A Finite Element method for Pricing Continuous-Installment Options under a Markov-modulated model

saghar heidari (s_heidari@sbu.ac.ir)
Shahid Beheshti University Faculty of Mathematical Sciences https://orcid.org/0000-0002-1639-1726

Research Article

Keywords: continuous-installment option, free boundary problem, regime-switching model, finite element method, linear complementarity problem, variational inequality

Posted Date: October 24th, 2022

DOI: https://doi.org/10.21203/rs.3.rs-1338217/v1

A FINITE ELEMENT METHOD FOR PRICING CONTINUOUS-INSTALLMENT OPTIONS UNDER A MARKOV-MODULATED MODEL

SAGHAR HEIDARI

Abstract. In this paper, we apply Markov-modulated models to value continuous-installment options of European style with partial differential equation approach. Under regime switching models and the opportunity for continuing or stopping to pay installments, the valuation problem can be formulated as coupled partial differential equations (CPDE) with free boundary features, which in many ways is similar to the free boundary problem for vanilla American options due to the possibility of early exercise. In this paper to value the continuous-installment options under the proposed model with numerical approach, we first express the truncated CPDE as a linear complementarity problem (LCP), then a finite element method is applied to solve the resulting variational inequality. Under some appropriate assumptions, we establish the stability of the method and illustrate some numerical results to examine the rate of convergence and accuracy of the proposed method for the pricing problem under regime-switching model.

1. Introduction

Nowadays, financial derivatives, in particular the installment options are widely considered in the modern financial markets. An European continuous-installment option is a type of path-dependent derivative instrument in which the buyer pays a low initial premium and a stream of further payments as installments at a given rate per unit time during the option lifetime, instead of the entire up-front premium. For these options, the holder has the right to lapse the contract through termination of payments at any time before the maturity, in which case the option payoff is zero (the present value of the expected payoff is less than the present value of the remaining payments). If all installments are paid until maturity, the holder can exercise the option and receive the exercise value. Due to this opportunity, this type of European option can be viewed as an American vanilla option with early exercise feature, and leads to a pricing problem with free boundary similar to that arising for American vanilla option. This flexibility in payments, also makes the total premium of the European installment option considerably greater than the corresponding European vanilla option's premium. However, this style options are more attractive for the option's holder to reduce the losses and increase the portfolio's liquidity by entering the contract at a low initial cost and flexibility to make a decision to stop the contract at any time before the maturity. On the other hand, the option's seller can use simple hedging strategy to eliminate financial risk.

Date: Sunday 20^{th} February, 2022.
Key words and phrases. continuous-installment option, free boundary problem, regime-switching model, finite element method, linear complementarity problem, variational inequality.

Installment options have been rather widely traded in financial markets. For example, in life insurance contracts, which the insurance beneficiary can lapse the contract by stopping the premium payments, or in large capital investment projects, which can be canceled at any time due to investment uncertainty [15]. In the last decades, application of installment options have been extended to the financial markets. For instance, the Deutsche Bank's 10-year warrant [11], the Canadian retail market [9], installment warrants on Australian stocks [16], employee stock options [24], compound options [18] and so on. Installment options can also be embedded in other contracts. For more examples see [13].

To value European installment options, due to existence of changing in time boundaries, we face more difficulties as we need to find the free boundaries as well as the solutions of the partial differential equations. There are no many research papers in the literature on installment options, especially for the European-style installment options. Davis et al. [11, 12] employed the martingale approach to study the European discrete-installment option in the Black Scholes framework with stochastic volatility and derived no-arbitrage bounds for the initial premium of the option using a robust static hedging strategy. They also presented numerical simulations, using binomial simulations and finite difference methods. In [17], authors derived a closed form solution to the initial premium of a European discrete-installment option in terms of multidimensional cumulative normal distribution functions and examined the limiting case of an installment option with a continuous payment plan independent of underlying asset model. For the European continuous case, it is worth mentioning these papers: Alobaidi et al. [2, 1] analyzed the European continuous-installment option using a partial Laplace transform to derive an integral equation for the stopping boundary and studied its asymptotic behavior close to expiry for the both call and put options. Fahuai Yi et al. [29, 28] investigated a parabolic variational inequality arising from European continuous-installment call and put options to prove the existence and uniqueness of the solution of pricing problem. They also determined regularity and the bounds of the free boundary. In the numerical valuation of the European continuous-installment options, we consider following papers: Using Laplace-Carson transform, Kimura [23] solved the integral representation arising from the pricing problem of European continuous-installment call and put options written on assets with dividends by PDE approach and obtained a closed form for the stopping boundary. Ciurlia [10] priced European continuous-installment options with constant continuous dividend under Black–Scholes model and applied Monte Carlo approach to the integral representation for both the initial premium and the optimal stopping boundary. Recently Beiranvand et al. [3] applied penalty method for pricing problem of European continuous-installment option written on the underlying asset which pays constant dividend. They also used finite element method to solve the linear complementarity problem resulting from pricing European continuous-installment options under Black-Scholes model [4].

In this paper, we consider European continuous-installment options written on a risky asset. In the last decades, concerning asset price dynamics, a number of alternative models have appeared in the finance literature to overcome the deficiencies in the standard geometric Brownian motion model that is known as the Black-Scholes-Merton model [5, 25]). In this respect, Regime switching (RS) models [7], in which the dynamics of the change of economic regimes is modeled by a Markov chain, produce better results in fitting market data because they explain the jump patterns exhibited by risky assets in real markets. However, they are more difficult to handle compared to the basic Black–Scholes model. Since the pioneering paper of Hamilton [19], regime switching models have been extensively applied to options pricing in financial markets [8, 6].

The standard Black–Scholes model has been widely used to describe the dynamics of underlying assets in pricing problem of installment options. Deng [14] has considered jump diffusion model as an alternative model to the classical Black–Scholes framework, to price American continuous installment put options. But to the best of our knowledge, there has been no progress applying regime switching models for installment options and they are restricted to the Black–Scholes model and jump diffusion model. This motivated us to consider regime switching models to pricing problem of European continuous-installment options with partial differential equation approach. For this purpose, volatility, dividend, interest rate and installment rate, which are assumed constant in the classical Black–Scholes model, are allowed to take diverse values by switching states governing by Markov process. For pricing installment option under this model, we write the obtained problem as a linear complementarity problem and proposed a finite element method [26] in order to solve the truncated problem. We illustrate the stability of the proposed method when the mesh ratio k/h^2 is chosen sufficiently small and obtain quadratic convergence rates. We also present our numerical results to examine the accuracy of the applied method.

The paper is organized as follows. In section 2, we introduce the free boundary problems for the European continuous-installment options under regime-switching model for both put and call. A finite element method is proposed in section 3 to solve the resulted linear complementarity problems. This section also contains the implementation of Crank–Nicolson discretization to obtain the systems of equations. In section 4, we analyze stability of the method under some appropriate assumptions. We illustrate numerical results to show the suitability of the proposed model and the accuracy of the applied method in section 5. Conclusion is presented in section 6.

2. European continuous-installment options under regime-switching model

In this section, we derive the system of partial differential equations(PDEs) satisfied by European continuous-installment option's price under a regime-switching model. For this, we borrow the idea and notations from paper [20, 21] to define the regime-switching model for the dynamic of underlying asset in installment options.

We define the state of system by a finite state continuous-time Markov chain $X = \{X(t), t \geq 0\}$ on probability space $(\Omega, \mathcal{F}, P)$ with the neutral-risk probability measure P.

Note In this paper, we consider the Markov chain with two regimes, since all the approaches can be extended for a problem with more than two regimes.

We assume the set of state vectors $\{e_1, e_2\}$ in $\mathbb{R}^2$, where $e_1 = (1, 0)$ and $e_2 = (0, 1)$.

Let the matrix $Q = (q_{i,j})_{2\times 2}$ be the rate matrix of the Markov chain. We know (see e.g. [21]) that $q_{i,j} > 0$ for $i \neq j$ and $\sum_{j=1}^{2} q_{i,j} = 0, \quad i = 1, 2$.

Discretizing the continuous-time Markov chain by partitioning the time into steps of size Δt, we obtain the infinitesimal transition probabilities:

$$P_{ij} = P\big(X(t+\Delta t) = j | X(t) = i\big) = \begin{cases} 1 + q_{i,i}\Delta t & i = j \\ q_{i,i}\Delta t & otherwise, \end{cases}$$

Therefore, the dynamics X is specified as the following semimartingale representation:

$$dX(t) = QX(t)dt + dM_t,$$

where $M = \{M_t, t \geq 0\}$ is a martingale with respect to the filtration $\mathcal{F}_t = \sigma\{X(u), u \leq t\}$.

Suppose the risk-free interest rate $r = \{r_t, t \geq 0\}$, the drift $\mu = \{\mu_t, t \geq 0\}$ and volatility $\sigma = \{\sigma_t, t \geq 0\}$ depend on the state X of the economy, so that

$$\begin{aligned} r_t &= r(X_t), \\ \mu_t &= \mu(X_t), \\ \sigma_t &= \sigma(X_t). \end{aligned}$$

Thus, the vectors r, μ and σ in $\mathbb{R}^2$ are as follows:

$$\begin{aligned} r &= (r_1, r_2)^T, \\ \mu &= (\mu_1, \mu_2)^T, \\ \sigma &= (\sigma_1, \sigma_2)^T. \end{aligned}$$

When the economy is in the i-th state (i.e., $X(t) = e_i$), the asset price process $S(t)$ is assumed to follow the stochastic differential equation:

$$\frac{dS(t)}{S(t)} = \mu_i dt + \sigma_i dW(t), \quad t > 0,$$

where $\mu_i = r_i - d_i$ with $r_i \geq 0$ the risk free interest rate and $d_i \geq 0$ the dividend, $\sigma_i \geq 0$ is the volatility rate and $W(t)$ is a standard Brownian motion, independent of the Markov chain $X(t)$.

Consider a European-style continuous-installment call option written on $S(t)$ with the vector of installment rates $q = (q_1, q_2)^T$, strike price \$$K$ and expiry date T years. Note that q is the continual input of cash via the premium, it means that at time t, the holder must pay an amount qt to keep the option alive.

Let $C(S, t; q)$ be the option price at time t such that

$$C(S, t; q) = \left(C_1(S, t; q), C_2(S, t; q)\right)^T,$$

where $C_i(S, t; q)$ be the option price when the economy is in the i-th state.

The opportunity to terminate the contract at any time, leads the pricing problem of installment options to an optimal stopping problem. Then under the risk-neutral probability measure, we have

$$C_i(S, t; q) = \sup_{\tau \in \mathscr{S}} \mathrm{E}\left[e^{-r(T-t)}\big(S(T) - K\big)^+ \mathbb{1}_{\{\tau \geq T\}} - \frac{q}{r}\big(1 - e^{-r(\tau \wedge (T-t))}\big) \Big| S(t) = S, X(t) = e_i\right],$$

where $\mathscr{S}$ is the set of all stopping times taking values in interval $[t, T]$ and $\tau \wedge T = \min\{\tau, T\}$.

Now by applying the Ito's rule to C_i we have

$$\begin{aligned} C_i(S, t; q) = C_i(0, t; q) &+ \int_0^t \frac{\partial C_i}{\partial u} du + \int_0^t \frac{\partial C_i}{\partial S} dS(u) + \frac{1}{2} \int_0^t \sigma_i^2 S^2 \frac{\partial^2 C_i}{\partial S^2} du \\ &- \int_0^t q_i du + \int_0^t (C, dX(u)). \end{aligned}$$

We know that the discounted price process is a martingale, therefore using no-arbitrage arguments, we get the following non-homogeneous partial differential equation for $C_i(S,t;q)$:

$$\begin{aligned}\frac{\partial C_i}{\partial t}(S,t;q) + \frac{1}{2}\sigma_i^2 S^2 \frac{\partial^2 C_i}{\partial S^2}(S,t;q) + \mu_i S \frac{\partial C_i}{\partial S}(S,t;q) - r_i C_i(S,t;q) \\ + (C(S,t;q), dX(t)) = q_i.\end{aligned}$$

From [20] we define the domain $D = \{(S,t) | S \in [0,\infty), t \in [0,T]\}$. When the economy is in the i-th state, if the present value of the expected payoff is less than the present value of the remaining installments, we have to let the contract to lapse. So, there is an optimal stopping time such that we should stop paying option premiums and the contract should be terminated. The asset price at the optimal stopping time, the stopping boundary $S_i^c(t;q)$, divides the domain D into subdomains $D_{1,i}$ and $D_{2,i}$ such that the set $D_{1,i} = \{(S,t) \in D | C_i(S,t;q) = 0\}$ is known the stopping region and the set $D_{2,i} = \{(S,t) \in D | C_i(S,t;q) > 0\}$ is called the continuation region, where it is optimal to continue paying the premiums. Therefore, the stopping boundary $S_i^c(t;q)$ is defined as:

$$S_i^c(t;q) = \inf\{S \in [0,\infty) | C_i(S,t;q) > 0\}, \quad t \in [0,T].$$

To apply the value matching and smooth contact conditions for installment option, due to the fact that option is worthless at the stopping region, the option's value and its derivative with respect to S (the option's delta) must be continuous across the boundary. This means that for $0 \le t < T$, they must be zero at the boundary:

$$\lim_{S \downarrow S_i^c(t;q)} C_i(S,t;q) = \lim_{S \downarrow S_i^c(t;q)} \frac{\partial C_i}{\partial S}(S,t;q) = 0.$$

Therefore, $C(S,t;q) = (C_1(S,t;q), C_2(S,t;q))^T$ and the stopping boundary $S_i^c(t;q)$ solve the following free boundary value problem consisting of the inhomogeneous partial differential equation:

$$\frac{\partial C_i}{\partial t}(S,t;q) + \mathcal{L}_i C(S,t;q) = q_i, \quad S > S_i^c(t;q), \tag{2.1}$$

with $\mathcal{L}_i = \mathcal{L}_i^{BS} + \mathcal{L}_i^{RS}$,
where the operators $\mathcal{L}_i^{BS}$ and $\mathcal{L}_i^{RS}$ for the Black-Scholes and regime-switching terms respectively are defined [20] as

$$\begin{aligned}\mathcal{L}_i^{BS} C_i(S,t;q) &= \frac{1}{2}\sigma_i^2 S^2 \frac{\partial^2 C_i}{\partial S^2}(S,t;q) + \mu_i S \frac{\partial C_i}{\partial S}(S,t;q) - r_i C_i(S,t;q), \\ \mathcal{L}_i^{RS} C(S,t;q) &= \sum_{j=1}^{2} q_{i,j} C_j(S,t;q),\end{aligned}$$

with terminal condition

$$C_i(S,T;q) = (S-K)^+, \quad S \ge 0, \tag{2.2}$$

and boundary conditions

$$\lim_{S\downarrow S_i^c(t;q)} \frac{\partial C_i}{\partial S}(S,t;q) = 0, \quad 0 \le t < T, \tag{2.3}$$

$$\lim_{S\downarrow S_i^c(t;q)} C_i(S,t;q) = 0, \quad 0 \le t < T, \tag{2.4}$$

$$\lim_{S\uparrow\infty} \frac{\partial C_i}{\partial S}(S,t;q) < \infty, \quad 0 \le t < T. \tag{2.5}$$

For put option problem, we have the similar free boundary value problem consisting of the inhomogeneous partial differential equation:

$$\frac{\partial P_i}{\partial t}(S,t;q) + \mathcal{L}_i P(S,t;q) = q_i, \quad S < S_i^p(t;q), \tag{2.6}$$

where the operators $\mathcal{L}_i, \mathcal{L}_i^{BS}$ and $\mathcal{L}_i^{RS}$ are defined as above:

$$\mathcal{L}_i = \mathcal{L}_i^{BS} + \mathcal{L}_i^{RS},$$
$$\mathcal{L}_i^{BS} P_i(S,t;q) = \frac{1}{2}\sigma_i^2 S^2 \frac{\partial^2 P_i}{\partial S^2}(S,t;q) + \mu_i S \frac{\partial P_i}{\partial S}(S,t;q) - r_i P_i(S,t;q),$$
$$\mathcal{L}_i^{RS} P(S,t;q) = \sum_{j=1}^{2} q_{i,j} P_j(S,t;q),$$

with terminal condition

$$P_i(S,T;q) = (K-S)^+, \quad S \ge 0, \tag{2.7}$$

and boundary conditions

$$\lim_{S\uparrow S_i^p(t;q)} \frac{\partial P_i}{\partial S}(S,t;q) = 0, \quad 0 \le t < T, \tag{2.8}$$

$$\lim_{S\uparrow S_i^p(t;q)} P_i(S,t;q) = 0, \quad 0 \le t < T, \tag{2.9}$$

$$\lim_{S\downarrow 0} \frac{\partial P_i}{\partial S}(S,t;q) < \infty, \quad 0 \le t < T. \tag{2.10}$$

where $P_i(S,t;q)$ and $S_i^p(t;q)$ are the put price and the stopping boundary in the i-th state of economy, respectively.

As we know the free boundary has the same value for both the put and the call installment options at expiry,

$$S_i^p(T;q) = S_i^c(T;q) = K,$$

and the value of q does not affect the position of the free boundary at expiry, but it affects the slope of the boundary. The competition between the premiums to be paid and the payoff to be received, makes the free boundary for installment option more complicated than a vanilla American option and leads to a boundary with non-monotonic behavior. For more details of installment option free boundary see [2].

Due to the well-known put-call duality, the put option price can be evaluated through the call option price. So we only consider call options. Using the variable transformation for the call option

$$S = Ke^x, \quad C_i(S,T-t;q) = Ku_i(x,t;q), \quad S_i^c(T-t;q) = Ke^{x_i^c(t;q)},$$

the linear complementarity problem [20] of the installment call option (2.1)–(2.5) is given by

$$\frac{\partial u_i}{\partial t}(x,t;q) + \mathcal{L}_i u(x,t;q) \geq -q_i, \quad x \in \mathbb{R}, t \in T, \tag{2.11}$$

$$u_i(x,t;q) \geq 0, \quad x \in \mathbb{R}, t \in T, \tag{2.12}$$

$$\left(\frac{\partial u_i}{\partial t}(x,t;q) + \mathcal{L}_i u(x,t;q) + q_i\right) u_i(x,t;q) = 0, \quad x \in \mathbb{R}, t \in T, \tag{2.13}$$

with the initial condition

$$u_i(x,0;q) = f_i(x), \quad x \in \mathbb{R}, \tag{2.14}$$

the operators

$$\mathcal{L}_i u(x,t;q) = -\gamma_i u_{i,xx} + \nu_i u_{i,x} + \kappa_i u_i - \mathcal{R}_i u, \tag{2.15}$$

$$\mathcal{R}_i u(x,t;q) = \sum_{j=1}^{2} q_{i,j} u_j(x,t;q), \tag{2.16}$$

and

$$f_i(x) = (e^x - 1)^+, \tag{2.17}$$

$$\gamma_i = \frac{1}{2}\sigma_i^2, \quad \nu_i = \gamma_i - \mu_i, \quad \kappa_i = r_i - q_{i,i}. \tag{2.18}$$

3. A Finite Element Method

In this section, a finite element method [22, 20] is developed for solving the pricing problem (2.11)–(2.14) for the installment call option.

We truncate the pricing problem (2.11)–(2.14) over a bounded domain $\Omega = (X_{\min}, X_{\max})$ for negative number $X_{\min}$ and positive number $X_{\max}$. Define $\bar{D} = \Omega \times T$, then we approximate the linear complementarity problem (2.11)–(2.14) for $i = 1, 2$ as follow:

$$\frac{\partial u_i}{\partial t}(x,t;q) + \mathcal{L}_i u(x,t;q) \geq -q_i, \quad (x,t) \in \bar{D}, \tag{3.1}$$

$$u_i(x,t;q) \geq 0, \quad (x,t) \in \bar{D}, \tag{3.2}$$

$$\left(\frac{\partial u_i}{\partial t}(x,t;q) + \mathcal{L}_i u(x,t;q) + q_i\right) u_i(x,t;q) = 0, \quad (x,t) \in \bar{D}, \tag{3.3}$$

with the boundary conditions

$$u_i(x,t;q) = f_i(x), \quad x \in \partial\Omega, t \in T, \tag{3.4}$$

and initial condition

$$u_i(x,0;q) = f_i(x), \quad x \in \Omega. \tag{3.5}$$

Since it is easier to deal with the homogeneous boundary conditions, we define functions $g_i \in C^2$ for $i = 1, 2$ such that $g_i(x)|_{\partial\Omega} = f_i(x)$. Then we consider the new unknown functions

$$v_i(x, t; q) = u_i(x, t; q) - g_i(x), \quad i = 1, 2, \quad (x, t) \in \bar{D}.$$

Thus, we rewrite the linear complementary problem (3.1)–(3.5) as

$$\frac{\partial v_i}{\partial t}(x, t; q) + \mathcal{L}_i v(x, t; q) \geq \mathcal{G}_i(x), \quad (x, t) \in \bar{D}, \tag{3.6}$$

$$v_i(x, t; q) \geq -g_i(x), \quad (x, t) \in \bar{D}, \tag{3.7}$$

$$\left(\frac{\partial v_i}{\partial t}(x, t; q) + \mathcal{L}_i v(x, t; q) - \mathcal{G}_i(x)\right) v_i(x, t; q) = 0, \quad (x, t) \in \bar{D}, \tag{3.8}$$

$$v_i(x, t; q) = 0, \quad x \in \partial\Omega, t \in T, \tag{3.9}$$

$$v_i(x, 0; q) = f_i(x) - g_i(x), \quad x \in \Omega, \tag{3.10}$$

where $\mathcal{G}_i = \mathcal{L}_i g_i$ for $i = 1, 2$.

Now we are ready to write the variational inequality. Let

$$W = \left\{ v : \ v \in L^2(0, T; H_0^1(\Omega)), \ \frac{\partial v}{\partial t} \in L^2(0, T; H^{-1}(\Omega)), \ v \geq -g \quad \text{a.e. in} \quad \bar{D} \right\}$$

and

$$V = \{ v \in H_0^1(\Omega), v \geq -g \quad \text{on} \quad \Omega \},$$

where we define as [20]:

$$H_0^1(\Omega) = \left\{ v : \ v \in L^2(\Omega), \ \frac{\partial v}{\partial x} \in L^2(\Omega), v|_{\partial\Omega} = 0 \right\},$$

and

$$L^2(0, T; H_0^1(\Omega)) = \left\{ v : \ v(., t) \in H_0^1(\Omega), \ \left(\int_0^T \|v(., t)\|^2_{H_0^1(\Omega)} dt \right)^{1/2} < \infty \right\}.$$

Then we have the variational problem for (3.6)–(3.10):

Find $w_i \in W$ such that

$$\left(\frac{\partial w_i}{\partial t}, v - w_i\right) + \mathcal{A}_i(w, v - w_i) \geq G_i(v - w_i), \quad \forall v \in V, \quad i = 1, 2, \tag{3.11}$$

where $w = (w_1, w_2)$, the operators

$$\begin{aligned}
\mathcal{A}_i(w, v) &= \mathcal{A}_{i,1}(w_i, v) + \mathcal{A}_{i,2}(w_{3-i}, v) \\
\mathcal{A}_{i,1}(w_i, v) &= \gamma_i(w_{i,x}, v_x) + (\nu_i w_{i,x} + \kappa_i w_i, v), \\
\mathcal{A}_{i,2}(w_{3-i}, v) &= -q_{i,3-i} \left(w_{3-i}, v\right),
\end{aligned}$$

and

$$\begin{aligned} G_i(v) &= \int_0^{X_{\max}} (\alpha_i e^x + \beta_i)\, v(x)dx - \int_{X_{\min}}^{X_{\max}} q_i v(x)dx, \\ \alpha_i &= \gamma_i - \nu_i - \kappa_i - q_{ii}, \\ \beta_i &= \kappa_i + q_{ii}. \end{aligned}$$

Let $\Pi_t : 0 = t_0 < t_1 < \cdots < t_M = T$ and $\Pi_x :\ X_{\min} = x_0 < x_1 < \cdots < x_N = X_{\max}$ be partitions of T and $\bar{D}$, respectively, where $M,\ N$ are positive integers. Let V_h be the piecewise linear element subspace of V with respect to partition Π_x, where $h = \max_{1\le j\le N}(x_j - x_{j-1})$. Let $w_{i,h}^0 = 0$. The finite element approximation to the variational problem (3.11) is:

For $m = 1, 2, \ldots, M$, find $w_{1,h}^m,\ w_{2,h}^m \in V_h$ such that

$$(\delta_t w_{i,h}^m, v - w_{i,h}^m) + \mathcal{A}_i\left(w_h^{m-\frac{1}{2}}, v - w_{i,h}^m\right) \ge G_i(v - w_{i,h}^m), \quad \forall v \in V_h, \quad i = 1, 2, \tag{3.12}$$

where $w_h^m = \left(w_{1,h}^m,\ w_{2,h}^m\right)$, and

$$\delta_t w_{i,h}^m = \frac{w_{i,h}^m - w_{i,h}^{m-1}}{k_m}, \quad w_{i,h}^{m-\frac{1}{2}} = \frac{1}{2}\left(w_{i,h}^m + w_{i,h}^{m-1}\right), \quad k_m = t_m - t_{m-1}.$$

Here we have only considered the Crank-Nicolson scheme since it has the highest order of the truncation error.

Denote the linear basis functions of V_h by $\phi_1, \ldots, \phi_{N-1}$, i.e., $\phi_j(x_l) = \delta_{l,j}$ for $j = 1, 2, \ldots, N-1$ and $l = 0, 1, \ldots, N$, where $\delta_{l,j}$ is the Kronecker delta. Write

$$w_{i,h}^m(x) = \sum_{j=1}^{N-1} w_{ij}^m \phi_j(x),$$

and let

$$W_i^m = \left(w_{i1}^m, \ldots, w_{i(N-1)}^m\right)^T, \quad i = 1, 2.$$

Then the matrix form of the linear complementarity problem (3.12) can be written as:

$$\begin{cases} A_{i,1}W_1^m + A_{i,2}W_2^m \ge F_i^m, \quad W_i^m \ge 0, \\ \left(A_{i,1}W_1^m + A_{i,2}W_2^m - F_i^m\right)^T W_i^m = 0, \end{cases} \tag{3.13}$$

for $i = 1, 2,\ m = 1, 2, \ldots, M$, where

$$\begin{aligned} A_{i,i} &= \left((\phi_k, \phi_l) + \frac{1}{2}k_m \mathcal{A}_{i,1}(\phi_k, \phi_l)\right)_{(N-1)\times(N-1)}, \\ A_{i,3-i} &= \left(\frac{1}{2}k_m \mathcal{A}_{i,2}(\phi_k, \phi_l)\right)_{(N-1)\times(N-1)}, \\ F_i^m &= k_m G_i^m + B_{i,i}W_i^{m-1} - A_{i,3-i}W_{3-i}^{m-1}, \\ G_i^m &= \left(G_i\left(\phi_1\right), \ldots, G_i\left(\phi_{N-1}\right)\right)^T, \\ B_{i,i} &= \left((\phi_k, \phi_l) - \frac{1}{2}k_m \mathcal{A}_{i,1}(\phi_k, \phi_l)\right)_{(N-1)\times(N-1)}. \end{aligned}$$

The inner products of linear basis functions can be computed exactly for $k, l = 1, 2, \ldots, N-1$:

$$(\phi_k, \phi_l) = \begin{cases} 2h/3, & k = l, \\ h/6, & |k-l| = 1, \\ 0, & otherwise, \end{cases}$$

$$(\phi'_k, \phi_l) = \begin{cases} 1/2, & k-l = 1, \\ -1/2, & k-l = -1, \\ 0, & otherwise, \end{cases}$$

$$(\phi'_k, \phi'_l) = \begin{cases} 2/h, & k = l, \\ -1/h, & |k-l| = 1, \\ 0, & otherwise. \end{cases}$$

It is easy to see that $A_{1,1}$ and $A_{2,2}$ are positive definite tridiagonal matrices and that A_{12} and A_{21} are symmetric tridiagonal matrices. Let ϵ be the tolerance and W_1^{old} and W_2^{old} be the initial guesses. The projected block Gauss-Seidel algorithm for (3.13) reads as follows:

Step 1. Solve the following LCP for W_1^{new} by the projected Thomas algorithm:

$$\begin{cases} A_{1,1}W_1^{new} + A_{1,2}W_2^{old} \geq F_1, \quad W_1^{new} \geq 0, \\ \left(A_{1,1}W_1^{new} + A_{1,2}W_2^{old} - F_1\right)^T W_1^{new} = 0. \end{cases}$$

Step 2. Solve the following LCP for W_2^{new} by the projected Thomas algorithm:

$$\begin{cases} A_{2,1}W_1^{new} + A_{2,2}W_2^{new} \geq F_2, \quad W_2^{new} \geq 0, \\ \left(A_{2,1}W_1^{new} + A_{2,2}W_2^{new} - F_2\right)^T W_2^{new} = 0. \end{cases}$$

Step 3. If

$$|W_1^{new} - W_1^{old}|^2 + |W_2^{new} - W_2^{old}|^2 \leq \epsilon^2,$$

then stop. Otherwise, let $W_i^{old} = W_i^{new}$ for $i = 1, 2$, and then go to (1).

4. Stability

In this section we show the existence and uniqueness of the solution to problem (3.12) and analyze stability of the proposed method.

Letting $v = 0$ in (3.12), the finite element approximation for our problem can be simplified as:

$$(\delta_t w_{i,h}^m, w_{i,h}^m) + \mathcal{A}_i(w_h^{m-\frac{1}{2}}, w_{i,h}^m) \geq G_i(w_{i,h}^m), \quad i = 1, 2. \tag{4.1}$$

Now we define the bilinear form

$$\mathcal{A}(w_1, w_2; v_1, v_2) := \sum_{i=1}^{2} \mathcal{A}_{i,1}(w_i, v_i) + \mathcal{A}_{i,2}(w_{3-i}, v_i).$$

We shall prove the existence of a unique solution to the problem (3.12) by exploiting the following result from the Lions–Stampacchia Theorem:

For all $w_i, v_i, v \in H_0^1(\Omega)$, there are positive constants c_1, c_2 and c_3 such that

$$|\mathcal{A}(w_1, w_2; w_1, w_2)| \geq c_1 \sum_{i=1}^{2} \|w_{i,x}\|^2, \tag{4.2}$$

$$|\mathcal{A}(w_1, w_2; v_1, v_2)| \leq c_2 \sqrt{\|w_{1,x}\|^2 + \|w_{2,x}\|^2} \sqrt{\|v_{1,x}\|^2 + \|v_{2,x}\|^2}, \tag{4.3}$$

$$|G_i(v)| \leq c_3 \|v_x\|, \tag{4.4}$$

where $\|.\|$ denotes the norm of $L^2(\Omega)$.

From the definition of $\mathcal{A}$ and applying the *Cauchy–Schwarz* and the *Poincaré* inequalities we have

$$\begin{aligned}
\mathcal{A}(w_1, w_2; w_1, w_2) &= \sum_{i=1}^{2} \mathcal{A}_{i,1}(w_i, w_i) + \mathcal{A}_{i,2}(w_{3-i}, w_i) \\
&= \sum_i \gamma_i \|w_{i,x}\|^2 + \nu_i(w_{i,x}, w_i) + \kappa_i \|w_i\|^2 - (q_{12} + q_{21})(w_1, w_2) \\
&\geq \sum_i \gamma_i \|w_{i,x}\|^2 + \kappa_i \|w_i\|^2 - \epsilon \|w_1\| \|w_2\| \\
&\geq \sum_i \gamma_i \|w_{i,x}\|^2 + \zeta \|w_1\| \|w_2\| \\
&\geq \sum_i \gamma_i \|w_{i,x}\|^2.
\end{aligned}$$

Here we have considered that $\zeta \geq 0$, where

$$\zeta = 2\sqrt{\kappa_1 \kappa_2} - \epsilon,$$

and

$$\epsilon = q_{12} + q_{21}.$$

We can choose the parameters such that $\zeta \geq 0$, i.e.

$$4\kappa_1 \kappa_2 \geq \epsilon^2. \tag{4.5}$$

So by considering (4.5) and putting $c_1 = \max_{1 \leq i \leq 2} \gamma_i$ the bilinear form $\mathcal{A}$ is coercive.

For obtaining the required bound in (4.3) we know from [27] that there is positive constant c_2 such that

$$|\mathcal{A}(w_1, w_2; w_1, w_2)| \leq c_2 \sqrt{\|w_{1,x}\|^2 + \|w_{2,x}\|^2} \sqrt{\|v_{1,x}\|^2 + \|v_{2,x}\|^2}.$$

Also by the *Cauchy-Schwarz* inequality

$$\begin{aligned}
&\left| \int_0^{X_{\max}} H_i(x) v(x) dx \right| \leq \Big(\int_0^{X_{\max}} |H_i(x)|^2 dx \Big)^{1/2} \Big(\int_0^{X_{\max}} |v(x)|^2 dx \Big)^{1/2} \\
&= \|H_i\| \|v\| \leq \|H_i\| \|v_x\|,
\end{aligned}$$

where $H_i(x) = \alpha_i e^x + \beta_i$, and let $\xi_1 = \max_i \|H_i\|$.

By using the same approach, it is easy to see that

$$|\int_{X_{\min}}^{X_{\max}} q_i v(x)dx| \le \xi_2 \|v_x\|,$$

where $\xi_2 = \max_i |q_i|$.
So by letting $c_3 = \max(\xi_1, \xi_2)$ we obtain the required bound (4.4),

$$|G_i(v)| \le c_3 \|v_x\|.$$

Therefore, we have the following theorem about the solution existence and uniqueness of (3.12).

Theorem 1. *When inequality (4.5) holds, the variational problem (3.12) has a unique solution.*

Now for getting an estimate for stability we apply the argument in [21, 27]. For this, by summation for $i = 1, 2$ in (4.1), we will have

$$\sum_{i=1}^{2} (\delta_t w_{ih}^m, w_{ih}^m) + \sum_{i=1}^{2} \mathcal{A}_i(w_h^{m-\frac{1}{2}}, w_{ih}^m) \le \sum_{i=1}^{2} G_i(w_{ih}^m).$$

By assumption, $\max_{1\le m\le M} k_m/h^2$ is small enough, and using the bounds from Theorem (1) and result [27], we obtain

$$\sum_{i=1}^{2} \left(\|w_{ih}^m\|^2 - \|w_{ih}^{m-1}\|^2\right) + \frac{k_m}{2} \sum_{i=1}^{2} \gamma_i \left(\|w_{ihx}^m\|^2 + \|w_{ihx}^{m-1}\|^2\right) \le c k_m, \tag{4.6}$$

where $c = 4c_3^2/\min_i \gamma_i$.

By summing (4.6) for $m = 1, ..., M$ and knowing that $w_{ih}^0 = 0$ we will have

$$\sum_{i=1}^{2} \left[\|w_{ih}^M\|^2 + \gamma_i \sum_{m=1}^{M-1} k_m \|w_{ih,x}^m\|^2 + \frac{\gamma_i}{2} k_M \|w_{ih,x}^M\|^2 \right] \le c \sum_{m=1}^{M} k_m.$$

Therefore, we have the following theorem about the stability problem.

Theorem 2. *For the variational problem (3.12), we have the following stability estimate:*

$$\max_{m=1}^{M} \|w_h^m\|^2 + \sum_{m=1}^{M} k_m \|w_{h,x}^m\|^2 \le C,$$

provided that the mesh ration $\frac{k_m}{h^2}$ is sufficiently small, where $\|.\|$ denotes the norm of $(L^2(\Omega))^2$ and C is a positive constant independent of w_h^m, h and k_m.

5. Numerical Results

In this section, we examine the results of applying the finite element method for pricing European installment options to verify the accuracy of our computational scheme. Since exact solutions for the options are unknown, we propose a comparison with some recent works and consider both European call and put options under regime switching model with two regimes. Our results show accurate values with considerable computational time. The computations were carried out with a C++ class in a computer with 8.00 GB RAM and 2.5 GHz processor. For implementation, we consider different combinations of problem's parameters. We also set the tolerance $\epsilon = 10^{-9}$ and uniform partitions are used for both space and time in $[-2.5, 2.5] \times [0, 1]$.

Test : First of all we show the convergence rates of the proposed method by refining mesh sizes. For this purpose, we consider the European installment call options with strike price $K = \$100$ and expiration time $T = 1$ year. we examine three different cases:

Case I. regime–switching volatilities:
In this case, we consider different volatilities, the same interest rates and installment rates for the two states of economy. The parameters are set as follows:

$$\begin{aligned} &\sigma_1 = 0.3,\ \sigma_2 = 0.2, \quad r_1 = r_2 = 0.1, \\ &d_1 = d_2 = 0.08,\ 0.10,\ 0.12, \quad q_1 = q_2 = 1. \end{aligned}$$

Case II. regime–switching interest rates:
For this case we consider the same volatilities and installment rate, but different interest rates for the two states of economy. Thus, the parameters are set as follows:

$$\begin{aligned} &\sigma_1 = \sigma_2 = 0.2, \quad r_1 = 0.10,\ r_2 = 0.06, \\ &d_1 = d_2 = 0.04,\ 0.06,\ 0.08,\ 0.10,\ 0.12, \quad q_1 = q_2 = 1. \end{aligned}$$

Case III. regime–switching installment rates:
In this case, we consider different installment rates and same interest rates and volatilities for the two states of economy. The parameters are set as follows:

$$\begin{aligned} &\sigma_1 = \sigma_2 = 0.2, \quad r_1 = r_2 = 0.1, \\ &d_1 = d_2 = 0.08,\ 0.10,\ 0.12, \quad q_1 = 1, q_2 = 3. \end{aligned}$$

For all the cases we let

$$q_{12} = 5, \quad q_{21} = 3.$$

For each case in Figure 1 we displayed graphs of the error $w_{2h}^m - w_h^m$ in L^2–norm and L^∞–norm for the space mesh size h of the option prices. It can be observed that the method is converging quadratically for the option price, as we expected.

Now let us consider some examples to represent our accurate numerical experiments resulting the proposed method.

Example 1: As the first example for testing our method, we use this fact that when q_{ij} for $i, j = 1, 2$ are set to be zero, the problems (2.1)– (2.5) and (2.6)– (2.10) becomes the decoupled problems for the installment call and put options without regime switching under the corresponding Black-Scholes models, respectively. Hence, we can use our program to evaluate the European installment options prices under the classical Black-Scholes models.

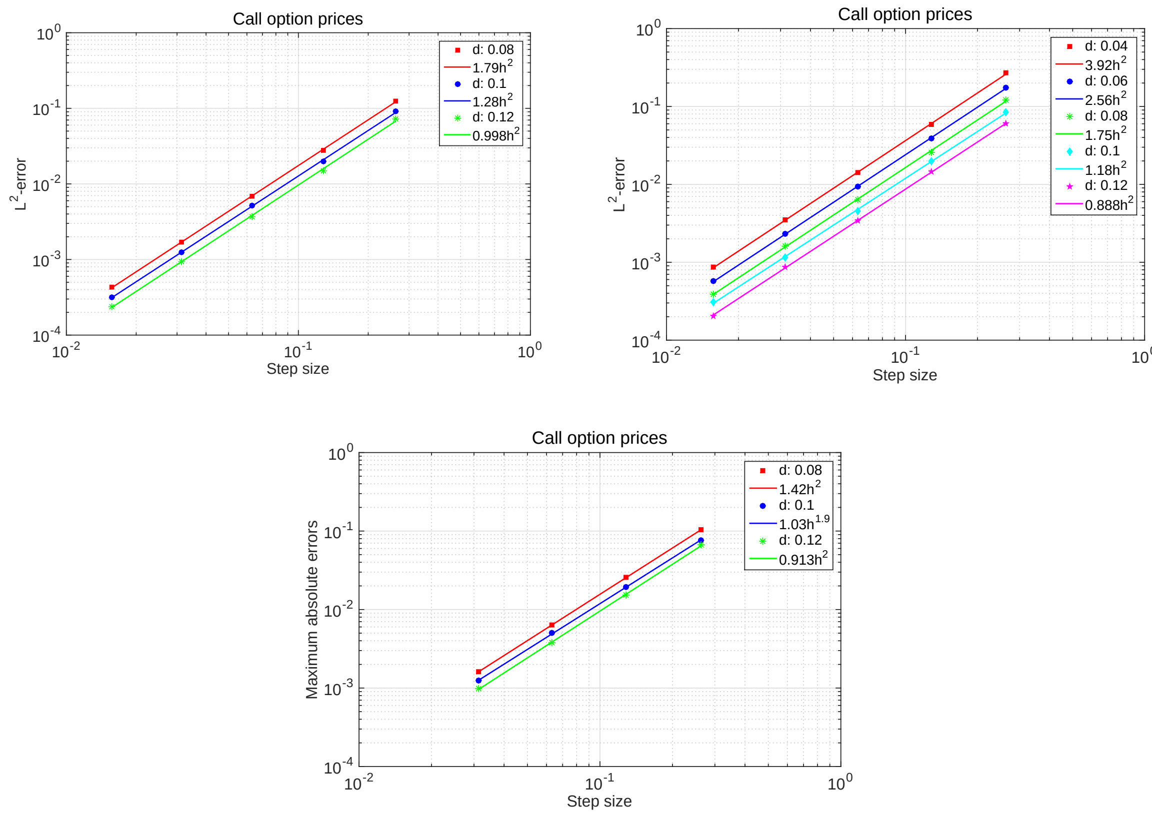

FIGURE 1. The convergence rates for prices of European installment call options under regime–switching volatilities, regime–switching interest rates and regime–switching installment rates, respectively.

For this purpose, we consider the European installment options with expiration date $T = 1$ year and strike price $K = \$100$. Other parameters are set as [23]:

$$r_1 = r_2 = 0.03; \quad d_1 = d_2 = 0.05; \quad \sigma_1 = \sigma_2 = 0.2.$$

Table 1 shows the price of European option under our model with no regime switching using finite element method and corresponding European option under Black-Scholes model using Laplace-Carson transform by Kimura in [23] for different values of initial stock prices S and installment rates. The data indicate that our algorithm can provide very accurate and reasonable results.

From Table 1, it can also be seen that the decrease in installment rates would lead to a higher price of installment option in our model. This proves that the premium of an installment option is less than the premium of a vanilla option. Thus, as installment rate q tends to zero in the first column of Table 1, installment option price approaches to price of its counterpart vanilla option.

Example 2: As the second example, we illustrate our numerical experiments for the installment European call and put options under regime switching of installment rates. For this purpose, we consider installment European call and put options with expiration date

TABLE 1. European installment call option prices by finite element method (FEM) and corresponding European installment call options by Laplace-Carson method (LCM) for different values of stock prices and installment rates.

	FEM				LCM		
S	q=0	q=1	q=3	q=6	q=1	q=3	q=6
95	4.5872	3.70712	2.22782	0.67600	3.7071	2.2280	0.6754
105	9.3451	8.40304	6.64219	4.25996	8.3994	6.6385	4.2745
115	15.8173	14.85289	12.96791	10.24892	14.8530	12.9687	10.2533

$T = 1$ year and strike price $K = \$100$. Other parameters are set as:

$$r_1 = r_2 = 0.05; \quad d_1 = d_2 = 0.04; \quad \sigma_1 = 0.3, \sigma_2 = 0.2;$$

and

$$q_1 = 3, \quad q_2 = 8,$$

where q_1 and q_2 are the installment rates in first and second regimes, respectively. For the regime switching parameters, let

$$q_{12} = 5, \quad q_{21} = 3.$$

Table 2 shows the price of European installment call and put options under our model with applying finite element method for different values of initial stock prices S in first and second regimes.

TABLE 2. Prices of European installment call and put options under regime–switching installment rates by finite element method for different values of stock prices.

	Regime 1		Regime 2	
S	Call	Put	Call	Put
96	7.6576	10.3571	1.2079	2.7891
100	10.0193	9.0013	2.6751	1.9902
104	11.9972	7.1290	4.0162	0.6731

6. Conclusion

In this paper, we studied pricing problem of European continuous-installment options under Markov-modulated model as an alternative model to the classical Black–Scholes model. For this purpose, the volatility, dividend, interest rate and installment rate, which are assumed constant in the Black–Scholes model, are allowed to take diverse values by switching states governed by Markov process. Then to price installment options under this model, we wrote the obtained free boundary problem as a linear complementarity problem and applied a finite element method to solve the arising coupled partial differential equations under regime switching models. We illustrated the stability of the method under some appropriate assumptions and examined its accuracy with some numerical results. From our numerical results, it is observed that the model produce better results in fitting market data, due to ability of regime switching models to capture the jump patterns exhibited by underlying assets. Our numerical experiments also show the stability and accuracy of the proposed method for pricing installment options under regime switching models. It is worth mentioning that the applied method converges quadratically, as we expected.

Acknowledgements

This research was supported by the Shahid Beheshti University.

Ethical Approval and Informed Consent

This article does not contain any studies with human or animal subjects.

Author Contributions

Author contributed to all aspects of this work.

Conflicts of Interest

The author declares no conflict of interest.

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Department of Mathematical Science , Shahid Beheshti University, Tehran, Iran
Email address: s_heidari@sbu.ac.ir, https://orcid.org/0000-0002-1639-1726