Modified approaches to solve matrix games with payoffs of single-valued trapezoidal neutrosophic numbers

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Modified approaches to solve matrix games with payoffs of single-valued trapezoidal neutrosophic numbers

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Abstract Seikh and Dutta (Soft Computing (2022) 26:921–936) pointed out that there does not exist any approach to solve single-valued trapezoidal neutrosophic (SVTN) matrix games (matrix games in which each payoff is represented by a SVTN number). To fill this gap, Seikh and Dutta proposed two approaches to solve SVTN matrix games. Brikaa (Soft Computing (2022) 26:9137–9139) pointed out that it is inappropriate to use Seikh and Dutta's first approach as a mathematically incorrect result is considered in it. Brikaa also modified Seikh and Dutta's first approach to resolve its inappropriateness. In this paper, it is pointed out that it is also inappropriate to use Brikaa's approach as some mathematically incorrect results are considered in it. Also, it is pointed out that it is inappropriate to use Seikh and Dutta's second approach as a mathematically incorrect result is considered in it. Furthermore, Brikaa's approach and Seikh and Dutta's second approach are modified to resolve their inappropriateness. Finally, the correct results of a SVTN matrix game, considered by Seikh and Dutta to illustrate their proposed approaches, are obtained by the modified approaches.

Keywords Matrix games, Single-valued trapezoidal neutrosophic numbers, Payoffs

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1 Introduction

On the basis of the existing results (Bector and Chandra, 2005, Section 1.4, pp. 6-7), it can be easily concluded that if in the payoff matrix $A = (a_{ij})_{m\times n}$ of a matrix game, a_{ij} represents the payoff of Player-I corresponding to the i^{th} strategy of Player-I and the j^{th} strategy of Player-II, m represents the number of strategies of Player-I and n represents the number of strategies of Player-II. Then, the optimal strategies $u_i, i = 1,2,\dots,m$ of Player-I can be obtained by solving the crisp mathematical programming problem (MPP) (P1) and the optimal strategies $v_j, j = 1,2,\dots,n$ of Player-II can be obtained by solving the crisp MPP (P2).

Problem (P1)

$$Maximize \left\{ \begin{array}{c} Minimize\left((\sum_{i=1}^{m} a_{i1}u_i)v_1 + (\sum_{i=1}^{m} a_{i2}u_i)v_2 + \cdots + (\sum_{i=1}^{m} a_{in}u_i)v_n\right) \\ \text{Subject to} \\ \sum_{j=1}^{n} v_j = 1,\ v_j \geq 0, j = 1,2,\dots,n \end{array} \right\}$$

Subject to

$\sum_{i=1}^{m} u_i = 1,\ u_i \geq 0, i = 1,2,\dots,m.$

Problem (P2)

$$Minimize \left\{ \begin{array}{c} Maximize\left((\sum_{j=1}^{n} a_{1j}v_j)u_1 + (\sum_{j=1}^{n} a_{2j}v_j)u_2 + \cdots + (\sum_{j=1}^{n} a_{mj}v_j)u_m\right) \\ \text{Subject to} \\ \sum_{i=1}^{m} u_i = 1,\ u_i \geq 0, i = 1,2,\dots,m \end{array} \right\}$$

Subject to

$\sum_{j=1}^{n} v_j = 1,\ v_j \geq 0, j = 1,2,\dots,n.$

In the literature (Bector and Chandra, 2005, Section 1.4, pp. 6-7), it is also shown that the crisp MPPs (P1) and (P2) are equivalent to the crisp linear programming problems (CLPPs) (P3) and (P4) respectively. So, to find the optimal strategies $u_i, i = 1,2,\dots,m$ of Player I is equivalent to find an optimal solution of the CLPP (P3) and to find the optimal strategies $v_j, j = 1,2,\dots,n$ of Player II is equivalent to find an optimal solution of the CLPP (P4).

Problem (P3)

$Maximize\ (\theta)$

Subject to

$\sum_{i=1}^{m} a_{ij} u_i \geq \theta, i = 1,2,\dots,m,$

$\sum_{i=1}^{m} u_i = 1,$

$u_i \geq 0, i = 1,2,\dots,m.$

Problem (P4)

$Minimize\ (\phi)$

Subject to

$\sum_{j=1}^{n} a_{ij} v_j \leq \phi, j = 1,2,\dots,n,$

$\sum_{j=1}^{n} v_j = 1,$

$v_j \geq 0, j = 1,2,\dots,n.$

On the same direction, Seikh and Dutta (2022) claimed that if in the payoff matrix $\tilde{A} = \left(\tilde{a}_{ij}\right)_{m\times n}$ of a SVTN matrix game, the SVTN number $\tilde{a}_{ij}$ represents the payoff of Player-I corresponding to the i^{th} strategy of Player-I and the j^{th} strategy of Player-II, m represent the number of strategies of Player-I and n represent the number of strategies of Player-II. Then, the optimal strategies $u_i, i = 1,2,\dots,m$ of Player-I can be obtained by solving the SVTN MPP (P5) and the optimal strategies $v_j, j = 1,2,\dots,n$ of Player-II can be obtained by solving the SVTN MPP (P6).

Problem (P5)

$$Maximize\begin{Bmatrix} Minimize\left((\sum_{i=1}^{m}\tilde{a}_{i1}u_i)v_1+(\sum_{i=1}^{m}\tilde{a}_{i2}u_i)v_2+\cdots+(\sum_{i=1}^{m}\tilde{a}_{in}u_i)v_n\right) \\ \text{Subject to} \\ \sum_{j=1}^{n}v_j=1, v_j\geq 0, j=1,2,\dots,n \end{Bmatrix}$$

Subject to

$\sum_{i=1}^{m}u_i=1, u_i\geq 0, i=1,2,\dots,m.$

Problem (P6)

$$Minimize\begin{Bmatrix} Maximize\left(\left(\sum_{j=1}^{n}\tilde{a}_{1j}v_j\right)u_1+\left(\sum_{j=1}^{n}\tilde{a}_{2j}v_j\right)u_2+\cdots+\left(\sum_{j=1}^{n}\tilde{a}_{mj}v_j\right)u_m\right) \\ \text{Subject to} \\ \sum_{i=1}^{m}u_i=1, u_i\geq 0, i=1,2,\dots,m \end{Bmatrix}$$

Subject to

$\sum_{j=1}^{n}v_j=1, v_j\geq 0, j=1,2,\dots,n.$

It is obvious that to find an optimal solution of the SVTN MPPs (P5) and (P6), there is a need to compare SVTN numbers. Keeping the same in mind, Seikh and Dutta (2022) considered the following two approaches for comparing two SVTN numbers $\tilde{A}_1=\langle(A_{11},A_{12},A_{13},A_{14});l_1,m_1,n_1\rangle$ and $\tilde{A}_2=\langle(A_{21},A_{22},A_{23},A_{24});l_2,m_2,n_2\rangle$.

First comparing approach

According to the first comparing approach, for a specific value of $\rho\in[0,min\{l_1,l_2\}]$,

$\sigma\in[max\{m_1,m_2\},1], \tau\in[max\{n_1,n_2\},1]$.

$\tilde{A}_1\preccurlyeq\tilde{A}_2$ if $\left(\tilde{A}_1\right)_\rho\leq\left(\tilde{A}_2\right)_\rho, \left(\tilde{A}_1\right)_\sigma\leq\left(\tilde{A}_2\right)_\sigma, \left(\tilde{A}_1\right)_\tau\leq\left(\tilde{A}_2\right)_\tau$,

i.e., $\left[(A_1)_{l\rho},(A_1)_{r\rho}\right]\leq\left[(A_2)_{l\rho},(A_2)_{r\rho}\right], [(A_1)_{l\sigma},(A_1)_{r\sigma}]\leq[(A_2)_{l\sigma},(A_2)_{r\sigma}], [(A_1)_{l\tau},(A_1)_{r\tau}]\leq$ $[(A_2)_{l\tau},(A_2)_{r\tau}]$

i.e.,

$(A_1)_{l\rho}\leq(A_2)_{l\rho}, (A_1)_{r\rho}\leq(A_2)_{r\rho}, (A_1)_{l\sigma}\leq(A_2)_{l\sigma}, (A_1)_{r\sigma}\leq(A_2)_{r\sigma}, (A_1)_{l\tau}\leq(A_2)_{l\tau}, (A_1)_{r\tau}\leq$ $(A_2)_{r\tau}$

where,

$(A_1)_{l\rho} = \frac{(l_1-\rho)A_{11}+\rho A_{12}}{l_1}$, $(A_1)_{r\rho} = \frac{(l_1-\rho)A_{14}+\rho A_{13}}{l_1}$, $(A_1)_{l\sigma} = \frac{(1-\sigma)A_{12}+(\sigma-m_1)A_{11}}{1-m_1}$,

$(A_1)_{r\sigma} = \frac{(1-\sigma)A_{13}+(\sigma-m_1)A_{14}}{1-m_1}$, $(A_1)_{l\tau} = \frac{(1-\tau)A_{12}+(\tau-n_1)A_{11}}{1-n_1}$, $(A_1)_{r\tau} = \frac{(1-\tau)A_{13}+(\tau-n_1)A_{14}}{1-n_1}$,

$(A_2)_{l\rho} = \frac{(l_2-\rho)A_{21}+\rho A_{22}}{l_2}$, $(A_2)_{r\rho} = \frac{(l_2-\rho)A_{24}+\rho A_{23}}{l_2}$, $(A_2)_{l\sigma} = \frac{(1-\sigma)A_{22}+(\sigma-m_2)A_{21}}{1-m_2}$,

$(A_2)_{r\sigma} = \frac{(1-\sigma)A_{23}+(\sigma-m_2)A_{24}}{1-m_2}$ and $(A_2)_{l\tau} = \frac{(1-\tau)A_{22}+(\tau-n_2)A_{21}}{1-n_2}$, $(A_2)_{r\tau} = \frac{(1-\tau)A_{23}+(\tau-n_2)A_{24}}{1-n_2}$.

Second comparing approach

According to the second comparing approach, for a specific value of $\alpha \in [0,1]$

$\tilde{A}_1 \preccurlyeq \tilde{A}_2$ if $\Pi_\alpha(\tilde{A}_1) \leq \Pi_\alpha(\tilde{A}_2)$.

where,

$\Pi_\alpha(\tilde{A}_1) = \frac{(A_{11}+2(A_{12}+A_{13})+A_{14})}{6}\left[\alpha(l_1{}^2) + (1-\alpha)(1-m_1)^2 + (1-\alpha)(1-n_1)^2\right]$,

$\Pi_\alpha(\tilde{A}_2) = \frac{(A_{21}+2(A_{22}+A_{23})+A_{24})}{6}\left[\alpha(l_2{}^2) + (1-\alpha)(1-m_2)^2 + (1-\alpha)(1-n_2)^2\right]$.

It is pertinent to mention that

(i) According to the first comparing approach, to find an optimal solution of the SVTN MPP (P5) is equivalent to find an optimal solution of the interval MPP (P7) and to find an optimal solution of the SVTN MPP (P6) is equivalent to find an optimal solution of the interval MPP (P8).

Problem (P7)

$$Maximize \left\{\begin{matrix} Minimize\left((\sum_{i=1}^m \tilde{a}_{i1}u_i)v_1 + (\sum_{i=1}^m \tilde{a}_{i2}u_i)v_2 + \cdots + (\sum_{i=1}^m \tilde{a}_{in}u_i)v_n\right)_\rho \\ Minimize\left((\sum_{i=1}^m \tilde{a}_{i1}u_i)v_1 + (\sum_{i=1}^m \tilde{a}_{i2}u_i)v_2 + \cdots + (\sum_{i=1}^m \tilde{a}_{in}u_i)v_n\right)_\sigma \\ Minimize\left((\sum_{i=1}^m \tilde{a}_{i1}u_i)v_1 + (\sum_{i=1}^m \tilde{a}_{i2}u_i)v_2 + \cdots + (\sum_{i=1}^m \tilde{a}_{in}u_i)v_n\right)_\tau \\ \text{Subject to} \\ \sum_{j=1}^n v_j = 1, v_j \geq 0, j = 1,2,\ldots,n \end{matrix}\right\}$$

Subject to

$\sum_{i=1}^m u_i = 1, u_i \geq 0, i = 1,2,\ldots,m.$

Problem (P8)

$$Minimize\left\{\begin{array}{c}Maximize\left(\left(\sum_{j=1}^{n}\tilde{a}_{1j}v_j\right)u_1+\left(\sum_{j=1}^{n}\tilde{a}_{2j}v_j\right)u_2+\cdots+\left(\sum_{j=1}^{n}\tilde{a}_{mj}v_j\right)u_m\right)_{\rho}\\ Maximize\left(\left(\sum_{j=1}^{n}\tilde{a}_{1j}v_j\right)u_1+\left(\sum_{j=1}^{n}\tilde{a}_{2j}v_j\right)u_2+\cdots+\left(\sum_{j=1}^{n}\tilde{a}_{mj}v_j\right)u_m\right)_{\sigma}\\ Maximize\left(\left(\sum_{j=1}^{n}\tilde{a}_{1j}v_j\right)u_1+\left(\sum_{j=1}^{n}\tilde{a}_{2j}v_j\right)u_2+\cdots+\left(\sum_{j=1}^{n}\tilde{a}_{mj}v_j\right)u_m\right)_{\tau}\\ \text{Subject to}\\ \sum_{i=1}^{m}u_i=1,u_i\geq 0,i=1,2,\ldots,m\end{array}\right\}$$

Subject to

$\sum_{j=1}^{n}v_j=1,v_j\geq 0,j=1,2,\ldots,n.$

(ii) According to second comparing approach, to find an optimal solution of the SVTN MPP (P5) is equivalent to find an optimal solution of the crisp MPP (P9) and to find an optimal solution of the SVTN MPP (P6) is equivalent to find an optimal solution of the crisp MPP (P10).

Problem (P9)

$$Maximize\left\{\begin{array}{c}Minimize\left(\Pi_{\alpha}\left((\sum_{i=1}^{m}\tilde{a}_{i1}u_i)v_1+(\sum_{i=1}^{m}\tilde{a}_{i2}u_i)v_2+\cdots+(\sum_{i=1}^{m}\tilde{a}_{in}u_i)v_n\right)\right)\\ \text{Subject to}\\ \sum_{j=1}^{n}v_j=1,v_j\geq 0,j=1,2,\ldots,n\end{array}\right\}$$

Subject to

$\sum_{i=1}^{m}u_i=1,u_i\geq 0,i=1,2,\ldots,m.$

Problem (P10)

$$Minimize\left\{\begin{array}{c}Maximize\left(\Pi_{\alpha}\left(\left(\sum_{j=1}^{n}\tilde{a}_{1j}v_j\right)u_1+\left(\sum_{j=1}^{n}\tilde{a}_{2j}v_j\right)u_2+\cdots+\left(\sum_{j=1}^{n}\tilde{a}_{mj}v_j\right)u_m\right)\right)\\ \text{Subject to}\\ \sum_{i=1}^{m}u_i=1,u_i\geq 0,i=1,2,\ldots,m\end{array}\right\}$$

Subject to

$\sum_{j=1}^{n}v_j=1,v_j\geq 0,j=1,2,\ldots,n.$

Seikh and Dutta (2022) shown that to find an optimal solution of

(i) The interval MPPs (P7) and (P8) is equivalent to find an optimal solution of the CLPPs (P11) and (P12) respectively.

(ii) The crisp MPPs (P9) and (P10) is equivalent to find an optimal solution of the CLPPs (P13) and (P14) respectively.

Problem (P11)

$$Maximize\left\{\frac{1}{2}\left(\frac{\theta_{l\rho}+\theta_{l\sigma}+\theta_{l\tau}}{3}+\frac{\theta_{l\rho}+\theta_{r\rho}+\theta_{l\sigma}+\theta_{r\sigma}+\theta_{l\tau}+\theta_{r\tau}}{6}\right)\right\}$$

Subject to

$\sum_{i=1}^{m}\Big(\big(l_{ij}-\rho\big)a_{ij1}+\rho\big(a_{ij2}\big)\Big)u_i \geq \big(l_{ij}\big)\theta_{l\rho}\,,\quad j=1,2,\dots,n,$

$\sum_{i=1}^{m}\Big(\big(l_{ij}-\rho\big)a_{ij4}+\rho\big(a_{ij3}\big)\Big)u_i \geq \big(l_{ij}\big)\theta_{r\rho}\,,\quad j=1,2,\dots,n,$

$\sum_{i=1}^{m}\Big((1-\sigma)a_{ij2}+\big(\sigma-m_{ij}\big)a_{ij1}\Big)u_i \geq \big(1-m_{ij}\big)\theta_{l\sigma}\,,\quad j=1,2,\dots,n,$

$\sum_{i=1}^{m}\Big((1-\sigma)a_{ij3}+\big(\sigma-m_{ij}\big)a_{ij4}\Big)u_i \geq \big(1-m_{ij}\big)\theta_{r\sigma}\,,\quad j=1,2,\dots,n,$

$\sum_{i=1}^{m}\Big((1-\tau)a_{ij2}+\big(\tau-n_{ij}\big)a_{ij1}\Big)u_i \geq \big(1-n_{ij}\big)\theta_{l\tau}\,,\quad j=1,2,\dots,n,$

$\sum_{i=1}^{m}\Big((1-\tau)a_{ij3}+\big(\tau-n_{ij}\big)a_{ij4}\Big)u_i \geq \big(1-n_{ij}\big)\theta_{r\tau}\,,\quad j=1,2,\dots,n,$

$\sum_{i=1}^{m}u_i=1,$

$u_i \geq 0,\quad i=1,2,\dots,m.$

Problem (P12)

$$Minimize\left\{\frac{1}{2}\left(\frac{\phi_{l\rho}+\phi_{l\sigma}+\phi_{l\tau}+\phi_{r\rho}+\phi_{r\sigma}+\phi_{r\tau}}{6}+\frac{\phi_{r\rho}+\phi_{r\sigma}+\phi_{r\tau}}{3}\right)\right\}$$

Subject to

$\sum_{j=1}^{n}\Big(\big(l_{ij}-\rho\big)a_{ij1}+\rho\big(a_{ij2}\big)\Big)v_j \leq \big(l_{ij}\big)\phi_{l\rho}\,,\quad i=1,2,\dots,m,$

$\sum_{j=1}^{n}\Big(\big(l_{ij}-\rho\big)a_{ij4}+\rho\big(a_{ij3}\big)\Big)v_j \leq \big(l_{ij}\big)\phi_{r\rho}\,,\quad i=1,2,\dots,m,$

$\sum_{j=1}^{n}\Big((1-\sigma)a_{ij2}+\big(\sigma-m_{ij}\big)a_{ij1}\Big)v_j \leq \big(1-m_{ij}\big)\phi_{l\sigma}\,,\quad i=1,2,\dots,m,$

$\sum_{j=1}^{n}\Big((1-\sigma)a_{ij3}+\big(\sigma-m_{ij}\big)a_{ij4}\Big)v_j \leq \big(1-m_{ij}\big)\phi_{r\sigma}\,,\quad i=1,2,\dots,m,$

$\sum_{j=1}^{n}\Big((1-\tau)a_{ij2}+\big(\tau-n_{ij}\big)a_{ij1}\Big)v_j \leq \big(1-n_{ij}\big)\phi_{l\tau}\,,\quad i=1,2,\dots,m,$

$\sum_{j=1}^{n}\Big((1-\tau)a_{ij3}+\big(\tau-n_{ij}\big)a_{ij4}\Big)v_j \leq \big(1-n_{ij}\big)\phi_{r\tau}\,,\quad i=1,2,\dots,m,$

$\sum_{j=1}^{n}v_j=1,$

$v_j \geq 0, j = 1,2, \ldots, n.$

where,

(a) $\rho \in \left[0, minimum_{\substack{1\leq i\leq m \\ 1\leq j\leq n}}(l_{ij})\right]$,

(b) $\sigma \in \left[maximum_{\substack{1\leq i\leq m \\ 1\leq j\leq n}}(m_{ij}), 1\right]$,

(c) $\tau \in \left[maximum_{\substack{1\leq i\leq m \\ 1\leq j\leq n}}(n_{ij}), 1\right]$.

Problem (P13)

$Maximize\ \{P_1\}$

Subject to

$\eta_j\left(\sum_{i=1}^{m} A_{ij}\, u_i\right) \geq P_1, j = 1,2, \ldots, n$

$\sum_{i=1}^{m} u_i = 1, u_i \geq 0, i = 1,2, \ldots, m.$

where,

$$\eta_j = \left[\alpha\left(minimum_{\,1\leq i\leq m}\{l_{ij}\}\right)^2 + (1-\alpha)\left(1 - maximum_{\,1\leq i\leq m}\{m_{ij}\}\right)^2 + (1-\alpha)\left(1 - maximum_{\,1\leq i\leq m}\{n_{ij}\}\right)^2\right],$$

$A_{ij} = \frac{a_{ij1}+2a_{ij2}+2a_{ij3}+a_{ij4}}{6}.$

Problem (P14)

$Minimize\ \{P_2\}$

Subject to

$\eta_i\left(\sum_{j=1}^{n} A_{ij}\, v_j\right) \leq P_2, i = 1,2, \ldots, m,$

$\sum_{j=1}^{n} v_j = 1, v_j \geq 0, j = 1,2, \ldots, n.$

where,

$$\eta_i = \left[\alpha\left(minimum_{\,1\leq j\leq n}\{l_{ij}\}\right)^2 + (1-\alpha)\left(1 - maximum_{\,1\leq j\leq n}\{m_{ij}\}\right)^2 + (1-\alpha)\left(1 - maximum_{\,1\leq j\leq n}\{n_{ij}\}\right)^2\right],$$

$A_{ij} = \frac{a_{ij1}+2a_{ij2}+2a_{ij3}+a_{ij4}}{6}$.

Brikaa (2022) pointed out that as Seikh and Dutta (2022) have considered a mathematically incorrect result to transform the interval MPP (P7) into the CLPP (P11) as well as to transform the interval MPP (P8) into the CLPP (P12). So, the crisp MPPs (P11) and (P12) are not equivalent to interval MPPs (P7) and (P8) respectively.

Brikaa (2022) also pointed out that in actual case

(i) The CLPP (P15) is equivalent to the interval MPP (P7).

(ii) The CLPP (P16) is equivalent to the interval MPP (P8).

Problem (P15)

$$Maximize\ \left\{\frac{1}{2}\left(\frac{\theta_{l\rho}+\theta_{l\sigma}+\theta_{l\tau}}{3}+\frac{\theta_{l\rho}+\theta_{r\rho}+\theta_{l\sigma}+\theta_{r\sigma}+\theta_{l\tau}+\theta_{r\tau}}{6}\right)\right\}$$

Subject to

$$\left(minimum_{1\le i\le m}\{l_{ij}\}-\rho\right)\sum_{i=1}^{m}\left(a_{ij1}\right)u_i+\rho\sum_{i=1}^{m}\left(a_{ij2}\right)u_i \ge \left(minimum_{1\le i\le m}\{l_{ij}\}\right)\theta_{l\rho},\quad j=1,2,\dots,n,$$

$$\left(minimum_{1\le i\le m}\{l_{ij}\}-\rho\right)\sum_{i=1}^{m}\left(a_{ij4}\right)u_i+\rho\sum_{i=1}^{m}\left(a_{ij3}\right)u_i \ge \left(minimum_{1\le i\le m}\{l_{ij}\}\right)\theta_{r\rho},\quad j=1,2,\dots,n,$$

$$(1-\sigma)\sum_{i=1}^{m}\left(a_{ij2}\right)u_i+\left(\sigma-maximum_{1\le i\le m}\{m_{ij}\}\right)\sum_{i=1}^{m}\left(a_{ij1}\right)u_i \ge (1-maximum_{1\le i\le m}\{m_{ij}\})\theta_{l\sigma},\quad j=1,2,\dots,n,$$

$$(1-\sigma)\sum_{i=1}^{m}\left(a_{ij3}\right)u_i+\left(\sigma-maximum_{1\le i\le m}\{m_{ij}\}\right)\sum_{i=1}^{m}\left(a_{ij4}\right)u_i \ge (1-maximum_{1\le i\le m}\{m_{ij}\})\theta_{r\sigma},\quad j=1,2,\dots,n,$$

$$(1-\tau)\sum_{i=1}^{m}\left(a_{ij2}\right)u_i+\left(\tau-maximum_{1\le i\le m}\{n_{ij}\}\right)\sum_{i=1}^{m}\left(a_{ij1}\right)u_i \ge (1-maximum_{1\le i\le m}\{n_{ij}\})\theta_{l\tau},\quad j=1,2,\dots,n,$$

$$(1-\tau)\sum_{i=1}^{m}\left(a_{ij3}\right)u_i+\left(\tau-maximum_{1\le i\le m}\{n_{ij}\}\right)\sum_{i=1}^{m}\left(a_{ij4}\right)u_i \ge (1-maximum_{1\le i\le m}\{n_{ij}\})\theta_{r\tau},\quad j=1,2,\dots,n,$$

$\sum_{i=1}^{m} u_i = 1,\ u_i \geq 0, \quad i = 1,2,\dots,m.$

Problem (P16)

$$Minimize\ \left\{\frac{1}{2}\left(\frac{\phi_{l\rho}+\phi_{l\sigma}+\phi_{l\tau}+\phi_{r\rho}+\phi_{r\sigma}+\phi_{r\tau}}{6}+\frac{\phi_{r\rho}+\phi_{r\sigma}+\phi_{r\tau}}{3}\right)\right\}$$

Subject to

$$\left(minimum_{1\leq j\leq n}\{\,l_{ij}\} - \rho\right)\sum_{j=1}^{n}\left(a_{ij1}\right)v_j + \rho\sum_{j=1}^{n}\left(a_{ij2}\right)v_j \leq \left(minimum_{1\leq j\leq n}\{\,l_{ij}\}\right)\phi_{l\rho}, \quad i = 1,2,\dots,m,$$

$$\left(minimum_{1\leq j\leq n}\{\,l_{ij}\} - \rho\right)\sum_{j=1}^{n}\left(a_{ij4}\right)v_j + \rho\sum_{j=1}^{n}\left(a_{ij3}\right)v_j \leq \left(minimum_{1\leq j\leq n}\{\,l_{ij}\}\right)\phi_{r\rho}, \quad i = 1,2,\dots,m,$$

$$(1-\sigma)\sum_{j=1}^{n}\left(a_{ij2}\right)v_j + \left(\sigma - maximum_{\ 1\leq j\leq n}\{m_{ij}\}\right)\sum_{j=1}^{n}\left(a_{ij1}\right) v_j \leq \left(1 - maximum_{\ 1\leq j\leq n}\{m_{ij}\}\right)\phi_{l\sigma},\ i = 1,2,\dots,m,$$

$$(1-\sigma)\sum_{j=1}^{n}\left(a_{ij3}\right)v_j + \left(\sigma - maximum_{\ 1\leq j\leq n}\{m_{ij}\}\right)\sum_{j=1}^{n}\left(a_{ij4}\right)v_j \leq \left(1 - maximum_{\ 1\leq j\leq n}\{m_{ij}\}\right)\phi_{r\sigma},\ i = 1,2,\dots,m,$$

$$(1-\tau)\sum_{j=1}^{n}\left(a_{ij2}\right)v_j + \left(\tau - maximum_{\ 1\leq j\leq n}\{n_{ij}\}\right)\sum_{j=1}^{n}\left(a_{ij1}\right)v_j \leq \left(1 - maximum_{\ 1\leq j\leq n}\{n_{ij}\}\right)\phi_{l\tau},\ i = 1,2,\dots,m,$$

$$(1-\tau)\sum_{j=1}^{n}\left(a_{ij3}\right)v_j + \left(\tau - maximum_{\ 1\leq j\leq n}\{n_{ij}\}\right)\sum_{j=1}^{n}\left(a_{ij4}\right)v_j \leq \left(1 - maximum_{\ 1\leq j\leq n}\{n_{ij}\}\right)\phi_{r\tau},\ i = 1,2,\dots,m,$$

$\sum_{j=1}^{n} v_j = 1,\ v_j \geq 0, j = 1,2,\dots,n.$

where,

(a) $\rho \in [0, minimum_{\substack{1\leq i\leq m\\ 1\leq j\leq n}}\left(l_{ij}\right)]$

(b) $\sigma \in [maximum_{\substack{1\leq i\leq m\\ 1\leq j\leq n}}\left(m_{ij}\right), 1]$

(c) $\tau \in [maximum_{\substack{1\leq i\leq m\\ 1\leq j\leq n}}\left(n_{ij}\right), 1]$

In this paper, it is pointed out that

(i) As Brikaa (2022) have considered some mathematically incorrect results to transform the interval MPPs (P7) and (P8) into the CLPPs (P15) and (P16) respectively. So, Brikaa's (2022) approach is not valid.

(ii) As Seikh and Dutta (2022) have considered a mathematically incorrect result to transform the crisp MPPs (P9) and (P10) into the CLPPs (P13) and (P14) respectively. So, Seikh and Dutta's (2022) second approach is not valid.

(iii) A modified approach is proposed corresponding to Brikka's (2022) approach.

(iv) A modified approach is proposed corresponding to Seikh and Dutta's (2022) second approach.

(v) The correct results of the existing SVTN matrix games (Seikh and Dutta, 2022, Section 5.1, Example 1, p. 929) are obtained by both the modified approaches.

2 Brikaa's approach

Brikaa (2022) proposed the following approach to find the optimal strategies $u_i, i = 1,2,\ldots,m$ of Player I, $v_j, j = 1,2,\ldots,n$ of Player II, the minimum expected gain of Player I and the maximum expected loss of Player II with the help of the interval MPPs (P7) and (P8).

Step 1: Using the expression,

$\left((\sum_{i=1}^{m}\tilde{a}_{i1}u_i)v_1 + (\sum_{i=1}^{m}\tilde{a}_{i2}u_i)v_2 + \cdots + (\sum_{i=1}^{m}\tilde{a}_{in}u_i)v_n\right)_k = (\sum_{i=1}^{m}\tilde{a}_{i1}u_i)_k v_1 + (\sum_{i=1}^{m}\tilde{a}_{i2}u_i)_k v_2 + \cdots + (\sum_{i=1}^{m}\tilde{a}_{in}u_i)_k v_n, k = \rho,\sigma,\tau,$ to find an optimal solution of the interval MPP (P7) is equivalent to find an optimal solution of the interval MPP (P17) and using the expression $\left((\sum_{j=1}^{n}\tilde{a}_{1j}v_j)u_1 + (\sum_{j=1}^{n}\tilde{a}_{2j}v_j)u_2 + \cdots + (\sum_{j=1}^{n}\tilde{a}_{mj}v_j)u_m\right)_k = (\sum_{j=1}^{n}\tilde{a}_{1j}v_j)_k u_1 + (\sum_{j=1}^{n}\tilde{a}_{2j}v_j)_k u_2 + \cdots + (\sum_{j=1}^{n}\tilde{a}_{mj}v_j)_k u_m, k = \rho,\sigma,\tau,$ to find an optimal solution of the interval MPP (P8) is equivalent to find an optimal solution of the interval MPP (P18).

Problem (P17)

$$Maximize\begin{Bmatrix} Minimize\big((\sum_{i=1}^{m}\tilde{a}_{i1}u_i)_\rho v_1+(\sum_{i=1}^{m}\tilde{a}_{i2}u_i)_\rho v_2+\cdots+(\sum_{i=1}^{m}\tilde{a}_{in}u_i)_\rho v_n\big)\\ Minimize((\sum_{i=1}^{m}\tilde{a}_{i1}u_i)_\sigma v_1+(\sum_{i=1}^{m}\tilde{a}_{i2}u_i)_\sigma v_2+\cdots+(\sum_{i=1}^{m}\tilde{a}_{in}u_i)_\sigma v_n)\\ Minimize((\sum_{i=1}^{m}\tilde{a}_{i1}u_i)_\tau v_1+(\sum_{i=1}^{m}\tilde{a}_{i2}u_i)_\tau v_2+\cdots+(\sum_{i=1}^{m}\tilde{a}_{in}u_i)_\tau v_n)\\ \text{Subject to}\\ \sum_{j=1}^{n}v_j=1, v_j\geq 0,\ j=1,2,\dots,n \end{Bmatrix}$$

Subject to

$\sum_{i=1}^{m}u_i=1, u_i\geq 0,\ i=1,2,\dots,m.$

Problem (P18)

$$Minimize\begin{Bmatrix} Maximize\left(\left(\sum_{j=1}^{n}\tilde{a}_{1j}v_j\right)_\rho u_1+\left(\sum_{j=1}^{n}\tilde{a}_{2j}v_j\right)_\rho u_2+\cdots+\left(\sum_{j=1}^{n}\tilde{a}_{mj}v_j\right)_\rho u_m\right)\\ Maximize\left(\left(\sum_{j=1}^{n}\tilde{a}_{1j}v_j\right)_\sigma u_1+\left(\sum_{j=1}^{n}\tilde{a}_{2j}v_j\right)_\sigma u_2+\cdots+\left(\sum_{j=1}^{n}\tilde{a}_{mj}v_j\right)_\sigma u_m\right)\\ Maximize\left(\left(\sum_{j=1}^{n}\tilde{a}_{1j}v_j\right)_\tau u_1+\left(\sum_{j=1}^{n}\tilde{a}_{2j}v_j\right)_\tau u_2+\cdots+\left(\sum_{j=1}^{n}\tilde{a}_{mj}v_j\right)_\tau u_m\right)\\ \text{Subject to}\\ \sum_{i=1}^{m}u_i=1, u_i\geq 0, j=1,2,\dots,n \end{Bmatrix}$$

Subject to

$\sum_{j=1}^{n}v_j=1, v_j\geq 0, j=1,2,\dots,n.$

Step 2: Since $(\sum_{i=1}^{m}\tilde{a}_{i1}u_i)_k v_1+(\sum_{i=1}^{m}\tilde{a}_{i2}u_i)_k v_2+\cdots+(\sum_{i=1}^{m}\tilde{a}_{in}u_i)_k v_n$ is a convex linear combination of $(\sum_{i=1}^{m}\tilde{a}_{i1}u_i)_k, (\sum_{i=1}^{m}\tilde{a}_{i1}u_i)_k, \dots, (\sum_{i=1}^{m}\tilde{a}_{in}u_i)_k,\ k=\rho,\sigma,\tau$ and $\left(\sum_{j=1}^{n}\tilde{a}_{1j}v_j\right)_k u_1+\left(\sum_{j=1}^{n}\tilde{a}_{2j}v_j\right)_k u_2+\cdots+\left(\sum_{j=1}^{n}\tilde{a}_{mj}v_j\right)_k u_m$ is a convex linear combination of $\left(\sum_{j=1}^{n}\tilde{a}_{1j}v_j\right)_k, \left(\sum_{j=1}^{n}\tilde{a}_{2j}v_j\right)_k, \dots, \left(\sum_{j=1}^{n}\tilde{a}_{mj}v_j\right)_k, k=\rho,\sigma,\tau$. So, to find an optimal solution of the interval MPP (P17) is equivalent to find an optimal solution of the interval MPP (P19) and to find an optimal solution of the interval MPP (P18) is equivalent to find an optimal solution of the interval MPP (P20).

Problem (P19)

$$Maximize\begin{Bmatrix} Minimum\big((\sum_{i=1}^{m}\tilde{a}_{i1}u_i)_\rho, (\sum_{i=1}^{m}\tilde{a}_{i2}u_i)_\rho, \dots, (\sum_{i=1}^{m}\tilde{a}_{in}u_i)_\rho\big)\\ Minimum((\sum_{i=1}^{m}\tilde{a}_{i1}u_i)_\sigma, (\sum_{i=1}^{m}\tilde{a}_{i2}u_i)_\sigma, \dots, (\sum_{i=1}^{m}\tilde{a}_{in}u_i)_\sigma)\\ Minimum((\sum_{i=1}^{m}\tilde{a}_{i1}u_i)_\tau, (\sum_{i=1}^{m}\tilde{a}_{i2}u_i)_\tau, \dots, (\sum_{i=1}^{m}\tilde{a}_{in}u_i)_\tau) \end{Bmatrix}$$

Subject to

$\sum_{i=1}^{m} u_i = 1, u_i \geq 0, i = 1,2, \dots, m.$

Problem (P20)

$$Minimize \begin{Bmatrix} Maximum\left(\left(\sum_{j=1}^{n} \tilde{a}_{1j} v_j\right)_\rho, \left(\sum_{j=1}^{n} \tilde{a}_{2j} v_j\right)_\rho, \dots, \left(\sum_{j=1}^{n} \tilde{a}_{mj} v_j\right)_\rho\right) \\ Maximum\left(\left(\sum_{j=1}^{n} \tilde{a}_{1j} v_j\right)_\sigma, \left(\sum_{j=1}^{n} \tilde{a}_{2j} v_j\right)_\sigma, \dots, \left(\sum_{j=1}^{n} \tilde{a}_{mj} v_j\right)_\sigma\right) \\ Maximum\left(\left(\sum_{j=1}^{n} \tilde{a}_{1j} v_j\right)_\tau, \left(\sum_{j=1}^{n} \tilde{a}_{2j} v_j\right)_\tau, \dots, \left(\sum_{j=1}^{n} \tilde{a}_{mj} v_j\right)_\tau\right) \end{Bmatrix}$$

Subject to

$\sum_{j=1}^{n} v_j = 1, v_j \geq 0, j = 1,2, \dots, n.$

Step 3: Assuming, $Minimum((\sum_{i=1}^{m} \tilde{a}_{i1} u_i)_k, (\sum_{i=1}^{m} \tilde{a}_{i2} u_i)_k, \dots, (\sum_{i=1}^{m} \tilde{a}_{in} u_i)_k) = \tilde{\theta}_k,\ k = \rho, \sigma, \tau$

to find an optimal solution of the interval MPP (P19) is equivalent to find an optimal solution of the interval MPP (P21). Also assuming,

$Maximum\left(\left(\sum_{j=1}^{n} \tilde{a}_{1j} v_j\right)_k, \left(\sum_{j=1}^{n} \tilde{a}_{2j} v_j\right)_k, \dots, \left(\sum_{j=1}^{n} \tilde{a}_{mj} v_j\right)_k\right) = \tilde{\phi}_k,\ k = \rho, \sigma, \tau,$ to find an optimal solution of the interval MPP (P20) is equivalent to find an optimal solution of the interval MPP (P22).

Problem (P21)

$Maximize \{\tilde{\theta}_\rho, \tilde{\theta}_\sigma, \tilde{\theta}_\tau\}$

Subject to

$\left(\sum_{i=1}^{m} \tilde{a}_{ij} u_i\right)_\rho \geq \tilde{\theta}_\rho,\ \ j = 1,2, \dots, n,$

$\left(\sum_{i=1}^{m} \tilde{a}_{ij} u_i\right)_\sigma \geq \tilde{\theta}_\sigma,\ \ j = 1,2, \dots, n,$

$\left(\sum_{i=1}^{m} \tilde{a}_{ij} u_i\right)_\tau \geq \tilde{\theta}_\tau,\ \ j = 1,2, \dots, n,$

$\sum_{i=1}^{m} u_i = 1, u_i \geq 0,\ i = 1,2, \dots, m.$

Problem (P22)

$Minimize \{\tilde{\phi}_\rho, \tilde{\phi}_\sigma, \tilde{\phi}_\tau\}$

Subject to

$\left(\sum_{j=1}^{n} \tilde{a}_{ij} v_j\right)_\rho \leq \tilde{\phi}_\rho,\ i = 1,2, \dots, m,$

$\left(\sum_{j=1}^{n} \tilde{a}_{ij} v_j\right)_\sigma \le \tilde{\phi}_\sigma,\ i = 1,2,\dots,m,$

$\left(\sum_{j=1}^{n} \tilde{a}_{ij} v_j\right)_\tau \le \tilde{\phi}_\tau,\ i = 1,2,\dots,m,$

$\sum_{j=1}^{n} v_j = 1, v_j \ge 0,\ j = 1,2,\dots,n.$

Step 4: Using the expression

$\sum_{i=1}^{m} \tilde{a}_{ij} u_i = \sum_{i=1}^{m} (a_{ij1}, a_{ij2}, a_{ij3}, a_{ij4}; l_{ij}, m_{ij}, n_{ij}) u_i =$

$\begin{pmatrix} \sum_{i=1}^{m}(a_{ij1})u_i, \sum_{i=1}^{m}(a_{ij2})u_i, \sum_{i=1}^{m}(a_{ij3})u_i, \sum_{i=1}^{m}(a_{ij4})u_i; \\ minimum_{1\le i\le m}\{l_{ij}\}, maximum_{1\le i\le m}\{m_{ij}\}, maximum_{1\le i\le m}\{n_{ij}\} \end{pmatrix}$ to find an optimal solution of the interval MPP (P21) is equivalent to find an optimal solution of the interval MPP (P23). Also, using the expression

$\sum_{j=1}^{n} \tilde{a}_{ij} v_j = \sum_{j=1}^{n} (a_{ij1}, a_{ij2}, a_{ij3}, a_{ij4}; l_{ij}, m_{ij}, n_{ij}) v_j =$

$\begin{pmatrix} \sum_{j=1}^{n}(a_{ij1})v_j, \sum_{j=1}^{n}(a_{ij2})v_j, \sum_{j=1}^{n}(a_{ij3})v_j, \sum_{j=1}^{n}(a_{ij4})v_j; \\ minimum_{1\le j\le n}\{l_{ij}\}, maximum_{1\le j\le n}\{m_{ij}\}, maximum_{1\le j\le n}\{n_{ij}\} \end{pmatrix}$ to find an optimal solution of the interval MPP (P22) is equivalent to find an optimal solution of the interval MPP (P24).

Problem (P23)

$Maximize\ \{\tilde{\theta}_\rho, \tilde{\theta}_\sigma, \tilde{\theta}_\tau\}$

Subject to

$$\begin{pmatrix} \sum_{i=1}^{m}(a_{ij1})u_i, \sum_{i=1}^{m}(a_{ij2})u_i, \sum_{i=1}^{m}(a_{ij3})u_i, \sum_{i=1}^{m}(a_{ij4})u_i; \\ minimum_{1\le i\le m}\{l_{ij}\}, maximum_{1\le i\le m}\{m_{ij}\}, maximum_{1\le i\le m}\{n_{ij}\} \end{pmatrix}_\rho \ge \tilde{\theta}_\rho, j = 1,2,\dots,n,$$

$$\begin{pmatrix} \sum_{i=1}^{m}(a_{ij1})u_i, \sum_{i=1}^{m}(a_{ij2})u_i, \sum_{i=1}^{m}(a_{ij3})u_i, \sum_{i=1}^{m}(a_{ij4})u_i; \\ minimum_{1\le i\le m}\{l_{ij}\}, maximum_{1\le i\le m}\{m_{ij}\}, maximum_{1\le i\le m}\{n_{ij}\} \end{pmatrix}_\sigma \ge \tilde{\theta}_\sigma, j = 1,2,\dots,n,$$

$$\begin{pmatrix} \sum_{i=1}^{m}(a_{ij1})u_i, \sum_{i=1}^{m}(a_{ij2})u_i, \sum_{i=1}^{m}(a_{ij3})u_i, \sum_{i=1}^{m}(a_{ij4})u_i; \\ minimum_{1\le i\le m}\{l_{ij}\}, maximum_{1\le i\le m}\{m_{ij}\}, maximum_{1\le i\le m}\{n_{ij}\} \end{pmatrix}_\tau \ge \tilde{\theta}_\tau, j = 1,2,\dots,n,$$

$\sum_{i=1}^{m} u_i = 1, u_i \ge 0, i = 1,2,\dots,m.$

Problem (P24)

$Minimize\ \{\tilde{\phi}_\rho, \tilde{\phi}_\sigma, \tilde{\phi}_\tau\}$

Subject to

$$\begin{pmatrix} \sum_{j=1}^{n}(a_{ij1})v_j, \sum_{j=1}^{n}(a_{ij2})v_j, \sum_{j=1}^{n}(a_{ij3})v_j, \sum_{j=1}^{n}(a_{ij4})v_j; \\ minimum_{1\le j\le n}\{l_{ij}\}, maximum_{1\le j\le n}\{m_{ij}\}, maximum_{1\le j\le n}\{n_{ij}\} \end{pmatrix}_\rho \le \tilde{\phi}_\rho, i = 1,2,\ldots,m,$$

$$\begin{pmatrix} \sum_{j=1}^{n}(a_{ij1})v_j, \sum_{j=1}^{n}(a_{ij2})v_j, \sum_{j=1}^{n}(a_{ij3})v_j, \sum_{j=1}^{n}(a_{ij4})v_j; \\ minimum_{1\le j\le n}\{l_{ij}\}, maximum_{1\le j\le n}\{m_{ij}\}, maximum_{1\le j\le n}\{n_{ij}\} \end{pmatrix}_\sigma \le \tilde{\phi}_\sigma, i = 1,2,\ldots,m,$$

$$\begin{pmatrix} \sum_{j=1}^{n}(a_{ij1})v_j, \sum_{j=1}^{n}(a_{ij2})v_j, \sum_{j=1}^{n}(a_{ij3})v_j, \sum_{j=1}^{n}(a_{ij4})v_j; \\ minimum_{1\le j\le n}\{l_{ij}\}, maximum_{1\le j\le n}\{m_{ij}\}, maximum_{1\le j\le n}\{n_{ij}\} \end{pmatrix}_\tau \le \tilde{\phi}_\tau, i = 1,2,\ldots,m,$$

$\sum_{j=1}^{n} v_j = 1, v_j \ge 0,\ j = 1,2,\ldots,n.$

Step 5: Using the expressions

$(\tilde{\beta}_{ij})_\rho = \left[(\beta_{ij})_{l\rho}, (\beta_{ij})_{r\rho}\right], (\tilde{\beta}_{ij})_\sigma = \left[(\beta_{ij})_{l\sigma}, (\beta_{ij})_{r\sigma}\right]\ and\ (\tilde{\beta}_{ij})_\tau = \left[(\beta_{ij})_{l\tau}, (\beta_{ij})_{r\tau}\right]$ to find an optimal solution of the interval MPP (P23) is equivalent to find an optimal solution of the interval MPP (P25) and to find an optimal solution of the interval MPP (P24) is equivalent to find an optimal solution of the interval MPP (P26).

Problem (P25)

$Maximize\ \{[\theta_{l\rho}, \theta_{r\rho}], [\theta_{l\sigma}, \theta_{r\sigma}], [\theta_{l\tau}, \theta_{r\tau}]\}$

Subject to

$$\begin{bmatrix} \begin{pmatrix} \sum_{i=1}^{m}(a_{ij1})u_i, \sum_{i=1}^{m}(a_{ij2})u_i, \sum_{i=1}^{m}(a_{ij3})u_i, \sum_{i=1}^{m}(a_{ij4})u_i; \\ minimum_{1\le i\le m}\{l_{ij}\}, maximum_{1\le i\le m}\{m_{ij}\}, maximum_{1\le i\le m}\{n_{ij}\} \end{pmatrix}_{l\rho}, \\ \begin{pmatrix} \sum_{i=1}^{m}(a_{ij1})u_i, \sum_{i=1}^{m}(a_{ij2})u_i, \sum_{i=1}^{m}(a_{ij3})u_i, \sum_{i=1}^{m}(a_{ij4})u_i; \\ minimum_{1\le i\le m}\{l_{ij}\}, maximum_{1\le i\le m}\{m_{ij}\}, maximum_{1\le i\le m}\{n_{ij}\} \end{pmatrix}_{r\rho} \end{bmatrix} \ge [\theta_{l\rho}, \theta_{r\rho}],$$

$$j = 1,2,\ldots,n,$$

$$\left[\begin{pmatrix} \sum_{i=1}^{m}(a_{ij1})u_i\,,\sum_{i=1}^{m}(a_{ij2})u_i\,,\sum_{i=1}^{m}(a_{ij3})u_i\,,\sum_{i=1}^{m}(a_{ij4})u_i\,; \\ minimum_{1\le i\le m}\{\,l_{ij}\}, maximum\,_{1\le i\le m}\{m_{ij}\}, maximum\,_{1\le i\le m}\{n_{ij}\} \end{pmatrix}_{l\sigma}, \begin{pmatrix} \sum_{i=1}^{m}(a_{ij1})u_i\,,\sum_{i=1}^{m}(a_{ij2})u_i\,,\sum_{i=1}^{m}(a_{ij3})u_i\,,\sum_{i=1}^{m}(a_{ij4})u_i\,; \\ minimum_{1\le i\le m}\{\,l_{ij}\}, maximum\,_{1\le i\le m}\{m_{ij}\}, maximum\,_{1\le i\le m}\{n_{ij}\} \end{pmatrix}_{r\sigma}\right] \ge [\theta_{l\sigma},\theta_{r\sigma}],$$

$$j = 1,2,\dots,n,$$

$$\left[\begin{pmatrix} \sum_{i=1}^{m}(a_{ij1})u_i\,,\sum_{i=1}^{m}(a_{ij2})u_i\,,\sum_{i=1}^{m}(a_{ij3})u_i\,,\sum_{i=1}^{m}(a_{ij4})u_i\,; \\ minimum_{1\le i\le m}\{\,l_{ij}\}, maximum\,_{1\le i\le m}\{m_{ij}\}, maximum\,_{1\le i\le m}\{n_{ij}\} \end{pmatrix}_{l\tau}, \begin{pmatrix} \sum_{i=1}^{m}(a_{ij1})u_i\,,\sum_{i=1}^{m}(a_{ij2})u_i\,,\sum_{i=1}^{m}(a_{ij3})u_i\,,\sum_{i=1}^{m}(a_{ij4})u_i\,; \\ minimum_{1\le i\le m}\{\,l_{ij}\}, maximum\,_{1\le i\le m}\{m_{ij}\}, maximum\,_{1\le i\le m}\{n_{ij}\} \end{pmatrix}_{r\tau}\right] \ge [\theta_{l\tau},\theta_{r\tau}],$$

$$j = 1,2,\dots,n,$$

$\sum_{i=1}^{m} u_i = 1, u_i \ge 0, i = 1,2,\dots,m.$

Problem (P26)

$Minimize\ \{[\phi_{l\rho},\phi_{r\rho}],[\phi_{l\sigma},\phi_{r\sigma}],[\phi_{l\tau},\phi_{r\tau}]\}$

Subject to

$$\left[\begin{pmatrix} \sum_{j=1}^{n}(a_{ij1})v_j\,,\sum_{j=1}^{n}(a_{ij2})v_j\,,\sum_{j=1}^{n}(a_{ij3})v_j\,,\sum_{j=1}^{n}(a_{ij4})v_j\,; \\ minimum_{1\le j\le n}\{\,l_{ij}\}, maximum\,_{1\le j\le n}\{m_{ij}\}, maximum\,_{1\le j\le n}\{n_{ij}\} \end{pmatrix}_{l\rho}, \begin{pmatrix} \sum_{j=1}^{n}(a_{ij1})v_j\,,\sum_{j=1}^{n}(a_{ij2})v_j\,,\sum_{j=1}^{n}(a_{ij3})v_j\,,\sum_{j=1}^{n}(a_{ij4})v_j\,; \\ minimum_{1\le j\le n}\{\,l_{ij}\}, maximum\,_{1\le j\le n}\{m_{ij}\}, maximum\,_{1\le j\le n}\{n_{ij}\} \end{pmatrix}_{r\rho}\right] \le [\phi_{l\rho},\phi_{r\rho}],$$

$$i = 1,2,\dots,m,$$

$$\left[\begin{pmatrix} \sum_{j=1}^{n}(a_{ij1})v_j\,,\sum_{j=1}^{n}(a_{ij2})v_j\,,\sum_{j=1}^{n}(a_{ij3})v_j\,,\sum_{j=1}^{n}(a_{ij4})v_j\,; \\ minimum_{1\le j\le n}\{\,l_{ij}\}, maximum\,_{1\le j\le n}\{m_{ij}\}, maximum\,_{1\le j\le n}\{n_{ij}\} \end{pmatrix}_{l\sigma}, \begin{pmatrix} \sum_{j=1}^{n}(a_{ij1})v_j\,,\sum_{j=1}^{n}(a_{ij2})v_j\,,\sum_{j=1}^{n}(a_{ij3})v_j\,,\sum_{j=1}^{n}(a_{ij4})v_j\,; \\ minimum_{1\le j\le n}\{\,l_{ij}\}, maximum\,_{1\le j\le n}\{m_{ij}\}, maximum\,_{1\le j\le n}\{n_{ij}\} \end{pmatrix}_{r\sigma}\right] \le [\phi_{l\sigma},\phi_{r\sigma}],$$

$$i = 1,2,\dots,m,$$

$$\left[\begin{matrix}\begin{pmatrix}\sum_{j=1}^{n}(a_{ij1})v_j\,,\sum_{j=1}^{n}(a_{ij2})v_j\,,\sum_{j=1}^{n}(a_{ij3})v_j\,,\sum_{j=1}^{n}(a_{ij4})v_j\,;\\ minimum_{1\le j\le n}\{\,l_{ij}\},maximum\,_{1\le j\le n}\{m_{ij}\},maximum\,_{1\le j\le n}\{n_{ij}\}\end{pmatrix}_{l\tau},\\ \begin{pmatrix}\sum_{j=1}^{n}(a_{ij1})v_j\,,\sum_{j=1}^{n}(a_{ij2})v_j\,,\sum_{j=1}^{n}(a_{ij3})v_j\,,\sum_{j=1}^{n}(a_{ij4})v_j\,;\\ minimum_{1\le j\le n}\{\,l_{ij}\},maximum\,_{1\le j\le n}\{m_{ij}\},maximum\,_{1\le j\le n}\{n_{ij}\}\end{pmatrix}_{r\tau}\end{matrix}\right]\le[\phi_{l\tau},\phi_{r\tau}],$$

$$i=1,2,\dots,m,$$

$\sum_{j=1}^{n}v_j=1, v_j\ge 0, j=1,2,\dots,n.$

Step 6: Using the relation $\left[(\beta_{ij})_{l\rho},(\beta_{ij})_{r\rho}\right]\ge\left[(\gamma_{ij})_{l\rho},(\gamma_{ij})_{r\rho}\right]\Rightarrow\ (\beta_{ij})_{l\rho}\ge(\gamma_{ij})_{l\rho},(\beta_{ij})_{r\rho}\ge(\gamma_{ij})_{r\rho},\left[(\beta_{ij})_{l\sigma},(\beta_{ij})_{r\sigma}\right]\ge\left[(\gamma_{ij})_{l\sigma},(\gamma_{ij})_{r\sigma}\right]\Rightarrow\ (\beta_{ij})_{l\sigma}\ge(\gamma_{ij})_{l\sigma},(\beta_{ij})_{r\sigma}\ge(\gamma_{ij})_{r\sigma}$ *and* $\left[(\beta_{ij})_{l\tau},(\beta_{ij})_{r\tau}\right]\ge\left[(\gamma_{ij})_{l\tau},(\gamma_{ij})_{r\tau}\right]\Rightarrow\ (\beta_{ij})_{l\tau}\ge(\gamma_{ij})_{l\tau},(\beta_{ij})_{r\tau}\ge(\gamma_{ij})_{r\tau}$ to find an optimal solution of the interval MPP (P25) is equivalent to find an optimal solution of the interval MPP (P27) and to find an optimal solution of the interval MPP (P26) is equivalent to find an optimal solution of the interval MPP (P28).

Problem (P27)

$Maximize\ \{[\theta_{l\rho},\theta_{r\rho}],[\theta_{l\sigma},\theta_{r\sigma}],[\theta_{l\tau},\theta_{r\tau}]\}$

Subject to

$$\begin{pmatrix}\sum_{i=1}^{m}(a_{ij1})u_i\,,\sum_{i=1}^{m}(a_{ij2})u_i\,,\sum_{i=1}^{m}(a_{ij3})u_i\,,\sum_{i=1}^{m}(a_{ij4})u_i\,;\\ minimum_{1\le i\le m}\{\,l_{ij}\},maximum\,_{1\le i\le m}\{m_{ij}\},maximum\,_{1\le i\le m}\{n_{ij}\}\end{pmatrix}_{l\rho}\ge\theta_{l\rho}, j=1,2,\dots,n,$$

$$\begin{pmatrix}\sum_{i=1}^{m}(a_{ij1})u_i\,,\sum_{i=1}^{m}(a_{ij2})u_i\,,\sum_{i=1}^{m}(a_{ij3})u_i\,,\sum_{i=1}^{m}(a_{ij4})u_i\,;\\ minimum_{1\le i\le m}\{\,l_{ij}\},maximum\,_{1\le i\le m}\{m_{ij}\},maximum\,_{1\le i\le m}\{n_{ij}\}\end{pmatrix}_{r\rho}\ge\theta_{r\rho}, j=1,2,\dots,n,$$

$$\begin{pmatrix}\sum_{i=1}^{m}(a_{ij1})u_i\,,\sum_{i=1}^{m}(a_{ij2})u_i\,,\sum_{i=1}^{m}(a_{ij3})u_i\,,\sum_{i=1}^{m}(a_{ij4})u_i\,;\\ minimum_{1\le i\le m}\{\,l_{ij}\},maximum\,_{1\le i\le m}\{m_{ij}\},maximum\,_{1\le i\le m}\{n_{ij}\}\end{pmatrix}_{l\sigma}\ge\theta_{l\sigma}, j=1,2,\dots,n,$$

$$\begin{pmatrix}\sum_{i=1}^{m}(a_{ij1})u_i\,,\sum_{i=1}^{m}(a_{ij2})u_i\,,\sum_{i=1}^{m}(a_{ij3})u_i\,,\sum_{i=1}^{m}(a_{ij4})u_i\,;\\ minimum_{1\le i\le m}\{\,l_{ij}\},maximum\,_{1\le i\le m}\{m_{ij}\},maximum\,_{1\le i\le m}\{n_{ij}\}\end{pmatrix}_{r\sigma}\ge\theta_{r\sigma}, j=1,2,\dots,n,$$

$$\begin{pmatrix} \sum_{i=1}^{m}(a_{ij1})u_i\,, \sum_{i=1}^{m}(a_{ij2})u_i\,, \sum_{i=1}^{m}(a_{ij3})u_i\,, \sum_{i=1}^{m}(a_{ij4})u_i\,; \\ minimum_{1\le i\le m}\{\,l_{ij}\}, maximum\,_{1\le i\le m}\{m_{ij}\}, maximum\,_{1\le i\le m}\{n_{ij}\} \end{pmatrix}_{l\tau} \ge \theta_{l\tau}, j = 1,2, \dots, n,$$

$$\begin{pmatrix} \sum_{i=1}^{m}(a_{ij1})u_i\,, \sum_{i=1}^{m}(a_{ij2})u_i\,, \sum_{i=1}^{m}(a_{ij3})u_i\,, \sum_{i=1}^{m}(a_{ij4})u_i\,; \\ minimum_{1\le i\le m}\{\,l_{ij}\}, maximum\,_{1\le i\le m}\{m_{ij}\}, maximum\,_{1\le i\le m}\{n_{ij}\} \end{pmatrix}_{r\tau} \ge \theta_{r\tau}, j = 1,2, \dots, n,$$

$\sum_{i=1}^{m} u_i = 1, u_i \ge 0, i = 1,2, \dots, m.$

Problem (P28)

$Minimize\ \{[\phi_{l\rho}, \phi_{r\rho}], [\phi_{l\sigma}, \phi_{r\sigma}], [\phi_{l\tau}, \phi_{r\tau}]\}$

Subject to

$$\begin{pmatrix} \sum_{j=1}^{n}(a_{ij1})v_j\,, \sum_{j=1}^{n}(a_{ij2})v_j\,, \sum_{j=1}^{n}(a_{ij3})v_j\,, \sum_{j=1}^{n}(a_{ij4})v_j\,; \\ minimum_{1\le j\le n}\{\,l_{ij}\}, maximum\,_{1\le j\le n}\{m_{ij}\}, maximum\,_{1\le j\le n}\{n_{ij}\} \end{pmatrix}_{l\rho} \le \phi_{l\rho}, i = 1,2, \dots, m,$$

$$\begin{pmatrix} \sum_{j=1}^{n}(a_{ij1})v_j\,, \sum_{j=1}^{n}(a_{ij2})v_j\,, \sum_{j=1}^{n}(a_{ij3})v_j\,, \sum_{j=1}^{n}(a_{ij4})v_j\,; \\ minimum_{1\le j\le n}\{\,l_{ij}\}, maximum\,_{1\le j\le n}\{m_{ij}\}, maximum\,_{1\le j\le n}\{n_{ij}\} \end{pmatrix}_{r\rho} \le \phi_{r\rho}, i = 1,2, \dots, m,$$

$$\begin{pmatrix} \sum_{j=1}^{n}(a_{ij1})v_j\,, \sum_{j=1}^{n}(a_{ij2})v_j\,, \sum_{j=1}^{n}(a_{ij3})v_j\,, \sum_{j=1}^{n}(a_{ij4})v_j\,; \\ minimum_{1\le j\le n}\{\,l_{ij}\}, maximum\,_{1\le j\le n}\{m_{ij}\}, maximum\,_{1\le j\le n}\{n_{ij}\} \end{pmatrix}_{l\sigma} \le \phi_{l\sigma}, i = 1,2, \dots, m,$$

$$\begin{pmatrix} \sum_{j=1}^{n}(a_{ij1})v_j\,, \sum_{j=1}^{n}(a_{ij2})v_j\,, \sum_{j=1}^{n}(a_{ij3})v_j\,, \sum_{j=1}^{n}(a_{ij4})v_j\,; \\ minimum_{1\le j\le n}\{\,l_{ij}\}, maximum\,_{1\le j\le n}\{m_{ij}\}, maximum\,_{1\le j\le n}\{n_{ij}\} \end{pmatrix}_{r\sigma} \le \phi_{r\sigma}, i = 1,2, \dots, m,$$

$$\begin{pmatrix} \sum_{j=1}^{n}(a_{ij1})v_j\,, \sum_{j=1}^{n}(a_{ij2})v_j\,, \sum_{j=1}^{n}(a_{ij3})v_j\,, \sum_{j=1}^{n}(a_{ij4})v_j\,; \\ minimum_{1\le j\le n}\{\,l_{ij}\}, maximum\,_{1\le j\le n}\{m_{ij}\}, maximum\,_{1\le j\le n}\{n_{ij}\} \end{pmatrix}_{l\tau} \le \phi_{l\tau}, i = 1,2, \dots, m,$$

$$\begin{pmatrix} \sum_{j=1}^{n}(a_{ij1})v_j\,, \sum_{j=1}^{n}(a_{ij2})v_j\,, \sum_{j=1}^{n}(a_{ij3})v_j\,, \sum_{j=1}^{n}(a_{ij4})v_j\,; \\ minimum_{1\le j\le n}\{\,l_{ij}\}, maximum\,_{1\le j\le n}\{m_{ij}\}, maximum\,_{1\le j\le n}\{n_{ij}\} \end{pmatrix}_{r\tau} \le \phi_{r\tau}, i = 1,2, \dots, m,$$

$\sum_{j=1}^{n} v_j = 1, v_j \ge 0, j = 1,2, \dots, n.$

Step 7: Using the expressions $(A_i)_{l\rho} = \frac{(l_i-\rho)A_{i1}+\rho A_{i2}}{l_i}, (A_i)_{r\rho} = \frac{(l_i-\rho)A_{i4}+\rho A_{i3}}{l_i},$

$(A_i)_{l\sigma} = \frac{(1-\sigma)A_{i2}+(\sigma-m_i)A_{i1}}{1-m_i}, (A_i)_{r\sigma} = \frac{(1-\sigma)A_{i3}+(\sigma-m_i)A_{i4}}{1-m_i}$ and $(A_i)_{l\tau} = \frac{(1-\tau)A_{i2}+(\tau-n_i)A_{i1}}{1-n_i},$

$(A_i)_{r\tau} = \frac{(1-\tau)A_{i3}+(\tau-n_i)A_{i4}}{1-n_i}$, to find an optimal solution of the interval MPP (P27) is equivalent to find an optimal solution of the interval MPP (P29) and to find an optimal solution of the interval MPP (P28) is equivalent to find an optimal solution of the interval MPP (P30).

Problem (P29)

$Maximize \{[\theta_{l\rho}, \theta_{r\rho}], [\theta_{l\sigma}, \theta_{r\sigma}], [\theta_{l\tau}, \theta_{r\tau}]\}$

Subject to

$$\left(minimum_{1\le i\le m}\{ l_{ij}\} - \rho\right) \sum_{i=1}^{m}(a_{ij1})u_i + \rho \sum_{i=1}^{m}(a_{ij2})u_i \ge \left(minimum_{1\le i\le m}\{ l_{ij}\}\right)\theta_{l\rho}, \quad j = 1,2, \dots, n,$$

$$\left(minimum_{1\le i\le m}\{ l_{ij}\} - \rho\right) \sum_{i=1}^{m}(a_{ij4})u_i + \rho \sum_{i=1}^{m}(a_{ij3})u_i \ge \left(minimum_{1\le i\le m}\{ l_{ij}\}\right)\theta_{r\rho}, \quad j = 1,2, \dots, n,$$

$$(1-\sigma) \sum_{i=1}^{m}(a_{ij2})u_i + \left(\sigma - maximum_{\,1\le i\le m}\{m_{ij}\}\right) \sum_{i=1}^{m}(a_{ij1})u_i \ge (1 - maximum_{\,1\le i\le m}\{m_{ij}\})\theta_{l\sigma}, \; j = 1,2, \dots, n,$$

$$(1-\sigma) \sum_{i=1}^{m}(a_{ij3})u_i + \left(\sigma - maximum_{\,1\le i\le m}\{m_{ij}\}\right) \sum_{i=1}^{m}(a_{ij4})u_i \ge (1 - maximum_{\,1\le i\le m}\{m_{ij}\})\theta_{r\sigma}, \; j = 1,2, \dots, n,$$

$$(1-\tau) \sum_{i=1}^{m}(a_{ij2})u_i + \left(\tau - maximum_{\,1\le i\le m}\{n_{ij}\}\right) \sum_{i=1}^{m}(a_{ij1})u_i \ge (1 - maximum_{\,1\le i\le m}\{n_{ij}\})\theta_{l\tau}, \; j = 1,2, \dots, n,$$

$$(1-\tau) \sum_{i=1}^{m}(a_{ij3})u_i + \left(\tau - maximum_{\,1\le i\le m}\{n_{ij}\}\right) \sum_{i=1}^{m}(a_{ij4})u_i \ge (1 - maximum_{\,1\le i\le m}\{n_{ij}\})\theta_{r\tau}, \; j = 1,2, \dots, n,$$

$\sum_{i=1}^{m} u_i = 1, u_i \ge 0, i = 1,2, \dots, m.$

Problem (P30)

$Minimize \{[\phi_{l\rho}, \phi_{r\rho}], [\phi_{l\sigma}, \phi_{r\sigma}], [\phi_{l\tau}, \phi_{r\tau}]\}$

Subject to

$$\left(minimum_{1\le j\le n}\{ l_{ij}\} - \rho\right) \sum_{j=1}^{n}(a_{ij1})v_j + \rho \sum_{j=1}^{n}(a_{ij2})v_j \le \left(minimum_{1\le j\le n}\{ l_{ij}\}\right)\phi_{l\rho},$$

$$i = 1,2,\dots,m,$$

$$\left(minimum_{1\le j\le n}\{\, l_{ij}\} - \rho\right)\sum_{j=1}^{n}\left(a_{ij4}\right)v_j + \rho\sum_{j=1}^{n}\left(a_{ij3}\right)v_j \le \left(minimum_{1\le j\le n}\{\, l_{ij}\}\right)\phi_{r\rho},$$

$$i = 1,2,\dots,m,$$

$$(1-\sigma)\sum_{j=1}^{n}\left(a_{ij2}\right)v_j + \left(\sigma - maximum_{\ 1\le j\le n}\{m_{ij}\}\right)\sum_{j=1}^{n}\left(a_{ij1}\right)v_j \le \left(1 - maximum_{\ 1\le j\le n}\right.$$

$$\left.\{m_{ij}\}\right)\phi_{l\sigma},\ i = 1,2,\dots,m,$$

$$(1-\sigma)\sum_{j=1}^{n}\left(a_{ij3}\right)v_j + \left(\sigma - maximum_{\ 1\le j\le n}\{m_{ij}\}\right)\sum_{j=1}^{n}\left(a_{ij4}\right)v_j \le \left(1 - maximum_{\ 1\le j\le n}\right.$$

$$\left.\{m_{ij}\}\right)\phi_{r\sigma},\ i = 1,2,\dots,m,$$

$$(1-\tau)\sum_{j=1}^{n}\left(a_{ij2}\right)v_j + \left(\tau - maximum_{\ 1\le j\le n}\{n_{ij}\}\right)\sum_{j=1}^{n}\left(a_{ij1}\right)v_j \le \left(1 - maximum_{\ 1\le j\le n}\right.$$

$$\left.\{n_{ij}\}\right)\phi_{l\tau},\ i = 1,2,\dots,m,$$

$$(1-\tau)\sum_{j=1}^{n}\left(a_{ij3}\right)v_j + \left(\tau - maximum_{\ 1\le j\le n}\{n_{ij}\}\right)\sum_{j=1}^{n}\left(a_{ij4}\right)v_j \le \left(1 - maximum_{\ 1\le j\le n}\right.$$

$$\left.\{n_{ij}\}\right)\phi_{r\tau},\ i = 1,2,\dots,m,$$

$$\sum_{j=1}^{n} v_j = 1, v_j \ge 0, j = 1,2,\dots,n.$$

Step 8: Aggregating the objective functions of interval MPPs (P29) and (P30), to find an optimal solution of the interval MPP (P29) is equivalent to find an optimal solution of the interval MPP (P31) and to find an optimal solution of the interval MPP (P30) is equivalent to find an optimal solution of the interval MPP (P32).

Problem (P31)

$$Maximize\left\{\left[\frac{\theta_{l\rho}+\theta_{l\sigma}+\theta_{l\tau}}{3}, \frac{\theta_{r\rho}+\theta_{r\sigma}+\theta_{r\tau}}{3}\right]\right\}$$

Subject to

Constraints of the Problem (P29).

Problem (P32)

$$Minimize\left\{\left[\frac{\phi_{l\rho}+\phi_{l\sigma}+\phi_{l\tau}}{3}, \frac{\phi_{r\rho}+\phi_{r\sigma}+\phi_{r\tau}}{3}\right]\right\}$$

Subject to

Constraints of the Problem (P30).

Step 8: Applying the concept of Ishibuchi and Tanaka (1990), to find an optimal solution of the interval MPP (P31) is equivalent to find an optimal solution of the interval MPP (P33) and to find an optimal solution of the interval MPP (P32) is equivalent to find an optimal solution of the interval MPP (P34).

Problem (P33)

$Maximize\left\{\frac{\theta_{l\rho}+\theta_{l\sigma}+\theta_{l\tau}}{3},\frac{\theta_{l\rho}+\theta_{r\rho}+\theta_{l\sigma}+\theta_{r\sigma}+\theta_{l\tau}+\theta_{r\tau}}{6}\right\}$

Subject to

Constraints of the Problem (P29).

Problem (P34)

$Minimize\left\{\frac{\phi_{l\rho}+\phi_{r\rho}+\phi_{l\sigma}+\phi_{r\sigma}+\phi_{l\tau}+\phi_{r\tau}}{6},\frac{\phi_{r\rho}+\phi_{r\sigma}+\phi_{r\tau}}{3}\right\}$

Subject to

Constraints of the Problem (P30).

Step 9: Using the weighted average method, to find an optimal solution of the interval MPP (P33) is equivalent to find an optimal solution of the CLPP (P15) and to find an optimal solution of the interval MPP (P34) is equivalent to find an optimal solution of the CLPP (P16).

Step 10: Find the optimal values of $u_i, i = 1,2,\dots,m, \theta_{l\rho}, \theta_{r\rho}, \theta_{l\sigma}, \theta_{r\sigma}, \theta_{l\tau}, \theta_{r\tau}$ and $\frac{1}{2}\left(\frac{\theta_{l\rho}+\theta_{l\sigma}+\theta_{l\tau}}{3}+\frac{\theta_{l\rho}+\theta_{r\rho}+\theta_{l\sigma}+\theta_{r\sigma}+\theta_{l\tau}+\theta_{r\tau}}{6}\right)$ by solving the CLPP (P15) for some specific values of

$\rho \in \left[0, minimum_{\substack{1\le i\le m\\1\le j\le n}}\left(l_{ij}\right)\right], \sigma \in [maximum_{\substack{1\le i\le m\\1\le j\le n}}\left(m_{ij}\right), 1]$ and $\tau \in \left[maximum_{\substack{1\le i\le m\\1\le j\le n}}\left(n_{ij}\right), 1\right]$.

Step 11: Find the optimal values of $v_j, j = 1,2,\ldots,n, \phi_{l\rho}, \phi_{r\rho}, \phi_{l\sigma}, \phi_{r\sigma}, \phi_{l\tau}, \phi_{r\tau}$ and $\frac{1}{2}\left(\frac{\phi_{l\rho}+\phi_{r\rho}+\phi_{l\sigma}+\phi_{r\sigma}+\phi_{l\tau}+\phi_{r\tau}}{6}+\frac{\phi_{r\rho}+\phi_{r\sigma}+\phi_{r\tau}}{3}\right)$ by solving the CLPP (P16) for some specific values of $\rho \in \left[0, minimum_{\substack{1\le i\le m\\1\le j\le n}}(l_{ij})\right], \sigma \in [maximum_{\substack{1\le i\le m\\1\le j\le n}}(m_{ij}), 1]$ and $\tau \in \left[maximum_{\substack{1\le i\le m\\1\le j\le n}}(n_{ij}), 1\right]$.

Step 12: The optimal values of $u_i, i = 1,2,\ldots,m$, obtained in Step 10, represents the optimal strategies of Player I for the considered SVTN matrix games and the optimal values of $v_j, j = 1,2,\ldots,n$, obtained in Step 11, represents the optimal strategies of Player II for the considered SVTN matrix games.

Step 13: The optimal value of $\frac{1}{2}\left(\frac{\theta_{l\rho}+\theta_{l\sigma}+\theta_{l\tau}}{3}+\frac{\theta_{l\rho}+\theta_{r\rho}+\theta_{l\sigma}+\theta_{r\sigma}+\theta_{l\tau}+\theta_{r\tau}}{6}\right)$, obtained in Step 10, represents the minimum expected gain of Player I and the optimal value of $\frac{1}{2}\left(\frac{\phi_{l\rho}+\phi_{r\rho}+\phi_{l\sigma}+\phi_{r\sigma}+\phi_{l\tau}+\phi_{r\tau}}{6}+\frac{\phi_{r\rho}+\phi_{r\sigma}+\phi_{r\tau}}{3}\right)$, obtained in Step 11, represents the maximum expected loss of Player II.

3 Invalidity of Brikaa's approach

If Brikaa's (2022) approach is valid then the minimum expected gain of Player I should be equal to the maximum expected loss of Player II. However, in this section with the help of a numerical example, it is shown that the minimum expected gain of Player I is not equal to the maximum expected loss of Player II. Hence, Brikaa's (2022) approach is not valid.

Seikh and Dutta (2022, Section 5.1, Example 1, p. 929) considered the payoff matrix $\tilde{A} = \begin{pmatrix} \langle(175,177,180,190);0.6,0.3,0.3\rangle & \langle(150,153,156,158);0.5,0.2,0.3\rangle \\ \langle(125,128,132,140);0.9,0.1,0.5\rangle & \langle(175,185,195,200);0.5,0.4,0.5\rangle \end{pmatrix}$ of a SVTN matrix game to illustrate their proposed approaches.

It can be easily verified that for the considered SVTN matrix game, the CLPPs (P15) and (P16) are transformed into the CLPPs (P35) and (P36) respectively.

Problem (P35)

$$Maximize\ \left\{\frac{1}{2}\left(\frac{\theta_{l\rho}+\theta_{l\sigma}+\theta_{l\tau}}{3}+\frac{\theta_{l\rho}+\theta_{r\rho}+\theta_{l\sigma}+\theta_{r\sigma}+\theta_{l\tau}+\theta_{r\tau}}{6}\right)\right\}$$

Subject to

$$(0.6-\rho)(175u_1+125u_2)+\rho(177u_1+128u_2)\geq 0.6\ \theta_{l\rho},$$

$$(0.6-\rho)(190u_1+140u_2)+\rho(180u_1+132u_2)\geq 0.6\ \theta_{r\rho},$$

$$(0.5-\rho)(150u_1+175u_2)+\rho(153u_1+185u_2)\geq 0.5\ \theta_{l\rho},$$

$$(0.5-\rho)(158u_1+200u_2)+\rho(156u_1+195u_2)\geq 0.5\ \theta_{r\rho},$$

$$(1-\sigma)(177u_1+128u_2)+(\sigma-0.3)(175u_1+125u_2)\geq 0.7\theta_{l\sigma},$$

$$(1-\sigma)(180u_1+132u_2)+(\sigma-0.3)(190u_1+140u_2)\geq 0.7\theta_{r\sigma},$$

$$(1-\sigma)(153u_1+185u_2)+(\sigma-0.4)(150u_1+175u_2)\geq 0.6\theta_{l\sigma},$$

$$(1-\sigma)(156u_1+195u_2)+(\sigma-0.4)(158u_1+200u_2)\geq 0.6\theta_{r\sigma},$$

$$(1-\tau)(177u_1+128u_2)+(\tau-0.5)(175u_1+125u_2)\geq 0.5\theta_{l\tau},$$

$$(1-\tau)(180u_1+132u_2)+(\tau-0.5)(190u_1+140u_2)\geq 0.5\theta_{r\tau},$$

$$(1-\tau)(153u_1+185u_2)+(\tau-0.5)(150u_1+175u_2)\geq 0.5\theta_{l\tau},$$

$$(1-\tau)(153u_1+195u_2)+(\tau-0.5)(158u_1+200u_2)\geq 0.5\theta_{r\tau},$$

$$u_1+u_2=1,\ \ u_1\geq 0, u_2\geq 0.$$

Problem (P36)

$$Minimize\ \left\{\frac{1}{2}\left(\frac{\phi_{l\rho}+\phi_{r\rho}+\phi_{l\sigma}+\phi_{r\sigma}+\phi_{l\tau}+\phi_{r\tau}}{6}+\frac{\phi_{r\rho}+\phi_{r\sigma}+\phi_{r\tau}}{3}\right)\right\}$$

Subject to

$$(0.5-\rho)(175v_1+150v_2)+\rho(177v_1+153v_2)\leq 0.5\ \phi_{l\rho},$$

$$(0.5-\rho)(190v_1+158v_2)+\rho(180v_1+156v_2)\leq 0.5\ \phi_{r\rho},$$

$$(0.5-\rho)(125v_1+175v_2)+\rho(128v_1+185v_2)\leq 0.5\ \phi_{l\rho},$$

$$(0.5-\rho)(140v_1+200v_2)+\rho(132v_1+195v_2)\leq 0.5\ \phi_{r\rho},$$

$$(1-\sigma)(177v_1+153v_2)+(\sigma-0.3)(175v_1+150v_2)\leq 0.7\ \phi_{l\sigma},$$

$$(1-\sigma)(180v_1+156v_2)+(\sigma-0.3)(190v_1+158v_2)\leq 0.7\ \phi_{r\sigma},$$

$$(1-\sigma)(128v_1+185v_2)+(\sigma-0.4)(125v_1+175v_2)\leq 0.6\ \phi_{l\sigma},$$

$$(1-\sigma)(132v_1+195v_2)+(\sigma-0.4)(140v_1+200v_2)\leq 0.6\ \phi_{r\sigma},$$

$(1-\tau)(177v_1+153v_2)+(\tau-0.4)(175v_1+150v_2)\leq 0.6\ \phi_{l\tau}$,

$(1-\tau)(180v_1+156v_2)+(\tau-0.4)(190v_1+158v_2)\leq 0.6\ \phi_{r\tau}$,

$(1-\tau)(128v_1+185v_2)+(\tau-0.5)(125v_1+175v_2)\leq 0.5\ \phi_{l\tau}$,

$(1-\tau)(132v_1+195v_2)+(\tau-0.5)(140v_1+200v_2)\leq 0.5\ \phi_{r\tau}$,

$v_1+v_2=1,\quad v_1\geq 0, v_2\geq 0.$

It is pertinent to mention that the CLPP (P35) is same as the existing problem (Seikh and Dutta, 2022, Section 5.1.1, Problem 13, p. 930) and the CLPP (P36) is same as the existing problem (Seikh and Dutta, 2022, Section 5.1.1, Problem 14, pp. 930-931). So, on solving

(i) The CLPP (P35), the existing results (Seikh and Dutta, 2022, Section 5.1.1, Table 1, p. 931), shown in Table 1, will be obtained

Table 1 Optimal values of u_1, u_2 and $\frac{1}{2}\left(\frac{\theta_{l\rho}+\theta_{l\sigma}+\theta_{l\tau}}{3}+\frac{\theta_{l\rho}+\theta_{r\rho}+\theta_{l\sigma}+\theta_{r\sigma}+\theta_{l\tau}+\theta_{r\tau}}{6}\right)$

(ρ,σ,τ)	(u_1,u_2)	$[\theta_{l\rho},\theta_{r\rho}]$	$[\theta_{l\sigma},\theta_{r\sigma}]$	$[\theta_{l\tau},\theta_{r\tau}]$	$\frac{1}{2}\left(\frac{\theta_{l\rho}+\theta_{l\sigma}+\theta_{l\tau}}{3}+\frac{\theta_{l\rho}+\theta_{r\rho}+\theta_{l\sigma}+\theta_{r\sigma}+\theta_{l\tau}+\theta_{r\tau}}{6}\right)$
(0,1,1)	(0.6667, 0.3333)	[158.3333, 172]	[158.3333, 172]	[158.3333, 172]	161.749975
(0,0.8,1)	(0.6667, 0.3333)	[158.3333, 172]	[159, 170.6667]	[158.3333, 172]	161.91665
(0,0.6,1)	(0.6667, 0.3333)	[158.3333, 172]	[160.3333, 165.33333]	[158.3333, 172]	162.249975
(0,1,0.8)	(0.6667, 0.3333)	[158.3333, 172]	[158.3333, 172]	[159.2667, 169.6]	161.783325
(0,1,0.6)	(0.6667, 0.3333)	[158.3333, 172]	[158.3333, 172]	[160.2, 165.8667]	161.705541
(0.2,1,1)	(0.6667, 0.3333)	[159.1111, 170.2222]	[158.3333, 172]	[158.3333, 172]	161.796125
(0.4,1,1)	(0.6667, 0.3333)	[159.8889, 167.1111]	[158.3333, 172]	[158.3333, 172]	161.73146

(0.3,0.4,0.5)	(0.7077, 0.2922)	[160.3338, 168.5486]	[162.3521, 167.3169]	[162.3521, 165.9718]	163.079275
(0.5,0.4,0.5)	(0.7084, 0.2916)	[162.3306, 167.3717]	[162.3306, 167.3494]	[162.3306, 166.0041]	163.47505

(ii) The CLPP (P36), the existing results (Seikh and Dutta, 2022, Section 5.1.1, Table 2, p. 931), shown in Table 2, will be obtained.

Table 2 Optimal values of v_1, v_2 and $\frac{1}{2}\left(\frac{\phi_{l\rho}+\phi_{r\rho}+\phi_{l\sigma}+\phi_{r\sigma}+\phi_{l\tau}+\phi_{r\tau}}{6}+\frac{\phi_{r\rho}+\phi_{r\sigma}+\phi_{r\tau}}{3}\right)$

(ρ,σ,τ)	(v_1,v_2)	$[\phi_{l\rho},\phi_{r\rho}]$	$[\phi_{l\sigma},\phi_{r\sigma}]$	$[\phi_{l\tau},\phi_{r\tau}]$	$\frac{1}{2}\left(\frac{\phi_{l\rho}+\phi_{r\rho}+\phi_{l\sigma}+\phi_{r\sigma}+\phi_{l\tau}+\phi_{r\tau}}{6}+\frac{\phi_{r\rho}+\phi_{r\sigma}+\phi_{r\tau}}{3}\right)$
(0,1,1)	(0.4565, 0.5435)	[161.413, 172.6087]	[161.413, 172.6087]	[161.413, 172.6087]	169.809775
(0,0.8,1)	(0.4565, 0.5435)	[161.413, 172.6087]	[162.1398, 170.9938]	[161.413, 172.6087]	169.466617
(0,0.6,1)	(0.4565, 0.5435)	[161.413, 172.6087]	[162.8665, 169.3789]	[161.413, 172.6087]	169.12345
(0,0.4,1)	(0.4565, 0.5435)	[161.413, 172.6087]	[163.5932, 167.764]	[161.413, 172.6087]	168.780283
(0,1,0.8)	(0.4565, 0.5435)	[161.413, 172.6087]	[161.413, 172.6087]	[162.2609, 170.7246]	169.409408
(0,1,0.6)	(0.4565, 0.5435)	[161.413, 172.6087]	[161.413, 172.6087]	[163.1087, 168.8406]	169.48005
(0.3,0.4, 0.5)	(0.4517, 0.5483)	[162.8211, 169.0858]	[163.4764, 167.6424]	[163.4157, 167.776]	166.9354
(0.5,0.4, 0.5)	(0.4483, 0.5517)	[163.7586, 166.7586]	[163.3941, 167.5567]	[163.3333, 167.6897]	166.37508

It is obvious from the results, shown in Table 1 and Table 2, that for a specific value of ρ, σ, τ, the optimal value of the CLPP (P35) is not equal to the optimal value of the CLPP (P36) i.e., the minimum expected gain of the Player I is not equal to the maximum expected loss of Player II.

For example, for $\rho = 0, \sigma = 1, \tau = 1$, the optimal value of the CLPP (P35) is 161.749975 i.e., the minimum expected gain of the Player I is 161.749975 and the optimal value of the CLPP (P36) is 169.809775 i.e., maximum expected loss of Player II is169.809775.

This clearly indicates that the minimum expected gain of the Player I is not equal to the maximum expected loss of Player II. So, Brikaa's (2022) approach is not valid.

4 Reasons for the invalidity

Brikaa's (2022) approach is not valid as the following two mathematically incorrect results are considered in it.

4.1 First mathematically incorrect result

It is obvious that in Step 1 of Brikaa's approach (2022), discussed in Section 2, the relation $\left((\sum_{i=1}^{m} \tilde{a}_{i1}u_i)v_1 + (\sum_{i=1}^{m} \tilde{a}_{i2}u_i)v_2 + \cdots + (\sum_{i=1}^{m} \tilde{a}_{in}u_i)v_n\right)_k = (\sum_{i=1}^{m} \tilde{a}_{i1}u_i)_k v_1 + (\sum_{i=1}^{m} \tilde{a}_{i2}u_i)_k v_2 + \cdots + (\sum_{i=1}^{m} \tilde{a}_{in}u_i)_k v_n, k = \rho, \sigma, \tau$, is used to transform the MPP (P7) into interval MPP (P17).

However, the following example clearly indicates that $\left((\sum_{i=1}^{m} \tilde{a}_{i1}u_i)v_1 + (\sum_{i=1}^{m} \tilde{a}_{i2}u_i)v_2 + \cdots + (\sum_{i=1}^{m} \tilde{a}_{in}u_i)v_n\right)_k \neq (\sum_{i=1}^{m} \tilde{a}_{i1}u_i)_k v_1 + (\sum_{i=1}^{m} \tilde{a}_{i2}u_i)_k v_2 + \cdots + (\sum_{i=1}^{m} \tilde{a}_{in}u_i)_k v_n, k = \rho, \sigma, \tau.$

Let $\tilde{A} = \begin{pmatrix} \langle(175,177,180,190); 0.6,0.3,0.3\rangle & \langle(150,153,156,158); 0.5,0.2,0.3\rangle \\ \langle(125,128,132,140); 0.9,0.1,0.5\rangle & \langle(175,185,195,200); 0.5,0.4,0.5\rangle \end{pmatrix}$ be the payoff matrix of a SVTN matrix game. Then,

$\left((\sum_{i=1}^{m} \tilde{a}_{i1}u_i)v_1 + (\sum_{i=1}^{m} \tilde{a}_{i2}u_i)v_2 + \cdots + (\sum_{i=1}^{m} \tilde{a}_{in}u_i)v_n\right) = (\sum_{i=1}^{2} \tilde{a}_{i1}u_i)v_1 + (\sum_{i=1}^{2} \tilde{a}_{i2}u_i)v_2$

$= \big((\langle(175,177,180,190); 0.6,0.3,0.3\rangle\, u_1 + \langle(125,128,132,140); 0.9,0.1,0.5\rangle\, u_2)v_1 +$

$(\langle(150,153,156,158); 0.5,0.2,0.3\rangle\, u_1 + \langle(175,185,195,200); 0.5,0.4,0.5\rangle\, u_2)v_2\big)$

$= (\langle (175u_1 + 125u_2, 177u_1 + 128u_2, 180u_1 + 132u_2, 190u_1 + 140u_2); 0.6,0.3,0.5\rangle\ )v_1 +$

$(\langle (150u_1 + 175u_2, 153u_1 + 185u_2, 156u_1 + 195u_2, 158u_1 + 200u_2); 0.5,0.4,0.5\rangle\ \ )v_2$

$$= \langle \begin{pmatrix} 175u_1v_1 + 125u_2v_1 + 150u_1v_2 + 175u_2v_2, \\ 177u_1v_1 + 128u_2v_1 + 153u_1v_2 + 185u_2v_2, \\ 180u_1v_1 + 132u_2v_1 + 156u_1v_2 + 195u_2v_2, \\ 190u_1v_1 + 140u_2v_1 + 158u_1v_2 + 200u_2v_2 \end{pmatrix}; 0.5,0.4,0.5\rangle$$

Substituting, $u_1 = \frac{1}{3}, u_2 = \frac{2}{3}, v_1 = \frac{2}{3},\ v_2 = \frac{1}{3}$

$= \langle (149.985, 154.3179, 159.3174, 166.4278); 0.5,0.4,0.5\rangle$

Therefore, $\left((\sum_{i=1}^{m} \tilde{a}_{i1}u_i)v_1 + (\sum_{i=1}^{m} \tilde{a}_{i2}u_i)v_2 + \cdots + (\sum_{i=1}^{m} \tilde{a}_{in}u_i)v_n\right)_\rho = \left((\sum_{i=1}^{2} \tilde{a}_{i1}u_i)v_1 + \right.$

$\left.(\sum_{i=1}^{2} \tilde{a}_{i2}u_i)v_2\right)_\rho$

$= (\langle (149.985, 154.3179, 159.3174, 166.4278); 0.5,0.4,0.5\rangle)_\rho$

$$= \left[\frac{74.9925 + 4.3329\rho}{0.5}, \frac{83.2139 - 7.1104\,\rho}{0.5}\right] \tag{1}$$

While,

$(\sum_{i=1}^{m} \tilde{a}_{i1}u_i)_\rho v_1 + (\sum_{i=1}^{m} \tilde{a}_{i2}u_i)_\rho v_2 + \cdots + (\sum_{i=1}^{m} \tilde{a}_{in}u_i)_\rho v_n = (\sum_{i=1}^{2} \tilde{a}_{i1}u_i)_\rho v_1 + (\sum_{i=1}^{2} \tilde{a}_{i2}u_i)_\rho v_2$

$= (\langle (175, 177, 180, 190); 0.6,0.3,0.3\rangle\ u_1 + \langle (125, 128, 132, 140); 0.9,0.1,0.5\rangle\ u_2)_\rho v_1 +$

$(\langle (150, 153, 156, 158); 0.5,0.2,0.3\rangle\ u_1 + \langle (175, 185, 195, 200); 0.5,0.4,0.5\rangle\ u_2)_\rho v_2$

$= (\langle (175u_1 + 125u_2, 177u_1 + 128u_2, 180u_1 + 132u_2, 190u_1 + 140u_2); 0.6,0.3,0.5\rangle)_\rho v_1 +$

$(\langle (150u_1 + 175u_2, 153u_1 + 185u_2, 156u_1 + 195u_2, 158u_1 + 200u_2); 0.5,0.4,0.5\rangle)_\rho v_2$

Substituting, $u_1 = \frac{1}{3}, u_2 = \frac{2}{3}, v_1 = \frac{2}{3}, v_2 = \frac{1}{3}$

$= (\langle (141.666, 144.333, 148, 156.666); 0.6,0.3,0.5\rangle)_\rho \left(\frac{2}{3}\right) +$

$(\langle (166.666, 174.333, 182\ , 185.999); 0.5,0.4,0.5\rangle)_\rho \left(\frac{1}{3}\right)$

$$= \begin{bmatrix} \frac{(0.6-\rho)141.666+144.333\,\rho}{0.6}, \\ \frac{(0.6-\rho)156.666+148\,\rho}{0.6} \end{bmatrix}\left(\frac{2}{3}\right) + \begin{bmatrix} \frac{(0.5-\rho)166.666+174.333\,\rho}{0.5}, \\ \frac{(0.5-\rho)185.999+182\,\rho}{0.5} \end{bmatrix}\left(\frac{1}{3}\right)$$

$$= \left[\frac{56.6603+1.7778\,\rho}{0.6}, \frac{62.6537-5.7767\,\rho}{0.6}\right] + \left[\frac{27.7748+2.5554\,\rho}{0.5}, \frac{30.9965-1.3328\,\rho}{0.5}\right]$$

$$= \left[\frac{44.9950+2.4221\rho}{0.3}, \frac{49.9247-3.6879\rho}{0.3}\right] \tag{2}$$

It is obvious from (1) and (2) that $\left((\sum_{i=1}^{m}\tilde{a}_{i1}u_i)v_1 + (\sum_{i=1}^{m}\tilde{a}_{i2}u_i)v_2 + \cdots + (\sum_{i=1}^{m}\tilde{a}_{in}u_i)v_n\right)_k \neq (\sum_{i=1}^{m}\tilde{a}_{i1}u_i)_k v_1 + (\sum_{i=1}^{m}\tilde{a}_{i2}u_i)_k v_2 + \cdots + (\sum_{i=1}^{m}\tilde{a}_{in}u_i)_k v_n, k = \rho, \sigma, \tau$. So, the interval MPP (P7) is not equivalent to the interval MPP (P17).

Furthermore, it is obvious that in Step 1 of Brikaa's approach (2022), discussed in Section 2, the relation $\left((\sum_{j=1}^{n}\tilde{a}_{1j}v_j)u_1 + (\sum_{j=1}^{n}\tilde{a}_{2j}v_j)u_2 + \cdots + (\sum_{j=1}^{n}\tilde{a}_{mj}v_j)u_m\right)_k = (\sum_{j=1}^{n}\tilde{a}_{1j}v_j)_k u_1 + (\sum_{j=1}^{n}\tilde{a}_{2j}v_j)_k u_2 + \cdots + (\sum_{j=1}^{n}\tilde{a}_{mj}v_j)_k u_m, k = \rho, \sigma, \tau,$ is used to transform the MPP (P8) into interval MPP (P18). While, it can be easily verified that $\left((\sum_{j=1}^{n}\tilde{a}_{1j}v_j)u_1 + (\sum_{j=1}^{n}\tilde{a}_{2j}v_j)u_2 + \cdots + (\sum_{j=1}^{n}\tilde{a}_{mj}v_j)u_m\right)_k \neq (\sum_{j=1}^{n}\tilde{a}_{1j}v_j)_k u_1 + (\sum_{j=1}^{n}\tilde{a}_{2j}v_j)_k u_2 + \cdots + (\sum_{j=1}^{n}\tilde{a}_{mj}v_j)_k u_m, k = \rho, \sigma, \tau.$

So, the interval MPP (P8) is not equivalent to the interval MPP (P18).

4.2 Second mathematically incorrect result

It is a well known fact that if $a_1, a_2, \ldots, a_n$ are real numbers. Then,

(i) $\theta = Minimum\{a_1, a_2, \ldots, a_n\}$ will be one of the real numbers $a_1, a_2, \ldots, a_n$. Therefore, the relations $a_1 \geq \theta, a_2 \geq \theta, \ldots, a_n \geq \theta$ are valid.

(ii) $\phi = Maximum\{a_1, a_2, \ldots, a_n\}$ will be one of the real numbers $a_1, a_2, \ldots, a_n$. Therefore, the relations $a_1 \leq \phi, a_2 \leq \phi, \ldots, a_n \leq \phi$ are valid.

On the same direction, in Step 3 of Brikaa's (2022) approach it is assumed that

(i) $\tilde{\theta}_k = Minimum((\sum_{i=1}^{m} \tilde{a}_{i1}u_i)_k, (\sum_{i=1}^{m} \tilde{a}_{i2}u_i)_k, \dots, (\sum_{i=1}^{m} \tilde{a}_{in}u_i)_k), k = \rho, \sigma, \tau$ will be one of the intervals $(\sum_{i=1}^{m} \tilde{a}_{i1}u_i)_k, (\sum_{i=1}^{m} \tilde{a}_{i2}u_i)_k, \dots, (\sum_{i=1}^{m} \tilde{a}_{in}u_i)_k$. Therefore, the relations $(\sum_{i=1}^{m} \tilde{a}_{i1}u_i)_k \geq \tilde{\theta}_k, (\sum_{i=1}^{m} \tilde{a}_{i2}u_i)_k \geq \tilde{\theta}_k, \dots, (\sum_{i=1}^{m} \tilde{a}_{in}u_i)_k \geq \tilde{\theta}_k$ are valid.

(ii) $\tilde{\phi}_k = Maximum\left(\left(\sum_{j=1}^{n} \tilde{a}_{1j}v_j\right)_k, \left(\sum_{j=1}^{n} \tilde{a}_{2j}v_j\right)_k, \dots, \left(\sum_{j=1}^{n} \tilde{a}_{mj}v_j\right)_k\right), k = \rho, \sigma, \tau$ will be one of the intervals $\left(\sum_{j=1}^{n} \tilde{a}_{1j}v_j\right)_k, \left(\sum_{j=1}^{n} \tilde{a}_{2j}v_j\right)_k, \dots, \left(\sum_{j=1}^{n} \tilde{a}_{mj}v_j\right)_k$. Therefore, the relations $\left(\sum_{j=1}^{n} \tilde{a}_{1j}v_j\right)_k \leq \tilde{\phi}_k, \left(\sum_{j=1}^{n} \tilde{a}_{2j}v_j\right)_k \leq \tilde{\phi}_k, \dots, \left(\sum_{j=1}^{n} \tilde{a}_{mj}v_j\right)_k \leq \tilde{\phi}_k$ are valid.

While, in actual case, if $\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n$ are SVTN numbers. Then,

(i) $\tilde{\theta}_k = Minimum\{\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n\}, where\ k = \rho, \sigma, \tau$ will not necessarily be one of the SVTN numbers $\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n$. Therefore, the relations $\tilde{a}_1 \geq \theta, \tilde{a}_2 \geq \theta, \dots, \tilde{a}_n \geq \theta$ are not always valid.

(ii) $\tilde{\phi}_k = Maximum\{\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n\}, where\ k = \rho, \sigma, \tau$ will not necessarily be one of the SVTN numbers $\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n$. Therefore, the relations $\tilde{a}_1 \leq \phi, \tilde{a}_2 \leq \phi, \dots, \tilde{a}_n \leq \phi$ are not always valid.

The following clearly indicates that the above-mentioned claim is valid.

Let $\widetilde{M} = \langle(2,4,8,10); 0.6,0.5,0.4\rangle$ and $\widetilde{N} = \langle(1,5,7,9); 0.6,0.5,0.4\rangle$ be two SVTN numbers. Then,

$\widetilde{M}_\rho = [M_{l\rho}, M_{r\rho}] = \left[2 + \frac{2\rho}{0.6}, 10 - \frac{2\rho}{0.6}\right], \quad \widetilde{N}_\rho = [N_{l\rho}, N_{r\rho}] = \left[1 + \frac{4\rho}{0.6}, 9 - \frac{2\rho}{0.6}\right].$

Substituting $\rho = 0.6, \widetilde{M}_\rho = [4,8], \widetilde{N}_\rho = [5,7]$

Therefore,

$Minimum\{\widetilde{M}_\rho, \widetilde{N}_\rho\} = [Minimum\ \{M_{l\rho}, N_{l\rho}\}, Minimum\{M_{r\rho}, N_{r\rho}\}] =$

$[Minimum\ \{4,5\}, Minimum\{8,7\}] = [4,7]$

$Maximum\{\widetilde{M}_\rho, \widetilde{N}_\rho\} = [Maximum\ \{M_{l\rho}, N_{l\rho}\}, Maximum\{M_{r\rho}, N_{r\rho}\}] =$

$[Maximum\ \{4,5\}, Maximum\{8,7\}] = [5,8]$

It is obvious that $Minimum\{\widetilde{M}_\rho, \widetilde{N}_\rho\} \neq \widetilde{M}_\rho, Minimum\{\widetilde{M}_\rho, \widetilde{N}_\rho\} \neq \widetilde{N}_\rho$ and $Maximum\{\widetilde{M}_\rho, \widetilde{N}_\rho\} \neq \widetilde{M}_\rho, Maximum\{\widetilde{M}_\rho, \widetilde{N}_\rho\} \neq \widetilde{N}_\rho$.

Similarly, it can be easily verified that $Minimum\{\widetilde{M}_\sigma, \widetilde{N}_\sigma\} \neq \widetilde{M}_\sigma, Minimum\{\widetilde{M}_\sigma, \widetilde{N}_\sigma\} \neq \widetilde{N}_\sigma$ and

$Maximum\{\widetilde{M}_\sigma, \widetilde{N}_\sigma\} \neq \widetilde{M}_\sigma, Maximum\{\widetilde{M}_\sigma, \widetilde{N}_\sigma\} \neq \widetilde{N}_\sigma$ and

$Minimum\{\widetilde{M}_\tau, \widetilde{N}_\tau\} \neq \widetilde{M}_\tau, Minimum\{\widetilde{M}_\tau, \widetilde{N}_\tau\} \neq \widetilde{N}_\tau, Maximum\{\widetilde{M}_\tau, \widetilde{N}_\tau\} \neq$

$\widetilde{M}_\tau, Maximum\{\widetilde{M}_\tau, \widetilde{N}_\tau\} \neq \widetilde{N}_\tau$.

5 Seikh and Dutta's second approach

Seikh and Dutta (2022) proposed the following approach to find the optimal strategies $u_i, i = 1,2,\ldots,m$ of Player I, $v_j, j = 1,2,\ldots,n$ of Player II, the minimum expected gain of Player I and the maximum expected loss of Player II with the help of the crisp MPPs (P9) and (P10).

Step 1: Using the relation $\Pi_\alpha\left(\sum_{j=1}^{n}\left(\sum_{i=1}^{m}\tilde{a}_{ij}u_i\right)v_j\right) = \sum_{j=1}^{n}\Pi_\alpha\left(\sum_{i=1}^{m}\tilde{a}_{ij}u_i\right)v_j$, to find an optimal solution of the crisp MPP (P9) is equivalent to find an optimal solution of the crisp MPP (P37) and using the relation $\Pi_\alpha\left(\sum_{i=1}^{m}\left(\sum_{j=1}^{n}\tilde{a}_{ij}v_j\right)u_i\right) = \sum_{i=1}^{m}\Pi_\alpha\left(\sum_{j=1}^{n}\tilde{a}_{ij}v_j\right)u_i$, to find an optimal solution of the crisp MPP (P10) is equivalent to find an optimal solution of the crisp MPP (P38).

Problem (P37)

$$Maximize \begin{Bmatrix} Minimize\left(\sum_{j=1}^{n}\Pi_\alpha\left(\sum_{i=1}^{m}\tilde{a}_{ij}u_i\right)v_j\right) \\ \text{Subject to} \\ \sum_{j=1}^{n}v_j = 1, v_j \geq 0, j = 1,2,\ldots,n \end{Bmatrix}$$

Subject to

$\sum_{i=1}^{m}u_i = 1, u_i \geq 0, i = 1,2,\ldots,m.$

Problem (P38)

$$Minimize \begin{Bmatrix} Maximize\left(\sum_{i=1}^{m}\Pi_\alpha\left(\sum_{j=1}^{n}\tilde{a}_{ij}v_j\right)u_i\right) \\ \text{Subject to} \\ \sum_{i=1}^{m}u_i = 1, u_i \geq 0, i = 1,2,\ldots,m \end{Bmatrix}$$

Subject to

$\sum_{j=1}^{n}v_j = 1, v_j \geq 0, j = 1,2,\ldots,n.$

Step 2: Since, $\sum_{j=1}^{n} \Pi_\alpha\left(\sum_{i=1}^{m} \tilde{a}_{ij} u_i\right) v_j$ is a convex linear combination of $\Pi_\alpha(\sum_{i=1}^{m} \tilde{a}_{i1} u_i), \Pi_\alpha(\sum_{i=1}^{m} \tilde{a}_{i2} u_i), \dots, \Pi_\alpha(\sum_{i=1}^{m} \tilde{a}_{in} u_i)$ and $\sum_{i=1}^{m} \Pi_\alpha\left(\sum_{j=1}^{n} \tilde{a}_{ij} v_j\right) u_i$ is a convex linear combination of $\Pi_\alpha\left(\sum_{j=1}^{n} \tilde{a}_{1j} v_j\right), \Pi_\alpha\left(\sum_{j=1}^{n} \tilde{a}_{2j} v_j\right), \dots, \Pi_\alpha\left(\sum_{j=1}^{n} \tilde{a}_{mj} v_j\right)$. So, to find an optimal solution of the crisp MPP (P37) is equivalent to find an optimal solution of the crisp MPP (P39) and to find an optimal solution of the crisp MPP (P38) is equivalent to find an optimal solution of the crisp MPP (P40).

Problem (P39)

$Maximize \left\{Minimum_{1\le j\le n} \left(\Pi_\alpha\left(\sum_{i=1}^{m} \tilde{a}_{ij} u_i\right)\right)\right\}$

Subject to

$\sum_{i=1}^{m} u_i = 1, u_i \ge 0, i = 1,2,\dots,m.$

Problem (P40)

$Minimize \left\{Maximum_{1\le i\le m} \left(\Pi_\alpha\left(\sum_{j=1}^{n} \tilde{a}_{ij} v_j\right)\right)\right\}$

Subject to

$\sum_{j=1}^{n} v_j = 1, v_j \ge 0, j = 1,2,\dots,n.$

Step 3: Assuming $Minimum_{1\le j\le n} \left(\Pi_\alpha\left(\sum_{i=1}^{m} \tilde{a}_{ij} u_i\right)\right) = P_1$ and $Maximum_{1\le i\le m} \left(\Pi_\alpha\left(\sum_{j=1}^{n} \tilde{a}_{ij} v_j\right)\right)$ $= P_2$, to find an optimal solution of the crisp MPP (P39) is equivalent to find an optimal solution of the CLPP (P41) and to find an optimal solution of the crisp MPP (P40) is equivalent to find an optimal solution of the CLPP (P42).

Problem (P41)

$Maximize \{P_1\}$

Subject to

$\Pi_\alpha\left(\sum_{i=1}^{m}\left(\langle(a_{ij1}, a_{ij2}, a_{ij3}, a_{ij4}); l_{ij}, m_{ij}, n_{ij}\rangle\right) u_i\right) \ge P_1, j = 1,2,\dots,n,$

$\sum_{i=1}^{m} u_i = 1, u_i \ge 0, i = 1,2,\dots,m.$

Problem (P42)

$Minimize\ \{P_2\}$

Subject to

$\Pi_\alpha\left(\sum_{j=1}^{n}\left(\langle(a_{ij1}, a_{ij2}, a_{ij3}, a_{ij4}); l_{ij}, m_{ij}, n_{ij}\rangle\right)v_j\right) \leq P_2, i = 1,2, \dots, m,$

$\sum_{j=1}^{n} v_j = 1, v_j \geq 0, j = 1,2, \dots, n.$

Step 4: Using the expression

$\sum_{i=1}^{m}\left(\langle(a_{ij1}, a_{ij2}, a_{ij3}, a_{ij4}); l_{ij}, m_{ij}, n_{ij}\rangle\right)u_i =$

$\left(\left\langle \begin{matrix} \left(\sum_{i=1}^{m} a_{ij1}u_i, \sum_{i=1}^{m} a_{ij2}u_i, \sum_{i=1}^{m} a_{ij3}u_i, \sum_{i=1}^{m} a_{ij4}u_i\right); \\ minimum_{1\leq i\leq m}\{l_{ij}\}, maximum_{1\leq i\leq m}\{m_{ij}\}, maximum_{1\leq i\leq m}\{n_{ij}\} \end{matrix} \right\rangle\right)$ to find an optimal solution

of the CLPP (P41) is equivalent to find an optimal solution of the CLPP (P43). Also, using the expression

$\sum_{j=1}^{n}\left(\langle(a_{ij1}, a_{ij2}, a_{ij3}, a_{ij4}); l_{ij}, m_{ij}, n_{ij}\rangle\right)v_j =$

$\left(\left\langle \begin{matrix} \left(\sum_{j=1}^{n} a_{ij1}v_j, \sum_{j=1}^{n} a_{ij2}v_j, \sum_{j=1}^{n} a_{ij3}v_j, \sum_{j=1}^{n} a_{ij4}v_j\right); \\ minimum_{1\leq j\leq n}\{l_{ij}\}, maximum_{1\leq j\leq n}\{m_{ij}\}, maximum_{1\leq j\leq n}\{n_{ij}\} \end{matrix} \right\rangle\right)$ to find an optimal solution

of the CLPP (P42) is equivalent to find an optimal solution of the CLPP (P44).

Problem (P43)

$Maximize\ \{P_1\}$

Subject to

$\Pi_\alpha\left(\left\langle \begin{matrix} \left(\sum_{i=1}^{m} a_{ij1}u_i, \sum_{i=1}^{m} a_{ij2}u_i, \sum_{i=1}^{m} a_{ij3}u_i, \sum_{i=1}^{m} a_{ij4}u_i\right); \\ minimum_{1\leq i\leq m}\{l_{ij}\}, maximum_{1\leq i\leq m}\{m_{ij}\}, maximum_{1\leq i\leq m}\{n_{ij}\} \end{matrix} \right\rangle\right) \geq P_1, j = 1,2, \dots, n,$

$\sum_{i=1}^{m} u_i = 1, u_i \geq 0, i = 1,2, \dots, m.$

Problem (P44)

$Minimize\ \{P_2\}$

Subject to

$$\Pi_\alpha\left(\left\langle \begin{matrix} \left(\sum_{j=1}^{n} a_{ij1}v_j, \sum_{j=1}^{n} a_{ij2}v_j, \sum_{j=1}^{n} a_{ij3}v_j, \sum_{j=1}^{n} a_{ij4}v_j\right); \\ minimum_{1\le j\le n}\{l_{ij}\}, maximum_{1\le j\le n}\{m_{ij}\}, maximum_{1\le j\le n}\{n_{ij}\} \end{matrix} \right\rangle\right) \le P_2, i = 1,2,\dots,m,$$

$\sum_{j=1}^{n} v_j = 1, v_j \ge 0, j = 1,2,\dots,n.$

Step 5: Using the expression

$$\Pi_\alpha\left(\sum_{i=1}^{m}\left(\langle(a_{ij1}, a_{ij2}, a_{ij3}, a_{ij4}); l_{ij}, m_{ij}, n_{ij}\rangle\right)u_i\right) = \eta_j\left(\frac{\sum_{i=1}^{m} a_{ij1}u_i + 2\left(\sum_{i=1}^{m} a_{ij2}u_i + \sum_{i=1}^{m} a_{ij3}u_i\right) + \sum_{i=1}^{m} a_{ij4}u_i}{6}\right)$$

where, $\eta_j = \left[\alpha\left(minimum_{1\le i\le m}\{l_{ij}\}\right)^2 + (1-\alpha)\left(1 - maximum_{1\le i\le m}\{m_{ij}\}\right)^2 + (1-\alpha)\left(1 - maximum_{1\le i\le m}\{n_{ij}\}\right)^2\right]$ to find an optimal solution of the CLPP (P43) is equivalent to find an optimal solution of the CLPP (P45). Also, using the expression

$$\Pi_\alpha\left(\sum_{j=1}^{n}\left(\langle(a_{ij1}, a_{ij2}, a_{ij3}, a_{ij4}); l_{ij}, m_{ij}, n_{ij}\rangle\right)v_j\right) = \eta_i\left(\frac{\sum_{j=1}^{n} a_{ij1}v_j + 2\left(\sum_{j=1}^{n} a_{ij2}v_j + \sum_{j=1}^{n} a_{ij3}v_j\right) + \sum_{j=1}^{n} a_{ij4}v_j}{6}\right)$$

where, $\eta_i = \left[\alpha\left(minimum_{1\le j\le n}\{l_{ij}\}\right)^2 + (1-\alpha)\left(1 - maximum_{1\le j\le n}\{m_{ij}\}\right)^2 + (1-\alpha)\left(1 - maximum_{1\le j\le n}\{n_{ij}\}\right)^2\right]$ to find an optimal solution of the CLPP (P44) is equivalent to find an optimal solution of CLPP (P46).

Problem (P45)

$Maximize\ \{P_1\}$

Subject to

$$\eta_j\left(\frac{\sum_{i=1}^{m} a_{ij1}u_i + 2\left(\sum_{i=1}^{m} a_{ij2}u_i + \sum_{i=1}^{m} a_{ij3}u_i\right) + \sum_{i=1}^{m} a_{ij4}u_i}{6}\right) \ge P_1,\ j = 1,2,\dots,n,$$

$\sum_{i=1}^{m} u_i = 1, u_i \ge 0, i = 1,2,\dots,m.$

Problem (P46)

$Minimize\ \{P_2\}$

Subject to

$$\eta_i\left(\frac{\sum_{j=1}^{n} a_{ij1}v_j+2\left(\sum_{j=1}^{n} a_{ij2}v_j+\sum_{j=1}^{n} a_{ij3}v_j\right)+\sum_{j=1}^{n} a_{ij4}v_j}{6}\right) \le P_2,\ i=1,2,\dots,m,$$

$\sum_{j=1}^{n} v_j = 1, v_j \ge 0, j = 1,2,\dots,n.$

Step 6: Using the expression

$\frac{\sum_{i=1}^{m} a_{ij1}u_i+2\left(\sum_{i=1}^{m} a_{ij2}u_i+\sum_{i=1}^{m} a_{ij3}u_i\right)+\sum_{i=1}^{m} a_{ij4}u_i}{6} = \sum_{i=1}^{m}\left(\frac{a_{ij1}+2a_{ij2}+2a_{ij3}+a_{ij4}}{6}\right)u_i = \sum_{i=1}^{m} A_{ij}u_i$, to find an optimal solution of the CLPP (P45) is equivalent to find an optimal solution of the CLPP (P13).

Also, using the expression

$\frac{\sum_{j=1}^{n} a_{ij1}v_j+2\left(\sum_{j=1}^{n} a_{ij2}v_j+\sum_{j=1}^{n} a_{ij3}v_j\right)+\sum_{j=1}^{n} a_{ij4}v_j}{6} = \sum_{j=1}^{n}\left(\frac{a_{ij1}+2a_{ij2}+2a_{ij3}+a_{ij4}}{6}\right)v_j = \sum_{j=1}^{n} A_{ij}\, v_j$, to find an optimal solution of the CLPP (P46) is equivalent to find an optimal solution of the CLPP (P14).

Step 7: Find the optimal values of $u_i, i = 1,2,\dots,m$ and P_1 by solving the CLPP (P13) for some specific values of $\alpha \in [0,1]$.

Step 8: Find the optimal values of $v_j, j = 1,2,\dots,n$ and P_2 by solving the CLPP (P14) for some specific values of $\alpha \in [0,1]$.

Step 9: The optimal values of $u_i, i = 1,2,\dots,m$, obtained in Step 7, represents the optimal strategies of Player I for the considered SVTN matrix games and the optimal values of $v_j, j = 1,2,\dots,n$, obtained in Step 8, represents the optimal strategies of Player II for the considered SVTN matrix games.

Step 10: The optimal value of P_1, obtained in Step 7, represents the minimum expected gain of Player I and the optimal value of P_2, obtained in Step 8, represents the maximum expected loss of Player II.

6 Invalidity of Seikh and Dutta's second approach

If Seikh and Dutta's (2022) approach is valid then the minimum expected gain of Player I should be equal to the maximum expected loss of Player II. However, in this section with the help of a numerical example, it is shown that the minimum expected gain of Player I is not equal to the maximum expected loss of Player II. Hence, Seikh and Dutta's (2022) approach is not valid.

Seikh and Dutta (2022, Section 5.1, Example 1, p. 929) considered the payoff matrix

$\tilde{A} = \begin{pmatrix} \langle(175, 177, 180, 190); 0.6,0.3,0.3\rangle & \langle(150, 153, 156, 158); 0.5,0.2,0.3\rangle \\ \langle(125, 128, 132, 140); 0.9,0.1,0.5\rangle & \langle(175, 185, 195, 200); 0.5,0.4,0.5\rangle \end{pmatrix}$ of a SVTN matrix game to illustrate their proposed approaches.

It can be easily verified that for the considered SVTN matrix game, the CLPPs (P13) and (P14) are transformed into the CLPPs (P47) and (P48) respectively.

Problem (P47)

$Maximize\ \{P_1\}$

Subject to

$\frac{(1079u_1+785u_2)(0.74-0.38\alpha)}{6} \geq P_1,$

$\frac{(926u_1+1135u_2)(0.61-0.36\alpha)}{6} \geq P_1,$

$u_1 + u_2 = 1,\ u_1 \geq 0, u_2 \geq 0.$

Problem (P48)

$Minimize\ \{P_2\}$

Subject to

$\frac{(1079v_1+926v_2)(0.98-0.73\alpha)}{6} \leq P_2,$

$\frac{(785v_1+1135v_2)(0.61-0.36\alpha)}{6} \leq P_2,$

$v_1 + v_2 = 1, v_1 \geq 0, v_2 \geq 0.$

It pertinent to mentioned that the CLPP (P47) is same as the existing problem (Seikh and Dutta, 2022, Section 5.1.1, Problem 15, p. 931) and the CLPP (P48) is same as the existing problem (Seikh and Dutta, 2022, Section 5.1.1, Problem 16, p. 931). So, on solving

(i) The CLPPs (P47) and (P48), the existing results (Seikh and Dutta, 2022, Section 5.1.1, Table 3, p. 932), shown in Table 3, will be obtained

Table 3 Optimal values of u_1, u_2, P_1 for Player I and v_1, v_2, P_2 for Player II

	For Player I		For Player II	
Values of α	P_1	(u_1, u_2)	P_2	(v_1, v_2)
0	108.5285	(0.3230,0.6770)	151.2467	(0,1)
0.1	102.4294	(0.3077,0.69230)	139.9803	(0,1)
0.2	96.3267	(0.2905,0.7094)	128.7140	(0,1)
0.3	90.2198	(0.2712,0.7288)	117.4477	(0,1)
0.4	84.1078	(0.2491,0.7509)	106.1813	(0,1)
0.5	77.9896	(0.2238,0.7762)	94.9150	(0,1)
0.6	71.8637	(0.1944,0.8056)	83.6487	(0,1)
0.7	65.7280	(0.1599,0.8401)	72.3823	(0,1)
0.8	59.5799	(0.1187,0.8813)	61.1160	(0,1)
0.9	53.4153	(0.0689,0.9311)	51.2550	(0.1706, 0.8294)
1	47.2283	(0.0073,0.9927)	41.2322	(0.4155, 0.5845)

It is obvious from the results shown in Table 3 that for a specific value of α, the optimal value of the CLPP (P47) is not equal to the optimal value of the CLPP (P48) i.e., the minimum expected gain of the Player I is not equal to the maximum expected loss of Player II.

For example, for $\alpha = 0.1$, the optimal value of CLPP (P47) is $P_1 = 102.4294$ i.e, the minimum expected gain of the Player I is 102.4294 and the optimal value of CLPP (P48) is $P_2 = 139.9803$ i.e, maximum expected loss of Player II is 139.9803.

This clearly indicates that the minimum expected gain of the Player I is not equal to the maximum expected loss of Player II. Hence, Seikh and Dutta's (2022) second approach is not valid.

7 Reason for invalidity

Seikh and Dutta's (2022) second approach is not valid as the following mathematically incorrect results are used in it.

It is obvious that in Step 1 of Seikh and Dutta's (2022) second approach, discussed in Section 5, the relation $\Pi_\alpha\left((\sum_{i=1}^{m}\tilde{a}_{i1}u_i)v_1+(\sum_{i=1}^{m}\tilde{a}_{i2}u_i)v_2+\cdots+(\sum_{i=1}^{m}\tilde{a}_{in}u_i)v_n\right)=(\Pi_\alpha(\sum_{i=1}^{m}\tilde{a}_{i1}u_i)v_1+\Pi_\alpha(\sum_{i=1}^{m}\tilde{a}_{i2}u_i)v_2+\cdots+\Pi_\alpha(\sum_{i=1}^{m}\tilde{a}_{in}u_i)v_n)$ is used to transform the CLPPs (P9) into the CLPPs (P37). However, the following example clearly indicates that

$\Pi_\alpha\left((\sum_{i=1}^{m}\tilde{a}_{i1}u_i)v_1+(\sum_{i=1}^{m}\tilde{a}_{i2}u_i)v_2+\cdots+(\sum_{i=1}^{m}\tilde{a}_{in}u_i)v_n\right)\neq$

$(\Pi_\alpha(\sum_{i=1}^{m}\tilde{a}_{i1}u_i)v_1+\Pi_\alpha(\sum_{i=1}^{m}\tilde{a}_{i2}u_i)v_2+\cdots+\Pi_\alpha(\sum_{i=1}^{m}\tilde{a}_{in}u_i)v_n)$.

Let $\tilde{A}=\begin{pmatrix}\langle(175,177,180,190);0.6,0.3,0.3\rangle & \langle(150,153,156,158);0.5,0.2,0.3\rangle\\ \langle(125,128,132,140);0.9,0.1,0.5\rangle & \langle(175,185,195,200);0.5,0.4,0.5\rangle\end{pmatrix}$ be the payoff matrix of SVTN matrix game. Then,

$\Pi_\alpha\left((\sum_{i=1}^{2}\tilde{a}_{i1}u_i)v_1+(\sum_{i=1}^{2}\tilde{a}_{i2}u_i)v_2\right)$

$=\Pi_\alpha\big((\langle(175,177,180,190);0.6,0.3,0.3\rangle\,u_1+\langle(125,128,132,140);0.9,0.1,0.5\rangle\,u_2)v_1+$

$(\langle(150,153,156,158);0.5,0.2,0.3\rangle\,u_1+\langle(175,185,195,200);0.5,0.4,0.5\rangle\,u_2)v_2\big)$

$$=\Pi_\alpha\begin{pmatrix}(\langle(175u_1v_1+125u_2v_1,177u_1v_1+128u_2v_1,180u_1v_1+132u_2v_1,190u_1v_1+140u_2v_1);0.6,0.3,0.5\rangle)\\ +\\ (\langle(150u_1v_2+175u_2v_2,153u_1v_2+185u_2v_2,156u_1v_2+195u_2v_2,158u_1v_2+200u_2v_2);0.5,0.4,0.5\rangle)\end{pmatrix}$$

$$=\Pi_\alpha\left(\left\langle\begin{pmatrix}175u_1v_1+125u_2v_1+150u_1v_2+175u_2v_2,\\177u_1v_1+128u_2v_1+153u_1v_2+185u_2v_2,\\180u_1v_1+132u_2v_1+156u_1v_2+195u_2v_2,\\190u_1v_1+140u_2v_1+158u_1v_2+200u_2v_2\end{pmatrix};0.5,0.4,0.5\right\rangle\right)$$

Assuming, $u_1=\frac{1}{3},u_2=\frac{2}{3},v_1=\frac{2}{3},v_2=\frac{1}{3}$

$$=\Pi_\alpha\left(\left\langle\begin{pmatrix}38.885+55.55+16.665+38.885,\\39.3294+56.8832+16.9983+41.107,\\39.996+58.6608+17.3316+43.329,\\42.218+62.216+17.5538+44.44\end{pmatrix};0.5,0.4,0.5\right\rangle\right)$$

$= \Pi_\alpha(\langle(149.985, 154.3179, 159.3174, 166.4278); 0.5,0.4,0.5\rangle)$

$= \frac{149.985+2(154.3179+159.3174)+166.4278}{6}[0.25\,\alpha + 0.36(1-\alpha) + 0.25(1-\alpha)]$

$= 157.2805[0.61 - 0.36\,\alpha] = 95.9411 - 56.6209\alpha$ (3)

and

$\Pi_\alpha(\sum_{i=1}^{2} \tilde{a}_{i1}u_i)v_1 + \Pi_\alpha(\sum_{i=1}^{2} \tilde{a}_{i2}u_i)v_2$

$= \Pi_\alpha\big((\langle(175, 177, 180, 190); 0.6,0.3,0.3\rangle\, u_1 + \langle(125, 128, 132, 140); 0.9,0.1,0.5\rangle\, u_2)v_1\big) +$

$\Pi_\alpha\big((\langle(150, 153, 156, 158); 0.5,0.2,0.3\rangle\, u_1 + \langle(175, 185, 195, 200); 0.5,0.4,0.5\rangle\, u_2)v_2\big)$

$$= \begin{pmatrix} \Pi_\alpha\left(\left\langle\begin{pmatrix}175u_1v_1 + 125u_2v_1, 177u_1v_1 + 128u_2v_1,\\ 180u_1v_1 + 132u_2v_1, 190u_1v_1 + 140u_2v_1\end{pmatrix}; 0.6,0.3,0.5\right\rangle\right) \\ + \\ \Pi_\alpha\left(\left\langle\begin{pmatrix}150u_1v_2 + 175u_2v_2, 153u_1v_2 + 185u_2v_2,\\ 156u_1v_2 + 195u_2v_2, 158u_1v_2 + 200u_2v_2\end{pmatrix}; 0.5,0.4,0.5\right\rangle\right) \end{pmatrix}$$

Assuming, $u_1 = \frac{1}{3}, u_2 = \frac{2}{3}, v_1 = \frac{2}{3}, v_2 = \frac{1}{3}$

$$= \begin{pmatrix} \Pi_\alpha(\langle(94.435, 96.2126, 98.6568, 104.434); 0.6,0.3,0.5\rangle) \\ + \\ \Pi_\alpha(\langle(55.55, 58.1053, 60.6606, 61.9938); 0.5,0.4,0.5\rangle) \end{pmatrix}$$

$$= \begin{pmatrix} \left(\frac{94.435+2(96.2126+98.6568)+104.434}{6}\right)[0.36\,\alpha + 0.49(1-\alpha) + 0.25(1-\alpha)] \\ + \\ \left(\frac{55.55+2(58.1053+60.6606)+61.9938}{6}\right)[0.25\,\alpha + 0.36(1-\alpha) + 0.25(1-\alpha)] \end{pmatrix}$$

$= 98.1013\,[0.74 - 0.38\,\alpha] + 59.1792[0.6 - 0.36\,\alpha] = 108.1024 - 58.5829\alpha$ (4)

It is obvious from (3) and (4) that $\Pi_\alpha\big((\sum_{i=1}^{m} \tilde{a}_{i1}u_i)v_1 + (\sum_{i=1}^{m} \tilde{a}_{i2}u_i)v_2 + \cdots + (\sum_{i=1}^{m} \tilde{a}_{in}u_i)v_n\big) \neq (\Pi_\alpha(\sum_{i=1}^{m} \tilde{a}_{i1}u_i)v_1 + \Pi_\alpha(\sum_{i=1}^{m} \tilde{a}_{i2}u_i)v_2 + \cdots + \Pi_\alpha(\sum_{i=1}^{m} \tilde{a}_{in}u_i)v_n)$. So, the CLPP (P9) is not equivalent to the CLPP (P37).

Furthermore, it is obvious that in Step 1 of Seikh and Dutta's (2022) second approach, discussed in Section 5, the relation

$\Pi_\alpha\left(\left(\sum_{j=1}^{n}\tilde{a}_{1j}v_j\right)u_1+\left(\sum_{j=1}^{n}\tilde{a}_{2j}v_j\right)u_2+\cdots+\left(\sum_{j=1}^{n}\tilde{a}_{mj}v_j\right)u_m\right)=\left(\Pi_\alpha\left(\sum_{j=1}^{n}\tilde{a}_{1j}v_j\right)u_1+\right.$ $\left.\Pi_\alpha\left(\sum_{j=1}^{n}\tilde{a}_{2j}v_j\right)u_2+\cdots+\Pi_\alpha\left(\sum_{j=1}^{n}\tilde{a}_{mj}v_j\right)u_m\right)$ is used to transform the CLPP (P10) into the CLPP (P38). While, it can be easily verified that $\Pi_\alpha\left(\left(\sum_{j=1}^{n}\tilde{a}_{1j}v_j\right)u_1+\left(\sum_{j=1}^{n}\tilde{a}_{2j}v_j\right)u_2+\cdots+\left(\sum_{j=1}^{n}\tilde{a}_{mj}v_j\right)u_m\right)\neq\left(\Pi_\alpha\left(\sum_{j=1}^{n}\tilde{a}_{1j}v_j\right)u_1+\Pi_\alpha\left(\sum_{j=1}^{n}\tilde{a}_{2j}v_j\right)u_2+\cdots+\Pi_\alpha\left(\sum_{j=1}^{n}\tilde{a}_{mj}v_j\right)u_m\right)$. So, the CLPP (P10) is not equivalent to the CLPP (P38).

8 Modified approaches

It is obvious from Section 4 and Section 7 that neither Brikaa's (2022) approach is valid nor Seikh and Dutta's (2022) second approach is valid. In this section, an approach by modifying Brikaa's (2022) approach as well as an approach by modifying Seikh and Dutta's (2022) second approach are proposed.

8.1 Modified approach-I

The following modified approach, corresponding to Brikaa's (2022) approach, is proposed to find the optimal strategies $u_i, i=1,2,\ldots,m$ of Player I, $v_j, j=1,2,\ldots,n$ of Player II, the minimum expected gain of Player I and the maximum expected loss of Player II with the help of the interval MPPs (P7) and (P8).

Step 1: Using the relation $\left(\sum_{j=1}^{n}\sum_{i=1}^{m}\tilde{a}_{ij}\right)_\rho=\left(\sum_{j=1}^{n}\sum_{i=1}^{m}\langle(a_{ij1},a_{ij2},a_{ij3},a_{ij4});l_{ij},m_{ij},n_{ij}\rangle\right)_\rho=$ $\sum_{j=1}^{n}\sum_{i=1}^{m}\left(\tilde{b}_{ij}\right)_\rho$

where,

$\tilde{b}_{ij}=\langle(a_{ij1},a_{ij2},a_{ij3},a_{ij4});minimum_{\substack{1\le i\le m\\1\le j\le n}}\{l_{ij}\},maximum_{\substack{1\le i\le m\\1\le j\le n}}\{m_{ij}\},maximum_{\substack{1\le i\le m\\1\le j\le n}}\{n_{ij}\}\rangle$

to find an optimal solution of the interval MPP (P7) is equivalent to find an optimal solution of the interval MPP (P49) and to find an optimal solution of the interval MPP (P8) is equivalent to find an optimal solution of the interval MPP (P50).

Problem (P49)

$$Maximize\begin{Bmatrix} Minimize\left(\sum_{j=1}^{n}\left(\sum_{i=1}^{m}(\tilde{b}_{ij})_{\rho}u_i\right)v_j\right)\\ Minimize\left(\sum_{j=1}^{n}\left(\sum_{i=1}^{m}(\tilde{b}_{ij})_{\sigma}u_i\right)v_j\right)\\ Minimize\left(\sum_{j=1}^{n}\left(\sum_{i=1}^{m}(\tilde{b}_{ij})_{\tau}u_i\right)v_j\right)\\ \text{Subject to}\\ \sum_{j=1}^{n}v_j=1, v_j\geq 0, j=1,2,\dots,n \end{Bmatrix}$$

Subject to

$\sum_{i=1}^{m}u_i = 1, u_i \geq 0,\ i = 1,2,\dots,m.$

Problem (P50)

$$Minimize\begin{Bmatrix} Maximize\left(\sum_{i=1}^{m}\left(\sum_{j=1}^{n}(\tilde{b}_{ij})_{\rho}v_j\right)u_i\right)\\ Maximize\left(\sum_{i=1}^{m}\left(\sum_{j=1}^{n}(\tilde{b}_{ij})_{\sigma}v_j\right)u_i\right)\\ Maximize\left(\sum_{i=1}^{m}\left(\sum_{j=1}^{n}(\tilde{b}_{ij})_{\tau}v_j\right)u_i\right)\\ \text{Subject to}\\ \sum_{i=1}^{m}u_i=1, u_i\geq 0, i=1,2,\dots,m \end{Bmatrix}$$

Subject to

$\sum_{j=1}^{n}v_j = 1, v_j \geq 0,\ j = 1,2,\dots,n.$

Step 2: Using the relation $(\tilde{b}_{ij})_{\rho} = \left[(b_{ij})_{l\rho}, (b_{ij})_{r\rho}\right], (\tilde{b}_{ij})_{\sigma} = \left[(b_{ij})_{l\sigma}, (b_{ij})_{r\sigma}\right], (\tilde{b}_{ij})_{\tau} = \left[(b_{ij})_{l\tau}, (b_{ij})_{r\tau}\right]$ to find an optimal solution of the interval MPP (P49) is equivalent to find an optimal solution of the interval MPP (P51) and to find an optimal solution of the interval MPP (P50) is equivalent to find an optimal solution of the interval MPP (P52).

Problem (P51)

$$Maximize\begin{Bmatrix} Minimize\left(\sum_{j=1}^{n}\left(\sum_{i=1}^{m}\left[(b_{ij})_{l\rho}, (b_{ij})_{r\rho}\right]u_i\right)v_j\right)\\ Minimize\left(\sum_{j=1}^{n}\left(\sum_{i=1}^{m}\left[(b_{ij})_{l\sigma}, (b_{ij})_{r\sigma}\right]u_i\right)v_j\right)\\ Minimize\left(\sum_{j=1}^{n}\left(\sum_{i=1}^{m}\left[(b_{ij})_{l\tau}, (b_{ij})_{r\tau}\right]u_i\right)v_j\right)\\ \text{Subject to}\\ \sum_{j=1}^{n}v_j=1, v_j\geq 0, j=1,2,\dots,n \end{Bmatrix}$$

Subject to

$\sum_{i=1}^{m}u_i = 1, u_i \geq 0,\ i = 1,2,\dots,m.$

Problem (P52)

$$\text{Minimize}\left\{\begin{matrix} \text{Maximize}\left(\sum_{i=1}^{m}\left(\sum_{j=1}^{n}\left[\left(b_{ij}\right)_{l\rho},\left(b_{ij}\right)_{r\rho}\right]v_j\right)u_i\right) \\ \text{Maximize}\left(\sum_{i=1}^{m}\left(\sum_{j=1}^{n}\left[\left(b_{ij}\right)_{l\sigma},\left(b_{ij}\right)_{r\sigma}\right]v_j\right)u_i\right) \\ \text{Maximize}\left(\sum_{i=1}^{m}\left(\sum_{j=1}^{n}\left[\left(b_{ij}\right)_{l\tau},\left(b_{ij}\right)_{r\tau}\right]v_j\right)u_i\right) \\ \text{Subject to} \\ \sum_{i=1}^{m}u_i=1, u_i\geq 0, i=1,2,\dots,m \end{matrix}\right\}$$

Subject to

$\sum_{j=1}^{n}v_j=1, v_j\geq 0, j=1,2,\dots,n.$

Step 3: Aggregating the objective function of the interval MPPs (P51) and (P52) to find an optimal solution of the interval MPP (P51) is equivalent to find an optimal solution of the interval MPP (P53) and to find an optimal solution of the interval MPP (P52) is equivalent to find an optimal solution of the interval MPP (P54).

Problem (P53)

$$\text{Maximize}\left\{\begin{matrix} \text{Minimize}\left(\sum_{j=1}^{n}\left(\sum_{i=1}^{m}\left[\frac{\left(b_{ij}\right)_{l\rho}+\left(b_{ij}\right)_{l\sigma}+\left(b_{ij}\right)_{l\tau}}{3},\frac{\left(b_{ij}\right)_{r\rho}+\left(b_{ij}\right)_{r\sigma}+\left(b_{ij}\right)_{r\tau}}{3}\right]u_i\right)v_j\right) \\ \text{Subject to} \\ \sum_{j=1}^{n}v_j=1, v_j\geq 0, j=1,2,\dots,n \end{matrix}\right\}$$

Subject to

$\sum_{i=1}^{m}u_i=1, u_i\geq 0, i=1,2,\dots,m.$

Problem (P54)

$$\text{Minimize}\left\{\begin{matrix} \text{Maximize}\left(\sum_{i=1}^{m}\left(\sum_{j=1}^{n}\left[\frac{\left(b_{ij}\right)_{l\rho}+\left(b_{ij}\right)_{l\sigma}+\left(b_{ij}\right)_{l\tau}}{3},\frac{\left(b_{ij}\right)_{r\rho}+\left(b_{ij}\right)_{r\sigma}+\left(b_{ij}\right)_{r\tau}}{3}\right]v_j\right)u_i\right) \\ \text{Subject to} \\ \sum_{i=1}^{m}u_i=1, u_i\geq 0, i=1,2,\dots,m \end{matrix}\right\}$$

Subject to

$\sum_{j=1}^{n}v_j=1, v_j\geq 0, j=1,2,\dots,n.$

Step 4: To find an optimal solution of the interval MPP (P53) is equivalent to find an optimal solution of the crisp bi-objective MPP (P55) and to find an optimal solution of the interval MPP (P54) is equivalent to find an optimal solution of the crisp bi-objective MPP (P56).

Problem (P55)

$$Maximize\left\{\begin{array}{c}Minimize\left(\sum_{j=1}^{n}\left(\sum_{i=1}^{m}\left\{\begin{array}{c}\frac{(b_{ij})_{l\rho}+(b_{ij})_{l\sigma}+(b_{ij})_{l\tau}}{3},\\ \frac{(b_{ij})_{l\rho}+(b_{ij})_{r\rho}+(b_{ij})_{l\sigma}+(b_{ij})_{r\sigma}+(b_{ij})_{l\tau}+(b_{ij})_{r\tau}}{6}\end{array}\right\}u_i\right)v_j\right)\\ \text{Subject to}\\ \sum_{j=1}^{n}v_j=1, v_j\geq 0, j=1,2,\ldots,n\end{array}\right\}$$

Subject to

$\sum_{i=1}^{m}u_i=1, u_i\geq 0, i=1,2,\ldots,m.$

Problem (P56)

$$Minimize\left\{\begin{array}{c}Maximize\left(\sum_{i=1}^{m}\left(\sum_{j=1}^{n}\left\{\begin{array}{c}\frac{(b_{ij})_{l\rho}+(b_{ij})_{l\sigma}+(b_{ij})_{l\tau}}{3},\\ \frac{(b_{ij})_{l\rho}+(b_{ij})_{r\rho}+(b_{ij})_{l\sigma}+(b_{ij})_{r\sigma}+(b_{ij})_{l\tau}+(b_{ij})_{r\tau}}{6}\end{array}\right\}v_j\right)u_i\right)\\ \text{Subject to}\\ \sum_{i=1}^{m}u_i=1, u_i\geq 0, i=1,2,\ldots,m\end{array}\right\}$$

Subject to

$\sum_{j=1}^{n}v_j=1, v_j\geq 0, j=1,2,\ldots,n.$

Step 5: Using the weighted average method, to find an optimal solution of the crisp bi-objective MPP (P55) is equivalent to find an optimal solution of the crisp MPP (P57) or the equivalent crisp MPP (P58) and to find an optimal solution of the crisp bi-objective MPP (P56) is equivalent to find an optimal solution of the crisp MPP (P59) or the equivalent crisp MPP (P60).

Problem (P57)

$$Maximize\left\{\begin{array}{c}Minimize\left(\sum_{j=1}^{n}\left(\sum_{i=1}^{m}\frac{1}{2}\left(\begin{array}{c}\frac{(b_{ij})_{l\rho}+(b_{ij})_{l\sigma}+(b_{ij})_{l\tau}}{3}+\\ \frac{(b_{ij})_{l\rho}+(b_{ij})_{r\rho}+(b_{ij})_{l\sigma}+(b_{ij})_{r\sigma}+(b_{ij})_{l\tau}+(b_{ij})_{r\tau}}{6}\end{array}\right)u_i\right)v_j\right)\\ \text{Subject to}\\ \sum_{j=1}^{n}v_j=1, v_j\geq 0, j=1,2,\ldots,n\end{array}\right\}$$

Subject to

$\sum_{i=1}^{m}u_i=1, u_i\geq 0,\ i=1,2,\ldots,m.$

Problem (P58)

$$Maximize\left\{\begin{array}{c} Minimize\left(\sum_{j=1}^{n}\left(\sum_{i=1}^{m}\left(\frac{3\left((b_{ij})_{l\rho}+(b_{ij})_{l\sigma}+(b_{ij})_{l\tau}\right)+(b_{ij})_{r\rho}+(b_{ij})_{r\sigma}+(b_{ij})_{r\tau}}{12}\right)u_i\right)v_j\right) \\ \text{Subject to} \\ \sum_{j=1}^{n}v_j=1, v_j\geq 0, j=1,2,\ldots,n \end{array}\right\}$$

Subject to

$\sum_{i=1}^{m}u_i=1, u_i\geq 0,\ i=1,2,\ldots,m.$

Problem (P59)

$$Minimize\left\{\begin{array}{c} Maximize\left(\sum_{i=1}^{m}\left(\sum_{j=1}^{n}\frac{1}{2}\left(\begin{array}{c}\frac{(b_{ij})_{l\rho}+(b_{ij})_{l\sigma}+(b_{ij})_{l\tau}}{3}+ \\ \frac{(b_{ij})_{l\rho}+(b_{ij})_{r\rho}+(b_{ij})_{l\sigma}+(b_{ij})_{r\sigma}+(b_{ij})_{l\tau}+(b_{ij})_{r\tau}}{6}\end{array}\right)v_j\right)u_i\right) \\ \text{Subject to} \\ \sum_{i=1}^{m}u_i=1, u_i\geq 0, i=1,2,\ldots,m \end{array}\right\}$$

Subject to

$\sum_{j=1}^{n}v_j=1, v_j\geq 0, j=1,2,\ldots,n.$

Problem (P60)

$$Minimize\left\{\begin{array}{c} Maximize\left(\sum_{i=1}^{m}\left(\sum_{j=1}^{n}\left(\frac{3\left((b_{ij})_{l\rho}+(b_{ij})_{l\sigma}+(b_{ij})_{l\tau}\right)+(b_{ij})_{r\rho}+(b_{ij})_{r\sigma}+(b_{ij})_{r\tau}}{12}\right)v_j\right)u_i\right) \\ \text{Subject to} \\ \sum_{i=1}^{m}u_i=1, u_i\geq 0, i=1,2,\ldots,m \end{array}\right\}$$

Subject to

$\sum_{j=1}^{n}v_j=1, v_j\geq 0, j=1,2,\ldots,n.$

Step 6: Since $\sum_{j=1}^{n}\left(\sum_{i=1}^{m}\left(\frac{3\left((b_{ij})_{l\rho}+(b_{ij})_{l\sigma}+(b_{ij})_{l\tau}\right)+(b_{ij})_{r\rho}+(b_{ij})_{r\sigma}+(b_{ij})_{r\tau}}{12}\right)u_i\right)v_j$ is a convex linear combination of

$\sum_{i=1}^{m}\left(\frac{3\left((b_{i1})_{l\rho}+(b_{i1})_{l\sigma}+(b_{i1})_{l\tau}\right)+(b_{i1})_{r\rho}+(b_{i1})_{r\sigma}+(b_{i1})_{r\tau}}{12}\right)u_i$, $\sum_{i=1}^{m}\left(\frac{3\left((b_{i2})_{l\rho}+(b_{i2})_{l\sigma}+(b_{i2})_{l\tau}\right)+(b_{i2})_{r\rho}+(b_{i2})_{r\sigma}+(b_{i2})_{r\tau}}{12}\right)u_i$, ...,

$\sum_{i=1}^{m}\left(\frac{3((b_{in})_{l\rho}+(b_{in})_{l\sigma}+(b_{in})_{l\tau})+(b_{in})_{r\rho}+(b_{in})_{r\sigma}+(b_{in})_{r\tau}}{12}\right)u_i$. So, to find an optimal solution of the crisp MPP (P58) is equivalent to find an optimal solution of the CLPP (P61). Also, as

$\sum_{i=1}^{m}\left(\sum_{j=1}^{n}\left(\frac{3\left((b_{ij})_{l\rho}+(b_{ij})_{l\sigma}+(b_{ij})_{l\tau}\right)+(b_{ij})_{r\rho}+(b_{ij})_{r\sigma}+(b_{ij})_{r\tau}}{12}\right)v_j\right)u_i$ is a convex linear combination of

$\sum_{j=1}^{n}\left(\frac{3\left((b_{1j})_{l\rho}+(b_{1j})_{l\sigma}+(b_{1j})_{l\tau}\right)+(b_{1j})_{r\rho}+(b_{1j})_{r\sigma}+(b_{1j})_{r\tau}}{12}\right)v_j, \sum_{j=1}^{n}\left(\frac{3\left((b_{2j})_{l\rho}+(b_{2j})_{l\sigma}+(b_{2j})_{l\tau}\right)+(b_{2j})_{r\rho}+(b_{2j})_{r\sigma}+(b_{2j})_{r\tau}}{12}\right)v_j,$

$\ldots, \sum_{j=1}^{n}\left(\frac{3\left((b_{mj})_{l\rho}+(b_{mj})_{l\sigma}+(b_{mj})_{l\tau}\right)+(b_{mj})_{r\rho}+(b_{mj})_{r\sigma}+(b_{mj})_{r\tau}}{12}\right)v_j$. So, to find an optimal solution of the crisp MPP (P60) is equivalent to find an optimal solution of the CLPP (P62).

Problem (P61)

$$Maximize\ \left\{Minimum_{1\le j\le n}\left(\sum_{i=1}^{m}\left(\frac{3\left((b_{ij})_{l\rho}+(b_{ij})_{l\sigma}+(b_{ij})_{l\tau}\right)+(b_{ij})_{r\rho}+(b_{ij})_{r\sigma}+(b_{ij})_{r\tau}}{12}\right)u_i\right)\right\}$$

Subject to

$\sum_{i=1}^{m}u_i = 1, u_i \ge 0.$

Problem (P62)

$$Minimize\ \left\{Maximum_{1\le i\le m}\left(\sum_{j=1}^{n}\left(\frac{3\left((b_{ij})_{l\rho}+(b_{ij})_{l\sigma}+(b_{ij})_{l\tau}\right)+(b_{ij})_{r\rho}+(b_{ij})_{r\sigma}+(b_{ij})_{r\tau}}{12}\right)v_j\right)\right\}$$

Subject to

$\sum_{j=1}^{n}v_j = 1, v_j \ge 0, j = 1,2,\ldots,n.$

Step 7: Assuming $Minimum_{1\le j\le n}\left(\sum_{i=1}^{m}\left(\frac{3\left((b_{ij})_{l\rho}+(b_{ij})_{l\sigma}+(b_{ij})_{l\tau}\right)+(b_{ij})_{r\rho}+(b_{ij})_{r\sigma}+(b_{ij})_{r\tau}}{12}\right)u_i\right) = \theta$ and $Maximum_{1\le i\le m}\left(\sum_{j=1}^{n}\left(\frac{3\left((b_{ij})_{l\rho}+(b_{ij})_{l\sigma}+(b_{ij})_{l\tau}\right)+(b_{ij})_{r\rho}+(b_{ij})_{r\sigma}+(b_{ij})_{r\tau}}{12}\right)v_j\right) = \phi$ to find an optimal solution of the CLPP (P61) is equivalent to find an optimal solution of the CLPP (P63) and to find an optimal solution of the CLPP (P62) is equivalent to find an optimal solution of the CLPP (P64).

Problem (P63)

$Maximize\ \{\theta\}$

Subject to

$\sum_{i=1}^{m}\left(\frac{3\left((b_{ij})_{l\rho}+(b_{ij})_{l\sigma}+(b_{ij})_{l\tau}\right)+(b_{ij})_{r\rho}+(b_{ij})_{r\sigma}+(b_{ij})_{r\tau}}{12}\right)u_i \geq \theta,\ \ j=1,2,\dots,n,$

$\sum_{i=1}^{m}u_i = 1, u_i \geq 0,\ \ i=1,2,\dots,m.$

Problem (P64)

$Minimize\ \{\phi\}$

Subject to

$\sum_{j=1}^{n}\left(\frac{3\left((b_{ij})_{l\rho}+(b_{ij})_{l\sigma}+(b_{ij})_{l\tau}\right)+(b_{ij})_{r\rho}+(b_{ij})_{r\sigma}+(b_{ij})_{r\tau}}{12}\right)v_j \leq \phi,\ \ i=1,2,\dots,m$

$\sum_{j=1}^{n}v_j = 1, v_j \geq 0, j=1,2,\dots,n.$

Step 8: Find the optimal values of $u_i, i=1,2,\dots,m$ and θ by solving the CLPP (P63) for some specific values of $\rho \in \left[0, minimum_{\substack{1\leq i\leq m\\1\leq j\leq n}}(l_{ij})\right], \sigma \in [maximum_{\substack{1\leq i\leq m\\1\leq j\leq n}}(m_{ij}), 1]$ and $\tau \in \left[maximum_{\substack{1\leq i\leq m\\1\leq j\leq n}}(n_{ij}), 1\right]$.

Step 9: Find the optimal values of $v_j, j=1,2,\dots,n$ and ϕ by solving the CLPP (P64) for some specific values of $\rho \in \left[0, minimum_{\substack{1\leq i\leq m\\1\leq j\leq n}}(l_{ij})\right], \sigma \in [maximum_{\substack{1\leq i\leq m\\1\leq j\leq n}}(m_{ij}), 1]$ and $\tau \in \left[maximum_{\substack{1\leq i\leq m\\1\leq j\leq n}}(n_{ij}), 1\right]$.

Step 10: The optimal values of $u_i, i=1,2,\dots,m,$ obtained in Step 8, represents the optimal strategies of Player I for the considered SVTN matrix game and the optimal values of $v_j, j=$

$1,2,\dots,n$, obtained in Step 9, represents the optimal strategies of Player II for the considered SVTN matrix game.

Step 11: The optimal value of θ, obtained in Step 8, represents the minimum expected gain of Player I and the optimal value of ϕ, obtained in Step 9, represents the maximum expected loss of Player II.

8.2 Modified approach-II

The following modified approach, corresponding to Seikh and Dutta's (2022) second approach, is proposed to find the optimal strategies $u_i, i = 1,2,\dots,m$ of Player I, $v_j, j = 1,2,\dots,n$ of Player II, the minimum expected gain of Player I and the maximum expected loss of Player II with the help of the crisp MPPs (P9) and (P10).

Step 1: Using the relation $\Pi_\alpha\left(\sum_{i=1}^{m}\widetilde{M}_{ij}\right) = \eta\sum_{i=1}^{m}\left(\frac{\Pi_\alpha(\widetilde{M}_{ij})}{\left[\alpha {l_{ij}}^2+(1-\alpha)\left(1-m_{ij}\right)^2+(1-\alpha)\left(\left(1-n_{ij}\right)^2\right)\right]}\right) = \eta\sum_{i=1}^{m}A_{ij}$

where,

$$\eta = \alpha\left(\min_{\substack{1\le i\le m\\1\le j\le n}}\{l_{ij}\}\right)^2 + (1-\alpha)\left(1-\max_{\substack{1\le i\le m\\1\le j\le n}}\{m_{ij}\}\right)^2 + (1-\alpha)\left(1-\max_{\substack{1\le i\le m\\1\le j\le n}}\{n_{ij}\}\right)^2$$

and $A_{ij} = \frac{\Pi_\alpha(\widetilde{M}_{ij})}{\left[\alpha {l_{ij}}^2+(1-\alpha)\left(1-m_{ij}\right)^2+(1-\alpha)\left(\left(1-n_{ij}\right)^2\right)\right]}$, to find an optimal solution of the crisp MPP (P9) is equivalent to find an optimal solution of the crisp MPP (P65) and to find an optimal solution of the crisp MPP (P10) is equivalent to find an optimal solution of the crisp MPP (P66).

Problem (P65)

$$Maximize\left\{\begin{matrix} Minimize\left(\sum_{j=1}^{n}\left(\left(\sum_{i=1}^{m}\eta\,A_{ij}\right)u_i\right)v_j\right) \\ \text{Subject to} \\ \sum_{j=1}^{n}v_j = 1, v_j \ge 0, j = 1,2,\dots,n \end{matrix}\right\}$$

Subject to

$\sum_{i=1}^{m}u_i = 1, u_i \ge 0, i = 1,2,\dots,m.$

Problem (P66)

$$Minimize \left\{ \begin{matrix} Maximize\left(\sum_{i=1}^{m}\left(\left(\sum_{j=1}^{n}\eta\ A_{ij}\right)v_j\right)u_i\right) \\ \text{Subject to} \\ \sum_{i=1}^{m}u_i = 1, u_i \geq 0, i = 1,2,\dots,m \end{matrix} \right\}$$

Subject to

$\sum_{j=1}^{n}v_j = 1, v_j \geq 0, j = 1,2,\dots,n.$

Step 2: Since, $\sum_{j=1}^{n}\left(\left(\sum_{i=1}^{m}\eta\ A_{ij}\right)u_i\right)v_j$ is a convex linear combination of $(\sum_{i=1}^{m}\eta\ A_{i1})u_i, (\sum_{i=1}^{m}\eta\ A_{i2})u_i, \dots, (\sum_{i=1}^{m}\eta\ A_{in})u_i$. So, to find an optimal solution of the crisp MPP (P65) is equivalent to find an optimal solution of the crisp MPP (P67). Also, $\sum_{i=1}^{m}\left(\left(\sum_{j=1}^{n}\eta\ A_{ij}\right)v_j\right)u_i$ is a convex linear combination of $\left(\sum_{j=1}^{n}\eta\ A_{1j}\right)v_j, \left(\sum_{j=1}^{n}\eta\ A_{2j}\right)v_j, \dots, \left(\sum_{j=1}^{n}\eta\ A_{mj}\right)v_j$. So, to find an optimal solution of the crisp MPP (P66) is equivalent to find an optimal solution of the crisp MPP (P68).

Problem (P67)

$Maximize\left\{Minimum_{1\leq j\leq n}\left(\sum_{i=1}^{m}\eta\ A_{ij}u_i\right)\right\}$

Subject to

$\sum_{i=1}^{m}u_i = 1, u_i \geq 0, i = 1,2,\dots,m.$

Problem (P68)

$Minimize\left\{Maximum_{1\leq i\leq m}\left(\sum_{j=1}^{n}\eta\ A_{ij}v_j\right)\right\}$

Subject to

$\sum_{j=1}^{n}v_j = 1, v_j \geq 0, j = 1,2,\dots,n.$

Step 3: Assuming, $Minimum_{1\leq j\leq n}\left(\sum_{i=1}^{m}\eta\ A_{ij}u_i\right) = P_1$ and $Maximum_{1\leq i\leq m}\left(\sum_{j=1}^{n}\eta\ A_{ij}v_j\right) = P_2$ to find an optimal solution of the crisp MPP (P67) is equivalent to find an optimal solution of the CLPP (P69) and to find an optimal solution of the crisp MPP (P68) is equivalent to find an optimal solution of the CLPP (P70).

Problem (P69)

$Maximize\ \{P_1\}$

Subject to

$\sum_{i=1}^{m} \eta\, A_{ij} u_i \geq P_1, j = 1,2, \dots, n,$

$\sum_{i=1}^{m} u_i = 1, u_i \geq 0, i = 1,2, \dots, m.$

Problem (P70)

$Minimize\ \{P_2\}$

Subject to

$\sum_{j=1}^{n} \eta\, A_{ij} v_j \leq P_2, i = 1,2, \dots, m,$

$\sum_{j=1}^{n} v_j = 1, v_j \geq 0,\ j = 1,2, \dots, n.$

Step 4: Find the optimal values of $u_i, i = 1,2, \dots, m$ and P_1 by solving the CLPP (P69) for some specific values of $\alpha \in [0, 1]$.

Step 5: Find the optimal values of $v_j, j = 1,2, \dots, n$ and P_2 by solving the CLPP (P70) for some specific values of $\alpha \in [0, 1]$.

Step 6: The optimal values of $u_i, i = 1,2, \dots, m,$ obtained in Step 4, represents the optimal strategies of Player I for the considered SVTN matrix game and the optimal values of $v_j,\ j = 1,2, \dots, n,$ obtained in Step 5, represents the optimal strategies of Player II for the considered SVTN matrix game.

Step 7: The optimal value of P_1, obtained in Step 4, represents the minimum expected gain of Player I and the optimal value of P_2, obtained in Step 5, represents the maximum expected loss of Player II.

9 Validity of the modified approaches

In this section, it is proved that the proposed modified approaches are valid.

9.1 Validity of the modified approach-I

It is pertinent to mention that the modified approach-I will be valid if the CLPPs (P63) and (P64) represents the primal-dual pair. Hence, the same is proved in this section.

Replacing the unrestricted decision variables θ and ϕ with $\theta_1 - \theta_2$ and $\phi_1 - \phi_2$ respectively, where $\theta_1 \geq 0, \theta_2 \geq 0, \phi_1 \geq 0, \phi_2 \geq 0$, the CLPPs (P63) and (P64) are transformed into the CLPPs (P63_1) and (P64_1) respectively.

Problem (P63_1)

$Maximize\ \{\theta_1 - \theta_2\}$

Subject to

$$\sum_{i=1}^{m}\left(\frac{3\left((b_{ij})_{l\rho}+(b_{ij})_{l\sigma}+(b_{ij})_{l\tau}\right)+(b_{ij})_{r\rho}+(b_{ij})_{r\sigma}+(b_{ij})_{r\tau}}{12}\right)u_i \geq \theta_1 - \theta_2,\ \ j = 1,2,\ldots,n,$$

$\sum_{i=1}^{m} u_i \leq 1, \sum_{i=1}^{m} u_i \geq 1, u_i \geq 0, i = 1,2,\ldots,m,$

$\theta_1 \geq 0, \theta_2 \geq 0.$

Problem (P64_1)

$Minimize\ \{\phi_1 - \phi_2\}$

Subject to

$$\sum_{j=1}^{n}\left(\frac{3\left((b_{ij})_{l\rho}+(b_{ij})_{l\sigma}+(b_{ij})_{l\tau}\right)+(b_{ij})_{r\rho}+(b_{ij})_{r\sigma}+(b_{ij})_{r\tau}}{12}\right)v_j \leq \phi_1 - \phi_2,\ \ i = 1,2,\ldots,m,$$

$\sum_{j=1}^{n} v_j \leq 1, \sum_{j=1}^{n} v_j \geq 1, v_j \geq 0,\ j = 1,2,\ldots,n,$

$\phi_1 \geq 0, \phi_2 \geq 0.$

Converting the sign of all the constraints of the CLPP (P63_1) into $\leq$, it is transformed into the CLPP (P63_2) and converting the sign of all the constraints of the CLPP (P64_1) into $\geq$, it is transformed into the CLPP (P64_2).

Problem (P63_2)

$Maximize\ \{\theta_1 - \theta_2\}$

Subject to

$-\sum_{i=1}^{m}\left(\frac{3\left((b_{ij})_{l\rho}+(b_{ij})_{l\sigma}+(b_{ij})_{l\tau}\right)+(b_{ij})_{r\rho}+(b_{ij})_{r\sigma}+(b_{ij})_{r\tau}}{12}\right)u_i \leq -\theta_1 + \theta_2,\ \ j = 1,2,\dots,n,$

$\sum_{i=1}^{m} u_i \leq 1, -\sum_{i=1}^{m} u_i \leq -1,\ u_i \geq 0,\ \ i = 1,2,\dots,m,$

$\theta_1 \geq 0, \theta_2 \geq 0.$

Problem (P64_2)

$Minimize\ \{\phi_1 - \phi_2\}$

Subject to

$-\sum_{j=1}^{n}\left(\frac{3\left((b_{ij})_{l\rho}+(b_{ij})_{l\sigma}+(b_{ij})_{l\tau}\right)+(b_{ij})_{r\rho}+(b_{ij})_{r\sigma}+(b_{ij})_{r\tau}}{12}\right)v_j \geq -\phi_1 + \phi_2,\ \ i = 1,2,\dots,m,$

$\sum_{j=1}^{n} v_j \geq 1, -\sum_{j=1}^{n} v_j \geq -1, v_j \geq 0, j = 1,2,\dots,n,$

$\phi_1 \geq 0, \phi_2 \geq 0.$

Assuming $(b_{ij})_{l\rho} + (b_{ij})_{l\sigma} + (b_{ij})_{l\tau} = A_{ij}$, and $(b_{ij})_{r\rho} + (b_{ij})_{r\sigma} + (b_{ij})_{r\tau} = B_{ij}$ the CLPP (P63_2) is transformed into the CLPP (P63_3) and the CLPP (P64_2) is transformed into the CLPP (P64_3).

Problem (P63_3)

$Maximize\ \{\theta_1 - \theta_2\}$

Subject to

$-\sum_{i=1}^{m} \frac{1}{12}(3A_{ij} + B_{ij})u_i \leq -\theta_1 + \theta_2,\ \ j = 1,2,\dots,n,$

$\sum_{i=1}^{m} u_i \leq 1, -\sum_{i=1}^{m} u_i \leq -1,\ u_i \geq 0,\ \ i = 1,2,\dots,m,$

$\theta_1 \geq 0, \theta_2 \geq 0.$

Problem (P64_3)

$Minimize \{\phi_1 - \phi_2\}$

Subject to

$-\sum_{j=1}^{n} \frac{1}{12}\left(3A_{ij} + B_{ij}\right)v_j \geq -\phi_1 + \phi_2, \;\; i = 1,2, \dots, m,$

$\sum_{j=1}^{n} v_j \geq 1, -\sum_{j=1}^{n} v_j \geq -1, v_j \geq 0, j = 1,2, \dots, n,$

$\phi_1 \geq 0, \phi_2 \geq 0.$

The CLPP (P63_4) and the CLPP (P64_4) represents the matrix form of the CLPP (P63_3) and the CLPP (P64_3) respectively.

Problem (63_4)

$Maximize \{CX\}$

Subject to

$AX \leq b,$

$X \geq 0.$

Problem (64_4)

$Minimize \{b^T Y\}$

Subject to

$A^T Y \geq C^T,$

$Y \geq 0.$

where, $C = [0, 0, \dots, 0, 1 - 1\,]_{1\times(m+2)}$, $X = \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ 0 \\ \theta_1 \\ \theta_2 \end{bmatrix}_{(m+2)\times 1}$

$$A=\begin{bmatrix} -\frac{1}{12}(3A_{11}+B_{11}) & -\frac{1}{12}(3A_{21}+B_{21}) & \dots & -\frac{1}{12}(3A_{m1}+B_{m1}) & 1 & -1 \\ -\frac{1}{12}(3A_{12}+B_{12}) & -\frac{1}{12}(3A_{22}+B_{22}) & \dots & -\frac{1}{12}(3A_{m2}+B_{m2}) & 1 & -1 \\ \cdots & \dots & \dots & \dots & \dots & \dots \\ -\frac{1}{12}(3A_{1n}+B_{1n}) & -\frac{1}{12}(3A_{2n}+B_{2n}) & \dots & -\frac{1}{12}(3A_{mn}+B_{mn}) & 1 & -1 \\ 1 & 1 & \dots & 1 & 0 & 0 \\ -1 & -1 & \dots & -1 & 0 & 0 \end{bmatrix}_{(n+2)\times(m+2)}$$

$$b=\begin{bmatrix} 0 \\ 0 \\ . \\ . \\ 0 \\ 1 \\ -1 \end{bmatrix}_{(n+2)\times 1} \qquad Y=\begin{bmatrix} 0 \\ 0 \\ . \\ . \\ 0 \\ \phi_1 \\ \phi_2 \end{bmatrix}_{(n+2)\times 1}$$

The CLPP (P63_4) and the CLPP (P64_4) represents the primal-dual pair and hence, the CLPPs (P63) and (P64) represents the primal-dual pair.

9.2 Validity of the modified approach-II

It is pertinent to mention that the modified approach-II will be valid if the CLPPs (P69) and (P70) represents the primal-dual pair. Hence, the same is proved in this section.

Replacing the unrestricted decision variables P_1 and P_2 with $P_{11}-P_{12}$ and $P_{21}-P_{22}$ respectively, where $P_{11}\geq 0, P_{12}\geq 0, P_{21}\geq 0, P_{22}\geq 0$ the CLPPs (P69) and (P70) are transformed into the CLPPs (P69_1) and (P70_1) respectively.

Problems (P69_1)

$Maximize\ \{P_{11}-P_{12}\}$

Subject to

$\sum_{i=1}^{m}\eta\ A_{ij}u_i \geq P_{11}-P_{12}, j=1,2,\dots,n,$

$\sum_{i=1}^{m}u_i \leq 1, \sum_{i=1}^{m}u_i \geq 1, u_i \geq 0, i=1,2,\dots,m,$

$P_{11}\geq 0, P_{12}\geq 0.$

Problems (P70_1)

$Minimize\ \{P_{21} - P_{22}\}$

Subject to

$\sum_{j=1}^{n} \eta\, A_{ij} v_j \leq \{P_{21} - P_{22}\}, i = 1,2, \dots, m,$

$\sum_{j=1}^{n} v_j \leq 1, \sum_{j=1}^{n} v_j \geq 1, v_j \geq 0, j = 1,2, \dots, n,$

$P_{21} \geq 0, P_{22} \geq 0.$

Converting the sign of all the constraints of the CLPP (P69_1) into $\leq$, it is transformed into the CLPP (P70_2) and converting the sign of all the constraints of the CLPP (P70_1) into $\geq$, it is transformed into the CLPP (P70_2).

Problem (P69_2)

$Maximize\ \{P_{11} - P_{12}\}$

Subject to

$-\sum_{i=1}^{m} \eta\, A_{ij} u_i \leq -P_{11} + P_{12}, j = 1,2, \dots, n,$

$\sum_{i=1}^{m} u_i \leq 1, -\sum_{i=1}^{m} u_i \leq -1, u_i \geq 0, i = 1,2, \dots, m,$

$P_{11} \geq 0, P_{12} \geq 0.$

Problem (P70_2)

$Minimize\ \{P_{21} - P_{22}\}$

Subject to

$-\sum_{j=1}^{n} \eta\, A_{ij} v_j \geq -P_{21} + P_{22}, i = 1,2, \dots, m,$

$\sum_{j=1}^{n} v_j \geq 1, -\sum_{j=1}^{n} v_j \geq -1, v_j \geq 0, j = 1,2, \dots, n,$

$P_{21} \geq 0, P_{22} \geq 0.$

The CLPPs (P69_3) and (P70_3) represents the matrix form of the CLPPs (P69_2) and (P70_2) respectively.

Problem (69_3)

$Maximize\ \{CX\}$

Subject to

$AX \leq b,$

$X \geq 0.$

Problem (70_3)

$Minimize\ \{b^T Y\}$

Subject to

$A^T Y \geq C^T,$

$Y \geq 0.$

$$C = [0, 0, \dots, 0, 1 - 1\ ]_{1\times(m+2)}\ X = \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ 0 \\ P_{11} \\ P_{12} \end{bmatrix}_{(m+2)\times 1}$$

$$A = \begin{bmatrix} -\eta\, A_{11} & -\eta\, A_{21} & \dots & -\eta\, A_{m1} & 1 & -1 \\ -\eta\, A_{12} & -\eta\, A_{22} & \dots & -\eta\, A_{m2} & 1 & -1 \\ \cdots & \dots & \dots & \dots & \dots & \dots \\ -\eta\, A_{1n} & -\eta\, A_{2n} & \dots & -\eta\, A_{mn} & 1 & -1 \\ 1 & 1 & \dots & 1 & 0 & 0 \\ -1 & -1 & \dots & -1 & 0 & 0 \end{bmatrix}_{(n+2)\times(m+2)}$$

$$b = \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ 0 \\ 1 \\ -1 \end{bmatrix}_{(n+2)\times 1} \qquad Y = \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ 0 \\ P_{21} \\ P_{22} \end{bmatrix}_{(n+2)\times 1}$$

The CLPPs (P69_3) and (P70_3) represents the primal-dual pair and hence, the CLPPs (P69) and (P70) represent the primal-dual pair.

10 Correct results of the existing SVTN matrix games

As discussed in Section 3 and Section 6, the results of the existing SVTN matrix games (Seikh and Dutta, 2022, Section 5.1, Example 1, p. 929), obtained by Brikaa (2022) as well as obtained by Seikh and Dutta (2022), are not correct. In this section, its correct results are obtained by the modified approaches.

10.1 Correct results by modified approach-I

Using the modified approach-I, the correct results of existing SVTN matrix games (Seikh and Dutta, 2022, Section 5.1, Example 1, p. 929), can be obtained as follows:

According to the modified approach-I, an optimal solution $u_i, i = 1,2, \ldots, m$ of the CLPP (P63) represent the optimal strategies of the Player I for a SVTN matrix game and an optimal solution $v_j, j = 1,2, \ldots, n$ of the CLPP (P64) represents the optimal strategies of the Player II for a SVTN matrix game. Since, for the existing SVTN matrix games (Seikh and Dutta, 2022, Section 5.1, Example 1, p. 929), the CLPP (P63) is transformed into the CLPP (P71) and the CLPP (P64) is transformed into the CLPP (P72). So, to find the optimal strategies of Player I for the existing SVTN matrix games (Seikh and Dutta, 2022, Section 5.1, Example 1, p. 929), there is a need to solve the CLPP (P71) and to find the optimal strategies of Player II for the existing SVTN matrix games (Seikh and Dutta, 2022, Section 5.1, Example 1, p. 929), there is a need to solve the CLPP (P72).

Problem (P71)

$Maximize\ \{\theta\}$

Subject to

$$\frac{1}{4}\Bigg(\bigg(\frac{(0.5-\rho)(175u_1+125u_2)+\rho(177u_1+128u_2)}{0.5}+\frac{(1-\sigma)(177u_1+128u_2)+(\sigma-0.4)(175u_1+125u_2)}{1-0.4}+$$

$$\frac{(1-\tau)(177u_1+128u_2)+(\tau-0.5)(175u_1+125u_2)}{1-0.5}\bigg)+$$

$$\frac{1}{3}\left(\frac{(0.5-\rho)(190u_1+140u_2)+\rho(180u_1+132u_2)}{0.5}+\frac{(1-\sigma)(180u_1+132u_2)+(\sigma-0.4)(190u_1+140u_2)}{1-0.4}+\right.$$

$$\left.\left.\frac{(1-\tau)(180u_1+132u_2)+(\tau-0.5\,)(190u_1+140u_2)}{1-0.5}\right)\right)\geq\theta,$$

$$\frac{1}{4}\left(\left(\frac{(0.5-\rho)(150u_1+175u_2)+\rho(153u_1+185u_2)}{0.5}+\frac{(1-\sigma)(153u_1+185u_2)+(\sigma-0.4)(150u_1+175u_2)}{1-0.4}+\right.\right.$$

$$\left.\frac{(1-\tau)(153u_1+185u_2)+(\tau-0.5)(150u_1+175u_2)}{1-0.5}\right)+$$

$$\frac{1}{3}\left(\frac{(0.5-\rho)(158u_1+200u_2)+\rho(156u_1+195u_2)}{0.5}+\frac{(1-\sigma)(156u_1+195u_2)+(\sigma-0.4)(158u_1+200u_2)}{1-0.4}+\right.$$

$$\left.\left.\frac{(1-\tau)(156u_1+195u_2)+(\tau-0.5\,)(158u_1+200u_2)}{1-0.5}\right)\right)\geq\theta,$$

$u_1+u_2=1,$

$u_1\geq 0, u_2\geq 0.$

Problem (P72)

$Minimize\ \{\phi\}$

Subject to

$$\frac{1}{4}\left(\left(\frac{(0.5-\rho)(175v_1+150v_2)+\rho(177v_1+153v_2)}{0.5}+\frac{(1-\sigma)(177v_1+153v_2)+(\sigma-0.4)(175v_1+150v_2)}{1-0.4}+\right.\right.$$

$$\left.\frac{(1-\tau)(177v_1+153v_2)+(\tau-0.5)(175v_1+150v_2)}{1-0.5}\right)+$$

$$\frac{1}{3}\left(\frac{(0.5-\rho)(190v_1+158v_2)+\rho(180v_1+156v_2)}{0.5}+\frac{(1-\sigma)(180v_1+156v_2)+(\sigma-0.4\,)(190v_1+158v_2)}{1-0.4}+\right.$$

$$\left.\left.\frac{(1-\tau)(180v_1+156v_2)+(\tau-0.5\,)(190v_1+158v_2)}{1-0.5}\right)\right)\leq\phi,$$

$$\frac{1}{4}\left(\left(\frac{(0.5-\rho)(125v_1+175v_2)+\rho(128v_1+185v_2)}{0.5}+\frac{(1-\sigma)(128v_1+185v_2)+(\sigma-0.4)(125v_1+175v_2)}{1-0.4}+\right.\right.$$

$$\left.\frac{(1-\tau)(128v_1+185v_2)+(\tau-0.5)(125v_1+175v_2)}{1-0.5}\right)+$$

$$\frac{1}{3}\left(\frac{(0.5-\rho)(140v_1+200v_2)+\rho(132v_1+195v_2)}{0.5}+\frac{(1-\sigma)(132v_1+195v_2)+(\sigma-0.4)(140v_1+200v_2)}{1-0.4}+\right.$$

$$\left.\frac{(1-\tau)(132v_1+195v_2)+(\tau-0.5)(140v_1+200v_2)}{1-0.5}\right)\Bigg) \leq \phi,$$

$$v_1 + v_2 = 1,$$

$$v_1 \geq 0, v_2 \geq 0.$$

It can be easily verified that on solving

(i) The CLPP (P71) for different values of ρ, σ, τ, the optimal values of u_1, u_2 and θ, presented in Table 4, are obtained.

(ii) The CLPP (P72) for different values of ρ, σ, τ, the optimal values of v_1, v_2 and ϕ, presented in Table 4, are obtained.

Table 4 Optimal values of $u_1, u_2, \theta, v_1, v_2$ and ϕ

(ρ, σ, τ)	θ	(u_1, u_2)	ϕ	(v_1, v_2)
(0,1,1)	161.8730	(0.6624,0.3375)	161.8730	(0.3690,0.6309)
(0,0.8,1)	162.0765	(0.6678,0.3321)	162.0765	(0.3736,0.6263)
(0,0.6,1)	162.2760	(0.6731,0.3268)	162.2760	(0.3782,0.6217)
(0,1,0.8)	162.1167	(0.6688,0.3311)	162.1167	(0.3746,0.6253)
(0,1,0.6)	162.3548	(0.6752,0.3247)	162.3548	(0.3800,0.6199)
(0.2,1,1)	162.1167	(0.6688,0.3311)	162.1167	(0.3746,0.6253)
(0.4,1,1)	162.3548	(0.6752,0.3247)	162.3548	(0.3800,0.6199)
(0.3,0.4,0.5)	163.3593	(0.7030,0.2969)	163.3593	(0.4039,0.5960)
(0.5,0.4,0.5)	163.5681	(0.7090,0.2909)	163.5681	(0.4090,0.5909)

10.2 Correct results by modified approach-II

Using the modified approach-II, the correct results of existing SVTN matrix games (Seikh and Dutta, 2022, Section 5.1, Example 1, p. 929) can be obtained as follows:

According to the modified approach-II, an optimal solution $u_i, i = 1,2, \ldots, m$ of the CLPP (P69) represents the optimal strategies of the Player I for a SVTN matrix game and an optimal solution $v_j, j = 1,2, \ldots, n$ of the CLPP (P70) represents the optimal strategies of the Player II for a SVTN matrix game. Since, for the existing SVTN matrix games (Seikh and Dutta, 2022, Section 5.1, Example 1, p. 929), the CLPP (P69) is transformed into the CLPP (P73) and the CLPP (P70) is transformed into the CLPP (P74). So, to find the optimal strategies of Player I for the existing SVTN matrix games (Seikh and Dutta, 2022, Section 5.1, Example 1, p. 929), there is a need to solve the CLPP (P73) and to find the optimal strategies of Player II for the existing SVTN matrix games (Seikh and Dutta, 2022, Section 5.1, Example 1, p. 929), there is a need to solve the CLPP (P74).

Problem (P73)

$Maximize\ \{P_1\}$

Subject to

$\frac{(1079u_1+785u_2)(0.61-0.36\alpha)}{6} \geq P_1,$

$\frac{(926u_1+1135u_2)(0.61-0.36\alpha)}{6} \geq P_1,$

$u_1 + u_2 = 1,\ u_1 \geq 0, u_2 \geq 0.$

Problem (P74)

$Minimize\{P_2\}$

Subject to

$\frac{(1079v_1+926v_2)(0.61-0.36\alpha)}{6} \leq P_2,$

$\frac{(785v_1+1135v_2)(0.61-0.36\alpha)}{6} \leq P_2$,

$v_1 + v_2 = 1, v_1 \geq 0, v_2 \geq 0$.

It can be easily verified that on solving

(i) The CLPP (P73) for different values of α, the optimal values of u_1, u_2 and P_1, presented in Table 5, are obtained.

(ii) The CLPP (P74) for different values of α, the optimal values of v_1, v_2 and P_2, presented in Table 5, are obtained.

Table 5 Optimal values of u_1, u_2, P_1 for Player I and v_1, v_2, P_2 for Player II

α	P_1	(u_1, u_2)	P_2	(v_1, v_2)
0	100.6065	(0.6958, 0.3041)	100.6065	(0.4155, 0.5844)
0.1	94.6691	(0.6958, 0.3041)	94.6691	(0.4155, 0.5844)
0.2	88.7316	(0.6958, 0.3041)	88.7316	(0.4155, 0.5844)
0.3	82.7942	(0.6958, 0.3041)	82.7942	(0.4155, 0.5844)
0.4	76.8568	(0.6958, 0.3041)	76.8568	(0.4155, 0.5844)
0.5	70.9193	(0.6958, 0.3041)	70.9193	(0.4155, 0.5844)
0.6	64.9819	(0.6958, 0.3041)	64.9819	(0.4155, 0.5844)
0.7	59.0444	(0.6958, 0.3041)	59.0444	(0.4155, 0.5844)
0.8	53.1070	(0.6958, 0.3041)	53.1070	(0.4155, 0.5844)
0.9	47.1696	(0.6958, 0.3041)	47.1696	(0.4155, 0.5844)
1	41.2321	(0.6958, 0.3041)	41.2321	(0.4155, 0.5844)

11 Validity of obtained results

It is obvious from Table 4 that corresponding to all the values of ρ, σ, τ, the optimal value of θ is same as the optimal value of ϕ i.e., minimum expected gain of Player I is same as the maximum expected loss of Player II. Also, it is obvious from Table 5 that corresponding to all the values of

α, the optimal value of P_1 is same as the optimal value of P_2 i.e., minimum expected gain of Player I is same as the maximum expected loss of Player II. Hence, the obtained results are valid.

12 Conclusion

It is shown that some mathematically incorrect results are considered in Brikka's (2022) approach as well as in Seikh and Dutta's (2022) second approach to solve SVTN matrix games. So, it is inappropriate to use Brikka's (2022) approach as well as in Seikh and Dutta's (2022) second approach to solve SVTN matrix games. Furthermore, a modified approach corresponding to the Brikaa's (2022) approach and a modified approach corresponding to the Seikh and Dutta's (2022) second approach are proposed. Finally, correct results of the existing SVTN matrix games (Seikh and Dutta, 2022, Section 5.1, Example 1, p. 929) are obtained by both the modified approaches.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

References

Bector CR, Chandra S (2005) Fuzzy mathematical programming and fuzzy matrix games, vol 169. Springer Verlag, Berlin

Brikaa MG (2022) A Note on "Solution of matrix games with payoffs of single-valued trapezoidal neutrosophic numbers". Soft Comput. 26:9137–9139

Ishibuchi H, Tanaka H (1990) Multiobjective programming in optimization of the interval objective function. Eur J Op Res 48(2):219–225

Seikh MR, Dutta S (2022) Solution of matrix games with payoffs of single-valued trapezoidal neutrosophic numbers. Soft Comput. 26:921-936