121} <= 1000 * S_1, X_{122} <= 0, X_{122} >= -1000 + 1000 * S_1$ $X_{211} + X_{212} >= 5.8, X_{22} <= 5.8, X_{211} >= 0, X_{211} <= 1000 * S_2, X_{212} <= 0, X_{212} >= -1000 + 1000 * S_2$ $X_{311} + X_{312} >= -5.062553, X_{32} <= 5.062553, X_{311} >= 0, X_{311} <= 1000 * S_3, X_{312} <= -0.1, X_{312} >= -1000 + 1000 * S_3$ $X_{11} <= -0.87, X_{121} + X_{122} >= -0.87, X_{211} + X_{212} <= 5.8, X_{22} >= 5.8, X_{311} + X_{312} <= 1, X_{32} >= 1$	$\lambda_1 = 0.7000000$ $\lambda_2 = 16.27231$ $X_{11} = -3.207526$ $X_{121} = 3.207526$ $X_{122} = 0.000000$ $X_{22} = 5.800000$ $X_{211} = 0.000000$ $X_{212} = -5.800000$ $X_{32} = 4.344056$ $X_{311} = 0.000000$ $X_{312} = -4.344056$

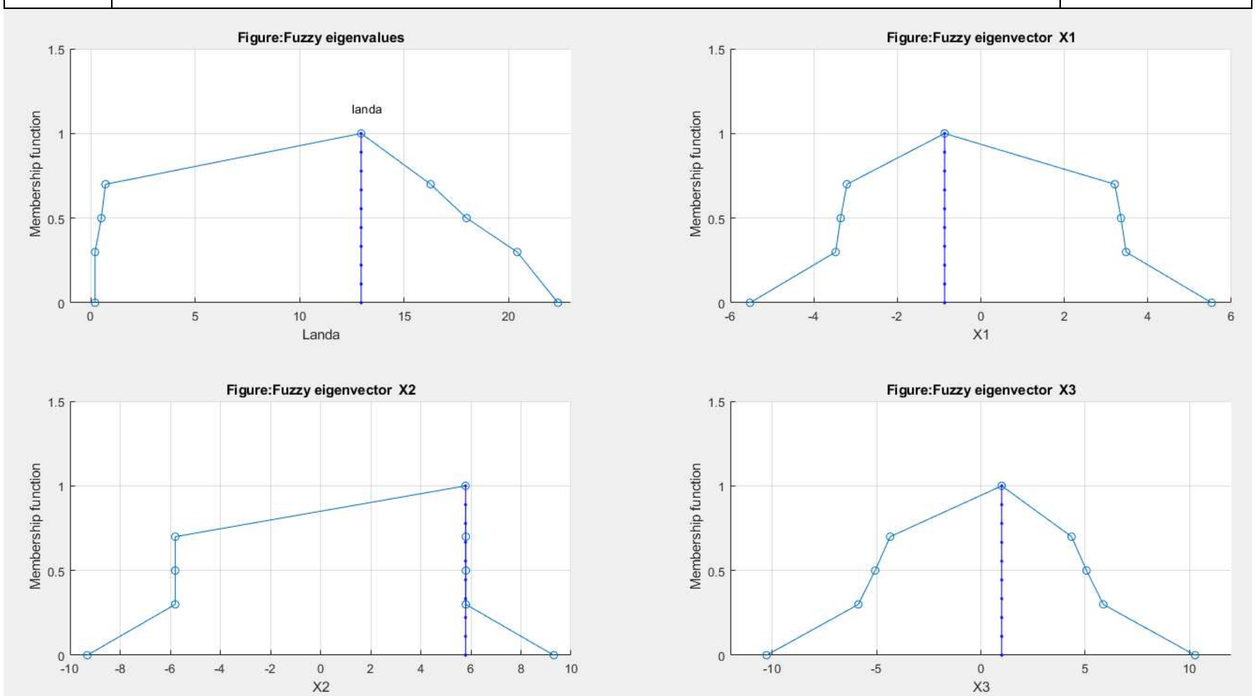

FIGURE 1:

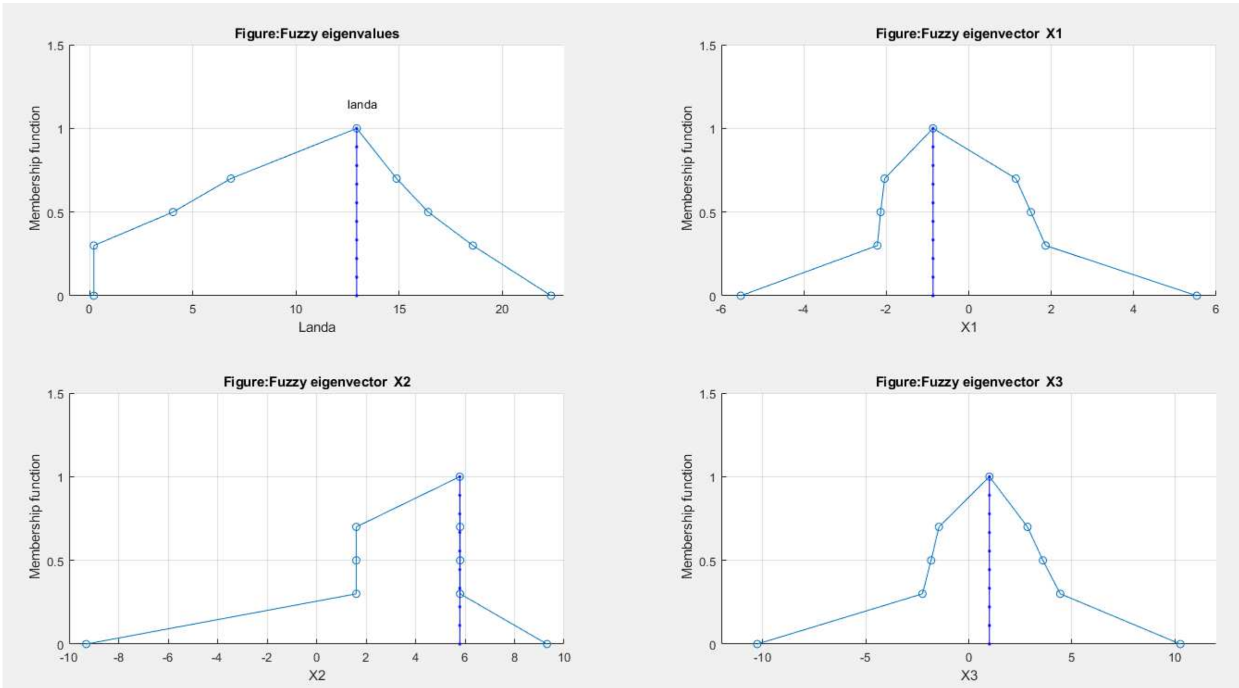

Figure:Fuzzy eigenvalues
landa
Membership function
Landa
Figure:Fuzzy eigenvector X1
X1
Figure:Fuzzy eigenvector X2
X2
Figure:Fuzzy eigenvector X3
X3

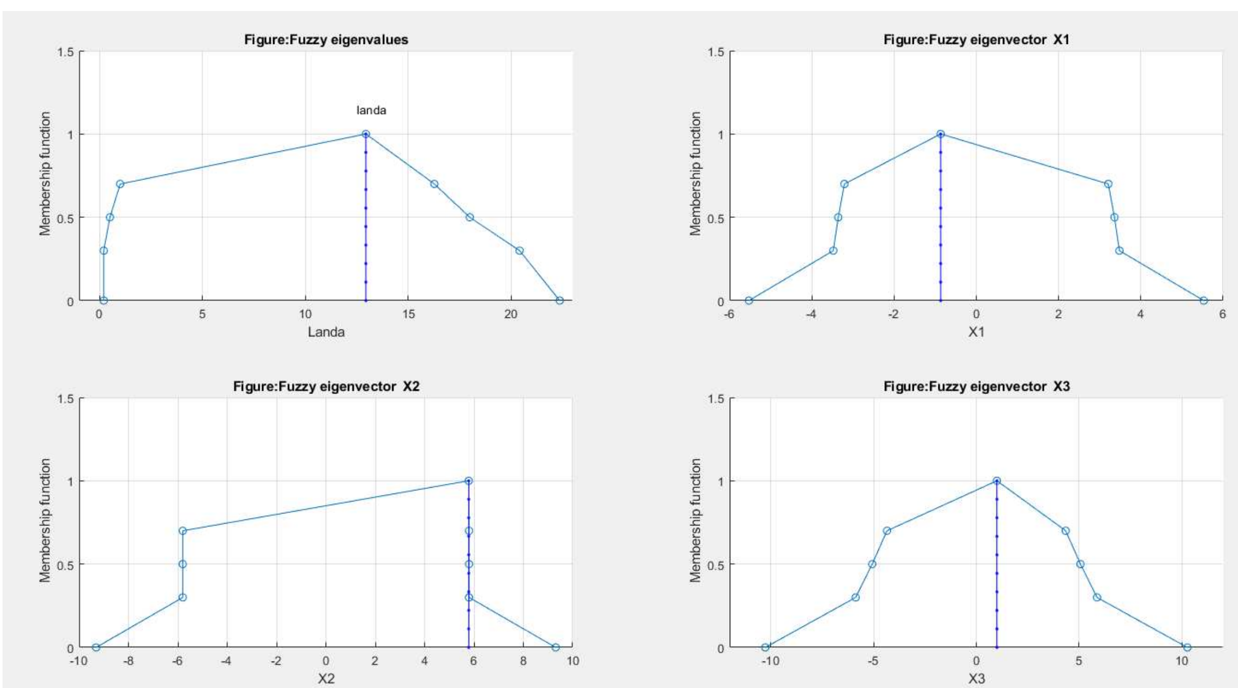

Figure:Fuzzy eigenvalues
landa
Membership function
Landa
Figure:Fuzzy eigenvector X1
X1
Figure:Fuzzy eigenvector X2
X2
Figure:Fuzzy eigenvector X3
X3

Example 5.3. Consider a fuzzy triangular matrix $\tilde{A}$, assuming that the fuzzy triangular number $\tilde{\lambda} = (\underline{\lambda}, \lambda, \overline{\lambda})$ is a fuzzy eigenvalue and the fuzzy triangular vector $\tilde{X} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4, \tilde{x}_5)$ is a fuzzy eigenvector of matrix $\tilde{A}$, we have:

$$\begin{cases} (26,38,45)x_1 + (6,16,25)x_2 + (-22,-20,-18)x_3 + (16,17,19)x_4 + (15,26,38)x_5 = (\underline{\lambda}, \overline{\lambda}) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \\ (10,15,23)x_1 + (25,45,70)x_2 + (20,29,37)x_3 + (-12,-11,-9)x_4 + (18,25,36)x_5 = (\underline{\lambda}, \overline{\lambda}) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \\ (-25,-23,-20)x_1 + (21,30,43)x_2 + (27,36,46)x_3 + (9,18,28)x_4 + (23,32,47)x_5 = (\underline{\lambda}, \overline{\lambda}) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \\ (10,21,38)x_1 + (-20,-9,-8)x_2 + (6,14,28)x_3 + (36,50,65)x_4 + (-25,-23,-22)x_5 = (\underline{\lambda}, \overline{\lambda}) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \\ (15,26,34)x_1 + (17,25,41)x_2 + (14,22,40)x_3 + (-50,-35,-23)x_4 + (44,65,87)x_5 = (\underline{\lambda}, \overline{\lambda}) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \end{cases} \tag{5.55}$$

So, the crisp solution $(1 - cut)$ obtained as follows:

$$\lambda = \begin{bmatrix} -27.0795 & 0 & 0 & 0 & 0 \\ 0 & 111.8871 & 0 & 0 & 0 \\ 0 & 0 & 24.0648 & 0 & 0 \\ 0 & 0 & 0 & 60.7752 & 0 \\ 0 & 0 & 0 & 0 & 64.3524 \end{bmatrix}$$

$$X = \begin{bmatrix} -1.4039 & 0.3004 & -0.0899 & 1.8516 & -0.3265 \\ 0.7980 & 0.7239 & -1.4366 & -1.3523 & -2.4186 \\ -1.7081 & 0.5578 & 0.3290 & -3.0913 & -4.0015 \\ 1.0843 & -0.2488 & 0.2835 & -1.4129 & -4.4669 \\ 1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 \end{bmatrix}$$

$$\lambda_1 = -27.0795,\ \lambda_2 = 111.8871,\ \lambda_3 = 24.0648,\ \lambda_4 = 60.7752,\ \lambda_5 = 64.3524,$$

And

$$x^1 = \begin{bmatrix} -1.4039 \\ 0.7980 \\ -1.7081 \\ 1.0843 \\ 1.0000 \end{bmatrix}, x^2 = \begin{bmatrix} 0.3004 \\ 0.7239 \\ 0.5578 \\ -0.2488 \\ 1.0000 \end{bmatrix}, x^3 = \begin{bmatrix} -0.0899 \\ -1.4366 \\ 0.3290 \\ 0.2835 \\ 1.0000 \end{bmatrix}, \quad x^4 = \begin{bmatrix} 1.8516 \\ -1.3523 \\ -3.0913 \\ -1.4129 \\ 1.0000 \end{bmatrix}, \quad x^5 = \begin{bmatrix} -0.3265 \\ -2.4186 \\ -4.0015 \\ -4.4669 \\ 1.0000 \end{bmatrix}$$

By using model (3.25) for $\lambda_4 = 60.7752$, $x^4 = \begin{bmatrix} 1.8516 \\ -1.3523 \\ -3.0913 \\ -1.4129 \\ 1.0000 \end{bmatrix}$, problem (5.55), written as follows:

$$\text{MAX} \quad z = \lambda_2 - \lambda_1$$

$$s.t \begin{cases} 26 * x_{111} + 45 * x_{112} + 25 * x_{21} - 22 * x_{321} - 18 * x_{322} + 19 * x_{41} + 15 * x_{511} + 38 * x_{512} = \lambda_1 * x_{111} + \lambda_2 * x_{112} \\ 45 * x_{12} + 25 * x_{221} + 6 * x_{222} - 22 * x_{31} + 19 * x_{421} + 16 * x_{422} + 38 * x_{52} = \lambda_2 * x_{12} \\ 10 * x_{111} + 23 * x_{112} + 70 * x_{21} + 37 * x_{31} - 12 * x_{421} - 9 * x_{422} + 18 * x_{511} + 36 * x_{512} = \lambda_2 * x_{21} \\ 23 * x_{12} + 70 * x_{221} + 25 * x_{222} + 37 * x_{321} + 20 * x_{322} - 12 * x_{41} + 36 * x_{52} = \lambda_2 * x_{221} + \lambda_1 * x_{222} \\ -25 * x_{12} + 43 * x_{21} + 46 * x_{31} + 28 * x_{41} + 23 * x_{511} + 47 * x_{512} = \lambda_2 * x_{31} \\ -20 * x_{111} - 25 * x_{112} + 43 * x_{221} + 21 * x_{222} + 46 * x_{321} + 27 * x_{322} + 28 * x_{421} + 9 * x_{422} + 47 * x_{52} = \lambda_2 * x_{321} + \lambda_1 * x_{322} \\ 10 * x_{111} + 38 * x_{112} - 20 * x_{221} - 8 * x_{222} + 28 * x_{31} + 65 * x_{41} - 25 * x_{52} = \lambda_2 * x_{41} \\ 38 * x_{12} - 20 * x_{21} + 28 * x_{321} + 6 * x_{322} + 65 * x_{421} + 36 * x_{422} - 22 * x_{511} - 25 x_{512} = \lambda_2 * x_{421} + \lambda_1 * x_{422} \\ 15 * x_{111} + 34 * x_{112} + 41 * x_{21} + 40 * x_{31} - 50 * x_{421} - 23 * x_{422} + 44 * x_{511} + 87 * x_{512} = \lambda_1 * x_{511} + \lambda_2 * x_{512} \\ 34 * x_{12} + 41 * x_{221} + 17 * x_{222} + 40 * x_{321} + 14 * x_{322} - 50 * x_{41} + 87 * x_{52} = \lambda_2 * x_{52} \\ x_{11} = x_{111} + x_{112} \leq 1.8516, x_{12} \geq 1.8516, \\ x_{21} \leq -1.3523, x_{22} = x_{221} + x_{222} \geq -1.3523, \\ x_{31} \leq -3.0913, x_{32} = x_{321} + x_{322} \geq -3.0913, \\ x_{41} \leq -1.4129, x_{42} = x_{421} + x_{422} \geq -1.4129, \\ x_{51} = x_{511} + x_{512} \leq 1.0000, x_{52} \geq 1.0000, \\ x_{111} \geq 0, x_{111} \leq 1000 * s_{1,} x_{112} \leq 0, x_{112} \geq -1000 + 1000 * s_1 \\ x_{221} \geq 0, x_{221} \leq 1000 * s_{2,} x_{222} \leq 0, x_{222} \geq -1000 + 1000 * s_2 \\ x_{321} \geq 0, x_{321} \leq 1000 * s_{3,} x_{322} \leq 0, x_{322} \geq -1000 + 1000 * s_3 \\ x_{421} \geq 0, x_{421} \leq 1000 * s_{3,} x_{422} \leq 0, x_{422} \geq -1000 + 1000 * s_4 \\ x_{511} \geq 0, x_{511} \leq 1000 * s_{3,} x_{512} \leq 0, x_{512} \geq -1000 + 1000 * s_3 \\ \lambda_1 \leq 60.7752, \lambda_2 \geq 60.7752, \end{cases} \tag{5.56}$$

That here:

$$\tilde{\lambda} = (\lambda_1, \lambda, \lambda_2 \,) \ , \tilde{X} = \begin{bmatrix} (x_{111} + x_{112}, x_1, x_{12}) \\ (x_{21}, x_2, x_{221} + x_{222}) \\ (x_{31}, x_3, x_{321} + x_{322}) \\ (x_{41}, x_4, x_{421} + x_{422}) \\ (x_{511} + x_{512}, x_5, x_{52}) \end{bmatrix},$$

Then, using the LINGO 18 software package, we can obtain:

$$\begin{aligned} &\lambda_2 = 193.5974 \, , \lambda_1 = 0.000000, \\ &\mathrm{x}_{11} = -17.38445, \ \mathrm{x}_{12} = 17.38445, x_{111} = 0.000000, x_{112} = -17.38445, \\ &\mathrm{x}_{21} = -21.03200, \ \mathrm{x}_{22} = 21.03200, x_{221} = 21.03200, x_{222} = 0.000000, \\ &\mathrm{x}_{31} = -22.71764, \ \mathrm{x}_{32} = 22.71764, x_{321} = 22.71764, x_{322} = 0.000000, \\ &\mathrm{x}_{41} = -19.43440, \ \mathrm{x}_{42} = 19.43440, x_{421} = 19.43440, x_{422} = 0.000000, \\ &\mathrm{x}_{51} = -31.27477, \ \mathrm{x}_{52} = 31.27477, x_{511} = 0.000000, \ x_{512} = -31.27477, \end{aligned} \tag{5.57}$$

Besides, the eigenvector and the fuzzy triangular eigenvalue for matrix $\tilde{A}$ based on the compute above given by:

$$\begin{aligned}&\tilde{x}_1 = (-17.38445, 1.8516, 17.38445), \quad \tilde{x}_2 = (-21.03200, -1.3523, 21.03200),\\ &\tilde{x}_3 = (-22.71764, -3.0913, 22.71764), \quad \tilde{x}_4 = (-19.43440, -1.4129, 19.43440),\\ &\tilde{x}_5 = (-31.27477, 1.0000, 31.27477), \ \tilde{\lambda} = (\underline{\lambda}, \lambda, \overline{\lambda}) = (0.000000, 60.7752, 193.5974)\end{aligned} \tag{5.58}$$

Now, If we write models (3.16) and (3.20)of the proposed based on the results obtained in a 1-cut position (5.55) for (λ_4, x^4) and, execution by using the software package as LINGO 18 software package, then will see, that models are both infeasible. Moreover, models (3.16), (3.20), and (3.25) for $(\lambda_1, x^1), (\lambda_3, x^3)$, and (λ_5, x^5) are infeasible, but for (λ_2, x^2) model (3.25) has the following answers:

$$\begin{aligned}&\lambda_2 = 193.5974, \lambda_1 = 0.000000,\\ &\mathrm{x}_{11} = -0.5983550, \mathrm{x}_{12} = 0.5983550, x_{111} = 0.000000, x_{112} = -0.5983550,\\ &\mathrm{x}_{21} = -0.7239000, \mathrm{x}_{22} = 0.7239000, x_{211} = 0.000000, x_{212} = -0.7239000,\\ &\mathrm{x}_{31} = -0.7819182, \mathrm{x}_{32} = 0.7819182, x_{311} = 0.000000, x_{312} = -0.7819182,\\ &\mathrm{x}_{41} = -0.6689122, \mathrm{x}_{42} = 0.6689122, x_{421} = 0.6689122, x_{422} = 0.000000,\\ &\mathrm{x}_{51} = -1.0764460, \mathrm{x}_{52} = 1.0764460, x_{511} = 0.000000, x_{512} = -1.0764460,\end{aligned} \tag{5.59}$$

Besides, the eigenvector and the fuzzy triangular eigenvalue for matrix $\tilde{A}$ given by:

$$\begin{aligned}&\tilde{x}_1 = (-0.5983550, 0.3004, 0.5983550), \ \tilde{x}_2 = (-0.7239000, 0.7239, 0.7239000),\\ &\tilde{x}_3 = (-0.7819182, 0.5578, 0.7819182), \tilde{x}_4 = (-0.6689122, -0.2488, 0.6689122),\\ &\tilde{x}_5 = (-1.076446, 1.0000, 1.076446), \tilde{\lambda} = (\underline{\lambda}, \lambda, \overline{\lambda}) = (0.000000, 111.8871, 193.5974)\end{aligned} \tag{5.60}$$

Example 5.4. Consider a fuzzy triangular matrix $\tilde{A}$, assuming that the fuzzy triangular number $\tilde{\lambda} = (\underline{\lambda}, \lambda, \overline{\lambda})$ is a fuzzy eigenvalue and the fuzzy triangular vector $\tilde{X} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$ is a fuzzy eigenvector of matrix $\tilde{A}$, we have:

$$\begin{cases}(48, 53, 60)x_1 + (9, 12, 14)x_2 + (19, 24, 30)x_3 = (\underline{\lambda}, \overline{\lambda})\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}\\ (11, 15, 20)x_1 + (-987, -461, -451)x_2 + (351, 385, 572)x_3 = (\underline{\lambda}, \overline{\lambda})\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}\\ (85, 236, 473)x_1 + (21, 158, 362)x_2 + (36, 282, 1091)x_3 = (\underline{\lambda}, \overline{\lambda})\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}\end{cases} \tag{5.61}$$

Consider 1-cut fuzzy triangular matrix $\tilde{A}$ above. If we write models (3.16), (3.20), and (3.25) of the proposed based on the results obtained in a 1-cut position and, execution by using of the software package as LINGO 18 software package, then will see, that models are infeasible. Therefore, we obtain the fuzzy escribed eigenvalue, fuzzy peripheral eigenvalue, and fuzzy approximate eigenvalue of above, first, we solve the fuzzy triangular matrix $\tilde{A}$ in a 1-cut position. So, we obtain:

$$\lambda = \begin{bmatrix} 376.4468 & 0 & 0 \\ 0 & 31.8520 & 0 \\ 0 & 0 & -534.2988 \end{bmatrix} \quad X = \begin{bmatrix} 0.0913 & -1.5513 & 0.0667 \\ 0.4614 & 0.7340 & -5.2661 \\ 1.0000 & 1.0000 & 1.0000 \end{bmatrix},$$

$$\lambda_1 = 376.4468,\ \lambda_2 = 31.8520,\ \lambda_3 = -534.2988 \ \ and \ \ x^1 = \begin{bmatrix} 0.0913 \\ 0.4614 \\ 1.0000 \end{bmatrix}, x^2 = \begin{bmatrix} -1.5513 \\ 0.7340 \\ 1.0000 \end{bmatrix}, x^3 = \begin{bmatrix} 0.0667 \\ -5.2661 \\ 1.0000 \end{bmatrix}$$

By using model (3.20) for $\lambda_2 = 31.8520, x^2 = \begin{bmatrix} -1.5513 \\ 0.7340 \\ 1.0000 \end{bmatrix}$, problem (5.61), written as follows:

$$\begin{aligned} &MAX \ \ z = \lambda_2 - \lambda_3 - \lambda_4 \\ &s.t. \begin{cases} 60 * x_{11} + 9 * x_{21} + 19 * x_{31} = \lambda_2 * x_{11} \\ 48 * x_{12} + 14 * x_{22} + 30 * x_{32} = \ \lambda_3 * x_{12} + \lambda_4 * x_{11} \\ 20 * x_{11} - 987 * x_{22} + 351 * x_{31} = \lambda_3 * x_{21} + \lambda_4 * x_{22} \\ 11 * x_{12} - 451 * x_{21} + 572 * x_{32} = \lambda_2 * x_{22} \\ 473 * x_{11} + 21 * x_{21} + 36 * x_{31} = \lambda_3 * x_{31} + \lambda_4 * x_{32} \\ 85 * x_{12} + 362 * x_{22} + 1091 * x_{32} = \lambda_2 * x_{32} \\ \qquad x_{11} \leq -1.5513, x_{12} \geq -1.5513 \\ \qquad x_{21} \leq 0.7340, x_{22} \geq 0.7340 \\ \qquad x_{31} \leq 1.0000, x_{32} \geq 1.0000 \\ \quad \lambda_2 \geq 31.8520, \lambda_1 = \lambda_3 + \lambda_4 \leq 31.8520, \\ \lambda_3 \geq 0, \lambda_3 \leq 1000 * \mathrm{s}_1, \lambda_4 \leq 0, \lambda_4 \geq -1000 + \ 1000 * \mathrm{s}_1 \end{cases} \end{aligned} \qquad (5.62)$$

That here:

$$\tilde{\lambda} = (\lambda_3 + \lambda_4, \lambda, \lambda_2) \ \ , \tilde{X} = \begin{bmatrix} (x_{11}, x_1, x_{12}) \\ (x_{21}, x_2, x_{22}) \\ (x_{31}, x_2, x_{32}) \end{bmatrix}$$

That model (5.62) is infeasible.

Therefore, to find 0-cut of the fuzzy escribed eigenvalue, we write model (4.33) of the proposed for above fuzzy triangular matrix $\tilde{A}$ as follows:

$$
\begin{aligned}
&Min \quad T\\
&s.t.\begin{cases}
60 * x_{11} + 9 * x_{21} + 19 * x_{31} - \lambda_2 * x_{11} \le T\\
20 * x_{11} - 987 * x_{22} + 351 * x_{31} - \lambda_3 * x_{21} - \lambda_4 * x_{22} \le T\\
473 * x_{11} + 21 * x_{21} + 36 * x_{31} - \lambda_3 * x_{31} - \lambda_4 * x_{32} \le T\\
60 * x_{11} + 9 * x_{21} + 19 * x_{31} - \lambda_2 * x_{11} \ge 0\\
20 * x_{11} - 987 * x_{22} + 351 * x_{31} - \lambda_3 * x_{21} - \lambda_4 * x_{22} \ge 0\\
473 * x_{11} + 21 * x_{21} + 36 * x_{31} - \lambda_3 * x_{31} - \lambda_4 * x_{32} \ge 0\\
-48 * x_{12} - 14 * x_{22} - 30 * x_{32} + \lambda_3 * x_{12} + \lambda_4 * x_{11} \le T\\
-11 * x_{12} + 451 * x_{21} - 572 * x_{32} + \lambda_2 * x_{22} \le T\\
-85 * x_{12} - 362 * x_{22} - 1091 * x_{32} + \lambda_2 * x_{32} \le T\\
-48 * x_{12} - 14 * x_{22} - 30 * x_{32} + \lambda_3 * x_{12} + \lambda_4 * x_{11} \ge 0\\
-11 * x_{12} + 451 * x_{21} - 572 * x_{32} + \lambda_2 * x_{22} \ge 0\\
-85 * x_{12} - 362 * x_{22} - 1091 * x_{32} + \lambda_2 * x_{32} \ge 0\\
\quad x_{11} \le -1.5513, x_{12} \ge -1.5513\\
\quad x_{21} \le 0.7340, x_{22} \ge 0.7340\\
\quad x_{31} \le 1.0000, x_{32} \ge 1.0000\\
\quad \lambda_2 \ge 31.8520, \lambda_1 = \lambda_3 + \lambda_4 \le 31.8520,\\
\quad \lambda_3 \ge 0, \lambda_3 \le 1000 * \mathrm{s1}, \lambda_4 \le 0, \lambda_4 \ge -1000 + 1000 * \mathrm{s1}
\end{cases}
\end{aligned}
\tag{5.63}
$$

Now, by solve above NLP problem by using the LINGO 18 software package, we can obtain:

$$
\begin{aligned}
&T = 1706.472\\
&x_{11} = -1.551300, x_{12} = -1.551300,\\
&x_{21} = 0.7340000, x_{22} = 0.7340000,\\
&x_{31} = 0.6120798\mathrm{E}-01, x_{32} = 2.090704,\\
&\lambda_3 = 0.000000, \lambda_4 = -1000.000\,, \lambda_2 = 1155.019
\end{aligned}
\tag{5.64}
$$

Therefore, fuzzy escribed eigenvalue obtained as follows:

$$
\begin{aligned}
&\tilde{x}_1 = (-1.551300, -1.5513, -1.551300), \tilde{x}_2 = (0.7340000, 0.7340, 0.7340000),\\
&\tilde{x}_3 = (0.6120798\mathrm{E}-01, 1.0000, 2.090704), \tilde{\lambda} = \left(\underline{\lambda}, \lambda\,, \overline{\lambda}\right) = (-1000.000, 31.8520, 1155.019\,)
\end{aligned}
\tag{5.65}
$$

Indeed $\tilde{\lambda}, \tilde{X} \in \mathrm{TTFES}$

Equivalent, fuzzy peripheral eigenvalue obtained as follows:

$$
\begin{aligned}
&\tilde{x}_1 = (-2.462680, -1.5513, -1.551300),\ \tilde{x}_2 = (0.000000, 0.7340, 0.7340000),\\
&\tilde{x}_3 = (0.000000, 1.0000, 1.000000), \tilde{\lambda} = \left(\underline{\lambda}, \lambda\,, \overline{\lambda}\right) = (28.36870, 31.8520, 60.00000)
\end{aligned}
\tag{5.66}
$$

Indeed $\tilde{\lambda}, \tilde{X} \in \mathrm{CTFES}$

Equivalent, fuzzy approximate eigenvalue obtained as follows:

$$\tilde{x}_1 = (-1.551300, -1.5513, -1.551300), \tilde{x}_2 = (0.000000, 0.7340, 0.7340000),$$
$$\tilde{x}_3 = (0.000000, 1.0000, 1.000000), \tilde{\lambda} = (\underline{\lambda}, \lambda, \overline{\lambda}) = (-64.31428, 31.8520, 516.5702) \quad (5.67)$$

Indeed $\tilde{\lambda}, \tilde{X} \in \mathrm{ATFES}$

Clear the value of the objective function of each model is the distance between $\tilde{A}\tilde{X}$ *and* $\tilde{\lambda}\tilde{X}$ in 0-cut.

So here:

$$T(\tilde{\lambda}_{\mathrm{TTFES}}, \tilde{X}_{\mathrm{TTFES}}) = 1706.472, T(\tilde{\lambda}_{\mathrm{CTFES}}, \tilde{X}_{\mathrm{CTFES}}) = 1164.848, T(\tilde{\lambda}_{\mathrm{ATFES}}, \tilde{X}_{\mathrm{ATFES}}) = 708.2773$$

Thereupon:

$$T(\tilde{\lambda}_{\mathrm{TTFES}}, \tilde{X}_{\mathrm{TTFES}}) = 1706.472 \leq T(\tilde{\lambda}_{\mathrm{CTFES}}, \tilde{X}_{\mathrm{CTFES}}) = 1164.848 \leq T(\tilde{\lambda}_{\mathrm{ATFES}}, \tilde{X}_{\mathrm{ATFES}}) = 708.2773$$

Similarly, to obtain widths $\lambda_1,\ x^1$ *and* λ_3, x^3, we behavior as above.

Therefore, the tables of values of T for proposed models assumed as follows:

Table 2

Values of $\tilde{\lambda}$ and T $(the\ distance\ \tilde{A}\tilde{X}\ and\ \tilde{\lambda}\tilde{X}\ in\ 0-cut)$ for

$\lambda = 376.4468\ and\ x = (0.0913, 0.4614, 1.0000)^t$

Models	Escribed		Peripheral		Approximate	
	$\tilde{\lambda}$	T	$\tilde{\lambda}$	T	$\tilde{\lambda}$	T
(3.16)	infeasible	infeasible	$\underline{\lambda} = 1.234568$ $\lambda = 376.4468$ $\overline{\lambda} = 459.3384$	841.8733	$\underline{\lambda} = 0.000000$ $\lambda = 376.4468$ $\overline{\lambda} = 1197.813$	103.3975
(3.20)	$\underline{\lambda} = -226.2718$ $\lambda = 376.4468$ $\overline{\lambda} = 1365.309$	262.2718	$\underline{\lambda} = 1.250812$ $\lambda = 376.4468$ $\overline{\lambda} = 459.3384$	841.8733	$\underline{\lambda} = -46.80566$ $\lambda = 376.4468$ $\overline{\lambda} = 1218.406$	82.80567
(3.25)	$\underline{\lambda} = 0.000000$ $\lambda = 376.4468$ $\overline{\lambda} = 1505.592$	0.000000	$\underline{\lambda} = 1.238309$ $\lambda = 376.4468$ $\overline{\lambda} = 459.3384$	841.8733	$\underline{\lambda} = 0.000000$ $\lambda = 376.4468$ $\overline{\lambda} = 1505.593$	0.000000

Therefore, the best solution in table 1 is the escribed eigenvalue $\left(T\left(\tilde{\lambda}_{\mathrm{TTFES}}, \tilde{X}_{\mathrm{TTFES}}\right) = 0.000000\right)$ of model (3.25).

Table 3

Values $\tilde{\lambda}$ and of T $\left(the\ distance\ \tilde{A}\tilde{X}\ and\ \tilde{\lambda}\tilde{X}\ in\ 0 - cut\right)$ for

$\lambda = 31.8520\ and\ x = (-1.5513, 0.7340, 1.0000)^t$

Models	Escribed		Peripheral		Approximate	
	$\tilde{\lambda}$	T	$\tilde{\lambda}$	T	$\tilde{\lambda}$	T
(3.16)	infeasible	infeasible	$\underline{\lambda} = 22.03726$ $\lambda = 31.8520$ $\overline{\lambda} = 60.00000$	1164.848	$\underline{\lambda} = 0.000000$ $\lambda = 31.8520$ $\overline{\lambda} = 512.6980$	712.1495
(3.20)	$\underline{\lambda} = -1000.000$ $\lambda = 31.8520$ $\overline{\lambda} = 1155.019$	1706.472	$\underline{\lambda} = 28.36870$ $\lambda = 31.8520$ $\overline{\lambda} = 60.00000$	1164.848	$\underline{\lambda} = -64.31428$ $\lambda = 31.8520$ $\overline{\lambda} = 516.5702$	708.2773
(3.25)	$\underline{\lambda} = 0.000000$ $\lambda = 31.8520$ $\overline{\lambda} = 1505.592$	0.000000	$\underline{\lambda} = 22.03726$ $\lambda = 31.8520$ $\overline{\lambda} = 69.41576$	1155.432	$\underline{\lambda} = 0.000000$ $\lambda = 31.8520$ $\overline{\lambda} = 512.6980$	712.1495

Therefore, the best solution in table 2 is the approximate eigenvalue $\left(T\left(\tilde{\lambda}_{\mathrm{ATFES}}, \tilde{X}_{\mathrm{ATFES}}\right) = 708.2773\right)$ of model (3.20).

Table 4

Values of $\tilde{\lambda}$ and T $\left(the\ distance\ \tilde{A}\tilde{X}\ and\ \tilde{\lambda}\tilde{X}\ in\ 0 - cut\right)$

for $\lambda = -534.2988\ and\ x = (0.0667, -5.2661, 1.0000)^t$

Models	Escribed		Peripheral		Approximate	
	$\tilde{\lambda}$	T	$\tilde{\lambda}$	T	$\tilde{\lambda}$	T

(3.16)	infeasible	infeasible	$\underline{\lambda} = -917.4524$ $\lambda = -534.2988$ $\overline{\lambda} = 1976.095$	1122.549	$\underline{\lambda} = -903.7074$ $\lambda = -534.2988$ $\overline{\lambda} = -258.8352$	1011.961
(3.20)	infeasible	infeasible	$\underline{\lambda} = -943.7568$ $\lambda = -534.2988$ $\overline{\lambda} = 0.000000$	1122.549	$\underline{\lambda} = -1205.017$ $\lambda = -534.2988$ $\overline{\lambda} = 179.1463$	943.4028
(3.25)	infeasible	infeasible	$\underline{\lambda} = -1127.641$ $\lambda = -534.2988$ $\overline{\lambda} = -534.2988$	2957.304	$\underline{\lambda} = -4974.252$ $\lambda = -534.2988$ $\overline{\lambda} = -534.2988$	2957.304

Therefore, the best solution in table 3 is the approximate eigenvalue $\left(T(\tilde{\lambda}_{\mathrm{ATFES}}, \tilde{X}_{\mathrm{ATFES}}) = 943.4028\right)$ of model (3.20).

5. Conclusion

In this paper, we suggested a novel method for finding eigenvalues and fuzzy triangular eigenvectors of a fuzzy triangular matrix $\tilde{A}$. In our approach, we solve the 1-cut of a fuzzy triangular matrix, then, we obtained 0-cut of eigenvalues and eigenvectors. In addition, we introduced three different eigenvalues namely, fuzzy escribed eigenvalue, fuzzy peripheral eigenvalue, and fuzzy approximate eigenvalue for when a fuzzy triangular matrix $(\tilde{A})$ does not have any suitable solution. Hence, our proposed method always has a solution, and the best solution is the shortest distance between $\tilde{A}\tilde{X}$ *and* $\tilde{\lambda}\tilde{X}$ in a 0-cut mode. However, our method is not cost-effective for large matrices because the volume of calculations is large. Finally, the numerical examples show that the methods are effective and applicable for obtaining the eigenvalues and fuzzy triangular eigenvectors of a fuzzy triangular matrix $\tilde{A}$.

Statements and Declarations

Funding: The authors declare that no funds, grants, or other support were received during the preparation of this manuscript.

Competing Interests: The authors have no relevant financial or non-financial interests to disclose.

Author Contributions: All authors contributed to the study conception and design. Material preparation, data collection and analysis were performed by Somaye khosravi, Ali Asghar Hosseinzadeh, Armin Ghane Kanafi and Amir Hosein Refahi Sheikhani. The first draft of the manuscript was written by Ali Asghar Hosseinzadeh and all authors commented on previous versions of the manuscript. All authors read and approved the final manuscript.

Data Availability: The datasets generated during and/or analyzed during the current study are not publicly available due to "no permission from the company from which the information was obtained" but are available from the corresponding author on reasonable request.

references

1. S. Abbasbandy, R. Ezzati, A. Jafarian, LU decomposition method for solving fuzzy system of linear equations, Appl. Math. Comput. 172(2006) 663-643.
2. T. Allahviranloo, Numerical methods for fuzzy system of linear equations, Appl. Math. Comput. 155(2004) 493-502.
3. T. Allahviranloo, Successive over relaxation iterative method for fuzzy system of linear equations, Appl. Math. Comput. 162 (2005) 189-196.
4. T. Allahviranloo, M. Ghanbari, Solving Fuzzy Linear Systems by Homotopy Perturbation Method, International Journal of Computational Cognition 8(2) (2010) 24-30.
5. T. Allahviranloo, A. A. Hosseinzade, M. Ghanbari, E. Haghi, R. Nuraei, On the new solutions for a fully fuzzy linear system, Soft Computing 18 (2014).
6. T. Allahviranloo, S. Salahshour, Fuzzy symmetric solutions of fuzzy linear systems, Appl. Math. Comput. 235(2011) 4545-4553.
7. T. Allahviranloo, S. Salahshour, M. Khezerloo, Maximal- and minimal symmetric solutions of fully fuzzy linear systems, Appl. Math. Comput. 235(2011) 4652-4662.
8. J.J. Buckley, Y. Qu, Solving system of linear fuzzy equations, Fuzzy Sets and Systems 43 (1991) 33-43.
9. J.J. Buckley, Y. Qu, Solving linear and quadratic fuzzy equations, Fuzzy Sets and Systems 38 (1990) 43-59.
10. J.J. Buckley, Y. Qu, Solving fuzzy equations: a new concept Fuzzy Sets and Systems 39 (1991) 291-301.
11. M. Dehghan, B. Hashemi, M. Ghatee, Computational methods for solving fully fuzzy linear systems, Appl. Math. Comput. 179 (2006) 328-343.
12. M. Dehghan, B. Hashemi, M. Ghatee, Solution of the fully fuzzy linear systems using iterative techniques, Chaos Solitons Fractals 34 (2007) 316-336.
13. D. Dubois, H. Prade, Fuzzy Sets and Systems: Theory and Applications, Academic Press, 1980.

14. M. Friedman, Ma. Ming, A. Kandel, Fuzzy linear system, Fuzzy Sets and Systems 96 (1998) 209-261.
15. M. Friedman, Ma. Ming, A. Kandel, Duality in fuzzy linear systems, Fuzzy Sets and Systems 109 (2000) 55-58.
16. R. Goetschel, W. Voxman, Elementary calculus, Fuzzy Sets and Systems 18 (1986) 31-43.
17. S. Muzzioli, H. Reynaerts, Fuzzy linear systems of the form $A_1x + B_1 = A_2x + B_2$, Fuzzy Sets and Systems 157 (2006) 939-951.
18. A. Vroman, G. Deschrijver, E.E. Kerre, Solving systems of linear fuzzy equations by parametric functions- an improved algorithm, Fuzzy Sets and Systems 14 (2007) 1515-1534.
19. K. Wang, B. Zhend, Inconsistent fuzzy linear systems, Appl. Math. Comput. 181 (2006) 973-981.
20. X. Wanga, Z. Zhong, M. Ha, Iteration algorithms for solving a system of fuzzy linear equations, Fuzzy Sets and System 119 (2001) 121-128.
21. B. Zheng, K. Wang, General fuzzy systems, Appl. Math. Comput. 181 (2006) 1276-1286.
22. C. Wu, M. Ming, On embedding problem of fuzzy number space: Part 1, Fuzzy Sets and Systems 44 (1991) 33-38.
23. H.J. Zimmermann, Fuzzy Sets Theory and Applications, Kluwer, Dorrecht, 1985.