Moments Estimation for Multi-factor Uncertain Differential Equations Based on Residuals

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Moments Estimation for Multi-factor Uncertain Differential Equations Based on Residuals

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Abstract

Parameter estimation is always the focus of constructing differential equations to simulate dynamic systems. In order to solve the unknown parameters of multi-factor uncertain differential equations, a class of updated multi-factor uncertain differential equations is proposed. According to its uncertainty distribution, the concept of residual is proposed and its specific expression is given. Based on the characteristic that the residuals obey the linear uncertainty distribution, moment estimation and generalized moment estimation are performed on the unknown parameters. Some numerical examples are given to demonstrate.

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1 Introduction

In 1944, Ito [1] proposed a stochastic differential equation driven by the Wiener process to simulate dynamic systems affected by random noise. Afterwards, stochastic differential equations have been widely used in many fields and have been studied in depth by scholars. For example, in the aspect of parameter estimation of stochastic differential equations, scholars have proposed estimation methods such as the least square method (Kailath [2]), maximum likelihood estimation (Strasser [3]), moment estimation (Gallant [4]), etc. When faced with some unexpected events or lack of samples, using the framework of probability theory to deal with problems may not be applicable. In 2007, Liu [5] proposed the uncertainty theory based on normality, duality, subadditivity and product axiom in order to solve expert reliability, another factor affecting system uncertainty. Later, in the in-depth study of uncertainty theory, Liu [6] proposed the Liu process.

Uncertain differential equation (UDE) was proposed by Liu [7] in 2008 to model and analyze dynamical systems subject to uncertain factors. Subsequently, many scholars focused their attention on UDE. In terms of the properties of UDE, the stability theorem (Yao, Gao and Gao [8]), stability in p-th moment (Sheng and Wang [10]) and stability in mean (Yao, Ke and Sheng [9]) have been proved successively. Different types of UDE have been proposed one after another. Zhu and Ge [11] proposed the backward uncertain differential equations and proved the existence theorems of the solutions of these equations. Yao [12] proved proved the completely unique solution of the uncertain differential equation with jumps driven by the update process and the Liu process and its uncertainty measure in the sense of stability. Li, Peng and Zhang [13] proposed a multi-factor uncertain differential equation and a numerical method to solve the general multi-factor uncertain differential equation. Yao [14] proposed the high-order uncertain differential equations with high-order derivatives.

Parameter estimation has always been a hot research field of UDE. Sheng et al. [15] presented a least squares estimation method for estimating unknown parameters. Yao and Liu [16] proposed a moment estimation method based on a difference form. In order to solve the case that the equation system obtained by moment estimation has no solution, Liu [17] proposed a generalized moment estimation method to solve such problems with the idea of solving for the optimal value. In addition, Lio et al. [18] proposed the uncertain maximum likelihood method. Later, Yang et al. [19] proposed a method to estimate unknown parameters of UDEs from discrete sampling data. Sheng and Zhang [20] also introduced three methods for parameter estimation based on different types of solutions. Liu et al. [21] first proposed the definition of residual error of uncertain differential equations and used residual error to solve unknown parameters. Zhang et al. [22] also estimated the parameters of high-order uncertain differential equations. In the study of unknown parameters of multi-factor uncertain differential equation, Zhang et al. [23] proposed a weighted method for moment estimation and least squares estimation of unknown parameters. However, that paper did not give a specific judgment on how to determine the rationality of the weighting method. In order to avoid complicated weighting and discuss the rationality of weighting, a new method based on residuals for estimating unknown parameters of multi-factor uncertain differential equations is proposed in this paper.

This paper introduces the idea of residuals into parameter estimation of multi-factor uncertain differential equation. The definition of residual is proposed and it is deduced that the residual obeys the linear uncertainty distribution $\mathcal{L}(0,1)$. The moment estimation and generalized moment estimation are performed on the unknown parameters from the residuals as samples from the linear uncertainty distribution. Section 2, we will introduce some basic definitions and theorems of uncertainty theory. Section 3, the concept of residuals of multi-factor uncertain differential equations is proposed, and the analytical expressions of residuals are deduced and demonstrated with examples. Section 4, based on the property that the residuals follow the linear uncertainty distribution $\mathcal{L}(0,1)$, the residual data are treated as a set of sample data from a linear distribution, and then the unknown parameters are estimated by moment estimation or generalized moment estimation. Section 5, an example of a multi-factor pharmacokinetic model and real data are given to demonstrate the feasibility of the estimation method proposed above. Section 6 is the summary of this paper.

2 Preliminary

In this section, some necessary definitions and theorems in uncertainty theory are introduced to help readers understand what follows.

Definition 1. (Liu [5, 6]) Let $\mathcal{L}$ be a σ-algebra on a nonempty set Γ. A set function $\mathcal{M} : \mathcal{L} \to [0, 1]$ is called an uncertainty measure if the four following axioms are satisfied:

Axiom 1: (normality Axiom) $\mathcal{M}\{\Gamma\} = 1$ for the universal set Γ.

Axiom 2: (duality Axiom) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event Λ.

Axiom 3: (subadditivity Axiom) For every countable sequence of events $\Lambda_1, \Lambda_2, \cdots$, we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \le \sum_{i=1}^{\infty} \mathcal{M}\left\{\Lambda_i\right\}.$$

The triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space. Besides, the product uncertain measure on the product σ-algebra $\mathcal{L}$ was defined by Liu as follows:

Axiom 4: (product Axiom) Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \cdots$, the product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=0}^{\infty} \mathcal{M}_k\{\Lambda_k\}$$

where Λ_k are arbitrarily chosen events from $\mathcal{L}_k$ for $k = 1, 2, \cdots$, respectively.

Definition 2. (Liu [6]) An uncertain process C_t is called a Liu process if

(i) $C_0 = 0$ and almost all sample paths are Lipschitz continuous,

(ii) C_t has stationary and independent increments,

(iii) the increment $C_{s+t} - C_s$ has a normal uncertainty distribution

$$\Phi_t(x) = \left(1 + \exp\left(-\frac{\pi x}{\sqrt{3}t}\right)\right)^{-1}, \quad x \in \Re.$$

Definition 3. (Liu [5]) Let ξ be an uncertain variable and its uncertainty distribution is defined by

$$\Phi(x) = \mathcal{M}\{\xi \le x\}$$

for any real number x.

Common uncertainty distributions include linear uncertainty distributions $\mathcal{L}(a, b)$, zigzag uncertainty distributions $\mathcal{Z}(a, b, c)$ and normal uncertainty differentials $\mathcal{N}(e, \sigma)$. For example, suppose the uncertain variable $\xi(x) = x$, it follows $\mathcal{L}(0, 1)$ uncertainty distribution

$$\Phi(x)=\begin{cases}0, & if \quad x\le 0\\ x, & if \quad 0<x<1\\ 1, & if \quad x\ge 1\end{cases}\cdot$$

Definition 4. (Li, Peng and Zhang [13]) Let C_{1t},C_{2t},$\cdots$,C_{nt} be independent Liu processes, and f and $g_1, g_2, \cdots, g_n$ are given functions. The multi-factor uncertain differential equation with respect to C_{jt} $(i = 1, 2, \cdots, n)$

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + \sum_{j=1}^{n} g_j(t, X_t)\mathrm{d}C_{jt}$$

is said to have an $\alpha-$path X_t^α if it solves the corresponding ordinary differential equation

$$\mathrm{d}X_t^\alpha = f(t, X_t^\alpha)\mathrm{d}t + \sum_{j=1}^{n} |g(t, X_t^\alpha)|\Phi^{-1}(\alpha)\mathrm{d}t$$

where

$$\Phi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}, \quad \alpha \in (0, 1).$$

3 Residuals of Multi-factor Uncertain Differential Equations

In this section, the definition of the residuals of multi-factor uncertain differential equation is proposed, and the analytical formulas that are convenient for calculation are given to prepare for the estimation of unknown parameters.

Consider a multi-factor uncertain differential equation with n observations (t_i, x_{t_i}) $(i = 1, 2, \cdots, n)$

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + \sum_{j=1}^{n} g_j(t, X_t)\mathrm{d}C_{jt} \tag{1}$$

where f and g_j $(j = 1, 2, \cdots, n)$ are given continuous functions and C_{jt} $(j = 1, 2, \cdots, n)$ is the Liu process.

From Equation (1) and its observations, the i-th corresponding updated uncertain differential equation can be obtained:

$$\begin{cases}\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + \sum_{j=1}^{n} g_j(t, X_t)\mathrm{d}C_{jt}\\ X_{t_{i-1}} = x_{t_{i-1}}\end{cases} \tag{2}$$

where $2 \le i \le n$ and $x_{t_{i-1}}$ is the new initial value at the new initial time t_{i-1}. The analytical formula of the uncertainty distribution $\Phi_{t_i}(X_i)$ of X_{t_i} can be obtained by Equation (2), and the specific value $\Phi_{t_i}(x_i)$ can be obtained by replacing X_{t_i} with the observed value x_{t_i}.

Definition 5. For any multi-factor uncertain differential equation with discrete observations (t_i, x_{t_i}) $(i = 1, 2, \cdots, n)$, the i-th residual is defined as ε_i and can be obtained by the uncertainty distribution

$\Phi_{t_i}(X_i)$ of the uncertain variables X_i in Equation (2),

$$\varepsilon_i = \Phi_{t_i}(x_i)$$

where $2 \le i \le n$.

The residuals of the multi-factor uncertain differential equations proposed above have the following two important properties.

Property 1. The uncertainty distribution $\Phi_{t_i}(X_i)$ $(2 \le i \le n)$ is also an uncertain variable and $0 \le \Phi_{t_i}(X_i) \le 1$. For any $0 < x < 1$, we can always get

$$\mathcal{M}\{\Phi_{t_i}(X_{t_i}) \le x\} = \mathcal{M}\{X_{t_i} \le \Phi^{-1}(x)\} = \Phi_{t_i}(\Phi_{t_i}^{-1}(x)) = x.$$

Obviously, the distribution of $\Phi_{t_i}(X_i)$ is as follows

$$\Phi(x) = \begin{cases} 0, & if \quad x \le 0 \\ x, & if \quad 0 < x < 1 \\ 1, & if \quad x \ge 1 \end{cases} \cdot$$

Therefore, the uncertain variable $\Phi_{t_i}(X_i)$ follows a linear uncertainty distribution $\mathcal{L}(0,1)$, and the discrete data $\varepsilon_i, (i = 1, 2, \cdots, n)$ can be used as a set of sample data from the linear uncertainty distribution $\mathcal{L}(0,1)$.

Property 2. According to Equation (2), we can obtain the corresponding updated ordinary differential equations

$$\begin{cases} \mathrm{d}X_t^\alpha = f(t, X_t^\alpha)\mathrm{d}t + |\sum_{j=1}^{n} g_j(t, X_t^\alpha)|\Phi^{-1}(\alpha)\mathrm{d}t \\ X_{t_{i-1}}^\alpha = x_{t_{i-1}} \end{cases}, \tag{3}$$

where

$$\Phi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}, \quad \alpha \in (0,1),$$

and $X_{t_i}^\alpha$ as the $\alpha-$path of X_{t_i} can be obtained by solving Equation (3).

Since for any $\alpha \in (0,1)$, we have

$$\mathcal{M}\{X_{t_i} \le \Phi^{-1}(\alpha)\} = \Phi_{t_i}(\Phi_{t_i}^{-1}(\alpha)) = \alpha,$$

then there must be an inverse uncertainty distribution $\Phi_{t_i}^{-1}$ of X_{t_i}.

Writing $x = \Phi_{t_i}^{-1}(\alpha)$, we can get $\alpha = \Phi_{t_i}$ and

$$\mathcal{M}\{X_{t_i} \le x\} = \alpha = \Phi_{t_i}(X_i).$$

Therefore, ε_i can be regarded as the corresponding α value of $\alpha-$path $X_{t_i}^\alpha$ at time t_i.

3.1 Analytical Expressions for Residuals

For some special multi-factor uncertain differential equations, we can get the analytical expressions of uncertain variables by solving the equations. Some examples will be given below.

Example 1. Consider a multi-factor uncertain differential equation with discrete observations (t_i, x_{t_i}) $(i = 1, 2, \cdots, n)$,

$$\mathrm{d}X_t = \mu \mathrm{d}t + \sum_{j=1}^{m} \sigma_j \mathrm{d}C_{jt}$$

where μ and σ_j are constants. By solving the i-th multi-factor updated uncertain differential equation below $(2 < i < n)$

$$\begin{cases} \mathrm{d}X_t = \mu \mathrm{d}t + \sum\limits_{j=1}^{m} \sigma_j \mathrm{d}C_{jt} \\ X_{t_{i-1}} = x_{t_{i-1}} \end{cases},$$

we can get the solution

$$X_{t_i} = X_{t_{i-1}} + \mu(t_i - t_{i-1}) + \sum_{j=1}^{m} \sigma_j (C_{t_i} - C_{t_{i-1}})$$

and the uncertainty distribution of X_{t_i}

$$\Phi_{t_i}(x) = \left(1 + \exp\left(\frac{\pi(x_{t_{i-1}} + \mu(t_i - t_{i-1}) - x)}{\sqrt{3} \sum\limits_{j=1}^{m} \sigma_j (t_i - t_{i-1})} \right) \right)^{-1}.$$

By Definition 5, we get the i-th residual corresponding to $\Phi_{t_i}(x_i)$

$$\varepsilon_i = \left(1 + \exp\left(\frac{\pi(x_{t_{i-1}} + \mu(t_i - t_{i-1}) - x_i)}{\sqrt{3} \sum\limits_{j=1}^{m} \sigma_j (t_i - t_{i-1})} \right) \right)^{-1}.$$

Example 2. Consider a multi-factor uncertain differential equation with discrete observations (t_i, x_{t_i}) $(i = 1, 2, \cdots, n)$,

$$\mathrm{d}X_t = \mu X_t \mathrm{d}t + \sum_{j=1}^{m} \sigma_j X_t \mathrm{d}C_{jt}$$

where μ and σ_j are constants. By solving the i-th multi-factor updated uncertain differential equation below $(2 < i < n)$

$$\begin{cases} \mathrm{d}X_t = \mu X_t \mathrm{d}t + \sum\limits_{j=1}^{m} \sigma_j X_t \mathrm{d}C_{jt} \\ X_{t_{i-1}} = x_{t_{i-1}} \end{cases},$$

we can get

$$\ln X_{t_i} = \ln X_{t_{i-1}} + \mu(t_i - t_{i-1}) + \sum_{j=1}^{m} \sigma_j (C_{t_i} - C_{t_{i-1}})$$

and the uncertainty distribution of X_{t_i}

$$\Phi_{t_i}(x) = \left(1 + \exp\left(\frac{\pi(\ln x_{t_{i-1}} + \mu(t_i - t_{i-1}) - \ln x)}{\sqrt{3}\sum\limits_{j=1}^{m}\sigma_j(t_i - t_{i-1})}\right)\right)^{-1}.$$

By Definition 5, we get the i-th residual corresponding to $\Phi_{t_i}(x_i)$

$$\varepsilon_i = \left(1 + \exp\left(\frac{\pi(\ln x_{t_{i-1}} + \mu(t_i - t_{i-1}) - \ln x_i)}{\sqrt{3}\sum\limits_{j=1}^{m}\sigma_j(t_i - t_{i-1})}\right)\right)^{-1}.$$

Example 3. Consider a multi-factor uncertain differential equation with discrete observations (t_i, x_{t_i}) $(i = 1, 2, \cdots, n)$

$$\mathrm{d}X_t = \mu t X_t \mathrm{d}t + \sum_{j=1}^{m}\sigma_j t X_t \mathrm{d}C_{jt}$$

where μ and σ_j are constants. By solving the i-th multi-factor updated uncertain differential equation below $(2 < i < n)$

$$\begin{cases} \mathrm{d}X_t = \mu t X_t \mathrm{d}t + \sum\limits_{j=1}^{m}\sigma_j t X_t \mathrm{d}C_{jt} \\ X_{t_{i-1}} = x_{t_{i-1}} \end{cases},$$

we can get

$$\ln X_{t_i} = \ln X_{t_{i-1}} + \mu(\frac{t_i^2 - t_{i-1}^2}{2}) + \sum_{j=1}^{m}\sigma_j \int_{t_{i-1}}^{t_i} t\mathrm{d}C_{jt}.$$

Since the Liu integral

$$\int_{t_{i-1}}^{t_i} t\mathrm{d}C_{jt} \sim \mathcal{N}(0, \frac{t_i^2 - t_{i-1}^2}{2})$$

we can get the uncertainty distribution of X_{t_i}

$$\Phi_{t_i}(x) = \left(1 + \exp\left(\frac{\pi(\ln x_{t_{i-1}} + \mu(\frac{t_i^2 - t_{i-1}^2}{2}) - \ln x)}{\sqrt{3}\sum\limits_{j=1}^{m}\sigma_j(\frac{t_i^2 - t_{i-1}^2}{2})}\right)\right)^{-1}.$$

By Definition 5, we get the i-th residual corresponding to $\Phi_{t_i}(x_i)$

$$\varepsilon_i = \left(1 + \exp\left(\frac{\pi(\ln x_{t_{i-1}} + \mu(\frac{t_i^2 - t_{i-1}^2}{2}) - \ln x_i)}{\sqrt{3}\sum\limits_{j=1}^{m}\sigma_j(\frac{t_i^2 - t_{i-1}^2}{2})}\right)\right)^{-1}.$$

Example 4. Consider a multi-factor uncertain differential equation with discrete observations (t_i, x_{t_i}) $(i = 1, 2, \cdots, n)$

$$\mathrm{d}X_t = \mu \mathrm{d}t + \sigma_1 t \mathrm{d}C_{1t} + \sigma_2(2 + t)^{(-\alpha)}\mathrm{d}C_{2t}$$

where μ, σ_j and α $(0 < \alpha < 1)$ are constants. By solving the i-th multi-factor updated uncertain differential equation below $(2 < i < n)$

$$\begin{cases} \mathrm{d}X_t = \mu \mathrm{d}t + \sigma_1 t \mathrm{d}C_{1t} + \sigma_2 (2+t)^{(-\alpha)} \mathrm{d}C_{2t} \\ X_{t_{i-1}} = x_{t_{i-1}} \end{cases},$$

we can get

$$X_{t_i} = X_{t_{i-1}} + \mu(t_i - t_{i-1}) + \sigma_1 \int_{t_i}^{t_{i-1}} t \mathrm{d}C_{1t} + \sigma_2 \int_{t_i}^{t_{i-1}} (2+t)^{(-\alpha)} \mathrm{d}C_{2t}$$

Since the Liu integrals

$$\int_{t_{i-1}}^{t_i} t \mathrm{d}C_{jt} \sim \mathcal{N}(0, \frac{t_i^2 - t_{i-1}^2}{2}) \quad \text{and} \quad \int_{t_{i-1}}^{t_i} (2-t)^{(-\alpha)} \mathrm{d}C_{jt} \sim \mathcal{N}(0, \frac{(2+t_i)^{(1-\alpha)} - (2+t_{i-1})^{(1-\alpha)}}{1-\alpha})$$

we can get the uncertainty distribution of X_{t_i}

$$\Phi_{t_i}(x) = \left(1 + \exp\left(\frac{\pi(x_{t_{i-1}} + \mu(t_i - t_{i-1}) - x)}{\sqrt{3}(\sigma_1 \frac{t_i^2 - t_{i-1}^2}{2} + \sigma_2 \frac{(2+t_i)^{1-\alpha} - (2+t_{i-1})^{1-\alpha}}{1-\alpha})}\right)\right)^{-1}.$$

By Definition 5, we get the i-th residual corresponding to $\Phi_{t_i}(x_i)$

$$\varepsilon_i = \left(1 + \exp\left(\frac{\pi(x_{t_{i-1}} + \mu(t_i - t_{i-1}) - x)}{\sqrt{3}(\sigma_1 \frac{t_i^2 - t_{i-1}^2}{2} + \sigma_2 \frac{(2+t_i)^{1-\alpha} - (2+t_{i-1})^{1-\alpha}}{1-\alpha})}\right)\right)^{-1}.$$

3.2 Approximate analytical expression of residuals

For a general multi-factor uncertain differential equation, if the uncertainty distribution cannot be obtained by solving the equation, the approximate expression of the residual can be obtained by the following method.

According to Equation (2), we can obtain the corresponding updated ordinary differential equations

$$\begin{cases} \mathrm{d}X_t^\alpha = f(t, X_t^\alpha) \mathrm{d}t + |\sum_{j=1}^n g_j(t, X_t^\alpha)| \Phi^{-1}(\alpha) \mathrm{d}t \\ X_{t_{i-1}}^\alpha = x_{t_{i-1}} \end{cases}.$$

By using the Euler differential method, $X_{t_i}^\alpha$ can be expressed as

$$X_{t_i}^\alpha = X_{t_{i-1}}^\alpha + f(t, X_{t_i}^\alpha)(t_i - t_{i-1}) + |\sum_{j=1}^n g_j(t, X_{t_i}^\alpha)| \Phi^{-1}(\alpha)(t_i - t_{i-1}), \tag{4}$$

where

$$\Phi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}.$$

Since the α satisfies the following minimization problem

$$\min_\alpha \mid X_{t_i}^\alpha - X_{t_i} \mid$$

and the α is equivalent to ε_i from Property 2, so we get ε_i satisfying $X_{t_i}^{\alpha} \approx X_{t_i}$.

Using the approximate relationship between $X_{t_i}^{\alpha}$ and X_{t_i}, we can transform the Equation (4) into

$$X_{t_i} = X_{t_{i-1}} + f(t, X_{t_i})(t_i - t_{i-1}) + |\sum_{j=1}^{n} g_j(t, X_{t_i})| \frac{\sqrt{3}}{\pi} \ln \frac{\varepsilon_i}{1-\varepsilon_i}(t_i - t_{i-1}).$$

After tidying up, the expression for the residuals ε_i is

$$\varepsilon_i = 1 - \left(1 + \exp\left(\frac{\pi(X_{t_i} - X_{t_{i-1}} - f(t, X_t)(t_i - t_{i-1}))}{\sqrt{3}\left|\sum\limits_{j=1}^{n} g_j(t, X_t)\right|(t_i - t_{i-1})} \right) \right)^{-1}.$$

Example 5. Assuming a multi-factor uncertain differential equation,

$$\mathrm{d}X_t = 0.0305t\mathrm{d}t + 0.7441t\mathrm{d}C_{1t} + 0.5000(2+t)^{(-2)}\mathrm{d}C_{2t},$$

the updated uncertain differential equation can be obtained

$$\begin{cases} \mathrm{d}X_t = 0.0305t\mathrm{d}t + 0.7441t\mathrm{d}C_{1t} + 0.5000(2+t)^{(-2)}\mathrm{d}C_{2t} \\ X_{t_{i-1}} = x_{t_{i-1}} \end{cases}.$$

The corresponding updated ordinary differential equation is

$$\begin{cases} \mathrm{d}X_t^{\alpha} = 0.0305t\mathrm{d}t + \left|0.7441t + 0.5000(2+t)^{(-2)}\right| \frac{\sqrt{3}}{\pi} \ln \frac{\varepsilon_i}{1-\varepsilon_i}\mathrm{d}t \\ X_{t_{i-1}}^{\alpha} = x_{t_{i-1}} \end{cases}.$$

Using the method described above, we can get

$$\varepsilon_i = 1 - \left(1 + \exp\left(\frac{\pi(X_{t_i} - X_{t_{i-1}} - 0.0305t_i(t_i - t_{i-1}))}{\sqrt{3}\left|0.7441t_i + 0.5000(2+t_i)^{(-2)}\right|(t_i - t_{i-1})} \right) \right)^{-1}.$$

Therefore, when we have observational data, we can substitute it into the solution. The observed data of Example 5 and its residual calculation results are shown in Table 1.

Example 6. Assuming a multi-factor uncertain differential equation,

$$\mathrm{d}X_t = \frac{X_t}{2+t}\mathrm{d}t + t^2\mathrm{d}C_{1t} + (2+t)^{(-2)}\mathrm{d}C_{2t}.$$

The updated uncertain differential equation can be obtained

$$\begin{cases} \mathrm{d}X_t = \frac{X_t}{2+t}\mathrm{d}t + t^2\mathrm{d}C_{1t} + (2+t)^{(-2)}\mathrm{d}C_{2t} \\ X_{t_{i-1}} = x_{t_{i-1}} \end{cases}.$$

The corresponding updated ordinary differential equation is

$$\begin{cases} \mathrm{d}X_t^{\alpha} = \frac{X_t^{\alpha}}{2+t}\mathrm{d}t + \left|t^2 + (2+t)^{(-2)}\right| \frac{\sqrt{3}}{\pi} \ln \frac{\varepsilon_i}{1-\varepsilon_i}\mathrm{d}t \\ X_{t_{i-1}}^{\alpha} = x_{t_{i-1}} \end{cases}$$

Table 1 The observed datas and residual result in Example 5

n	1	2	3	4	5	6	7	8	9	10
i	0.1169	0.3421	0.4445	0.4697	0.7393	0.8870	0.9491	1.0323	1.0488	1.0499
x_{t_i}	3.5697	3.7697	3.9586	4.1011	5.0433	5.7954	6.0219	6.5893	6.7863	6.8925
ε_i	0.4504	0.5217	0.8384	0.5554	0.6118	0.5078	0.6594	0.8521	0.9832	0.5194
i	1.1867	1.2444	1.2751	1.6349	1.6967	1.7208	1.7555	1.8354	1.9544	2.0314
x_{t_i}	7.5486	8.0173	8.5121	10.8996	11.3680	12.0230	12.8299	13.4825	14.5079	15.2054
ε_i	0.6580	0.8885	0.5209	0.5450	0.9462	0.9059	0.5442	0.5419	0.5449	0.4503
i	2.3141	2.3796	2.4114	2.8449	2.9213	3.2041	3.6195	3.6935	3.7154	3.9190
x_{t_i}	17.0852	19.6936	20.2691	25.2093	27.0489	31.1364	39.5616	41.5116	42.1558	46.0395
ε_i	0.9581	0.7136	0.5181	0.7495	0.5486	0.6130	0.6966	0.7352	0.5700	

and the residual expression is

$$\varepsilon_i = 1 - \left(1 + \exp\left(\frac{\pi(X_{t_i} - X_{t_{i-1}} - \frac{X_{t_i}}{2+t_i})(t_i - t_{i-1}))}{\sqrt{3}\left|t_i^2 + (2+t_i)^{(-2)}\right|(t_i - t_{i-1})}\right)\right)^{-1}.$$

Using the above residual expression and known discrete data, the specific residual data can be easily obtained and the results are shown in Table 2.

Table 2 The observed datas and residual result in Example 6

n	1	2	3	4	5	6	7	8	9	10
i	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.3	1.4
x_{t_i}	15.5138	16.1254	16.7455	17.3758	18.0180	18.6738	19.2449	20.0330	20.7399	21.4674
ε_i	0.4924	0.4999	0.5105	0.5234	0.5377	0.4444	0.6706	0.5823	0.5963	0.6093
i	1.5	1.6	1.7	1.8	1.9	2.0	2.1	2.2	2.3	2.4
x_{t_i}	22.2172	22.9911	23.7909	24.6184	25.4755	26.3639	27.2856	28.2423	29.2359	30.2682
ε_i	0.6214	0.6326	0.6427	0.6521	0.6605	0.6682	0.6752	0.6815	0.6873	0.6925
i	2.5	2.6	2.7	2.8	2.9	3.0	3.1	3.2	3.3	3.4
x_{t_i}	31.3412	32.4567	33.6165	34.8226	36.0768	37.3810	38.7372	40.1471	41.6128	43.1360
ε_i	0.6973	0.7017	0.7057	0.7094	0.7127	0.7158	0.7186	0.7213	0.7237	

4 Parameter Estimation

In this section, we use moment estimation and generalized moment estimation methods to estimate unknown parameters of multi-factor uncertain differential equation by exploiting the property that residuals follow $\mathcal{L}(0,1)$.

Assume a multi-factor uncertain differential equation with unknown parameters driven by multiple

Liu process

$$\mathrm{d}X_t = f(t, X_t; \mu)\mathrm{d}t + \sum_{j=1}^{n} g_j(t, X_t; \sigma_j)\mathrm{d}C_{jt} \tag{5}$$

where μ and $\sigma_j (j = 1, 2, \cdots, n)$ are the parameters to be estimated. For Equation (5), we have the observed values $x_1, x_2, \cdots, x_n$ at time $t_1, t_2, \cdots, t_n$, respectively.

Through the analytical and numerical methods of residuals ε_i mentioned in Section 3, we can obtain a series of residuals $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_{n-1}$ and this set of data can be used as samples of the uncertainty distribution $\mathcal{L}(0, 1)$.

According to the method of moments, the p-th sample moments is

$$\frac{1}{N-1} \sum_{i=1}^{N-1} \varepsilon_i(\mu; \sigma_1, \sigma_2, \cdots, \sigma_n)^p$$

and the corresponding p-th population moments

$$\frac{1}{p+1}$$

where $p = 1, 2, \cdots, K$ (K is the number of unknown parameters). According to the principle of moment estimation method, we can obtain the following equations:

$$\begin{cases} \frac{1}{N-1} \sum\limits_{i=1}^{N-1} \varepsilon_i(\mu; \sigma_1, \sigma_2, \cdots, \sigma_n) = \frac{1}{1+1} \\ \frac{1}{N-1} \sum\limits_{i=1}^{N-1} (\varepsilon_i(\mu; \sigma_1, \sigma_2, \cdots, \sigma_n))^2 = \frac{1}{2+1} \\ \cdots \\ \frac{1}{N-1} \sum\limits_{i=1}^{N-1} (\varepsilon_i(\mu; \sigma_1, \sigma_2, \cdots, \sigma_n))^K = \frac{1}{K+1} \end{cases}. \tag{6}$$

The estimated value $(\hat{\mu}; \hat{\sigma_1}, \hat{\sigma_2}, \cdots, \hat{\sigma_n})$ of unknown parameters can be obtained by solving the equations.

However, with some observations, the moment estimation method is no longer applicable when the Equation system (6) based on moment estimation has no solution. In this case, the unknown parameters can be obtained by solving the following minimization problem based on the generalized estimation of moments principle:

$$\min_{(\mu; \sigma_1, \sigma_2, \cdots, \sigma_n)} \sum_{p=1}^{p} \left(\frac{1}{N-1} \sum_{i=1}^{N-1} \varepsilon_i(\mu; \sigma_1, \sigma_2, \cdots, \sigma_n)^p - \frac{1}{p+1} \right)^2. \tag{7}$$

Example 7. Consider a multi-factor uncertain differential equation with parameters μ, σ_1 and σ_2

$$\mathrm{d}X_t = \mu t X_t \mathrm{d}t + \sigma_1 t X_t \mathrm{d}C_{1t} + \sigma_2 t X_t \mathrm{d}C_{2t}.$$

Then we can get the related updated multi-factor uncertain differential equation

$$\begin{cases} \mathrm{d}X_t = \mu \mathrm{d}t + \sigma_1 \mathrm{d}C_{1t} + \sigma_2 \mathrm{d}C_{1t} \\ X_{t_{i-1}} = x_{t_{i-1}} \end{cases},$$

By solving the uncertain variable X_{t_i} and its uncertainty distribution $\Phi(X_{t_i})$, the residual can be expressed as

$$\varepsilon_i(\mu,\sigma_1,\sigma_2)=\left(1+\exp\left(\frac{\pi(\ln x_{t_{i-1}}+\mu(\frac{t_i^2-t_{i-1}^2}{2})-\ln x_i)}{\sqrt{3}(\sigma_1(\frac{t_i^2-t_{i-1}^2}{2})+\sigma_2(\frac{t_i^2-t_{i-1}^2}{2}))}\right)\right)^{-1},$$

and $\varepsilon_i\sim\mathcal{L}(0,1)$. The observed data are shown in Table 3.

Table 3 The observed datas in Example 7

n	1	2	3	4	5	6	7	8	9	10
i	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.3	1.4
x_{t_i}	2.8674	4.2431	5.9619	7.1445	9.6730	16.7927	18.2922	23.9932	47.1088	47.9922
i	1.5	1.6	1.7	1.8	1.9	2.0	2.1	2.2	2.3	2.4
x_{t_i}	48.9901	50.0472	52.1650	57.6722	60.9867	61.7666	68.1972	71.2694	80.5489	81.7547
i	2.5	2.6	2.7	2.8	2.9					
x_{t_i}	81.8149	85.9442	88.6512	90.4722	97.8681					

According to the principle of the moment estimation, we obtain the following equations

$$\begin{cases}\frac{1}{N-1}\sum\limits_{i=1}^{N-1}\varepsilon_i(\mu;\sigma_1,\sigma_2)=\frac{1}{2}\\ \frac{1}{N-1}\sum\limits_{i=1}^{N-1}(\varepsilon_i(\mu;\sigma_1,\sigma_2))^2=\frac{1}{3}\\ \frac{1}{N-1}\sum\limits_{i=1}^{N-1}(\varepsilon_i(\mu;\sigma_1,\sigma_2))^3=\frac{1}{4}\end{cases}.$$

By solving the above system of equations, the unknown parameter results are obtained

$$\hat{\mu}=-7.0861,\quad \hat{\sigma_1}=-3.2052\quad and\quad \hat{\sigma_2}=2.9596.$$

Finally, the multi-factor uncertain differential equation is obtained

$$\mathrm{d}X_t=-7.0861tX_t\mathrm{d}t-3.2052tX_t\mathrm{d}C_{1t}+2.9596tX_t\mathrm{d}C_{2t}.$$

Example 8. Assuming a multi-factor uncertain differential equation,

$$\mathrm{d}X_t=\frac{X_t}{\sigma_1+t}\mathrm{d}t+t^2\mathrm{d}C_{1t}+(\sigma_2+t)^{(-2)}\mathrm{d}C_{2t},$$

where σ_1, σ_2 are unknown parameter. The observed data are shown in Table 4. The updated uncertain differential equation can be obtained

$$\begin{cases}\mathrm{d}X_t=\dfrac{X_t}{\sigma_1+t}\mathrm{d}t+t^2\mathrm{d}C_{1t}+(\sigma_2+t)^{(-2)}\mathrm{d}C_{2t}\\ X_{t_{i-1}}=x_{t_{i-1}}\end{cases}.$$

The corresponding updated ordinary differential equation is

$$\begin{cases}\mathrm{d}X_t^\alpha=\frac{X_t^\alpha}{\sigma_1+t}\mathrm{d}t+\left|t^2+(\sigma_2+t)^{(-2)}\right|\frac{\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\mathrm{d}t\\ X_{t_{i-1}}^\alpha=x_{t_{i-1}}\end{cases}$$

and the residual expression is

$$\varepsilon_i(\sigma_1,\sigma_2)=1-\left(1+\exp\left(\frac{\pi(X_{t_i}-X_{t_{i-1}}-\frac{X_{t_i}}{\sigma_1+t})(t_i-t_{i-1}))}{\sqrt{3}\left|t_i^2+(\sigma_2+t_i)^{(-2)}\right|(t_i-t_{i-1})}\right)\right)^{-1}.$$

According to the principle of the moment estimation, we obtain the following equations

$$\begin{cases}\frac{1}{N-1}\sum\limits_{i=1}^{N-1}\varepsilon_i(\sigma_1,\sigma_2)=\frac{1}{2}\\ \frac{1}{N-1}\sum\limits_{i=1}^{N-1}(\varepsilon_i(\sigma_1,\sigma_2))^2=\frac{1}{3}\end{cases}.$$

By solving the above system of equations, we can obtain the estimated results of the unknown parameters

$$\hat{\sigma}_1=0.0011 \quad \text{and} \quad \hat{\sigma}_2=0.0010,$$

and the multi-factor uncertain differential equation is

$$\mathrm{d}X_t=\frac{X_t}{0.0011+t}\mathrm{d}t+t^2\mathrm{d}C_{1t}+(0.0010+t)^{(-2)}\mathrm{d}C_{2t}.$$

Table 4 The observed datas in Example 8

n	1	2	3	4	5	6	7	8	9	10
i	1	2	3	4	5	6	7	8	9	10
x_{t_i}	50.6495	51.3515	52.6250	55.4185	56.7131	60.4231	62.2069	64.4992	66.5161	67.2648
i	11	12	13	14	15	16	17	18	19	20
x_{t_i}	68.5184	69.8204	71.2035	72.8185	73.6952	74.3754	75.0168	76.3934	77.3171	79.7374

5 Numerical example

In this section, a meaningful model and actual data are presented to implement the proposed method.

Example 9. Considering a multi-factor uncertain pharmacokinetic model with unknown parameters proposed by Liu and Yang [24] is as follows:

$$\mathrm{d}X_t=(k_0-k_1X_t)\mathrm{d}t+\sigma_1X_t\mathrm{d}C_{1t}+\sigma_2\mathrm{d}C_{2t}$$

where X_t is the drug concentration at time t and k_0, k_1, σ_1, σ_2 are the unknown constant parameters.

The research data of the JNJ-53718678 drug by Huntjens, D.R.H et al. [25] was cited as the discrete data for the model. JNJ-53718678 is a small molecule fusion inhibitor for the treatment of respiratory diseases. A single injection of 250 mg of JNJ-53718678 was administered, and the plasma drug concentration was measured before injection and at 0.5h, 1.0h, 1.5h, 2.0h, 3.0h, 4.0h, 6.0h, 8.0h, 12.0h, 16.0h and 24.0h after injection. The specific data is shown in Table 5.

Table 5 Plasma drug concentration data at each time point in Example 9

n	1	2	3	4	5	6
$Time(h)$	0.0	0.5	1.0	1.5	2.0	3.0
$Drugconcentration(ng/ml)$	0.0	878.6	1967.2	1783.4	1678.8	1223.5
$Time(h)$	4.0	6.0	8.0	12.0	16.0	24.0
$Drugconcentration(ng/ml)$	1109.2	686.7	503.9	438.6	275.3	136.4

The corresponding updated multi-factor uncertain differential equation is

$$\begin{cases} \mathrm{d}X_t = (k_0 - k_1 X_t)\mathrm{d}t + \sigma_1 X_t \mathrm{d}C_{1t} + \sigma_2 \mathrm{d}C_{2t} \\ X_{t_{i-1}} = x_{t_{i-1}} \end{cases} \tag{8}$$

and the corresponding updated ordinary differential equation is

$$\begin{cases} \mathrm{d}X_t^\alpha = (k_0 - k_1 X_t^\alpha)\mathrm{d}t + |\sigma_1 X_t^\alpha + \sigma_2| \dfrac{\sqrt{3}}{\pi} \ln \dfrac{\alpha}{1-\alpha} \mathrm{d}t \\ X_{t_{i-1}}^\alpha = x_{t_{i-1}} \end{cases}. \tag{9}$$

Therefore, the residual expression can be obtained as

$$\varepsilon_i(k_0, k_1, \sigma_1, \sigma_2) = 1 - \left(1 + \exp\left(\frac{\pi(X_{t_i} - X_{t_{i-1}} - k_0 + k_1 X_{t_i})(t_i - t_{i-1}))}{\sqrt{3}\,|\sigma_1 X_{t_i} + \sigma_2|\,(t_i - t_{i-1})}\right)\right)^{-1}.$$

According to the principle of the moment estimation, we obtain the following equations

$$\begin{cases} \frac{1}{N-1} \sum\limits_{i=1}^{N-1} \varepsilon_i(k_0, k_1, \sigma_1, \sigma_2) = \frac{1}{2} \\ \frac{1}{N-1} \sum\limits_{i=1}^{N-1} (\varepsilon_i(k_0, k_1, \sigma_1, \sigma_2))^2 = \frac{1}{3} \\ \frac{1}{N-1} \sum\limits_{i=1}^{N-1} (\varepsilon_i(k_0, k_1, \sigma_1, \sigma_2))^3 = \frac{1}{4} \\ \frac{1}{N-1} \sum\limits_{i=1}^{N-1} (\varepsilon_i(k_0, k_1, \sigma_1, \sigma_2))^4 = \frac{1}{5} \end{cases}.$$

By solving the above system of equations, the moment estimates of the unknown parameters are:

$$\hat{k_0} = 0.9780, \quad \hat{k_1} = 0.3177, \quad \hat{\sigma_1} = 0.5830 \quad \text{and} \quad \hat{\sigma_2} = 0.7134.$$

We obtained the multi-factor uncertain pharmacokinetic model equation as

$$\mathrm{d}X_t = (0.9780 - 0.3177X_t)\mathrm{d}t + 0.5830X_t\mathrm{d}C_{1t} + 0.7134\mathrm{d}C_{2t}.$$

6 Conclusion

In this paper, the concept of residual of uncertain differential equations with multiple factors is proposed. For the multi-factor uncertain differential equation, which can obtain the uncertainty distribution directly, the concrete expression of the residual is given. When the uncertainty distribution cannot be obtained directly, the approximate expression of the residual is obtained by using the corresponding updated ordinary differential equation. Then, the unknown parameters are obtained by moment estimation and generalized moment estimation, which is based on the characteristic that the residual obeies the linear uncertainty distribution $\mathcal{L}(0,1)$. An example of multi-factor pharmacokinetics is given to verify the feasibility of residual method. Compared with previously proposed estimation methods, the residual method does not need to weight and normalize the unknown parameters. When the time interval is relatively large and the difference method is not applicable, residual method can also be used for estimation. In the future, the residual method can be used to estimate the parameters of high-order uncertain differential equations or multi-dimensional uncertain differential equations.

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Compliance with ethical standards

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Ethical approval This paper does not contain any studies with human participants or animals performed by any of the authors.

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