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Oscillation Criteria for Fractional Differential Equations with a Distributed Delay

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Abstract

This paper deals with obtaining some sufficient conditions for oscillation of high order neutral fractional integro-differential equations. The obtained results are mentioned for the first time in the literature for the oscillation of Caputo-Fabrizio fractional integro-differential equations. Finally, an illustrative example is given to verify our main results.

Keywords: Oscillation, Fractional Differential Equations, Caputo-Fabrizio Fractional Derivative, Distributed Delay

1 Introduction

The theory of fractional differential equations has become very important in recent years due to the increasing effects of their applications in various fields of science.For the fractional derivative, we refer the reader to the related books (Kilbas et al. 1996; Podlubny 1998; Samko et al. 1993). Other than the different areas of pure mathematics, fractional differential equations can be considered in modeling diverse areas of engineering and science such as self similar dynamical processes, viscoelasticity, fluid flows, electrochemistry, electromagnetic theory, control theory, and many other disciplines. To find out the details, see (Singh et al. 2019; Luo et al. 2018; Failla and Zingales 2020;

Sierociuk et al. 2013; Baleanu et al. 2020; Khan et al. 2019; Lazopoulos et al. 2016; Bushnaq et al. 2018; Owolabi 2019).

The Riemann-Liouville type is the most widely used definition for the fractional derivative and has many applications. But this type of fractional differential equation had some disadvantages. For example, the derivative of the constant function is not zero, and we need the initial values in practical examples. The disadvantages given above do not apply to the Caputo fractional derivative, and this is why it is considered one of the most influential definitions of fractional derivative. Therefore, it is applied in the fields of science and engineering. Following these studies, Caputo and Fabrizio introduced a new definition with all the characteristics of the old definitions (Caputo and Fabrizio 2015; Al-Refai and Pal 2019; Caputo and Fabrizio 2016). It supposes two different representations for the spatial and temporal variables. Caputo gives the classic definition, especially suitable for mechanical phenomena related to plasticity, fatigue, damage. They asserted that using Caputo-Fabrizio is more convenient than Caputo if such effects are not available. The essential advantage of the Caputo-Fabrizio method is the boundary conditions that admits the same form as for the integer-order differential equations.

It is well known that the oscillation theory is one of the most important topics for differential equations and dynamic equations. This theory first emerged thanks to the Sturm- Liouville theorems. Together with the spectral theory, these theorems have become quite worthy of the scientific world's attention. These days, there is a lack of studies on these topics in the literature (Zhang et al. 2012; Grace et al. 2010; Li et al. 2011; Öcalan and Öztürk 2015; Öcalan and Öztürk 2018; Öcalan et al. 2020; Cesarano and Bazighifan 2019; Tunc and Bazighifan 2019). In other words, in this regard, comprehensive studies have not been carried out yet. Furthermore, as it is known, the role of fractional differential equations in the theory of oscillation has started to increase considerably in the last few years (Yalçın Uzun et al. 2019; Yalçın Uzun 2020; Grace et al. 2012; Abdalla and Abdeljawad 2019; Bayram et al. 2016). In particular, many articles have appeared on fractional integrodifferential equations with Riemann-Liouville, Caputo and Caputo-Fabrizio types. Nonetheless, there are many open problems with developing the oscillation theory of fractional integrodifferential equations for such types (Aslıyüce et al. 2017; Feng and Sun 2022; Restrepo and Suragan 2020; Restrepo and Suragan 2021).

Cesarano and Bazighifan (2019) studied the oscillation of fourth-order functional differential equations with distributed delay

$$\left[\mu(x)\left(\eta'''(x)\right)^{\lambda}\right]' + \int_c^d \gamma(x,\xi) f\left(y\left(\delta(x,\xi)\right)\right) d\xi = 0,\ x \geq x_0.$$

Tunç and Bazighifan (2019) have considered the fourth order neutral differential equation with a continuously distributed delay

$$\left[\mu(x)\left(\eta'''(x)\right)^{\lambda}\right]' + \int_c^d \gamma(x,\xi) f\left(y\left(\delta(x,\xi)\right)\right) d\xi = 0, x \geq x_0.$$

Restrepo and Suragan (2020), studied a high order neutral differential fractional equation with a continuously distributed delay

$${}^{C}D_{a^+}^{\alpha,\psi}\left(\mu\left(.\right)\left(\left(\eta'''\left(.\right)\right)^{\lambda}\right)\right)(x) + \int_c^d \gamma(x,\xi) h\left(y\left(\delta\left(x,\xi\right)\right)\right) d\xi = 0$$

where ${}^{C}D_{a^+}^{\alpha,\psi}$ is Caputo fractional derivative with respect to another function. Restrepo and Suragan (2021) focused on a high order neutral differential fractional equation with a continuously distributed delay of the form

$${}^{C}D_{\theta,\beta,\omega;\,a^+}^{\alpha}\left(\mu\left(.\right)\left(\left(\eta'''\left(.\right)\right)^{\lambda}\right)\right)(x) + \int_c^d \gamma(x,\xi) h\left(y\left(\delta\left(x,\xi\right)\right)\right) d\xi = 0$$

where ${}^{C}D_{\theta,\beta,\omega;\,a^+}^{\alpha}$ is regularized Prabhakar derivative. And also they considered similar higher-order neutral differential equation with a continuously distributed delay for AtanganaBaleanu-Caputo fractional derivative operator.

$${}^{ABC}D_{a^+}^{\alpha}\left(\mu\left(.\right)\left(\left(\eta'''\left(.\right)\right)^{\lambda}\right)\right)(x) + \int_c^d \gamma(x,\xi) h\left(y\left(\delta\left(x,\xi\right)\right)\right) d\xi = 0.$$

This paper aims to obtain oscillation criteria for solutions of high order neutral Caputo-Fabrizio fractional differential equations with a distributed delay. For our proofs, we use techniques from the literature (Restrepo and Suragan 2020; Restrepo and Suragan 2021; Tunc and Bazighifan 2019; Cesarano and Bazighifan 2019). The paper organized as follows: In Section 2 we focus on some basic definitions and lemmas, and auxiliary results about fractional calculus. Section 3 is dedicated to obtain of oscillation criteria for the solutions of Caputo-Fabrizio fractional integrodifferential equations and a concrete example is given to conclude the paper.

2 Preliminaries

This section will briefly focus on some basic definitions and auxiliary results about fractional calculus.

Definition 1 (Caputo and Fabrizio 2015) Let x be a function $H^1(a,b)$, $b > a$ and $\lambda \in [0,1]$ then the Caputo-Fabrizio fractional derivative (CFD) of order λ is defined as

$${}^{CF}D_a^{\alpha} x(t) = \frac{M(\alpha)}{1-\alpha} \int_a^t x'(\tau) \exp\left[-\frac{\alpha(t-\tau)}{1-\alpha}\right] d\tau \tag{1}$$

where $M(\alpha)$ is normalization function such that $M(0) = M(1) = 1$.

Definition 2 Let $a, b, \alpha \in \mathbb{R}$ such that $0 < \alpha < 1$. The Caputo-Fabrizio fractional integral of order α of a function $x \in H^1[a, b]$ is a linear operator defined by

$$^{CF}I_a^\alpha x(t) = \frac{(1-\alpha)}{M(\alpha)} x(t) + \frac{\alpha}{M(\alpha)} \int_a^t x(\tau) d\tau. \tag{2}$$

Lemma 1 (Nchama 2020) Let $x \in H^1(a, b)$, $a < b$ and $0 < \alpha < 1$. Then

$$^{CF}I_a^\alpha \left(^{CF}D_a^\alpha \right) x(t) = x(t) - x(a). \tag{3}$$

Losada and Nieto (2015) modified CFD as

$$^{CF}\mathcal{D}_a^\alpha = \frac{(2-\alpha)M(\alpha)}{2(1-\alpha)} \int_a^t x'(\tau) \exp\left[-\frac{\alpha(t-\tau)}{1-\alpha} \right] d\tau \tag{4}$$

and defined the fractional integral associated to the CFD as

Definition 3 Assume $0 < \lambda < 1$. The fractional integral of order λ of a function is defined by

$$^{CF}\mathcal{I}_a^\alpha x(t) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} x(t) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_a^t x(\tau) d\tau, \, t \geq 0. \tag{5}$$

We consider high order neutral fractional integro-differential equation

$$^{CF}D_a^\alpha \left(\mu(.) \left((\eta'''(.))^\lambda \right) \right)(x) + \int_c^d \gamma(x, \xi) h\left(y\left(\delta\left(x, \xi \right) \right) \right) d\xi = 0 \tag{6}$$

with the condition

$$\int_a^{+\infty} \frac{dx}{\mu^{1/\lambda}(x)} = +\infty \tag{7}$$

where $a \geq 0$, $h : \mathbb{R}^+ \to \mathbb{R}^+$, $0 < \alpha < 1$ and

$$\eta(x) = y(x) + \int_{x_0}^{x_1} \rho\left(x, \tau \right) y\left(\omega\left(x, \tau \right) \right) d\tau,$$

and the conditions below are met.

(i) λ is a quotient of odd positive integers.
(ii) $\mu(x) \in C\left([a, +\infty), \mathbb{R}^+ \right)$ and $\mu'(x) \geq 0$.
(iii) $\rho(x, \tau), \omega(x, \tau) \in C\left([a, +\infty) \times [x_0, x_1], \mathbb{R}^+ \right)$,
(iv) $\gamma(x, \xi), \delta(x, \xi) \in C\left([a, +\infty) \times [c, d], \mathbb{R}^+ \right)$, $\delta(x, \xi)$ is nondecreasing in ξ, $a \leq \delta(x, \xi) \leq x$ for any $x \in [a, +\infty)$.
(v) for any $x \neq 0$ and $\kappa > 0$, $h(x) \geq \kappa x^\lambda$.

A solution $y(x)$ of (6), is a nontrivial real function in $C\left([\tilde{x},+\infty),\mathbb{R}\right)$, $\tilde{x}\geq a$ and provides the Eq. (6) on $[\tilde{x},+\infty)$, such that $\eta(x)$, $\eta'(x)$, $\eta''(x)$, $\mu(x)\left(\eta'''(x)\right)^{\lambda}\in C\left([\tilde{x},+\infty)\right)$.

Lemma 2 Let $z\in C^{m}\left([a,\infty),\mathbb{R}^{+}\right)$ be a function with $z^{(m)}(x)\leq 0$ for $x\in[a,\infty)$ and not identically zero on any interval $[b,+\infty)$, $b\geq a$. Then there exists an integer k, $0\leq k\leq m-1$, with $k+m$ odd and such that for some $b\geq a$, we have

$$\begin{aligned}(-1)^{k+i}z^{(i)}&>0 \text{ on } [b,+\infty)\ ,\ i=1,\ldots,m-1,\\ z^{(j)}&>0 \text{ on } [b,+\infty)\ ,\ j=1,\ldots,k-1,\ \text{ when } k>1.\end{aligned}$$

Lemma 3 If h satisfies $h^{(k)}(x)>0$ and $h^{(m+1)}(x)<0$ for any $k=0,1,\ldots,m$ and $x\in[a,\infty)$, then

$$kh(x)\geq xh'(x),\ x\in[a,\infty).$$

Lemma 4 Let $h\in C^{m}\left([a,\infty)\right)$. Assume that $h^{(m)}$ is of fixed sign and not identically zero on $[a,+\infty)$. If $h(x)>0$, $h^{(m-1)}(x)h^{(m)}(x)\leq 0$ and $\lim_{x\to+\infty}f(x)\neq 0$, then for every $\alpha\in(0,1)$, there exist $x_{\alpha}\geq a$ such that

$$h(x)\geq\frac{\alpha}{(m-1)!}x^{m-1}|h^{(m-1)}|,\ x\in[x_{\alpha},+\infty).$$

3 Main Results

Theorem 1 *Let λ be a quotient of odd positive integers. Suppose that the conditions (7) and (I)-(V) satisfy and $\alpha\in(0,1)$ and $0<\nu\leq 1/3$. For arbitrarily large values of x_{ν} such that $x_{\nu}>a$ and $x_{\nu}\leq\delta(x,c)\leq x$, if the following equation holds*

$$\limsup_{x\to+\infty}{}^{CF}I_{a}^{\alpha}\left(\frac{\delta^{\lambda/\nu}(x,c)}{x^{\lambda/\nu}}\bar{\gamma}(x)\right)=+\infty,\ x\in[x_{\nu},+\infty), \tag{8}$$

where $\bar{\gamma}(x)=\int_{c}^{d}\gamma(x,\xi)d\xi$, then all the solutions of Eq. (6) are oscillatory.

Proof From the supposition (8), we have

$$\limsup_{x\to+\infty}{}^{CF}I_{a}^{\alpha}\left(\frac{\delta^{3\lambda}(x,c)}{x^{3\lambda}}\bar{\gamma}(x)\right)=+\infty, \tag{9}$$

since $3\lambda\leq\lambda/\nu$ and $\delta(x,c)/x\leq 1$ for any $x\in[a,+\infty)$. Now, we consider that $y(x)$ is an non-oscillatory solution of Eq. (6) over $[a,+\infty)$. With no loos of generality, we suppose that $y(x)$ is eventually positive. In view of (6) with the initial conditions, we obtain

$${}^{CF}D_{a}^{\alpha}\left(\mu\left(.\right)\left(\left(\eta'''\left(.\right)\right)^{\lambda}\right)\right)(x)=-\int_{c}^{d}\gamma(x,\xi)h\left(y\left(\delta\left(x,\xi\right)\right)\right)d\xi<0.$$

For any $\xi\geq a$, the above equation implies that $\left(\mu(\xi)\left(\eta'''(\xi)\right)^{\lambda}\right)'<0$. Let us assume $\left(\mu(\xi)\left(\eta'''(\xi)\right)^{\lambda}\right)'\geq 0$. Then multiplying by some positive additional terms and integrating we can obtain ${}^{CF}D_a^{\alpha}\left(\mu\left(.\right)\left(\left(\eta'''\left(.\right)\right)^{\lambda}\right)\right)(x) \geq 0$. So this leads us to a contradiction. For any $\xi \geq a$, we have $\mu(\xi)\left(\eta'''(\xi)\right)^{\lambda} \leq \mu(a)\left(\eta'''(a)\right)^{\lambda}$. We prove that $\eta'''(\xi) > 0$ holds for any $\xi \geq a$ under the condition (7). Thus, we assume the contrary, i.e. $\eta'''(\xi) < 0$. Hence, it is obvious that $\eta''(\xi)$ is decreasing. In addition, we can write

$$\eta''(\xi) - \eta''(a) = \int_a^{\xi} \eta'''(t)dt \leq \mu^{1/\lambda}(a)\eta'''(a)\int_a^{\xi}\frac{dt}{\mu^{1/\lambda}(t)},$$

or what is the same as

$$\eta''(a) - \eta''(\xi) \geq \mu^{1/\lambda}(a)\left(-\eta'''(a)\right)\int_a^{\xi}\frac{dt}{\mu^{1/\lambda}(t)}.$$

Letting $\xi \to +\infty$ and by using (7) we get contradiction. Thus we have $\eta'''(\xi) > 0$ for any $\xi \geq a$. Notice that $\eta^{(4)}(\xi) < 0$ since $\left(\mu(\xi)\left(\eta'''(\xi)\right)^{\lambda}\right)' < 0$, $\mu(\xi) \geq 0$, $\mu'(\xi) \geq 0$ and $\eta'''(s) > 0$. By Lemma 2, we have one of the two following cases:

$$\eta'(\xi) > 0,\ \eta''(\xi) < 0,\ \eta'''(\xi) > 0,\ \eta^{(4)}(\xi) < 0,\ \left(\mu(\xi)\left(\eta'''(\xi)\right)^{\lambda}\right)' < 0, \tag{10}$$

$$\eta'(\xi) > 0,\ \eta''(\xi) > 0,\ \eta'''(\xi) > 0,\ \eta^{(4)}(\xi) < 0,\ \left(\mu(\xi)\left(\eta'''(\xi)\right)^{\lambda}\right)' < 0, \tag{11}$$

for any $\xi \geq a$. First let us consider the case (11). Not that $\eta(x)$ is eventually increasing since $\eta'(x) > 0$. Also, $\omega(x,\tau) \leq \omega(x,x_1)$ due to $\tau < x_1$ and $\omega(x,\tau)$ is a nondecreasing function with respect to the variable τ. Hence, $-\eta\left(\omega(x,\tau)\right) \geq -\eta\left(\omega(x,x_1)\right)$. Moreover, $\omega(x,\tau) \leq x$, hence $-\eta\left(\omega(x,x_1)\right) \geq -\eta(x)$. By the estimates $-y(x) > -\eta(x)$, $-\eta\left(\omega(x,\tau)\right) \geq -\eta\left(\omega(x,x_1)\right)$, $-\eta\left(\omega(x,x_1)\right) \geq -\eta(x)$ and $-\int_{x_0}^{x_1}\rho(x,\tau)d\tau \geq -P$, it follows that

$$\begin{aligned} y(x) &= \eta(x) - \int_{x_0}^{x_1}\rho(x,\tau)y\left(\omega(x,\tau)\right)d\tau \geq \eta(x) - \int_{x_0}^{x_1}\rho(x,\tau)\eta\left(\omega(x,\tau)\right)d\tau \\ &\geq \eta(x) - \eta\left(\omega(x,x_1)\right)\int_{x_0}^{x_1}\rho(x,\tau)d\tau \geq \eta(x)\left(1 - \int_{x_0}^{x_1}\rho(x,\tau)d\tau\right) \geq \eta(x)(1-P). \end{aligned} \tag{12}$$

Eq. (6) and the condition $h(x) \geq \kappa x^{\lambda}$ $(x \neq= 0)$ yield

$$\begin{aligned} {}^{CF}D_a^{\alpha}\left(\mu\left(.\right)\left(\left(\eta'''\left(.\right)\right)^{\lambda}\right)\right)(x) &= -\int_c^d \gamma(x,\xi)h\left(y\left(\delta\left(x,\xi\right)\right)\right)d\xi \\ &\leq -\kappa\int_c^d \gamma(x,\xi)y^{\lambda}\left(\delta(x,\xi)\right)d\xi. \end{aligned}$$

Combining this with (12) we get

$${}^{CF}D_a^{\alpha}\left(\mu\left(.\right)\left(\left(\eta'''\left(.\right)\right)^{\lambda}\right)\right)(x) \leq -\kappa\left(1-P\right)^{\lambda}\int_c^d \gamma(x,\xi)\eta^{\lambda}\left(\delta(x,\xi)\right)d\xi,\ x \geq a.$$

Since $\delta(x,\xi)$ is a nondecreasing function with respect to the variable ξ, we have $\delta(x,\xi) \geq \delta(x,c)$. Hence $-\eta\left(\delta(x,\xi)\right) \leq -\eta\left(\delta(x,c)\right)$. We then obtain

$$\begin{aligned} {}^{CF}D_a^{\alpha}\left(\mu\left(.\right)\left(\left(\eta'''\left(.\right)\right)^{\lambda}\right)\right)(x) &\leq -\kappa\left(1-P\right)^{\lambda}\int_c^d \gamma(x,\xi)\eta^{\lambda}\left(\delta(x,\xi)\right)d\xi \\ &= -\kappa\left(1-P\right)^{\lambda}\eta^{\lambda}\left(\delta(x,c)\right)\bar{\gamma}(x),\ x \geq a. \end{aligned} \tag{13}$$

By Lemma 3, it follows that $\eta(\xi) \geq \frac{\xi}{3}\eta'(\xi)$ on $[a,+\infty)$, so we have

$$\frac{3}{\xi} \geq \frac{\eta'(\xi)}{\eta(\xi)} \Rightarrow 3\int_{\delta(x,c)}^{x}\frac{d\xi}{\xi} \geq \int_{\delta(x,c)}^{x}\frac{\eta'(\xi)d\xi}{\eta(\xi)} \Rightarrow \frac{\eta\left(\delta(x,c)\right)}{\eta(x)} \geq \frac{\delta^3(x,c)}{x^3}, \tag{14}$$

for any $x \in [a,+\infty)$ in (13), we arrive at

$${}^{CF}D_a^{\alpha}\left(\mu\left(.\right)\left(\left(\eta'''\left(.\right)\right)^{\lambda}\right)\right)(x) \leq -\kappa\left(1-P\right)^{\lambda}\eta^{\lambda}(a)\frac{\delta^{3\lambda}(x,c)}{x^{3\lambda}}\bar{\gamma}(x),\ x \in [a,+\infty).$$

Thus, we have

$${}^{CF}I_a^{\alpha}\left({}^{CF}D_a^{\alpha}\left(\mu\left(.\right)\left(\left(\eta'''\left(.\right)\right)^{\lambda}\right)\right)\right)(x) \leq -\kappa\left(1-P\right)^{\lambda}\eta^{\lambda}(a){}^{CF}I_a^{\alpha}\left(\frac{\delta^{3\lambda}(x,c)}{x^{3\lambda}}\bar{\gamma}(x)\right).$$

$$\mu(x)\left(\eta'''(x)\right)^{\lambda} - \mu(a)\left(\eta'''(a)\right)^{\lambda} \leq -\kappa\left(1-P\right)^{\lambda}\eta^{\lambda}(a){}^{CF}I_a^{\alpha}\left(\frac{\delta^{3\lambda}(x,c)}{x^{3\lambda}}\bar{\gamma}(x)\right).$$

Hence, we obtain

$$\mu(a)\left(\eta'''(a)\right)^{\lambda} \geq \kappa\left(1-P\right)^{\lambda}\eta^{\lambda}(a){}^{CF}I_a^{\alpha}\left(\frac{\delta^{3\lambda}(x,c)}{x^{3\lambda}}\bar{\gamma}(x)\right). \tag{15}$$

This gives a contradiction to (9). Now, considering the case (10). By the help of Eq. (6), we have

$$\mu(x)\left(\eta'''(x)\right)^{\lambda} - \mu(a)\left(\eta'''(a)\right)^{\lambda} = -{}^{CF}I_a^{\alpha}\left(\int_c^d \gamma(x,\xi)h\left(y\left(\delta(x,\xi)\right)\right)d\xi\right).$$

Since $h(x) \geq \kappa x^{\lambda}$ $(x \neq 0)$, it follows that

$$\mu(x)\left(\eta'''(x)\right)^{\lambda} - \mu(a)\left(\eta'''(a)\right)^{\lambda} \leq -\kappa\ {}^{CF}I_a^{\alpha}\left(\int_c^d \gamma(x,\xi)y^{\lambda}\left(\delta(x,\xi)\right)d\xi\right).$$

Moreover we noted that inequality (12) can be obtained similarly under the case (10) by using only used the $\eta(x)$ function that is to be eventually increased $(\eta'(x) > 0)$. Thus, using inequality (12) and the fact that $\eta\left(\delta(x,c)\right) \leq \eta\left(\delta(x,\xi)\right)$ for any $c \leq \xi$, we obtain

$$\mu(x)\left(\eta'''(x)\right)^{\lambda} - \mu(a)\left(\eta'''(a)\right)^{\lambda} \leq -\kappa\left(1-P\right)^{\lambda}\ {}^{CF}I_a^{\alpha}\left(\eta^{\lambda}\left(\delta(x,c)\right)\right)\bar{\gamma}(x).$$

due to $\nu \in (0,1/3]$, by Lemma 4, there exists $x_{\theta} \geq a$ such that $\eta(x) \geq \nu x\eta'(x)$ for any $x \in [x_{\theta},+\infty)$. Hence

$$\frac{\eta\left(\delta(x,c)\right)}{\eta(x)} \geq \frac{\delta^{1/\nu}(x,c)}{x^{1/\nu}},\ x \in [x_{\nu},+\infty),$$

since $x \geq \delta(x,c) \geq x_{\nu} \geq x_{\theta}$. Therefore, we have

$$\begin{aligned}\mu(a)\left(\eta'''(a)\right)^{\lambda} &\geq \mu(a)\left(\eta'''(a)\right)^{\lambda} - \mu(x)\left(\eta'''(x)\right)^{\lambda}\\ &\geq \kappa\left(1-P\right)^{\lambda}\ {}^{CF}I_a^{\alpha}\left(\eta^{\lambda}\left(\delta(x,c)\right)\right)\bar{\gamma}(x)\\ &\geq \kappa\left(1-P\right)^{\lambda}\ {}^{CF}I_a^{\alpha}\left(\eta^{\lambda}(x)\frac{\delta^{\lambda/\nu}(x,c)}{x^{\lambda/\nu}}\bar{\gamma}(x)\right)\\ &\geq \kappa\left(1-P\right)^{\lambda}\eta^{\lambda}(a)\ {}^{CF}I_a^{\alpha}\left(\frac{\delta^{\lambda/\nu}(x,c)}{x^{\lambda/\nu}}\bar{\gamma}(x)\right)\end{aligned}$$

due to $\eta^{\lambda}(x) \geq \eta^{\lambda}(a)$ for any $x \in [a,+\infty)$, which contradicts (8). Finally, this ends the proof. □

Theorem 2 *Let us assume that the conditions (7) and (i)-(v) satisfy, where λ is a quotient of odd positive integers, $\alpha \in (0,1)$ and $1/3 < \nu \leq 1$. If*

$$\limsup_{x \to +\infty} {}^{CF}I_a^{\alpha} \left(\frac{\delta^{3\lambda}(x,c)}{x^{3\lambda}} \bar{\gamma}(t) \right) = +\infty, \ x \in [x_\nu, +\infty), \tag{16}$$

for some large values of x_ν such that $x_\nu > a$ and $x_\nu \leq \delta(x,c) \leq x$, where $\bar{\gamma}(x) = \int_c^d \gamma(x,\xi) d\xi$, then all the solutions of Eq. (6) are oscillatory.

Proof Form (16), we find that

$$\limsup_{x \to +\infty} {}^{CF}I_a^{\alpha} \left(\frac{\delta^{\lambda/\nu}(x,c)}{x^{\lambda/\nu}} \bar{\gamma}(x) \right) = +\infty, \ x \in [x_\nu, +\infty) \tag{17}$$

where $3\lambda > \lambda/\nu$ and $\delta(x,c)/x \leq 1$ for any $x \in [x_\nu, +\infty)$. If we suppose that $y(x)$ is eventually positive and a nonoscillatory solution of the Eq. (6) over $[a, +\infty)$, then we arrive at two possible cases using the same methodology to prove Theorem 1. In this case, we reach the desired result thanks to the contradictory arguments to be obtained. □

3.1 Example

Example 1 We consider the fractional integro-differential equation given below:

$$^{CF}D^{\alpha} \left(\left(\eta'''(x) \right)^3 \right) + \int_0^1 \frac{\xi}{x} y^3 \left(\frac{x\xi^2}{4} \right) d\xi = 0 \tag{18}$$

for any $x \in [a, +\infty)$, where

$$\eta(x) = y(x) + \int_1^2 \frac{1}{x} y \left(\frac{x+\xi}{3} \right) d\xi$$

and $\alpha \in (0,1)$. For $\mu(x) = 1$, $h(x) = x^3$, $\delta(x,\xi) = x\xi^2/4$, $\gamma(x,\xi) = \xi/x$, $\rho(x,\xi) = 1/x$, $\omega(x,\xi) = \frac{x+\xi}{3}$ and $\xi \in [1,2]$. It is easily seen that conditions (I)-(V) are met, and condition (7) is valid. Seeing that $\delta(x,\xi) = x\xi^2/4$, let us show that the condition (8) is held for some $0 < \nu \leq 1/3$. Namely, we have

$$\limsup_{x \to +\infty} {}^{CF}I^{\alpha} \left(\left(\frac{xc^2}{4} \right)^{\lambda/\nu} \frac{1}{x^{\lambda/\nu}} \frac{1}{2x} \right) = +\infty, \ x \in [a, +\infty).$$

There, from Theorem 1 all solutions of the Eq. (18) are oscillatory.

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Authors' contributions

TYU convinced the original idea. The authors discussed and agreed with the main focus and ideas of the manuscript. All authors developed the theory and performed the computations. The main text of the manuscript was written by TYU with support from SÖ. All authors read and approved the final manuscript.