A dynamic framework for updating approximations with increasing or decreasing objects in multi-granulation rough sets

Hong Wang (whdw218@163.com)
Shanxi Normal University https://orcid.org/0000-0002-3300-1638
Tao Guan
Shanxi Normal University

Research Article

Keywords: Multi-granulation rough sets, Knowledge discovery, Incremental updating, Approximations

Posted Date: August 24th, 2021

DOI: https://doi.org/10.21203/rs.3.rs-531454/v1

A dynamic framework for updating approximations with increasing or decreasing objects in multi-granulation rough sets

Hong Wang*, Jingtao Guan

School of Mathematics and Computer Science, Shanxi Normal University, Shanxi, Linfen, 041000, P.R. China

Abstract

The data we need to deal with is getting bigger and bigger in recent years, and the same happens to multi-granulation rough set, so updated schemes have been proposed with the variation of attributes or attribute values in multi-granulation rough sets, this paper puts forward a dynamic mechanism to update the approximations of multi-granulation rough sets when adding or deleting objects. Firstly, the relationships between the original approximations and updated approximations are explored when adding or deleting objects in multi-granulation rough sets, and the dynamic processes of updating optimistic and pessimistic multi-granulation rough approximations are proposed. Secondly, two corresponding dynamic algorithms are proposed to update the lower and upper approximations of optimistic and pessimistic multi-granulation rough sets. Finally, a great quantity of experiments had been implemented, and the results indicate that two dynamic algorithms proposed are more effective than the static algorithm.

Keywords: Multi-granulation rough sets, Knowledge discovery, Incremental updating, Approximations

1. Introduction

As a data analysing and processing theory, rough set theory, set up by Polish scientist Z.Pawlak in 1982 [1], had made great progress in both theory and application practice as the scientific research of intelligent computing. Rough set theory is an effective tool to deal with imprecise, inconsistent and incomplete information without any prior knowledge, and it had been used in data mining [2], knowledge discovery [3], machine learning [4-6] and so on. As the basic computation of rough set, calculating the lower approximation and the upper approximation is an essential step for knowledge discovery and attribute reduction.

With the development of big data, the data we need to deal with in recent years is updating constantly, however, the original method can not recognise updated knowledge in time, which causes a problem: How do we get efficient results from fast data updating? Many scholars have done a lot of research about it, and they found that the updated data are linked to the original data. Thereby we can obtain updated knowledge via the relationship between them

*Corresponding author

Email addresses: whdw218@163.com (Hong Wang), 1121208261@qq.com (Jingtao Guan)

and existing knowledge, a lot of time and space will be saved. There are three main situations as follows and we can obtain updated knowledge via set operation or matrix operation.

Updating data with the variation of objects. Shu et al. proposed the incremental method of dynamic feature selection by updating the dependency function [7]. An attribute reduction and incremental algorithm of decision rule based on the decision matrix were proposed by Fan et al. [8]. Zhang et al. came up with a dynamic neighborhood rough set model to deal with the dynamic change of numerical information system [9]. Chen put forward an incremental algorithm for data when objects are added or deleted [10] based on the variable precision rough set model. An incremental updating algorithm of approximate set based on the dominance relationship in ordered information system was proposed by Li et al. [11]. Considering the data analysis of dual universes, Hu et al. studied the dynamic updating approximations [12]. Luo et al. explored an efficient updating approach of probabilistic rough set with incremental objects [13].

Updating data with the variation of attributes. Hu et al. proposed two matrix-based incremental strategies, which can dynamically update the upper and lower approximations of each decision class of multi-granulation rough set based on dominance-based relationship [14]. Lang et al. studied the incremental mechanism of updating the upper and lower approximations with the change of attributes in the dynamic covering information system [15]. Based on the definition of relation matrix, diagonal matrix and cut matrix in multi-granulation rough set, Hu et al. put forward the matrix representation of upper and lower approximations in optimistic and pessimistic multi-granulation rough set, and then gave the dynamic updated approximations based on matrix [16]. Cheng proposed an efficient incremental algorithm for rough fuzzy approximations and applied it to attribute reduction [17]. Zhang et al. proposed the concept of basic vector induced by relational matrix for set-valued information system and then put forward the static and dynamic methods of rough approximations for set-valued information systems [18]. Li further put forward a calculated method of approximations based on the dominant matrix for dominance rough set [19]. Yang et al. proposed an approach for updating dynamic approximations in multi-granulation rough sets variation of granular structures [20].

Updating data with the variation of attribute values, aiming at the dynamic updating of attribute values in information system, Chen et al. first defined the concept of attribute values coarsening and refining, and designed an efficient dynamic algorithm of approximations [21], further discussed the incremental updating of approximations based on the dominance relationship [22]. Wang et al. put forward an incremental algorithm of attribute reduction based on the rough set theory [23]. An incremental algorithm of updating decision rules was proposed for inconsistent decision table by Chen et al. [24]. Li et al. proposed a fast method of approximations by using matrix operation when attribute values are updated dynamically in ordered information system [25]. Luo et al.come up with a fast algorithm for computing rough approximations in set-valued decision systems [26]. Zeng et al. proposed a dynamical updating method of fuzzy rough approximations for hybrid data under the variation of attribute values [27]. Hu et al. put forward a dynamic algorithm of updating approximations in multi-granulation rough sets by using approximations's monotonicity directly [28].

Most of the existing dynamic updated studies occurred under the situation of single granule, but as demanded, there is a situation of multiple granules that we should explore its dynamic updating, i.e., multi-granulation rough set. Professor Qian Yuhua first proposed the multi-granulation rough set [29], optimistic multi-granulation rough set and pessimistic multi-granulation rough set were studied, approximating the concept and granulating the universe by combining or intersecting attributes. multi-granulation rough set is very important, it can be applied to multiple contexts and produce multiple types of multi-granulation rough set, such as Incomplete multi-granulation rough set [30], Neighborhood-based multi-granulation rough sets [31], Multi-granulation decision-theoretic rough sets [32], Intuitionistic fuzzy multi-granulation rough sets [33], Variable precision multi-granulation decision-theoretic fuzzy rough sets [34], Local multi-granulation decision-theoretic rough sets [35], Generalized multi-granulation rough sets [36], and so on.

There are few studies on the multi-granulation rough set about dynamic updating, and when adding or deleting objects, there is no studies on multi-granulation rough set model. Thus this paper mainly focuses on this problem. For dynamic multi-granulation rough set models, the approximations will change with the change of objects in the universe, and sometimes objects of the target concept will change from the change of the universe, and it makes us not able to seek approximations by using monotonicity directly. Given this situation, two dynamic algorithms are given. We first explore the relationships between the original approximations and updating approximations and propose the dynamic process of updating the lower and upper approximations when adding or deleting objects, and then the corresponding dynamic algorithms are proposed in the multi-granulation rough set. Finally, the effectiveness of two algorithms is verified by experiments.

The rest of this paper is organized as follows. The basic concepts and properties of rough sets and multi-granulation rough sets are briefly introduced in Section 2. In section 3, the methods of updating the lower and upper approximations are proposed when adding or deleting objects. Then the static algorithm and two dynamic algorithms are given in Section 4. In section 5, we verify the effectiveness of the proposed dynamic algorithm experimentally. Finally, Section 6 concludes this paper.

2. Preliminaries

In this section, we first review some concepts and propositions of rough sets and multi-granulation rough sets.

2.1. Rough sets

Definition 1. [1] Give an information system $IS = \langle U, AT, V, f \rangle$, where $U = \{x_1, x_2, \ldots, x_n\}$ is a non-empty finite set of objects called universe; AT is a non-empty finite set of attributes, $a \in AT$ is called an attribute; $V = \bigcup_{a \in AT} V_a$ is a set of attributes values; where V_a is a non-empty set of values of attribute $a \in AT$, called the domain of a; $f : U \times AT \to V$ is an information function that maps an object in U to exactly one value in V_a such that $\forall a \in AT, x_i \in U, f(x_i, a) \in V_a$.

Definition 2. [1] Give an information system $IS = \langle U, AT, V, f \rangle$, Each subset of attributes $B \subseteq AT$ determines an indiscernibility relation R_B as follows:

$R_B = \{(x_i, y_i) \in U \times U : f(x_i, a) = f(y_i, a), \forall a \in B\}$.

R_B is an equivalence relation on U. The equivalence relation R_B partitions the universe U into a family of disjoint subsets called equivalence classes, the equivalence class including x_i with respect to B is denoted as $[x_i]_B = \{y \in U : (x_i, y) \in R_B\}$.

Definition 3. [1] Give an information system $IS = \langle U, AT, V, f\rangle$, For any $X \subseteq U$, R is an equivalence relation, two subsets of objects, called lower and upper approximations of X with respect to R, are difened as:

$\underline{R}(X) = \{x_i \in U : [x_i]_R \subseteq X\}$,

$\overline{R}(X) = \{x_i \in U : [x_i]_R \cap X \neq \emptyset\}$.

where $[x_i]_R = \{y \in U | (x_i, y) \in R\}$ is the R-equivalence class containing x_i.

If $\underline{R}(X) = \overline{R}(X)$, we say that X is a definable set; Otherwise, X is a rough set.

2.2. Multi-granulation rough sets(MGRS)

Definition 4. [29] Give an information system $IS = \langle U, AT, V, f\rangle$, $A_1, A_2, \ldots, A_m \subseteq AT$, and $\forall X \subseteq U$, the optimistic multi-granulation lower and upper approximations of the set X with respect to $A_1, A_2, \ldots, A_m$ are denoted by $\underline{\sum_{k=1}^m A_k}^O(X)$ and $\overline{\sum_{k=1}^m A_k}^O(X)$, respectively, where

$\underline{\sum\limits_{k=1}^m A_k}^O(X) = \{x_i \in U : [x_i]_{A_1} \subseteq X \vee [x_i]_{A_2} \subseteq X \vee \ldots \vee [x_i]_{A_m} \subseteq X\}$,

$\overline{\sum\limits_{k=1}^m A_k}^O(X) = \sim \underline{\sum\limits_{k=1}^m A_k}^O(\sim X)$.

Proposition 1. [29] Give an information system $IS = \langle U, AT, V, f\rangle$, $A_1, A_2, \ldots, A_m \subseteq AT$, and $\forall X \subseteq U$, we have the following properties:

$\overline{\sum\limits_{k=1}^m A_k}^O(X) = \{x_i \in U : [x_i]_{A_1} \cap X \neq \emptyset \wedge [x_i]_{A_2} \cap X \neq \emptyset \wedge \ldots \wedge [x_i]_{A_m} \cap X \neq \emptyset\}$.

Proposition 2. [29] Give an information system $IS = \langle U, AT, V, f\rangle$, $A_1, A_2, \ldots, A_m \subseteq AT$, and $X, Y \subseteq U$. If $X \subseteq Y$, the following properties hold:

(1) $\underline{\sum\limits_{k=1}^m A_k}^O(X) \subseteq \underline{\sum\limits_{k=1}^m A_k}^O(Y)$,

(2) $\overline{\sum\limits_{k=1}^m A_k}^O(X) \subseteq \overline{\sum\limits_{k=1}^m A_k}^O(Y)$.

Definition 5. [29] Give an information system $IS = \langle U, AT, V, f\rangle$, $A_1, A_2, \ldots, A_m \subseteq AT$, and $\forall X \subseteq U$, the pessimistic multi-granulation lower and upper approximations of the set X with respect to $A_1, A_2, \ldots, A_m$ are denoted by $\underline{\sum_{k=1}^m A_k}^P(X)$ and $\overline{\sum_{k=1}^m A_k}^P(X)$, respectively, where

$\underline{\sum\limits_{k=1}^m A_k}^P(X) = \{x_i \in U : [x_i]_{A_1} \subseteq X \wedge [x_i]_{A_2} \subseteq X \wedge \ldots \wedge [x_i]_{A_m} \subseteq X\}$,

$\overline{\sum\limits_{k=1}^m A_k}^P(X) = \sim \underline{\sum\limits_{k=1}^m A_k}^P(\sim X)$.

Proposition 3. [29] Give an information system $IS = \langle U, AT, V, f\rangle$, $A_1, A_2, \ldots, A_m \subseteq AT$, and $\forall X \subseteq U$, we have the following properties:

$\overline{\sum_{k=1}^{m} A_k}^P (X) = \{x_i \in U : [x_i]_{A_1} \cap X \neq \emptyset \vee [x_i]_{A_2} \cap X \neq \emptyset \vee \ldots \vee [x_i]_{A_m} \cap X \neq \emptyset\}$.

Proposition 4. [29] Give an information system $IS = \langle U, AT, V, f\rangle$, $A_1, A_2, \ldots, A_m \subseteq AT$, and $X, Y \subseteq U$. If $X \subseteq Y$, the following properties hold:

(1) $\underline{\sum_{k=1}^{m} A_k}^P (X) \subseteq \underline{\sum_{k=1}^{m} A_k}^P (Y)$,

(2) $\overline{\sum_{k=1}^{m} A_k}^P (X) \subseteq \overline{\sum_{k=1}^{m} A_k}^P (Y)$.

Table 1: **An information system**

U	a_1	a_2	a_3	a_4
x_1	1	1	1	3
x_2	3	1	1	1
x_3	2	1	1	3
x_4	3	2	1	3
x_5	2	3	2	2
x_6	1	2	1	1
x_7	1	3	2	3
x_8	3	3	2	2

Example 1. Give an information system in Table 1, $U = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$, $AT = \{a_1, a_2, a_3, a_4\}$, $A_1 = \{a_1\}$, $A_2 = \{a_2\}$, $A_3 = \{a_3\}$, $A_4 = \{a_4\}$, $X = \{x_3, x_4, x_5, x_7, x_8\}$. According to Definition 4 and Definition 5, we can calculate the optimistic multi-granulation lower and upper approximations, the pessimistic multi-granulation lower and upper approximations of the set X as follows:

$\underline{\sum_{k=1}^{4} A_k}^O (X) = \{x_3, x_5, x_7, x_8\}$,

$\overline{\sum_{k=1}^{4} A_k}^O (X) = \{x_1, x_3, x_4, x_5, x_7, x_8\}$,

$\underline{\sum_{k=1}^{4} A_k}^P (X) = \{x_5\}$,

$\overline{\sum_{k=1}^{4} A_k}^P (X) = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$.

3. Updating multi-granulation rough approximations with increasing or decreasing of objects

3.1 Updating multi-granulation rough approximations while increasing objects

In this subsection, we define the concept of optimistic multi-granulation rough set and pessimistic multi-granulation rough set in the new information system after adding objects and then discuss the relationship between the original approximations and updated approximations.

Definition 6. Give an information system $IS = \langle U, AT, V, f\rangle$, $A_1, A_2, \ldots, A_m \subseteq AT$, $U = \{x_1, x_2, \ldots, x_n\}$, $X \subseteq U$, adding n' new objects to U, assume that the new universe is $U' = U \cup U^+$, where $U^+ = \{x_{n+1}, x_{n+2}, \ldots, x_{n+n'}\}$.

$\forall X' \subseteq U'$, the optimistic multi-granulation lower and upper approximation of X' in the new universe are denoted by $\underline{\sum_{k=1}^{m} A_k}^{O\vee}(X')$ and $\overline{\sum_{k=1}^{m} A_k}^{O\vee}(X')$, respectively,

$$\underline{\sum_{k=1}^{m} A_k}^{O\vee}(X') = \{x_i \in U' : [x_i]_{A_1}^{\vee} \subseteq X' \vee [x_i]_{A_2}^{\vee} \subseteq X' \vee \ldots \vee [x_i]_{A_m}^{\vee} \subseteq X'\},$$

$$\overline{\sum_{k=1}^{m} A_k}^{O\vee}(X') = \{x_i \in U' : [x_i]_{A_1}^{\vee} \cap X' \neq \emptyset \wedge [x_i]_{A_2}^{\vee} \cap X' \neq \emptyset \wedge \ldots \wedge [x_i]_{A_m}^{\vee} \cap X' \neq \emptyset\}.$$

where $[x_i]_{A_k}^{\vee}$ is the equivalence class including x_i with respect to a granular A_k in the new universe.

Definition 7. Give an information system $IS = \langle U, AT, V, f\rangle$, $A_1, A_2, \ldots, A_m \subseteq AT$, $U = \{x_1, x_2, \ldots, x_n\}$, $X \subseteq U$, adding n' new objects to U, assume that the new universe is $U' = U \cup U^+$, where $U^+ = \{x_{n+1}, x_{n+2}, \ldots, x_{n+n'}\}$. $\forall X' \subseteq U'$, the pessimistic multi-granulation lower and upper approximation of X' in the new universe are denoted by $\underline{\sum_{k=1}^{m} A_k}^{P\vee}(X')$ and $\overline{\sum_{k=1}^{m} A_k}^{P\vee}(X')$, respectively,

$$\underline{\sum_{k=1}^{m} A_k}^{P\vee}(X') = \{x_i \in U' : [x_i]_{A_1}^{\vee} \subseteq X' \wedge [x_i]_{A_2}^{\vee} \subseteq X' \wedge \ldots \wedge [x_i]_{A_m}^{\vee} \subseteq X'\},$$

$$\overline{\sum_{k=1}^{m} A_k}^{P\vee}(X') = \{x_i \in U' : [x_i]_{A_1}^{\vee} \cap X' \neq \emptyset \vee [x_i]_{A_2}^{\vee} \cap X' \neq \emptyset \vee \ldots \vee [x_i]_{A_m}^{\vee} \cap X' \neq \emptyset\}.$$

where $[x_i]_{A_k}^{\vee}$ is the equivalence class including x_i with respect to a granular A_k in the new universe.

Theorem 1. Give an information system $IS = \langle U, AT, V, f\rangle$, $A_1, A_2, \ldots, A_m \subseteq AT$, and $\forall X \subseteq U$, the following results hold:

(1) $\underline{\sum_{k=1}^{m} A_k}^{O}(X) \supseteq \underline{\sum_{k=1}^{m} A_k}^{O\vee}(X)$;

(2) $\overline{\sum_{k=1}^{m} A_k}^{O}(X) \subseteq \overline{\sum_{k=1}^{m} A_k}^{O\vee}(X)$.

Proof. (1) $\forall x \in \underline{\sum_{k=1}^{m} A_k}^{O\vee}(X)$, by Definition 6, we have $\exists k \in \{1, 2, \ldots, m\}$, $[x]_{A_k}^{\vee} \subseteq X$. When adding objects, by Definition 6, we have $[x]_{A_k} \subseteq [x]_{A_k}^{\vee}$. Thus, $\exists k \in \{1, 2, \ldots, m\}$, $[x]_{A_k} \subseteq X$. According to Definition 4, $x \in \underline{\sum_{k=1}^{m} A_k}^{O}(X)$. Therefore, $\underline{\sum_{k=1}^{m} A_k}^{O}(X) \supseteq \underline{\sum_{k=1}^{m} A_k}^{O\vee}(X)$;

(2) $\forall x \in \overline{\sum_{k=1}^{m} A_k}^{O}(X)$, by Proposition 1, we have $\forall k \in \{1, 2, \ldots, m\}$, $[x]_{A_k} \cap X \neq \emptyset$. When adding objects, by Definition 6, we have $[x]_{A_k} \subseteq [x]_{A_k}^{\vee}$. Thus, $\forall k \in \{1, 2, \ldots, m\}$, $[x]_{A_k}^{\vee} \cap X \neq \emptyset$. According to Definition 6, $x \in \overline{\sum_{k=1}^{m} A_k}^{O\vee}(X)$. Therefore, $\overline{\sum_{k=1}^{m} A_k}^{O}(X) \subseteq \overline{\sum_{k=1}^{m} A_k}^{O\vee}(X)$.

Theorem 2. Give an information system $IS = \langle U, AT, V, f\rangle$, $A_1, A_2, \ldots, A_m \subseteq AT$, let $X^+ \subseteq U^+$, and $\forall X \subseteq U$, $X' \subseteq U'$, if $X' = X \cup X^+$, the following results hold:

(1) $\underline{\sum_{k=1}^{m} A_k}^{O\vee}(X) = \underline{\sum_{k=1}^{m} A_k}^{O}(X) - \Delta H_1$,

$\Delta H_1 = \{x_i \in \underline{\sum_{k=1}^{m} A_k}^{O}(X) : [x_i]_{A_k}^{\vee} \nsubseteq X, \forall k \in \{1, 2, \ldots, m\}\}$;

(2) $\underline{\sum_{k=1}^{m} A_k}^{O\vee}(X') = \underline{\sum_{k=1}^{m} A_k}^{O\vee}(X) \cup \Delta H_2$,

$\Delta H_2 = \bigcup\{[x_j]_{A_k}^{\vee} : [x_j]_{A_k}^{\vee} \subseteq X', x_j \in X^+\}$;

(3) $\overline{\sum_{k=1}^{m} A_k}^{O\vee}(X') = \overline{\sum_{k=1}^{m} A_k}^{O}(X) \cup \Delta H_3$,

$\Delta H_3 = \{x_i \in U' - \overline{\sum_{k=1}^{m} A_k}^{O}(X) : [x_i]_{A_k}^{\vee} \cap X' \neq \emptyset, \forall k \in \{1, 2, \ldots, m\}\}$.

Proof. (1) $\forall x \in \underline{\sum_{k=1}^{m} A_k}^{O}(X)$, by Definition 4, we have $\exists k \in \{1, 2, \ldots, m\}$, $[x]_{A_k} \subseteq X$. If $\exists k \in \{1, 2, \ldots, m\}$, $[x]_{A_k}^{\vee} \subseteq X$, then $x \in \underline{\sum_{k=1}^{m} A_k}^{O\vee}(X)$; If $\forall k \in \{1, 2, \ldots, m\}$, $[x]_{A_k}^{\vee} \nsubseteq X$, then $x \in \Delta H_1$. Therefore, $\underline{\sum_{k=1}^{m} A_k}^{O\vee}(X) = \underline{\sum_{k=1}^{m} A_k}^{O}(X) - \Delta H_1$.

(2) "⇒" $\forall x \in \underline{\sum_{k=1}^{m} A_k}^{O\vee}(X^{'})$, by Definition 6, we have $\exists k \in \{1, 2, \ldots, m\}$, $[x]_{A_k}^{\vee} \subseteq X^{'}$. If $\exists k \in \{1, 2, \ldots, m\}$, $[x]_{A_k}^{\vee} \subseteq X$, then $x \in \underline{\sum_{k=1}^{m} A_k}^{O\vee}(X)$. Otherwise, $\exists x_j \in X^{+}$, $[x]_{A_k}^{\vee} = [x_j]_{A_k}^{\vee}$, i.e., $x \in [x_j]_{A_k}^{\vee}$. Therefore, $\underline{\sum_{k=1}^{m} A_k}^{O\vee}(X^{'}) \subseteq \underline{\sum_{k=1}^{m} A_k}^{O\vee}(X) \cup \Delta H_2$.

"⇐" It is obvious that $\Delta H_2 \subseteq \underline{\sum_{k=1}^{m} A_k}^{O\vee}(X^{'})$ by Definition 6. By Proposition 2, $\underline{\sum_{k=1}^{m} A_k}^{O\vee}(X) \subseteq \underline{\sum_{k=1}^{m} A_k}^{O\vee}(X^{'})$. Therefore, $\underline{\sum_{k=1}^{m} A_k}^{O\vee}(X) \cup \Delta H_2 \subseteq \underline{\sum_{k=1}^{m} A_k}^{O\vee}(X^{'})$.

In conclusion, $\underline{\sum_{k=1}^{m} A_k}^{O\vee}(X^{'}) = \underline{\sum_{k=1}^{m} A_k}^{O\vee}(X) \cup \Delta H_2$.

(3) $\forall x \in \overline{\sum_{k=1}^{m} A_k}^{O\vee}(X^{'})$, by Definition 6, we have $\forall k \in \{1, 2, \ldots, m\}$, $[x]_{A_k}^{\vee} \cap X^{'} \neq \emptyset$. If $\forall k \in \{1, 2, \ldots, m\}$, $[x]_{A_k} \cap X \neq \emptyset$, then $x \in \overline{\sum_{k=1}^{m} A_k}^{O}(X)$; Otherwise, $x \notin \overline{\sum_{k=1}^{m} A_k}^{O}(X)$, i.e., $x \in \Delta H_3$. Therefore, $\overline{\sum_{k=1}^{m} A_k}^{O\vee}(X^{'}) = \overline{\sum_{k=1}^{m} A_k}^{O}(X) \cup \Delta H_3$.

Table 2: **An information system after adding objects**

$U^{'}$	a_1	a_2	a_3	a_4
x_1	1	1	1	3
x_2	3	1	1	1
x_3	2	1	1	3
x_4	3	2	1	3
x_5	2	3	2	2
x_6	1	2	1	1
x_7	1	3	2	3
x_8	3	3	2	2
x_9	1	1	2	2
x_{10}	3	3	1	1

Example 2. (Continuation of Example 1) The Table 2 is the expansion of Table 1, let $U^{+} = \{x_9, x_{10}\}$, $X^{+} = \{x_9\}$, $X^{'} = X \cup X^{+}$. In the new universe, according to Theorem 2 and the results of Example 1, we can calculate the optimistic lower approximation and the upper approximation of $X^{'}$ as follows:

$U^{'} = U \cup U^{+} = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}\}$,

$X^{'} = X \cup X^{+} = \{x_3, x_4, x_5, x_7, x_8, x_9\}$;

(1) $\underline{\sum_{k=1}^{4} A_k}^{O}(X) = \{x_3, x_5, x_7, x_8\}$,

$[x_3]_{A_1}^{\vee} \subseteq X,\ x_3 \notin \Delta H_1$;

$[x_5]_{A_1}^{\vee} \subseteq X, [x_5]_{A_2}^{\vee} \nsubseteq X,\ x_5 \notin \Delta H_1$;

$[x_7]_{A_1}^{\vee} \nsubseteq X, [x_7]_{A_2}^{\vee} \nsubseteq X, [x_7]_{A_3}^{\vee} \nsubseteq X, [x_7]_{A_4}^{\vee} \nsubseteq X,\ x_7 \in \Delta H_1$;

$[x_8]_{A_1}^{\vee} \nsubseteq X, [x_8]_{A_2}^{\vee} \nsubseteq X, [x_8]_{A_3}^{\vee} \nsubseteq X, [x_8]_{A_4}^{\vee} \nsubseteq X,\ x_8 \in \Delta H_1$;

Then, $\Delta H_1 = \{x_7, x_8\}$, $\underline{\sum_{k=1}^{4} A_k}^{O\vee}(X) = \underline{\sum_{k=1}^{4} A_k}^{O}(X) - \Delta H_1 = \{x_3, x_5\}$.

$[x_9]_{A_1}^{\vee} \nsubseteq X^{'}, [x_9]_{A_2}^{\vee} \nsubseteq X^{'}, [x_9]_{A_3}^{\vee} \subseteq X^{'}, [x_9]_{A_4}^{\vee} \subseteq X^{'}$;

Therefore, $\Delta H_2 = [x_9]^{\vee}_{A_3} \cup [x_9]^{\vee}_{A_4} = \{x_5, x_7, x_8, x_9\}$, $\underline{\sum_{k=1}^{4} A_k}^{O\vee}(X') = \underline{\sum_{k=1}^{4} A_k}^{O\vee}(X) \cup \Delta H_2 = \{x_3, x_5, x_7, x_8, x_9\}$.

(2) $\overline{\sum_{k=1}^{4} A_k}^{O}(X) = \{x_1, x_3, x_4, x_5, x_7, x_8\}$,

$[x_2]^{\vee}_{A_1} \cap X' \neq \emptyset$, $[x_2]^{\vee}_{A_2} \cap X' \neq \emptyset$, $[x_2]^{\vee}_{A_3} \cap X' \neq \emptyset$, $[x_2]^{\vee}_{A_4} \cap X' = \emptyset$, $x_2 \notin \Delta H_3$;

$[x_6]^{\vee}_{A_1} \cap X' \neq \emptyset$, $[x_6]^{\vee}_{A_2} \cap X' \neq \emptyset$, $[x_6]^{\vee}_{A_3} \cap X' \neq \emptyset$, $[x_6]^{\vee}_{A_4} \cap X' = \emptyset$, $x_6 \notin \Delta H_3$;

$[x_9]^{\vee}_{A_1} \cap X' \neq \emptyset$, $[x_9]^{\vee}_{A_2} \cap X' \neq \emptyset$, $[x_9]^{\vee}_{A_3} \cap X' \neq \emptyset$, $[x_9]^{\vee}_{A_4} \cap X' \neq \emptyset$, $x_9 \in \Delta H_3$;

$[x_{10}]^{\vee}_{A_1} \cap X' \neq \emptyset$, $[x_{10}]^{\vee}_{A_2} \cap X' \neq \emptyset$, $[x_{10}]^{\vee}_{A_3} \cap X' \neq \emptyset$, $[x_{10}]^{\vee}_{A_4} \cap X' = \emptyset$, $x_{10} \notin \Delta H_3$;

Therefore, $\Delta H_3 = \{x_9\}$, $\overline{\sum_{k=1}^{4} A_k}^{O\vee}(X') = \overline{\sum_{k=1}^{4} A_k}^{O}(X) \cup \Delta H_3 = \{x_1, x_3, x_4, x_5, x_7, x_8, x_9\}$.

Theorem 3. Give an information system $IS = \langle U, AT, V, f\rangle$, $A_1, A_2, \ldots, A_m \subseteq AT$, and $\forall X \subseteq U$, the following results hold:

(1) $\underline{\sum_{k=1}^{m} A_k}^{P}(X) \supseteq \underline{\sum_{k=1}^{m} A_k}^{P\vee}(X)$;

(2) $\overline{\sum_{k=1}^{m} A_k}^{P}(X) \subseteq \overline{\sum_{k=1}^{m} A_k}^{P\vee}(X)$.

Proof. (1) $\forall x \in \underline{\sum_{k=1}^{m} A_k}^{P\vee}(X)$, by Definition 7, we have $\forall k \in \{1, 2, \ldots, m\}$, $[x]^{\vee}_{A_k} \subseteq X$. When adding objects, by Definition 7, we have $[x]_{A_k} \subseteq [x]^{\vee}_{A_k}$. Thus, $\forall k \in \{1, 2, \ldots, m\}$, $[x]_{A_k} \subseteq X$. According to Definition 5, $x \in \underline{\sum_{k=1}^{m} A_k}^{P}(X)$. Therefore, $\underline{\sum_{k=1}^{m} A_k}^{P}(X) \supseteq \underline{\sum_{k=1}^{m} A_k}^{P\vee}(X)$;

(2) $\forall x \in \overline{\sum_{k=1}^{m} A_k}^{P}(X)$, by Proposition 3, we have $\exists k \in \{1, 2, \ldots, m\}$, $[x]_{A_k} \cap X \neq \emptyset$. When adding objects, by Definition 7, we have $[x]_{A_k} \subseteq [x]^{\vee}_{A_k}$. Thus, $\exists k \in \{1, 2, \ldots, m\}$, $[x]^{\vee}_{A_k} \cap X \neq \emptyset$. According to Definition 7, $x \in \overline{\sum_{k=1}^{m} A_k}^{P\vee}(X)$. Therefore, $\overline{\sum_{k=1}^{m} A_k}^{P}(X) \subseteq \overline{\sum_{k=1}^{m} A_k}^{P\vee}(X)$.

Theorem 4. Give an information system $IS = \langle U, AT, V, f\rangle$, $A_1, A_2, \ldots, A_m \subseteq AT$, let $X^+ \subseteq U^+$, and $\forall X \subseteq U$, $X' \subseteq U'$, if $X' = X \cup X^+$, the following results hold:

(1) $\underline{\sum_{k=1}^{m} A_k}^{P\vee}(X) = \underline{\sum_{k=1}^{m} A_k}^{P}(X) - \Delta H_4$,

$\Delta H_4 = \{x_i \in \underline{\sum_{k=1}^{m} A_k}^{P}(X) : [x_i]^{\vee}_{A_k} \nsubseteq X, \exists k \in \{1, 2, \ldots, m\}\}$;

(2) $\underline{\sum_{k=1}^{m} A_k}^{P\vee}(X') = \underline{\sum_{k=1}^{m} A_k}^{P\vee}(X) \cup \Delta H_5$,

$\Delta H_5 = \{x_j \in U' - \underline{\sum_{k=1}^{m} A_k}^{P\vee}(X) : [x_j]^{\vee}_{A_k} \subseteq X', \forall k \in \{1, 2, \ldots, m\}\}$;

(3) $\overline{\sum_{k=1}^{m} A_k}^{P\vee}(X') = \overline{\sum_{k=1}^{m} A_k}^{P}(X) \cup \Delta H_6$,

$\Delta H_6 = \bigcup\{[x_i]^{\vee}_{A_k} : [x_i]^{\vee}_{A_k} \cap X' \neq \emptyset, x_i \in U^+\}$.

Proof. (1) $\forall x \in \underline{\sum_{k=1}^{m} A_k}^{P}(X)$, by Definition 5, we have $\forall k \in \{1, 2, \ldots, m\}$, $[x]_{A_k} \subseteq X$. If $\forall k \in \{1, 2, \ldots, m\}$, $[x]^{\vee}_{A_k} \subseteq X$, then $x \in \underline{\sum_{k=1}^{m} A_k}^{P\vee}(X)$; If $\exists k \in \{1, 2, \ldots, m\}$, $[x]^{\vee}_{A_k} \nsubseteq X$, then $x \in \Delta H_4$. Therefore, $\underline{\sum_{k=1}^{m} A_k}^{P\vee}(X) = \underline{\sum_{k=1}^{m} A_k}^{P}(X) - \Delta H_4$.

(2) $\forall x \in \underline{\sum_{k=1}^{m} A_k}^{P\vee}(X')$, by Definition 7, we have $\forall k \in \{1, 2, \ldots, m\}$, $[x]^{\vee}_{A_k} \subseteq X'$. If $\forall k \in \{1, 2, \ldots, m\}$, $[x]^{\vee}_{A_k} \subseteq X$, then $x \in \underline{\sum_{k=1}^{m} A_k}^{P\vee}(X)$. If $\exists k \in \{1, 2, \ldots, m\}$, $[x]^{\vee}_{A_k} \nsubseteq X$, then $x \in \Delta H_5$. Therefore, $\underline{\sum_{k=1}^{m} A_k}^{P\vee}(X') = \underline{\sum_{k=1}^{m} A_k}^{P\vee}(X) \cup \Delta H_5$.

(3) "$\Rightarrow$" $\forall x \in \overline{\sum_{k=1}^{m} A_k}^{P\vee}(X')$, by Definition 7, we have $\exists k \in \{1, 2, \ldots, m\}$, $[x]^{\vee}_{A_k} \cap X' \neq \emptyset$. If $\exists k \in \{1, 2, \ldots, m\}$, $[x]_{A_k} \cap X \neq \emptyset$, then $x \in \overline{\sum_{k=1}^{m} A_k}^{P}(X)$; Otherwise, $\exists x_i \in U^+$, $[x]^{\vee}_{A_k} = [x_i]^{\vee}_{A_k}$, i.e., $x \in [x]^{\vee}_{A_k} = [x_i]^{\vee}_{A_k}$. Therefore, $\overline{\sum_{k=1}^{m} A_k}^{P\vee}(X') \subseteq \overline{\sum_{k=1}^{m} A_k}^{P}(X) \cup \Delta H_6$.

"$\Leftarrow$" It is obvious that $\Delta H_6 \subseteq \overline{\sum_{k=1}^{m} A_k}^{P\vee}(X^{'})$ by Definition 7. By Theorem 3 and Proposition 4, $\overline{\sum_{k=1}^{m} A_k}^{P}(X) \subseteq \overline{\sum_{k=1}^{m} A_k}^{P\vee}(X) \subseteq \overline{\sum_{k=1}^{m} A_k}^{P\vee}(X^{'})$. Therefore, $\overline{\sum_{k=1}^{m} A_k}^{P}(X) \cup \Delta H_6 \subseteq \overline{\sum_{k=1}^{m} A_k}^{P\vee}(X^{'})$.

In conclusion, $\overline{\sum_{k=1}^{m} A_k}^{P\vee}(X^{'}) = \overline{\sum_{k=1}^{m} A_k}^{P}(X) \cup \Delta H_6$.

Example 3. (Continuation of Example 1) The Table 2 is the expansion of Table 1, let $U^+ = \{x_9, x_{10}\}$, $X^+ = \{x_9\}$, $X^{'} = X \cup X^+$. In the new universe, according to Theorem 4 and the results of Example 1, we can calculate the pessimistic lower approximation and the upper approximation of $X^{'}$ as follows:

$U^{'} = U \cup U^+ = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}\}$,

$X^{'} = X \cup X^+ = \{x_3, x_4, x_5, x_7, x_8, x_9\}$;

(1) $\underline{\sum_{k=1}^{4} A_k}^{P}(X) = \{x_5\}$,

$[x_5]_{A_1}^{\vee} \subseteq X$, $[x_5]_{A_2}^{\vee} \nsubseteq X$, $x_5 \in \Delta H_4$;

Then, $\Delta H_4 = \{x_5\}$, $\underline{\sum_{k=1}^{4} A_k}^{P\vee}(X) = \underline{\sum_{k=1}^{4} A_k}^{P}(X) - \Delta H_4 = \emptyset$.

$[x_1]_{A_1}^{\vee} \nsubseteq X^{'}$, $x_1 \notin \Delta H_5$;

$[x_2]_{A_1}^{\vee} \nsubseteq X^{'}$, $x_2 \notin \Delta H_5$;

$[x_3]_{A_1}^{\vee} \subseteq X^{'}$, $[x_3]_{A_2}^{\vee} \nsubseteq X^{'}$, $x_3 \notin \Delta H_5$;

$[x_4]_{A_1}^{\vee} \nsubseteq X^{'}$, $x_4 \notin \Delta H_5$;

$[x_5]_{A_1}^{\vee} \subseteq X^{'}$, $[x_5]_{A_2}^{\vee} \nsubseteq X^{'}$, $x_5 \notin \Delta H_5$;

$[x_6]_{A_1}^{\vee} \nsubseteq X^{'}$, $x_6 \notin \Delta H_5$;

$[x_7]_{A_1}^{\vee} \nsubseteq X^{'}$, $x_7 \notin \Delta H_5$;

$[x_8]_{A_1}^{\vee} \nsubseteq X^{'}$, $x_8 \notin \Delta H_5$;

$[x_9]_{A_1}^{\vee} \nsubseteq X^{'}$, $x_9 \notin \Delta H_5$;

$[x_{10}]_{A_1}^{\vee} \nsubseteq X^{'}$, $x_{10} \notin \Delta H_5$;

Therefore, $\Delta H_5 = \emptyset$, $\underline{\sum_{k=1}^{4} A_k}^{P\vee}(X^{'}) = \underline{\sum_{k=1}^{4} A_k}^{P\vee}(X) \cup \Delta H_5 = \emptyset$.

(2) $\overline{\sum_{k=1}^{4} A_k}^{P}(X) = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$,

$[x_9]_{A_1}^{\vee} \cap X^{'} \neq \emptyset$, $[x_9]_{A_2}^{\vee} \cap X^{'} \neq \emptyset$, $[x_9]_{A_3}^{\vee} \cap X^{'} \neq \emptyset$, $[x_9]_{A_4}^{\vee} \cap X^{'} \neq \emptyset$;

$[x_{10}]_{A_1}^{\vee} \cap X^{'} \neq \emptyset$, $[x_{10}]_{A_2}^{\vee} \cap X^{'} \neq \emptyset$, $[x_{10}]_{A_3}^{\vee} \cap X^{'} \neq \emptyset$, $[x_{10}]_{A_4}^{\vee} \cap X^{'} = \emptyset$;

Therefore, $\Delta H_6 = [x_9]_{A_1}^{\vee} \cup [x_9]_{A_2}^{\vee} \cup [x_9]_{A_3}^{\vee} \cup [x_9]_{A_4}^{\vee} \cup [x_{10}]_{A_1}^{\vee} \cup [x_{10}]_{A_2}^{\vee} \cup [x_{10}]_{A_3}^{\vee} = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}\}$, $\overline{\sum_{k=1}^{4} A_k}^{P\vee}(X^{'}) = \overline{\sum_{k=1}^{4} A_k}^{P}(X) \cup \Delta H_6 = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}\}$.

3.2 Updating multi-granulation rough approximations while decreasing objects

In this subsection, we define the concept of optimistic multi-granulation rough set and pessimistic multi-granulation rough set in the new information system after deleting objects and then discuss the relationship between the original approximations and the updated approximations.

Definition 8. Give an information system $IS = \langle U, AT, V, f \rangle$, $A_1, A_2, \ldots, A_m \subseteq AT$, $U = \{x_1, x_2, \ldots, x_n\}$, $X \subseteq U$, deleting $n^{'}$ objects from U, assume that the new universe is $U^{'} = U - U^-$, where U^- is the set that $n^{'}$ objects was deleted from U. $\forall X^{'} \subseteq U^{'}$, the optimistic multi-granulation lower and upper approximation of $X^{'}$ in the new universe are denoted by $\underline{\sum_{k=1}^{m} A_k}^{O\wedge}(X')$ and $\overline{\sum_{k=1}^{m} A_k}^{O\wedge}(X')$, respectively,

$$\underline{\sum_{k=1}^{m} A_k}^{O\wedge}(X') = \{x_i \in U' : [x_i]^{\wedge}_{A_1} \subseteq X' \vee [x_i]^{\wedge}_{A_2} \subseteq X' \vee \ldots \vee [x_i]^{\wedge}_{A_m} \subseteq X'\},$$

$$\overline{\sum_{k=1}^{m} A_k}^{O\wedge}(X') = \{x_i \in U' : [x_i]^{\wedge}_{A_1} \cap X' \neq \emptyset \wedge [x_i]^{\wedge}_{A_2} \cap X' \neq \emptyset \wedge \ldots \wedge [x_i]^{\wedge}_{A_m} \cap X' \neq \emptyset\}.$$

where $[x_i]^{\wedge}_{A_k}$ is the equivalence class including x_i with respect to a granular A_k in the new universe.

Definition 9. Give an information system $IS = \langle U, AT, V, f\rangle$, $A_1, A_2, \ldots, A_m \subseteq AT$, $U = \{x_1, x_2, \ldots, x_n\}$, $X \subseteq U$, deleting n' objects from U, assume that the new universe is $U' = U - U^-$, where U^- is the set that n' objects was deleted from U. $\forall X' \subseteq U'$, the pessimistic multi-granulation lower and upper approximation of X' in the new universe are denoted by $\underline{\sum_{k=1}^{m} A_k}^{P\wedge}(X')$ and $\overline{\sum_{k=1}^{m} A_k}^{P\wedge}(X')$, respectively,

$$\underline{\sum_{k=1}^{m} A_k}^{P\wedge}(X') = \{x_i \in U' : [x_i]^{\wedge}_{A_1} \subseteq X' \wedge [x_i]^{\wedge}_{A_2} \subseteq X' \wedge \ldots \wedge [x_i]^{\wedge}_{A_m} \subseteq X'\},$$

$$\overline{\sum_{k=1}^{m} A_k}^{P\wedge}(X') = \{x_i \in U' : [x_i]^{\wedge}_{A_1} \cap X' \neq \emptyset \vee [x_i]^{\wedge}_{A_2} \cap X' \neq \emptyset \vee \ldots \vee [x_i]^{\wedge}_{A_m} \cap X' \neq \emptyset\}.$$

where $[x_i]^{\wedge}_{A_k}$ is the equivalence class including x_i with respect to a granular A_k in the new universe.

Theorem 5. Give an information system $IS = \langle U, AT, V, f\rangle$, $A_1, A_2, \ldots, A_m \subseteq AT$, and $\forall X' \subseteq U'$, the following results hold:

(1) $\underline{\sum_{k=1}^{m} A_k}^{O}(X') \subseteq \underline{\sum_{k=1}^{m} A_k}^{O\wedge}(X')$;

(2) $\overline{\sum_{k=1}^{m} A_k}^{O}(X') \supseteq \overline{\sum_{k=1}^{m} A_k}^{O\wedge}(X')$.

Proof. (1) $\forall x \in \underline{\sum_{k=1}^{m} A_k}^{O}(X')$, by Definition 4, we have $\exists k \in \{1, 2, \ldots, m\}$, $[x]_{A_k} \subseteq X'$. When deleting objects, by Definition 8, we have $[x]^{\wedge}_{A_k} \subseteq [x]_{A_k}$. Thus, $\exists k \in \{1, 2, \ldots, m\}$, $[x]^{\wedge}_{A_k} \subseteq X'$. According to Definition 8, $x \in \underline{\sum_{k=1}^{m} A_k}^{O\wedge}(X')$. Therefore, $\underline{\sum_{k=1}^{m} A_k}^{O}(X') \subseteq \underline{\sum_{k=1}^{m} A_k}^{O\wedge}(X')$.

(2) $\forall x \in \overline{\sum_{k=1}^{m} A_k}^{O\wedge}(X')$, by Definition 8, we have $\forall k \in \{1, 2, \ldots, m\}$, $[x]^{\wedge}_{A_k} \cap X' \neq \emptyset$. When deleting objects, by Definition 8, we have $[x]^{\wedge}_{A_k} \subseteq [x]_{A_k}$. Thus, $\forall k \in \{1, 2, \ldots, m\}$, $[x]_{A_k} \cap X' \neq \emptyset$. According to Proposition 1, $x \in \overline{\sum_{k=1}^{m} A_k}^{O}(X')$. Therefore, $\overline{\sum_{k=1}^{m} A_k}^{O}(X') \supseteq \overline{\sum_{k=1}^{m} A_k}^{O\wedge}(X')$.

Theorem 6. Give an information system $IS = \langle U, AT, V, f\rangle$, $A_1, A_2, \ldots, A_m \subseteq AT$, let $X^- = U^- \cap X$, and $\forall X' \subseteq U'$, $X \subseteq U$, if $X' = X - X^-$, the following results hold:

(1) $\underline{\sum_{k=1}^{m} A_k}^{O}(X') = \underline{\sum_{k=1}^{m} A_k}^{O}(X) - \Delta H_7$,

$\Delta H_7 = \{x_i \in \underline{\sum_{k=1}^{m} A_k}^{O}(X) : [x_i]_{A_k} \nsubseteq X', \forall k \in \{1, 2, \ldots, m\}\}$;

(2) $\underline{\sum_{k=1}^{m} A_k}^{O\wedge}(X') = \underline{\sum_{k=1}^{m} A_k}^{O}(X') \cup \Delta H_8$,

$\Delta H_8 = \{x_j \in X' - \underline{\sum_{k=1}^{m} A_k}^{O}(X') : [x_j]^{\wedge}_{A_k} \subseteq X', \exists k \in \{1, 2, \ldots, m\}\}$;

(3) $\overline{\sum_{k=1}^{m} A_k}^{O\wedge}(X') = \overline{\sum_{k=1}^{m} A_k}^{O}(X) - U^- - \Delta H_9$,

$\Delta H_9 = \{x_i \in \overline{\sum_{k=1}^{m} A_k}^{O}(X) - U^- - X' : [x_i]^{\wedge}_{A_k} \cap X' = \emptyset, \exists k \in \{1, 2, \ldots, m\}\}$.

Proof. (1) $\forall x \in \underline{\sum_{k=1}^{m} A_k}^{O}(X)$, by Definition 4, we have $\exists k \in \{1, 2, \ldots, m\}$, $[x]_{A_k} \subseteq X$. If $\exists k \in \{1, 2, \ldots, m\}$, $[x]_{A_k} \subseteq X'$, then $x \in \underline{\sum_{k=1}^{m} A_k}^{O}(X')$; If $\forall k \in \{1, 2, \ldots, m\}$, $[x]_{A_k} \nsubseteq X'$, then $x \in \Delta H_7$. Therefore, $\underline{\sum_{k=1}^{m} A_k}^{O}(X') = \underline{\sum_{k=1}^{m} A_k}^{O}(X) - \Delta H_7$.

(2) $\forall x \in \underline{\sum_{k=1}^{m} A_k}^{O\wedge}(X')$, by Definition 8, we have $\exists k \in \{1, 2, \ldots, m\}$, $[x]_{A_k}^{\wedge} \subseteq X'$. If $\exists k \in \{1, 2, \ldots, m\}$, $[x]_{A_k} \subseteq X'$, then $x \in \underline{\sum_{k=1}^{m} A_k}^{O}(X')$. If $\forall k \in \{1, 2, \ldots, m\}$, $[x]_{A_k} \nsubseteq X'$, because of $\underline{\sum_{k=1}^{m} A_k}^{O\wedge}(X') \subseteq X'$, then $x \in \Delta H_8$. Therefore, $\underline{\sum_{k=1}^{m} A_k}^{O\wedge}(X') = \underline{\sum_{k=1}^{m} A_k}^{O}(X') \cup \Delta H_8$.

(3) $\forall x \in \overline{\sum_{k=1}^{m} A_k}^{O}(X) - U^-$, by Proposition 1, we have $\forall k \in \{1, 2, \ldots, m\}$, $[x]_{A_k} \cap X \neq \emptyset$. If $\forall k \in \{1, 2, \ldots, m\}$, $[x]_{A_k}^{\wedge} \cap X' \neq \emptyset$, then $x \in \overline{\sum_{k=1}^{m} A_k}^{O\wedge}(X')$; If $\exists k \in \{1, 2, \ldots, m\}$, $[x]_{A_k}^{\wedge} \cap X' = \emptyset$, then $x \notin \overline{\sum_{k=1}^{m} A_k}^{O\wedge}(X')$, and because $X' \subseteq \overline{\sum_{k=1}^{m} A_k}^{O\wedge}(X')$, then $x \notin X'$, thus $x \in \Delta H_9$. Therefore, $\overline{\sum_{k=1}^{m} A_k}^{O\wedge}(X') = \overline{\sum_{k=1}^{m} A_k^{O}}(X) - U^- - \Delta H_9$.

Table 3: **An information system with deleting objects**

U'	a_1	a_2	a_3	a_4
x_1	1	1	1	3
x_2	3	1	1	1
x_3	2	1	1	3
x_4	3	2	1	3
x_5	2	3	2	2
x_7	1	3	2	3

Example 4. (Continuation of Example 1) The Table 3 is the reduction of Table 1, let $U^- = \{x_6, x_8\}$, $X^- = U^- \cap X$, $X' = X - X^-$. According to Theorem 6 and the results of Example 1, we can calculate the optimistic lower approximation and the upper approximation of X' in the new universe as follows:

$U' = U - U^- = \{x_1, x_2, x_3, x_4, x_5, x_7\}$,

$X^- = U^- \cap X = \{x_8\}$,

$X' = X - X^- = \{x_3, x_4, x_5, x_7\}$,

(1) $\underline{\sum_{k=1}^{4} A_k}^{O}(X) = \{x_3, x_5, x_7, x_8\}$;

$[x_3]_{A_1} \subseteq X'$, $x_3 \notin \Delta H_7$;

$[x_5]_{A_1} \subseteq X'$, $x_5 \notin \Delta H_7$;

$[x_7]_{A_1} \nsubseteq X'$, $[x_7]_{A_2} \nsubseteq X'$, $[x_7]_{A_3} \nsubseteq X'$, $[x_7]_{A_4} \nsubseteq X'$, $x_7 \in \Delta H_7$;

$[x_8]_{A_1} \nsubseteq X'$, $[x_8]_{A_2} \nsubseteq X'$, $[x_8]_{A_3} \nsubseteq X'$, $[x_8]_{A_4} \nsubseteq X'$, $x_8 \in \Delta H_7$;

Then, $\Delta H_7 = \{x_7, x_8\}$, $\underline{\sum_{k=1}^{4} A_k}^{O}(X') = \underline{\sum_{k=1}^{4} A_k}^{O}(X) - \Delta H_7 = \{x_3, x_5\}$.

$[x_4]_{A_1}^{\wedge} \nsubseteq X'$, $[x_4]_{A_2}^{\wedge} \subseteq X'$, $x_4 \in \Delta H_8$;

$[x_7]_{A_1}^{\wedge} \nsubseteq X'$, $[x_7]_{A_2}^{\wedge} \subseteq X'$, $x_7 \in \Delta H_8$;

Therefore, $\Delta H_8 = \{x_4, x_7\}$, $\underline{\sum_{k=1}^{4} A_k}^{O\wedge}(X') = \underline{\sum_{k=1}^{4} A_k}^{O}(X') \cup \Delta H_8 = \{x_3, x_4, x_5, x_7\}$.

(2) $\overline{\sum_{k=1}^{4} A_k}^{O}(X) = \{x_1, x_3, x_4, x_5, x_7, x_8\}$,

$[x_1]_{A_1}^{\wedge} \cap X' \neq \emptyset$, $[x_1]_{A_2}^{\wedge} \cap X' \neq \emptyset$, $[x_1]_{A_3}^{\wedge} \cap X' \neq \emptyset$, $[x_1]_{A_4}^{\wedge} \cap X' \neq \emptyset$, $x_1 \notin \Delta H_9$;

Therefore, $\Delta H_9 = \emptyset$, $\overline{\sum_{k=1}^{4} A_k}^{O\wedge}(X') = \overline{\sum_{k=1}^{4} A_k}^{O}(X) - U^- - \Delta H_9 = \{x_1, x_3, x_4, x_5, x_7\}$.

Theorem 7. Give an information system $IS = \langle U, AT, V, f\rangle$, $A_1, A_2, \ldots, A_m \subseteq AT$, and $\forall X' \subseteq U'$, the following results hold:

(1) $\underline{\sum_{k=1}^{m} A_k}^{P}(X') \subseteq \underline{\sum_{k=1}^{m} A_k}^{P\wedge}(X')$,

(2) $\overline{\sum_{k=1}^{m} A_k}^{P}(X') \supseteq \overline{\sum_{k=1}^{m} A_k}^{P\wedge}(X')$.

Proof. (1) $\forall x \in \underline{\sum_{k=1}^{m} A_k}^{P}(X')$, by Definition 5, we have $\forall k \in \{1, 2, \ldots, m\}$, $[x]_{A_k} \subseteq X'$. When deleting objects, by Definition 9, we have $[x]^{\wedge}_{A_k} \subseteq [x]_{A_k}$. Thus, $\forall k \in \{1, 2, \ldots, m\}$, $[x]^{\wedge}_{A_k} \subseteq X'$. According to Definition 9, $x \in \underline{\sum_{k=1}^{m} A_k}^{P\wedge}(X')$. Therefore, $\underline{\sum_{k=1}^{m} A_k}^{P}(X') \subseteq \underline{\sum_{k=1}^{m} A_k}^{P\wedge}(X')$.

(2) $\forall x \in \overline{\sum_{k=1}^{m} A_k}^{P\wedge}(X')$, by Definition 9, we have $\exists k \in \{1, 2, \ldots, m\}$, $[x]^{\wedge}_{A_k} \cap X' \neq \emptyset$. When deleting objects, by Definition 9, we have $[x]^{\wedge}_{A_k} \subseteq [x]_{A_k}$. Thus, $\exists k \in \{1, 2, \ldots, m\}$, $[x]_{A_k} \cap X' \neq \emptyset$. According to Proposition 3, $x \in \overline{\sum_{k=1}^{m} A_k}^{P}(X')$. Therefore, $\overline{\sum_{k=1}^{m} A_k}^{P}(X') \supseteq \overline{\sum_{k=1}^{m} A_k}^{P\wedge}(X')$.

Theorem 8. Give an information system $IS = \langle U, AT, V, f\rangle$, $A_1, A_2, \ldots, A_m \subseteq AT$, let $X^- = U^- \cap X$, and $\forall X' \subseteq U'$, $X \subseteq U$, if $X' = X - X^-$, the following results hold:

(1) $\underline{\sum_{k=1}^{m} A_k}^{P}(X') = \underline{\sum_{k=1}^{m} A_k}^{P}(X) - \Delta H_{10}$,

$\Delta H_{10} = \{x_i \in \underline{\sum_{k=1}^{m} A_k}^{P}(X) : [x_i]_{A_k} \nsubseteq X',\ \exists k \in \{1, 2, \ldots, m\}\}$;

(2) $\underline{\sum_{k=1}^{m} A_k}^{P\wedge}(X') = \underline{\sum_{k=1}^{m} A_k}^{P}(X') \cup \Delta H_{11}$,

$\Delta H_{11} = \{x_i \in X' - \underline{\sum_{k=1}^{m} A_k}^{P}(X') : [x_i]^{\wedge}_{A_k} \subseteq X',\ \forall k \in \{1, 2, \ldots, m\}\}$;

(3) $\overline{\sum_{k=1}^{m} A_k}^{P\wedge}(X') = \overline{\sum_{k=1}^{m} A_k}^{P}(X) - U^- - \Delta H_{12}$,

$\Delta H_{12} = \{x_i \in \overline{\sum_{k=1}^{m} A_k}^{P}(X) - U^- - X' : [x_i]^{\wedge}_{A_k} \cap X' = \emptyset,\ \forall k \in \{1, 2, \ldots, m\}\}$.

Proof. (1) $\forall x \in \underline{\sum_{k=1}^{m} A_k}^{P}(X)$, by Definition 5, we have $\forall k \in \{1, 2, \ldots, m\}$, $[x]_{A_k} \subseteq X$. If $\forall k \in \{1, 2, \ldots, m\}$, $[x]_{A_k} \subseteq X'$, then $x \in \underline{\sum_{k=1}^{m} A_k}^{P}(X')$; If $\exists k \in \{1, 2, \ldots, m\}$, $[x]_{A_k} \nsubseteq X'$, then $x \in \Delta H_{10}$. Therefore, $\underline{\sum_{k=1}^{m} A_k}^{P}(X') = \underline{\sum_{k=1}^{m} A_k}^{P}(X) - \Delta H_{10}$.

(2) $\forall x \in \underline{\sum_{k=1}^{m} A_k}^{P\wedge}(X')$, by Definition 9, we have $\forall k \in \{1, 2, \ldots, m\}$, $[x]^{\wedge}_{A_k} \subseteq X'$. If $\forall k \in \{1, 2, \ldots, m\}$, $[x]_{A_k} \subseteq X'$, then $x \in \underline{\sum_{k=1}^{m} A_k}^{P}(X')$. If $\forall k \in \{1, 2, \ldots, m\}$, $[x]_{A_k} \nsubseteq X'$, because of $\underline{\sum_{k=1}^{m} A_k}^{P\wedge}(X') \subseteq X'$, then $x \in \Delta H_{11}$. Therefore, $\underline{\sum_{k=1}^{m} A_k}^{P\wedge}(X') = \underline{\sum_{k=1}^{m} A_k}^{P}(X') \cup \Delta H_{11}$.

(3) $\forall x \in \overline{\sum_{k=1}^{m} A_k}^{P}(X) - U^-$, by Proposition 3, we have $\exists k \in \{1, 2, \ldots, m\}$, $[x]_{A_k} \cap X \neq \emptyset$. If $\exists k \in \{1, 2, \ldots, m\}$, $[x]^{\wedge}_{A_k} \cap X' \neq \emptyset$, then $x \in \overline{\sum_{k=1}^{m} A_k}^{P\wedge}(X')$; If $\forall k \in \{1, 2, \ldots, m\}$, $[x]_{A_k} \cap X' = \emptyset$, then $x \notin \overline{\sum_{k=1}^{m} A_k}^{P\wedge}(X')$, and because $X' \subseteq \overline{\sum_{k=1}^{m} A_k}^{P\wedge}(X')$, then $x \notin X'$, thus $x \in \Delta H_{12}$. Therefore, $\overline{\sum_{k=1}^{m} A_k}^{P\wedge}(X') = \overline{\sum_{k=1}^{m} A_k}^{P}(X) - U^- - \Delta H_{12}$.

Example 5. (Continuation of Example 1) The Table 3 is the reduction of Table 1, let $U^- = \{x_6, x_8\}$, $X^- = U^- \cap X$, $X' = X - X^-$. According to Theorem 8 and the results of Example 1, we can calculate the pessimistic lower approximation and the upper approximation of X' in the new universe as follows:

$U' = U - U^- = \{x_1, x_2, x_3, x_4, x_5, x_7\}$,

$X^- = U^- \cap X = \{x_8\}$,

$X' = X - X^- = \{x_3, x_4, x_5, x_7\}$;

(1) $\underline{\sum_{k=1}^{4} A_k}^{P}(X) = \{x_5\}$,

$[x_5]_{A_1} \subseteq X'$, $[x_5]_{A_2} \nsubseteq X'$, $x_5 \in \Delta H_{10}$;

Then, $\Delta H_{10} = \{x_5\}$, $\underline{\sum_{k=1}^{4} A_k}^{P}(X') = \underline{\sum_{k=1}^{4} A_k}^{P}(X) - \Delta H_{10} = \emptyset$.

$[x_3]^{\wedge}_{A_1} \subseteq X'$, $[x_3]^{\wedge}_{A_2} \nsubseteq X'$, $x_3 \notin \Delta H_{11}$;

$[x_4]^{\wedge}_{A_1} \nsubseteq X'$, $x_4 \notin \Delta H_{11}$;

$[x_5]^{\wedge}_{A_1} \subseteq X'$, $[x_5]^{\wedge}_{A_2} \subseteq X'$, $[x_5]^{\wedge}_{A_3} \subseteq X'$, $[x_5]^{\wedge}_{A_4} \subseteq X'$, $x_5 \in \Delta H_{11}$;

$[x_7]^{\wedge}_{A_1} \nsubseteq X'$, $x_7 \notin \Delta H_{11}$;

Therefore, $\Delta H_{11} = \{x_5\}$, $\underline{\sum_{k=1}^{4} A_k}^{P\wedge}(X') = \underline{\sum_{k=1}^{4} A_k}^{P}(X') \cup \Delta H_{11} = \{x_5\}$.

(2) $\overline{\sum_{k=1}^{4} A_k}^{P}(X) = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$,

$[x_1]^{\wedge}_{A_1} \cap X' \neq \emptyset$, $x_1 \notin \Delta H_{12}$;

$[x_2]^{\wedge}_{A_1} \cap X' \neq \emptyset$, $x_2 \notin \Delta H_{12}$;

Therefore, $\Delta H_{12} = \emptyset$, $\overline{\sum_{k=1}^{4} A_k}^{P\wedge}(X') = \overline{\sum_{k=1}^{4} A_k}^{P}(X) - U^{-} - \Delta H_{12} = \{x_1, x_2, x_3, x_4, x_5, x_7\}$.

4. The algorithms for updating multi-granulation approximations while adding or deleting objects

4.1. The static algorithm for computing multi-granulation approximations

In this subsection, we introduce a static algorithm for multi-granulation rough approximations, which is outlined in Algorithm 1 [20].

Algorithm 1: Static algorithm for computing multi-granulation rough approximations

Input: $IS = \langle U, AT, V, f\rangle$, X

Output: $\underline{\sum_{k=1}^{m} A_k}^{O}(X)$, $\overline{\sum_{k=1}^{m} A_k}^{O}(X)$, $\underline{\sum_{k=1}^{m} A_k}^{P}(X)$, $\overline{\sum_{k=1}^{m} A_k}^{P}(X)$

$\underline{\sum_{k=1}^{m} A_k}^{O}(X) = \emptyset$, $\overline{\sum_{k=1}^{m} A_k}^{O}(X) = \emptyset$, $\underline{\sum_{k=1}^{m} A_k}^{P}(X) = \emptyset$, $\overline{\sum_{k=1}^{m} A_k}^{P}(X) = \emptyset$
For each $x \in U$
    For k= 1 to m
        Compute $[x]_{A_k}$;
    End
End
For each $x \in U$
    For k= 1 to m
        If $[x]_{A_k} \subseteq X$ **then** $\underline{\sum_{k=1}^{m} A_k}^{O}(X) = \underline{\sum_{k=1}^{m} A_k}^{O}(X) \cup \{x\}$; break;
    End
End
For each $x \in U$
    int flag=1;
        For k= 1 to m
            If $[x]_{A_k} \cap X \neq \emptyset$ **then** flag=1;
            else flag=0; break;
        End
    If flag==1 **then** $\overline{\sum_{k=1}^{m} A_k}^{O}(X) = \overline{\sum_{k=1}^{m} A_k}^{O}(X) \cup \{x\}$;
End
For each $x \in U$
    int flag=1;
        For k= 1 to m
            If $[x]_{A_k} \subseteq X$ **then** flag=1;
            else flag=0; break;

25 **End**
26 **If** flag==1 **then** $\underline{\sum_{k=1}^{m} A_k}^{P}(X) = \underline{\sum_{k=1}^{m} A_k}^{P}(X) \cup \{x\}$;
27 **End**
28 **For each** $x \in U$
29 **For** k= 1 to m
30 **If** $[x]_{A_k} \cap X \neq \emptyset$ **then** $\overline{\sum_{k=1}^{m} A_k}^{P}(X) = \overline{\sum_{k=1}^{m} A_k}^{P}(X) \cup \{x\}$; break;
31 **End**
32 **End**
33 **Return** $\underline{\sum_{k=1}^{m} A_k}^{O}(X)$, $\overline{\sum_{k=1}^{m} A_k}^{O}(X)$, $\underline{\sum_{k=1}^{m} A_k}^{P}(X)$, $\overline{\sum_{k=1}^{m} A_k}^{P}(X)$.

For Algorithm 1, Step 1 initialize the approximations and its time complexity is $O(1)$; Steps 2-6 calculate the equivalence class according to Definition 2, and its time complexity is $O(m|U|^2)$; Steps 7-11 calculate the lower approximation of optimistic multi-granulation rough sets according to Definition 4, and its time complexity is $O(m|U|)$; Steps 12-19 calculate the upper approximation of optimistic multi-granulation rough sets according to Proposition 1, and its time complexity is $O(m|U|)$; Steps 20-27 calculate the lower approximation of pessimistic multi-granulation rough sets according to Definition 5, and its time complexity is $O(m|U|)$; Steps 28-32 calculate the upper approximation of pessimistic multi-granulation rough sets according to Proposition 4, and its time complexity is $O(m|U|)$. Hence, the total time complexity of Algorithm 1 is $O(m|U|^2)$.

4.2 The incremental algorithm for updating multi-granulation approximations while adding objects

In this subsection, we introduce an incremental algorithm for updating multi-granulation rough approximations while adding objects, which is outlined in Algorithm 2.

Algorithm 2: Incremental algorithm for updating multi-granulation rough approximations while adding objects

Input: $IS = \langle U, AT, V, f\rangle$, X, U^+, X^+, $\underline{\sum_{k=1}^{m} A_k}^{O}(X)$, $\overline{\sum_{k=1}^{m} A_k}^{O}(X)$, $\underline{\sum_{k=1}^{m} A_k}^{P}(X)$, $\overline{\sum_{k=1}^{m} A_k}^{P}(X)$
Output: $\underline{\sum_{k=1}^{m} A_k}^{O\vee}(X')$, $\overline{\sum_{k=1}^{m} A_k}^{O\vee}(X')$, $\underline{\sum_{k=1}^{m} A_k}^{P\vee}(X')$, $\overline{\sum_{k=1}^{m} A_k}^{P\vee}(X')$.

$U' = U \cup U^+$, $X' = X \cup X^+$, $\underline{\sum_{k=1}^{m} A_k}^{O\vee}(X') = \underline{\sum_{k=1}^{m} A_k}^{O}(X)$, $\overline{\sum_{k=1}^{m} A_k}^{O\vee}(X') = \overline{\sum_{k=1}^{m} A_k}^{O}(X)$, $\underline{\sum_{k=1}^{m} A_k}^{P\vee}(X') = \underline{\sum_{k=1}^{m} A_k}^{P}(X)$, $\overline{\sum_{k=1}^{m} A_k}^{P\vee}(X') = \overline{\sum_{k=1}^{m} A_k}^{P}(X)$.
For each $x \in \underline{\sum_{k=1}^{m} A_k}^{O\vee}(X')$
int flag=1;
For k= 1 to m
Compute $[x]^{\vee}_{A_k}$;
If $[x]^{\vee}_{A_k} \nsubseteq X$ **then** flag=1;
else flag=0; break;
End
If flag==1 **then** $\underline{\sum_{k=1}^{m} A_k}^{O\vee}(X') = \underline{\sum_{k=1}^{m} A_k}^{O\vee}(X') - \{x\}$;
End
For each $x \in X^+$
For k= 1 to m
Compute $[x]^{\vee}_{A_k}$;
If $[x]^{\vee}_{A_k} \subseteq X'$ **then** $\underline{\sum_{k=1}^{m} A_k}^{O\vee}(X') = \underline{\sum_{k=1}^{m} A_k}^{O\vee}(X') \cup [x]^{\vee}_{A_k}$;
End
End
For each $x \in U' - \overline{\sum_{k=1}^{m} A_k}^{O\vee}(X')$
int flag=1;

19 **For** k= 1 to m
20 Compute $[x]^{\vee}_{A_k}$;
21 **If** $[x]^{\vee}_{A_k} \cap X' \neq \emptyset$ **then** flag=1;
22 **else** flag=0; break;
23 **End**
24 **If** flag==1 **then** $\overline{\sum_{k=1}^{m} A_k}^{O\vee}(X') = \overline{\sum_{k=1}^{m} A_k}^{O\vee}(X') \cup \{x\}$;
25 **End**
26 **For each** $x \in \underline{\sum_{k=1}^{m} A_k}^{P\vee}(X')$
27 **For** k= 1 to m
28 Compute $[x]^{\vee}_{A_k}$;
29 **If** $[x]^{\vee}_{A_k} \nsubseteq X$ **then** $\underline{\sum_{k=1}^{m} A_k}^{P\vee}(X') = \underline{\sum_{k=1}^{m} A_k}^{P\vee}(X') - \{x\}$; break;
30 **End**
31 **End**
32 **For each** $x \in U' - \underline{\sum_{k=1}^{m} A_k}^{P\vee}(X')$
33 int flag=1;
34 **For** k= 1 to m
35 Compute $[x]^{\vee}_{A_k}$;
36 **If** $[x]^{\vee}_{A_k} \subseteq X'$ **then** flag=1;
37 **else** flag=0; break;
38 **End**
39 **If** flag==1 **then** $\underline{\sum_{k=1}^{m} A_k}^{P\vee}(X') = \underline{\sum_{k=1}^{m} A_k}^{P\vee}(X') \cup \{x\}$;
40 **End**
41 **For each** $x \in U^{+}$
42 **For** k= 1 to m
43 Compute $[x]^{\vee}_{A_k}$;
44 **If** $[x]^{\vee}_{A_k} \cap X' \neq \emptyset$ **then** $\overline{\sum_{k=1}^{m} A_k}^{P\vee}(X') = \overline{\sum_{k=1}^{m} A_k}^{P\vee}(X') \cup [x]^{\vee}_{A_k}$;
45 **End**
46 **End**
47 **Return** $\underline{\sum_{k=1}^{m} A_k}^{O\vee}(X')$, $\overline{\sum_{k=1}^{m} A_k}^{O\vee}(X')$, $\underline{\sum_{k=1}^{m} A_k}^{P\vee}(X')$, $\overline{\sum_{k=1}^{m} A_k}^{P\vee}(X')$

For Algorithm 2, Step 1 calculate the new universe and the new target concept, initialize the approximate sets and its time complexity is $O(1)$; Steps 2-16 calculate the lower approximation of optimistic multi-granulation rough sets according to Definition 6, and its time complexity is $O(m|U'|^2)$; Steps 17-25 calculate the upper approximation of optimistic multi-granulation rough sets according to Definition 6, and its time complexity is $O(m|U'|^2)$; Steps 26-40 calculate the lower approximation of pessimistic multi-granulation rough sets according to Definition 7, and its time complexity is $O(m|U'|^2)$; Steps 41-46 calculate the upper approximation of pessimistic multi-granulation rough sets according to Definition 7, and its time complexity is $O(m|U'|^2)$. Hence, the total time complexity of Algorithm 2 is $O(m|U'|^2)$, i.e., $O(m|U \cup U^{+}|^2)$, which is no more than Algorithm 1.

4.3 The incremental algorithm for updating multi-granulation approximations while deleting objects

In this subsection, we introduce an incremental algorithm for updating multi-granulation rough approximations while deleting objects, which is outlined in Algorithm 3.

Algorithm 3: Incremental algorithm for updating multi-granulation rough approximations while deleting objects

Input: $IS = \langle U, AT, V, f \rangle$, X, U^{-}, X^{-}, $\underline{\sum_{k=1}^{m} A_k}^{O}(X)$, $\overline{\sum_{k=1}^{m} A_k}^{O}(X)$, $\underline{\sum_{k=1}^{m} A_k}^{P}(X)$, $\overline{\sum_{k=1}^{m} A_k}^{P}(X)$
Output: $\underline{\sum_{k=1}^{m} A_k}^{O\wedge}(X')$, $\overline{\sum_{k=1}^{m} A_k}^{O\wedge}(X')$, $\underline{\sum_{k=1}^{m} A_k}^{P\wedge}(X')$, $\overline{\sum_{k=1}^{m} A_k}^{P\wedge}(X')$.

1 $U' = U - U^-$, $X' = X - X^-$, $\underline{\sum_{k=1}^m A_k}^{O\wedge}(X') = \underline{\sum_{k=1}^m A_k}^{O}(X)$, $\overline{\sum_{k=1}^m A_k}^{O\wedge}(X') = \overline{\sum_{k=1}^m A_k}^{O}(X)$, $\underline{\sum_{k=1}^m A_k}^{P\wedge}(X') = \underline{\sum_{k=1}^m A_k}^{P}(X)$, $\overline{\sum_{k=1}^m A_k}^{P\wedge}(X') = \overline{\sum_{k=1}^m A_k}^{P}(X)$.
2 **For each** $x \in \underline{\sum_{k=1}^m A_k}^{O\wedge}(X')$
3 int flag=1;
4 **For** k= 1 to m
5 Compute $[x]^{\wedge}_{A_k}$;
6 **If** $[x]^{\wedge}_{A_k} \nsubseteq X'$ **then** flag=1;
7 **else** flag=0; break;
8 **End**
9 **If** flag==1 **then** $\underline{\sum_{k=1}^m A_k}^{O\wedge}(X') = \underline{\sum_{k=1}^m A_k}^{O\wedge}(X') - \{x\}$; break;
10 **End**
11 **For each** $x \in X' - \underline{\sum_{k=1}^m A_k}^{O\wedge}(X')$
12 **For** k= 1 to m
13 Compute $[x]^{\wedge}_{A_k}$;
14 **If** $[x]^{\wedge}_{A_k} \subseteq X'$ **then** $\underline{\sum_{k=1}^m A_k}^{O\wedge}(X') = \underline{\sum_{k=1}^m A_k}^{O\wedge}(X') \cup \{x\}$;
15 **End**
16 **End**
17 **For each** $x \in \overline{\sum_{k=1}^m A_k}^{O\wedge}(X') - U^- - X'$
18 **For** k= 1 to m
19 Compute $[x]^{\wedge}_{A_k}$;
20 **If** $[x]^{\wedge}_{A_k} \cap X' = \emptyset$ **then** $\overline{\sum_{k=1}^m A_k}^{O\wedge}(X') = \overline{\sum_{k=1}^m A_k}^{O\wedge}(X') - U^- - \{x\}$; break;
21 **End**
22 **End**
23 **For each** $x \in \underline{\sum_{k=1}^m A_k}^{P\wedge}(X')$
24 **For** k= 1 to m
25 Compute $[x]^{\wedge}_{A_k}$;
26 **If** $[x]^{\wedge}_{A_k} \nsubseteq X'$ **then** $\underline{\sum_{k=1}^m A_k}^{P\wedge}(X') = \underline{\sum_{k=1}^m A_k}^{P\wedge}(X') - \{x\}$; break;
27 **End**
28 **End**
29 **For each** $x \in X' - \underline{\sum_{k=1}^m A_k}^{P\wedge}(X')$
30 int flag=1;
31 **For** k= 1 to m
32 Compute $[x]^{\wedge}_{A_k}$;
33 **If** $[x]^{\wedge}_{A_k} \subseteq X'$ **then** flag=1;
34 **else** flag=0; break;
35 **End**
36 **If** flag==1 **then** $\underline{\sum_{k=1}^m A_k}^{P\wedge}(X') = \underline{\sum_{k=1}^m A_k}^{P\wedge}(X') \cup \{x\}$;
37 **End**
38 **For each** $x \in \overline{\sum_{k=1}^m A_k}^{P\wedge}(X') - U^- - X'$
39 int flag=1;
40 **For** k= 1 to m
41 Compute $[x]^{\wedge}_{A_k}$;
42 **If** $[x]^{\wedge}_{A_k} \cap X' = \emptyset$ **then** flag=1;
43 **else** flag=0; break;
44 **End**
45 **If** flag==1 **then** $\overline{\sum_{k=1}^m A_k}^{P\wedge}(X') = \overline{\sum_{k=1}^m A_k}^{P\wedge}(X') - U^- - \{x\}$;
46 **End**
47 **Return** $\underline{\sum_{k=1}^m A_k}^{O\wedge}(X')$, $\overline{\sum_{k=1}^m A_k}^{O\wedge}(X')$, $\underline{\sum_{k=1}^m A_k}^{P\wedge}(X')$, $\overline{\sum_{k=1}^m A_k}^{P\wedge}(X')$

For Algorithm 3, Step 1 calculate the new universe and the new target concept, initialize the approximate sets, and its time complexity is $O(1)$; Steps 2-16 calculate the lower approximation of optimistic multi-granulation rough sets according to Definition 8, and its time complexity is $O(m|U'|^2)$; Steps 17-22 calculate the upper approximation of optimistic multi-granulation rough sets according to Definition 8, and its time complexity is $O(m|U'|^2)$; Steps 23-37 calculate the lower approximation of pessimistic multi-granulation rough sets according to Definition 9, and its time complexity is $O(m|U'|^2)$; Steps 38-46 calculate the upper approximation of pessimistic multi-granulation rough sets according to Definition 9, and its time complexity is $O(m|U'|^2)$. Hence, the total time complexity of Algorithm 3 is $O(m|U'|^2)$, i.e., $O(m|U - U^-|)$, which is no more than Algorithm 1.

5. Experimental evaluation and analysis

Many experiments were conducted to evaluate the performance of proposed incremental algorithms. We use 6 date sets available from UCI, the detailed information of these data sets is outlined in Table 4. All the experiments are implemented on a PC with Windows 10, AMD Ryzen5 3550H CPU, 2.10 GHz and 16 GB memory. The experimental comparisons between Algorithm 1 and Algorithms 2, Algorithm 1 and Algorithms 3 are made in two aspects, respectively. One is to compare the computational time with different sized data sets, the other is to compare the computational time with different updating ratios of the adding or deleting for objects.

Table 4: **The description of data sets**

No.	Data sets	Objects	Attributes
1	Breast	286	10
2	Balance	625	5
3	Solar	1389	13
4	Chess	3196	37
5	Mushroom	8124	23
6	Nursery	12960	8

5.1 Experiments with different sized data sets when adding objects or deleting objects

In this subsection, we compare the computational time of the static and incremental algorithms with the same updating ratio but with different sized data sets when adding objects. We assume that the updating ratio of the incremental or reduced objects is equal to 5%. Each data set in Table 5 is divided into 10 parts of equal size firstly. Based on the 10 equal sized sub-data sets, and the first part is viewed as the first basic data set, the combination of first part and second part is regarded as the second basic data set, and so on. While adding objects, for each of 10 different basic data sets, we randomly select 5% of the size of the basic data set from the next subset of data as the new data set to be inserted. While deleting objects, for each of 10 different basic data sets, we randomly select 5% of the size of the basic data set from the basic data set as the data set to be deleted.

By comparing the computational time of the static and incremental algorithms, we show the efficiency of proposed incremental algorithms, and the experimental results for computing approximations of the static and incremental algorithms while adding or deleting objects are listed in Tables 5 and 6, respectively. More detailed change trend lines of two proposed algorithms with the increasing size of the data sets are shown in Fig.1 and 2, respectively.

Table 5: **The comparison of static and incremental algorithms versus the size of added objects**

	Breast		Balance		Solar		Chess		Mushroom		Nursery	
NO.	Static	Incre.	Static	Incre.	Static	Incre.	Static	Incre.	Static	Incre.	Static	Incre.
1	0.112	0.039	0.233	0.042	0.713	0.121	4.534	0.494	11.540	1.143	7.715	0.850
2	0.252	0.045	0.396	0.067	1.366	0.169	10.707	0.635	32.352	1.741	8.765	1.321
3	0.338	0.059	0.493	0.069	2.194	0.215	19.587	0.929	57.341	2.079	16.513	1.425
4	0.405	0.055	0.641	0.096	3.350	0.243	33.600	1.075	49.075	2.372	23.339	1.737
5	0.498	0.058	0.780	0.104	4.443	0.257	33.904	0.927	62.718	2.329	33.817	2.142
6	0.513	0.053	0.964	0.115	5.822	0.293	35.181	1.239	88.771	2.651	46.082	2.657
7	0.625	0.082	1.104	0.135	7.082	0.367	43.418	1.784	117.439	3.289	59.327	3.332
8	0.720	0.073	1.298	0.142	8.237	0.389	47.542	1.590	149.876	3.575	74.426	3.901
9	0.851	0.065	1.518	0.146	11.262	0.491	51.219	1.967	184.151	4.156	90.745	4.437
10	0.903	0.076	1.710	0.164	13.533	0.574	59.941	1.995	218.581	4.560	110.957	4.982

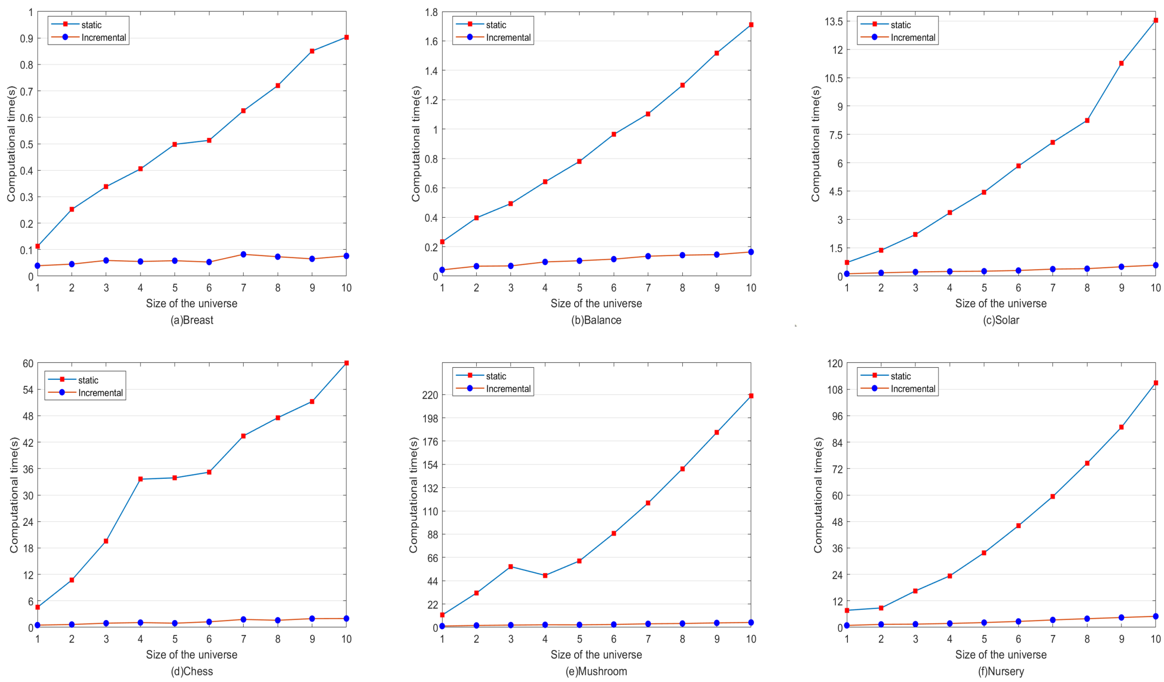

Figure 1: Computational time of static and incremental algorithms versus the size of added objects.

In each sub-figure of Fig.1 and 2, the x-coordinate is the size of the data set, the y-coordinate is the computational time, in addition, square and point lines denote the computational time of static and incremental algorithms,

Table 6: **The comparison of static and incremental algorithms versus the size of deleted objects**

	Breast		Balance		Solar		Chess		Mushroom		Nursery	
NO.	Static	Incre.	Static	Incre.	Static	Incre.	Static	Incre.	Static	Incre.	Static	Incre.
1	0.085	0.045	0.231	0.134	0.693	0.347	4.148	1.916	10.154	4.668	6.662	2.973
2	0.187	0.084	0.360	0.205	1.288	0.555	10.132	4.754	28.341	13.830	16.737	7.876
3	0.279	0.103	0.463	0.246	2.009	0.860	17.732	8.403	51.744	24.618	16.039	7.747
4	0.337	0.147	0.602	0.340	3.189	1.319	29.870	14.690	61.151	34.013	24.972	11.692
5	0.364	0.182	0.769	0.414	4.006	1.880	37.275	17.269	62.044	30.238	34.973	16.064
6	0.483	0.195	0.873	0.426	5.446	2.427	32.431	11.376	89.346	44.344	45.171	21.099
7	0.618	0.240	1.121	0.497	6.436	3.034	36.615	15.061	107.902	59.253	61.046	27.474
8	0.683	0.261	1.297	0.559	7.985	3.699	46.094	19.956	147.401	70.713	72.578	33.901
9	0.760	0.319	1.506	0.695	10.373	4.614	45.153	24.915	167.690	91.833	85.091	39.344
10	0.855	0.369	1.688	0.685	12.108	5.258	55.242	26.417	253.503	109.896	103.568	48.805

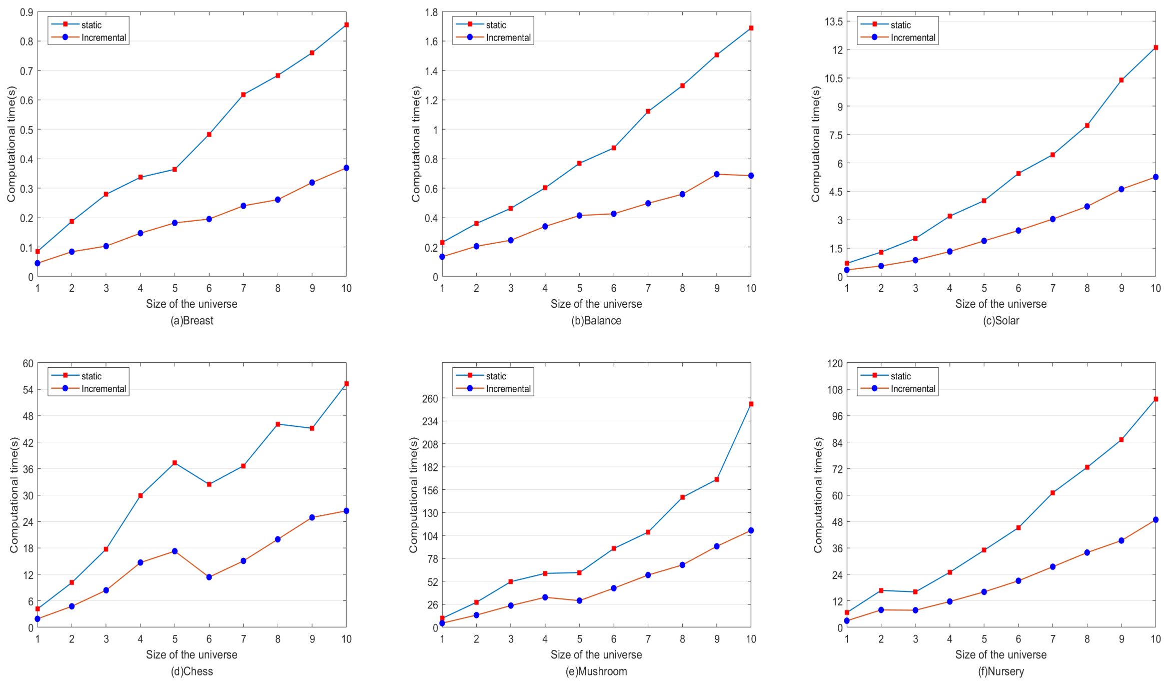

Figure 2: Computational time of static and incremental algorithms versus the size of deleted objects.

respectively. From Fig.1, it is easy to see the computation time of Algorithm 1 and Algorithm 2 usually increases with the increase of the size of data sets, while the proposed incremental algorithm is consistently faster than the static algorithm, and the differences of efficiency are profoundly larger when the size of the data sets increases. From Fig.2, it is easy to see the computation time of Algorithm 1 and Algorithm 3 usually increases with the increase of the size of data sets, while the proposed incremental algorithm is consistently faster than the static algorithm.

5.2 Experiments with different update rations of data sets when adding or deleting objects

In this subsection, we compare the computational time of the static and incremental algorithms with the same size of the basic data sets but with different updating ratios when adding or deleting objects. For the situation of adding objects, We select half of the universe randomly as the basic data sets, and then select R_a^i of the objects from the remaining 50% objects of the universe as the added objects, R_a^i is equal to 10%, 20%, . . ., 100%, respectively. For the situation of deleting objects, we select whole universe as the basic data sets, and then select R_a^i of the objects from the basic data sets as the deleted objects, R_a^i is equal to 10%, 20%, . . ., 90%, respectively. Algorithms 2 and 3 are applied to update the approximations, respectively.

By comparing the computational time of the static and incremental algorithms, we show the efficiency of proposed incremental algorithms, and the experimental results for computing approximations of the static and incremental algorithms while adding or deleting objects are listed in Tables 7 and 8, respectively. More detailed change trend lines of two proposed algorithms with the increasing size of the data sets are shown in Fig. 3 and 4, respectively.

Table 7: **The comparison of static and incremental algorithms versus the updating rations of added objects**

	Breast		Balance		Solar		Chess		Mushroom		Nursery	
NO.	Static	Incre.	Static	Incre.	Static	Incre.	Static	Incre.	Static	Incre.	Static	Incre.
1	0.412	0.117	0.870	0.193	4.896	0.962	36.793	7.029	98.507	17.244	42.701	8.057
2	0.501	0.192	1.070	0.326	5.490	1.890	35.280	9.570	112.667	33.899	59.677	16.957
3	0.531	0.254	1.118	0.422	6.155	2.753	39.778	16.161	125.729	53.827	60.791	22.318
4	0.639	0.321	1.199	0.555	6.856	3.804	41.056	27.417	138.285	70.254	70.967	32.738
5	0.671	0.358	1.307	0.756	7.584	4.810	40.320	25.279	151.907	83.778	82.254	40.761
6	0.754	0.467	1.271	0.749	8.469	6.014	42.591	30.225	168.730	95.654	88.581	48.797
7	0.835	0.569	1.442	0.913	9.247	6.946	54.640	41.294	168.863	113.764	93.026	64.467
8	0.843	0.650	1.479	1.028	11.863	9.504	59.906	49.553	182.185	133.639	99.211	72.585
9	0.883	0.726	1.567	1.143	12.513	10.347	67.339	57.147	202.941	172.190	112.888	85.562
10	0.914	0.996	1.680	1.303	14.948	11.730	80.699	71.612	218.138	186.150	126.808	107.642

Table 8: **The comparison of static and incremental algorithms versus the updating rations of deleted objects**

	Breast		Balance		Solar		Chess		Mushroom		Nursery	
NO.	Static	Incre.	Static	Incre.	Static	Incre.	Static	Incre.	Static	Incre.	Static	Incre.
1	0.824	0.413	1.640	0.711	11.195	5.275	63.688	31.734	200.307	88.628	98.763	45.732
2	0.734	0.387	1.364	0.576	8.627	3.794	50.901	24.277	154.009	80.044	97.667	39.605
3	0.727	0.334	1.122	0.457	7.185	3.218	44.443	20.893	133.876	64.992	75.115	34.136
4	0.550	0.323	1.005	0.435	5.545	2.680	30.674	17.629	103.105	51.651	62.277	29.929
5	0.479	0.299	0.731	0.348	4.534	2.031	35.032	14.685	72.416	36.850	44.945	22.278
6	0.407	0.259	0.589	0.273	3.312	1.426	33.109	14.601	52.367	25.079	30.884	15.973
7	0.315	0.167	0.486	0.240	2.308	1.027	22.072	9.678	44.052	28.147	21.062	10.301
8	0.256	0.143	0.376	0.133	1.447	0.649	12.432	5.529	26.338	16.277	16.507	9.002
9	0.113	0.089	0.202	0.086	0.657	0.300	5.758	1.907	12.638	5.132	8.544	3.274

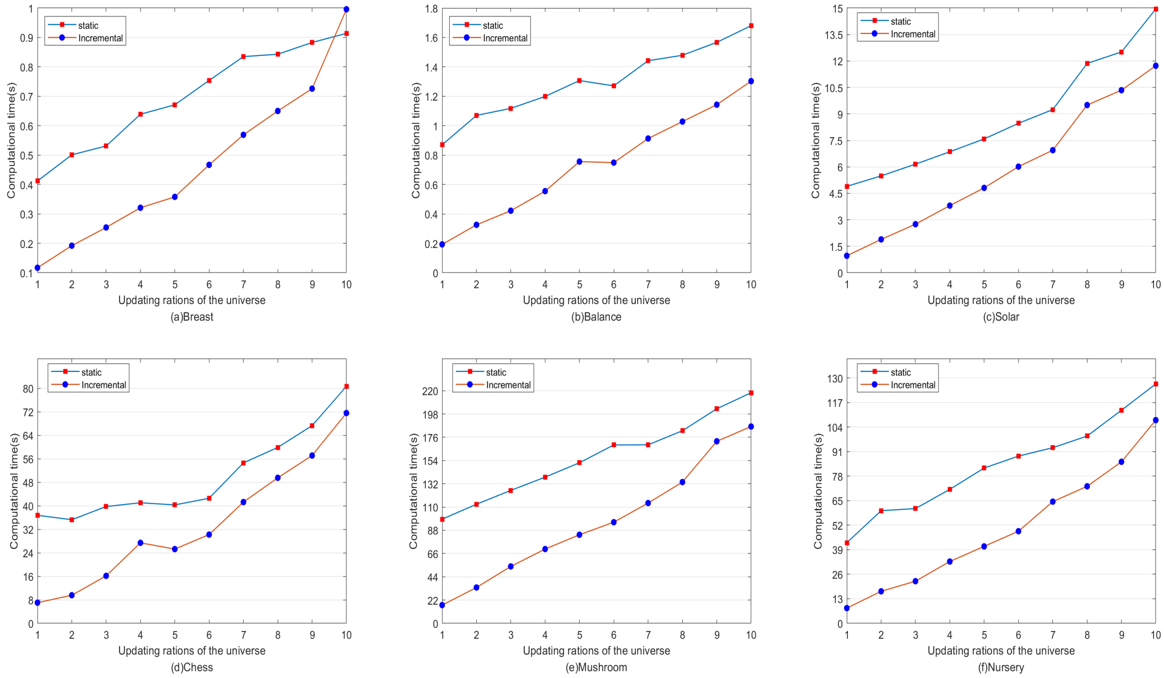

Figure 3: Computational time of static and incremental algorithms versus the updating rations of added objects.

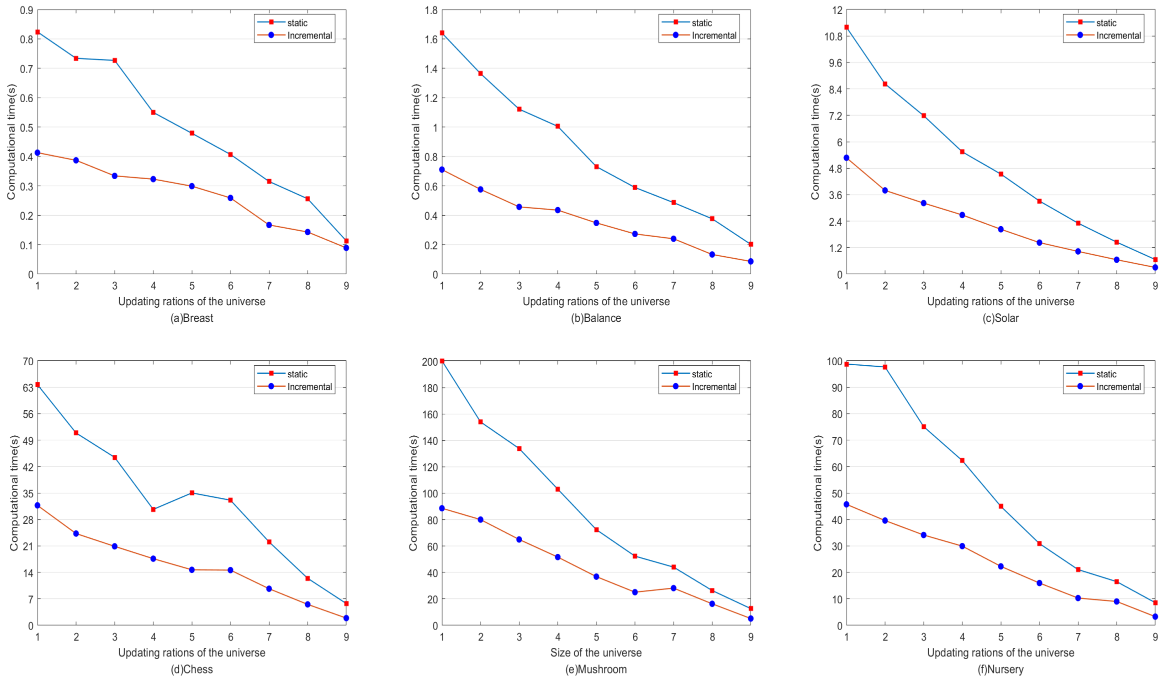

Figure 4: Computational time of static and incremental algorithms versus the updating rations of deleted objects.

In each sub-figure of Fig.3 and 4, the x-coordinate is the updating ratios of the data set, the y-coordinate is the computational time, in addition, square and point lines denote the computational time of static and incremental algorithms, respectively. From Fig.3, it is easy to see the computation time of Algorithm 1 and Algorithm 2 usually increases with the increase of the updating ratios of the data sets, while the proposed incremental algorithm is consistently faster than the static algorithm, and the differences of efficiency are profoundly larger when the updating ratios of the data sets decreases. From Fig.4, it is easy to see the computation time of Algorithm 1 and Algorithm 3 usually decreases with the increase of the updating ratios of data sets, while the proposed incremental algorithm is consistently faster than the static algorithm.

6. Conclusions

In this paper, incremental approximations are presented. With adding or deleting of objects, the approximations can be calculated incrementally without computing fully updated data set, they are used to design incremental algorithms for updating approximations. Our theoretical analysis guarantees the results of incremental algorithms are right, experimental studies using 6 UCI data sets shows that proposed incremental algorithms can significantly reduce computing time for calculating approximations with dynamic data. Further investigations are planned, such as tolerance-based multi-granulation incremental approximations, or multi-granulation incremental approximations with attributes and objects changed at the same time.

Acknowledgments This work is supported by the Nature Science Foundation of Shanxi Province(No. 201901D111280).

Compliance with ethical standards

Conflict of interest Author Hong Wang declares that he has no conflict of interest. Author Jingtao Guan declares that she has no conflict of interest.

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

References

[1] Z.Pawlak, Rough sets, International Journal of Computer and Information Sciences, 11(5)(1982) 341-356.

[2] V.S.Ananthanarayana, M.Narasimha Murty, D.K.Subramanian, Tree structure for efficient data mining using rough sets, Pattern Recognition Letters, 24(2003) 851-862.

[3] J.F.Peters, A.Skowron, A rough set approach to knowledge discovery, International Journal of Intelligent Systems, 17(2)(2002) 109-112.

[4] J.Stepaniuk, K.Kierzkowska, Hybrid classifier based on rough sets and neural networks, Electronic Notes in Theoretical Computer Science, 82(2003) 228-238.

[5] J.Stepaniu, Relational data and rough sets, Fundamenta Informaticae, 79(2007) 525-539.

[6] A.Skowron, J.Stepaniuk, R.Swiniarski, Approximation spaces in roughgranular computing, Fundamenta Informaticae, 100(2010) 141-157.

[7] W.H.Shu, W.B.Qian, Y.H.Xie, Incremental approaches for feature selection from dynamic data with the variation of multiple objects, Knowledge-Based Systems, 163(2019) 320-331.

[8] Y.Fan, T.Tseng, C.Chem, C.Huang, Rule induction based on an incremental rough set, Expert Systems with Applications, 36(9)(2009) 11439-11450.

[9] J.Zhang, T.R.Li, D.Ruan, D.Liu, Neighborhood rough sets for dynamic data mining, International Joumnal of Intelligence Systems, 27(2012) 317-342.

[10] H.M.Chen, T.R.Li, D.Ruan, J.H.Lin, C.X.Hu, A rough-set based incremental approach for updating approximations under dynamic maintenance environments, IEEE Trans-actions on Knowledge and Data Engineering, 25(2)(2013) 274-284.

[11] S.Y.Li, T.R.Li, D.Liu, Dynamic maintenance of approximations in dominance-basedrough set approach under the variation of the object set, International Journal of Intelligent Systems, 28(2013) 729-751.

[12] J.Hu, T.R.Li, H.M.Chen, A.P.Zeng, An incremental learning approach for updating approximations in rough set model over dual universes, International Journal of Intelligent Systems, 30(8)(2015) 923-947.

[13] C.Luo, T.R.Li, H.M.Chen, H.Fujita, Z.Yi, Efficient updating of probabilistic approximations with incremental objects, Knowledge-Based Systems, 109(2016) 71-83.

[14] C.X.Hu, L.Zhang, Efficient approaches for maintaining dominance-based multigranulation approximations with incremental granular structures, International Journal of Approximate Reasoning, 126(2020) 202-227.

[15] G.M.Lang, D.Q.Miao, M.J.Cai, Z.F.Zhang, Incremental approaches for updating reducts in dynamic covering information systems, Knowledge-Based Systems, 134(2017) 85-104.

[16] C.X.Hu, S.X.Liu, G.X.Liu, Matrix-based approaches for dynamic updating approximations in multigranulation rough sets, Knowledge-Based Systems, 122(2017) 51-63.

[17] Y.Cheng, The incremental method for fast computing the rough fuzzy approximations, Data & Knowledge Engineering, 70(1)(2011) 84-100.

[18] J.B.Zhang, T.R.Li, D.Ruan, D.Liu, Rough sets based matrix approaches with dynamic attribute variation in set-valued information systems, International Journal of Approximate Reasoning, 53(4)(2012) 620-635.

[19] S.Y.Li, T.R.Li, D.Liu, Incremental updating approximations in dominance-based rough sets approach under the variation of the attirbute set, Knowledge-Based Systems, 40(2013) 17-26.

[20] X.B.Yang, Y.Qi, H.L.Yu, X.N.Song, J.Y.Yang, Updating multigranulation rough approximations with increasing of granular structures, Knowledge-Based Systems, 64(2014) 59-69.

[21] H.M.Chen, T.R.Li, S.J.Qiao, A rough set based dynamic maintenance approach for approximations in coarsening and refining attribute values, International Journal of Intelligent Systems, 25(2010) 1005-1026

[22] H.M.Chen, T.R.Li, D.Ruan, Maintenance of approximations in incomplete ordered decision systems while attribute values coarsening or refining, Knowledge-Based Systems, 31(7)(2012) 140-161.

[23] F.Wang, J.Y.Liang, C.Y.Dang, Attribute reduction for dynamic data sets, Applied Soft Computing, 13(1)(2013) 676-689.

[24] H.M.Chen, T.R.Li, C.Luo, S.J.Horng, G.Y.Wang, A rough set-based method for updating decision rules on attribute values coarsening and refining, IEEE Transactions on Knowledge and Data Engineering, 26(12)(2014) 2886-2899.

[25] S.Y.Li, T.R.Li, Incremental update of approximations in dominance-based rough sets approach under the variation of attribute values, Information Sciences, 294(2015) 348-361.

[26] C.Luo, T.R.Li, H.M.Chen, L.X.Lu, Fast algorithms for computing rough approximations in set-valued decision systems while updating criteria values, Information Sciences, 299(2016) 221-242.

[27] A.P.Zeng, T.R.Li, J.Hu, H.M.Chen, C.Luo, Dynamical updating fuzzy rough approximations for hybrid data under the variation of attribute values, Information Sciences, 378(2017) 363-388.

[28] C.X.Hu, S.X.Liu, X.L.Huang, Dynamic updating approximations in multigranulation rough sets while refining or coarsening attribute values, Knowledge-Based Systems, 130(2017) 62-73.

[29] Y.H.Qian, J.Y.Liang, Y.Y.Yao, C.Y.Dang, MGRS: A multi-granulation rough set, Information Sciences, 180(2010) 949-970.

[30] Y.H.Qian, J.Y.Liang, C.Y.Dang, Incomplete multigranulation rough set, IEEE Transactions on Systems Man and Cybernetics Part A, 40(2)(2010) 420-431.

[31] G.P.Lin, Y.H.Qian, J.J.Li, NMGRS:Neighborhood-based multigranulation rough sets, International Journal of Approximate Reasoning, 53(7)(2012)1080-1093.

[32] Y.H.Qian, H.Zhang, Y.L.Sang, J.Y.Liang, Multigranulation decision-theoretic rough sets, International Journal of Approximate Reasoning, 55(1)(2014) 225-237.

[33] B.Huang, C.X.Guo, Y.L.Zhuang, H.X.Li, X.Z.Zhou, Intuitionistic fuzzy multigranulation rough sets, Information Sciences, 277(2014) 299-320.

[34] T.Feng, J.S.Mi, Variable precision multigranulation decision-theoretic fuzzy rough sets, Knowledge-Based Systems, 91(2016) 93-101.

[35] Y.H.Qian, X.Y.Liang, G.P.Lin, Q.Guo, J.Y.Liang, Local multigranulation decision-theoretic rough sets, International Journal of Approximate Reasoning, 82(2017) 119-137.

[36] W.H.Xu, W.T.Wang, X.T.Zhang, Generalized multigranulation rough sets and optimal granularity selection, Granular Computing, 2(2017) 271-288.