Application of a Novel Grey Model GM (1, 1, exp×sin, exp×cos) in China's GDP Per Capita Prediction

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Abstract: In the grey prediction, the GM (1, 1) model is an important type, but it sometimes shows big prediction errors and thus has limitations in applications. To improve the prediction precision of GM (1, 1) model, the paper improves from the following two aspects: (1) to improve the data's adaptability to the model, the paper transforms the accumulated generating sequence of original time sequence to make the transformed sequence meet the laws presented by the model; (2) because the traditional GM (1, 1) model's residual sequence generally shows a sine-cosine fluctuating state with a weak tendency, the paper extends the grey action of traditional GM (1, 1) model. The extended grey model is called the GM (1, 1, exp×sin, exp×cos) model. The paper gives the parameter optimization and time response equation of GM (1, 1, exp×sin, exp×cos) model. The traditional optimization method has its limitations, generally requiring the information such as the gradient value of objective function, and shows a slow convergence rate and poor precision. The paper gives a modern intelligent optimization algorithm, i.e. the particle swarm optimization algorithm (PSO), which has strong robustness and a fast convergence rate and can be realized easily and used flexibly. To improve the algorithm's convergence rate and precision, the paper improves the traditional PSO properly. According to the model and method proposed, the paper builds a GM (1, 1, exp×sin, exp×cos) model for China's GDP per capita. Results show that the model has high precision.
Keywords: GM (1, 1, exp×sin, exp×cos) model, parameter estimation, time response equation, prediction precision
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1. Introduction

The grey prediction is an important method of statistical prediction. Currently, the grey prediction has been used widely in the fields like industry, agriculture, commerce and economy (Cao et al. 2020; Liu et al. 2019), and other fields, such as environment, energy, society and military (Cai et al. 2021; Hu 2020; Huang et al. 2021; Qi and Cheng 2021; Xu and Dang 2018). The grey prediction models used widely include the GM (1, 1) model (Tong 2021; Wang and Zhao 2020), the GM (1, 1) power model (Cheng and liu 2021), the GM (1, N) model (Xie and Wu 2021) and the GM (N, 1) model (Xu and Dang 2015). In the models, the GM (1, 1) model is an important type, but it has big prediction errors sometimes and thus is limited in the applications. To improve the prediction precision of GM (1, 1) model, many scholars have made related studies. Wang and Lu (2020) constructed the background value as a variable using the Lagrange's mean value theorem. Meanwhile, they set the initial value as a variable and determined the minimum value of average relative error using the time response time to build a grey GM (1, 1) model. The example proved that the improved model was superior to other models. Chen and Zhu (2021)

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constructed the background value using a continuous monotonous piecewise rational linear interpolation spline and built a novel GM (1, 1) model, offering a more rational formula for the calculation of background value. The new GM (1, 1) model had higher effectiveness and accuracy compared with the classic GM (1, 1) model in terms of data processing. Mi et al. (2021), considering China's cross-border e-commerce as the typical research object, introduced the new information priority principle to the grey model, and proposed an improved grey GM (1, 1) model to predict the future development trend based on exploring the overall dynamic variation laws. Wang and Sun (2021) proposed a new method based on the X12-GM (1,1) combined model. The method first got the long-term trend factor, periodic variation factor and seasonal factor of data using the X12 model, and then fitted and predicted the long-term trend factor and periodic variation factor using the GM (1, 1) model, and finally multiplied the fitting value and prediction value of GM (1, 1) by the seasonal factor, respectively, to get the fitting value and prediction value of original sequence. According to the volatility and instability of modern finance, Rathnayaka and Seneviratna (2020) proposed an unbiased GM (1, 1) mixed method based on the Taylor series approximation to handle the noise and uncertain data in multidisciplinary systems. According to the exponential growth trend and seasonal fluctuating pattern of China's wind power generation, Qian and Wang (2020) proposed a new seasonal prediction method integrating the HP filter into the grey model GM (1, 1) on the basis that the grey GM (1, 1) model could capture the exponential growth trend and the Hodrick-Prescott filter could handle the seasonal factor. To improve the fitting and prediction result of unequal-interval GM (1, 1) model, on the basis of analyzing the model's main factors causing errors, Tang and Lu (2020) proposed an unequal-interval GM (1, 1) model based on grey derivative and accumulated generating method improvements. Finally, they verified the effectiveness and practicability of the improved unequal-interval GM (1, 1) model. The scholars mainly extended and optimized the grey model in terms of background value (Jiang and Zhang 2015; Xu et al. 2015), grey derivative (Li and Wei 2009; Wang and Li 2019), parameter optimization (Ding 2018; Xu et al.2016) and extrapolation (Ding 2019; Zeng and Li 2016), and further promoted modeling precision and application fields. To further improve the prediction precision of GM (1, 1) model and give data better adaptability to the model, the paper first transforms the accumulated generating sequence of original time sequence, and then, considering the traditional GM (1, 1) model's residual sequence is generally in a sine-cosine fluctuating state with the weak-tendency variation, extends the traditional GM (1, 1) model's grey action. The newly extended grey model is called the GM (1, 1, exp×sin, exp×cos) model. To avoid big average simulation relative error $(MAPE_1)$ or big average prediction relative error $(MAPE_2)$, the paper makes the objective function be the minimum of $\max(MAPE_1, MAPE_2)$. Because the optimization problem has many parameters and the traditional optimization has limitations, the paper gives a modern intelligent optimization algorithm, i.e. the particle swarm optimization (PSO). The PSO has strong robustness and a fast rate of convergence and can be realized easily and used flexibly. To improve the convergence rate and precision, the paper improves the traditional PSO properly. We get the model's parameter using the improved PSO and then make a simulation and prediction using the time response equation derived. With the model and method proposed, the paper builds a GM (1, 1, exp×sin, exp×cos) model for China's GDP per capita. Results show that the model has high precision.

2. The Method to Build the Traditional Grey GM (1, 1) Model

Definition 1: Suppose the original time sequence is $X^{(0)} = \left\{x^{(0)}(1), x^{(0)}(2), \cdots, x^{(0)}(n)\right\}$ and its first-order accumulated generating sequence ($1-AGO$) is $X^{(1)}=\left\{x^{(1)}(1),x^{(1)}(2),\cdots,x^{(1)}(n)\right\}$ in which $x^{(1)}(k)=\sum_{i=1}^{k}x^{(0)}(i),k=1,2,\cdots,n$.

Definition 2: Call $Z^{(1)}=\left\{z^{(1)}(1),z^{(1)}(2),\cdots,z^{(1)}(n)\right\}$ the background value sequence of $x^{(1)}(k)$ in which $z^{(1)}(k)=\frac{1}{2}\left(x^{(1)}(k)+x^{(1)}(k-1)\right),k=2,3,\cdots,n$.

Definition 3: Call $x^{(0)}(k)+az^{(1)}(k)=b$ the grey differential equation of grey GM (1, 1) model.

Definition 4: Call $\frac{dx^{(1)}(t)}{dt}+ax^{(1)}(t)=b$ the whitening equation of grey GM (1, 1) model.

Theorem 1 (Liu 2021)**:** The parameter estimate of grey GM (1, 1) model is

$$\begin{bmatrix} a \\ b \end{bmatrix}=(B'B)^{-1}B'Y$$

where

$$Y=\begin{bmatrix} x^{(0)}(2) \\ x^{(0)}(3) \\ \vdots \\ x^{(0)}(n) \end{bmatrix},B=\begin{pmatrix} -z^{(1)}(2) & 1 \\ -z^{(1)}(3) & 1 \\ \vdots & \vdots \\ -z^{(1)}(n) & 1 \end{pmatrix}.$$

Theorem 2 (Liu 2021)**:** The time response function of grey GM (1, 1) model is

$$\hat{x}^{(1)}(t)=(x^{(1)}(1)-\frac{b}{a})e^{-a(t-1)}+\frac{b}{a}.$$

According to the time response function sequence, we can get the simulation value and prediction value of original sequence from $\hat{x}^{(0)}(k)=\hat{x}^{(1)}(k)-\hat{x}^{(1)}(k-1)$.

3. The Form of Extended Grey Model GM (1, 1, exp×sin, exp×cos)

To make data have better adaptability to the model, the paper first improves the accumulated generating sequence of $x^{(0)}(t)$, i.e. making a new transformation, to make the new transformed sequence meet the laws presented by the model. Then, have the following definition.

Definition 5: Suppose the original time sequence is $x^{(0)}=\left\{x^{(0)}(1),x^{(0)}(2),\cdots,x^{(0)}(n)\right\}$ and the new accumulated generating sequence is $x^{(r)}=\left\{x^{(r)}(1),x^{(r)}(2),\cdots,x^{(r)}(n)\right\}$ in which $x^{(r)}(k)=\sum_{i=1}^{k}\frac{x^{(0)}(i)}{v+g\cdot r^{i}}$, $v\geq 0,g\geq 0,0<r\leq 1,k=1,2,\cdots,n$.

Then, considering the traditional GM (1, 1) model's residual sequence generally shows a sine-cosine fluctuating state with the weak-tendency variation, we extend the traditional grey model's structure to meet the requirements of this type of modeling. Then, have the following definitions.

Definition 6: Suppose the original time sequence is $x^{(0)}=\left\{x^{(0)}(1),x^{(0)}(2),\cdots,x^{(0)}(n)\right\}$, the new accumulated generating sequence is $x^{(r)} = \left\{x^{(r)}(1), x^{(r)}(2), \cdots, x^{(r)}(n)\right\}$ and the new background value is $z^{(r)}(k) = \int_{k-1}^{k} x^{(r)}(t)dt$. Call

$$x^{(r)}(k) + a_1 z^{(r)}(k) = c_0 + \sum_{i=1}^{p} e^{b_i k}(c_i \sin(s_i k) + d_i \cos(s_i k))$$

the grey differential equation of extended grey model GM (1, 1, exp×sin, exp×cos).

Definition 7: Call

$$\frac{dx^{(r)}(t)}{dt} + ax^{(r)}(t) = c_0 + \sum_{i=1}^{p} e^{d_i t}(c_i \sin(s_i t) + d_i \cos(s_i t))$$

the whitening equation of extended grey model GM (1, 1, exp×sin, exp×cos).

Especially, when $c_i = 0, d_i = 0$, the equation above is the whitening equation of traditional grey GM (1, 1) model.

4. The Time Response Equation of Extended Grey Model GM (1, 1, exp×sin, exp×cos)

Theorem 3: The sequence $x^{(r)} = \left\{x^{(r)}(1), x^{(r)}(2), \cdots, x^{(r)}(n)\right\}$, and then the whitening equation of extended grey model GM (1, 1, exp×sin, exp×cos) is

$$\frac{dx^{(r)}(t)}{dt} + ax^{(r)}(t) = c_0 + \sum_{i=1}^{p} e^{b_i t}(c_i \sin(s_i t) + d_i \cos(s_i t)),$$

and then its time response equation is

$$\begin{aligned} x^{(r)}(t) = e^{-a(t-1)} &\left\{ x^{(r)}(1) + \frac{c_0(e^{a(t-1)} - 1)}{a} \right. \\ &+ \sum_{i=1}^{p} \frac{c_i e^{-a} e^{(a+b_i)t}[(a+b_i)\sin(s_i t) - s_i \cos(s_i t)]}{(a+b_i)^2 + s_i^2} - \sum_{i=1}^{p} \frac{c_i e^{-a} e^{(a+b_i)}[(a+b_i)\sin(s_i) - s_i \cos(s_i)]}{(a+b_i)^2 + s_i^2} \\ &\left. + \sum_{i=1}^{p} \frac{d_i e^{-a} e^{(a+b_i)t}[(a+b_i)\cos(s_i t) + s_i \sin(s_i t)]}{(a+b_i)^2 + s_i^2} - \sum_{i=1}^{p} \frac{d_i e^{-a} e^{(a+b_i)}[(a+b_i)\cos(s_i) + s_i \sin(s_i)]}{(a+b_i)^2 + s_i^2} \right\} \end{aligned}.$$

Proof:

$$\frac{dx^{(r)}(t)}{dt} + ax^{(r)}(t) = c_0 + \sum_{i=1}^{p} e^{b_i t}(c_i \sin(s_i t) + d_i \cos(s_i t))$$

$$\frac{dx^{(r)}(t)}{dt} = -ax^{(r)}(t) + c_0 + \sum_{i=1}^{p} e^{b_i t}(c_i \sin(s_i t) + d_i \cos(s_i t))$$

Its solution is

$$
\begin{aligned}
x^{(r)}(t) &= e^{\int_1^t -a d\delta}[x^{(r)}(1)+\int_1^t e^{\int_1^\theta a d\delta}[c_0+\sum_{i=1}^{p} e^{b_i\theta}(c_i\sin(s_i\theta)+d_i\cos(s_i\theta))]d\theta] \\
&= e^{-a(t-1)}[x^{(r)}(1)+\int_1^t c_0 e^{a(\theta-1)}d\theta+e^{-a}\sum_{i=1}^{p} c_i\int_1^t e^{(b_i+a)\theta}\sin(s_i\theta)d\theta+e^{-a}\sum_{i=1}^{p} d_i\int_1^t e^{(b_i+a)\theta}\cos(s_i\theta)d\theta] \\
&= e^{-a(t-1)}\left\{x^{(r)}(1)+\frac{c_0(e^{a(t-1)}-1)}{a}\right. \\
&\quad+\sum_{i=1}^{p}\frac{c_i e^{-a}e^{(a+b_i)t}[(a+b_i)\sin(s_i t)-s_i\cos(s_i t)]}{(a+b_i)^2+s_i^2}-\sum_{i=1}^{p}\frac{c_i e^{-a}e^{(a+b_i)}[(a+b_i)\sin(s_i)-s_i\cos(s_i)]}{(a+b_i)^2+s_i^2} \\
&\quad\left.+\sum_{i=1}^{p}\frac{d_i e^{-a}e^{(a+b_i)t}[(a+b_i)\cos(s_i t)+s_i\sin(s_i t)]}{(a+b_i)^2+s_i^2}-\sum_{i=1}^{p}\frac{d_i e^{-a}e^{(a+b_i)}[(a+b_i)\cos(s_i)+s_i\sin(s_i)]}{(a+b_i)^2+s_i^2}\right\}.
\end{aligned}
$$

5. The Linear Parameter Estimation Method of Extended Grey Model GM (1, 1, exp×sin, exp×cos)

Theorem 4: The grey differential equation of extended grey model GM (1, 1, exp×sin, exp×cos) is

$$x^{(r)}(k)+a_1 z^{(r)}(k)=c_0+\sum_{i=1}^{p} e^{b_i k}(c_i\sin(s_i k)+d_i\cos(s_i k)).$$

For the given $v, g, r, \alpha, b_i, s_i$, have the parameter estimate

$$\hat{B}=\begin{bmatrix} a_1 \\ c_0 \\ c_1 \\ d_1 \\ \vdots \\ c_p \\ d_p \end{bmatrix}=(X'X)^{-1}X'Y,$$

where

$$X=\begin{pmatrix}
-\alpha\sum_{i=1}^{1}\frac{x^{(0)}(i)}{v+g\cdot r^i}-(1-\alpha)\sum_{i=1}^{2}\frac{x^{(0)}(i)}{v+g\cdot r^i} & 1 & z_1^{(1)}(2) & z_1^{(2)}(2) & \cdots & z_p^{(1)}(2) & z_p^{(2)}(2) \\
-\alpha\sum_{i=1}^{2}\frac{x^{(0)}(i)}{v+g\cdot r^i}-(1-\alpha)\sum_{i=1}^{3}\frac{x^{(0)}(i)}{v+g\cdot r^i} & 1 & z_1^{(1)}(3) & z_1^{(2)}(2) & \cdots & z_p^{(1)}(3) & z_p^{(2)}(3) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-\alpha\sum_{i=1}^{n-1}\frac{x^{(0)}(i)}{v+g\cdot r^i}-(1-\alpha)\sum_{i=1}^{n}\frac{x^{(0)}(i)}{v+g\cdot r^i} & 1 & z_1^{(1)}(n) & z_1^{(2)}(2) & \cdots & z_p^{(1)}(n) & z_p^{(2)}(n)
\end{pmatrix}$$

$$Y=\begin{bmatrix} \frac{x^{(0)}(2)}{v+g\cdot r^2} \\ \frac{x^{(0)}(3)}{v+g\cdot r^3} \\ \cdots \\ \frac{x^{(0)}(n)}{v+g\cdot r^n} \end{bmatrix}$$

$$z_i^{(1)}(k)=\frac{e^{b_i k}(b_i\sin(s_i k)-s_i\cos(s_i k))}{b_i^2+s_i^2}-\frac{e^{b_i(k-1)}(b_i\sin(s_i(k-1))-s_i\cos(s_i(k-1)))}{b_i^2+s_i^2}$$

$$z_i^{(2)}(k)=\frac{e^{b_ik}(b_i\cos(s_ik)+s_i\sin(s_ik))}{b_i^2+s_i^2}-\frac{e^{b_i(k-1)}(b_i\cos(s_i(k-1))+s_i\sin(s_i(k-1)))}{b_i^2+s_i^2}.$$

Proof: The whitening equation of extended grey model GM (1, 1, exp×sin, exp×cos) is

$$\frac{dx^{(r)}(t)}{dt}+ax^{(r)}(t)=c_0+\sum_{i=1}^{p}e^{b_it}(c_i\sin(s_it)+d_i\cos(s_it))$$

$$\frac{dx^{(r)}(t)}{dt}=-ax^{(r)}(t)+c_0+\sum_{i=1}^{p}e^{b_it}(c_i\sin(s_it)+d_i\cos(s_it))$$

i.e. $$\int_{k-1}^{k}\frac{dx^{(\mathrm{r})}(t)}{dt}dt=-a\int_{k-1}^{k}x^{(r)}(t)dt+c_0+\sum_{i=1}^{p}c_i\int_{k-1}^{k}e^{b_it}\sin(s_it)dt+\sum_{i=1}^{p}d_i\int_{k-1}^{k}e^{b_it}\cos(s_it)dt\cdot$$

Let the background value $z^{(r)}(k)=\int_{k-1}^{k}x^{(r)}(t)dt$,

$$z_i^{(1)}(k)=\int_{k-1}^{k}e^{b_it}\sin(s_it)dt$$
$$=\frac{e^{b_ik}(b_i\sin(s_ik)-s_i\cos(s_ik))}{b_i^2+s_i^2}-\frac{e^{b_i(k-1)}(b_i\sin(s_i(k-1))-s_i\cos(s_i(k-1)))}{b_i^2+s_i^2}$$

$$z_i^{(2)}(k)=\int_{k-1}^{k}e^{b_it}\cos(s_it)dt$$
$$=\frac{e^{b_ik}(b_i\cos(s_ik)+s_i\sin(s_ik))}{b_i^2+s_i^2}-\frac{e^{b_i(k-1)}(b_i\cos(s_i(k-1))+s_i\sin(s_i(k-1)))}{b_i^2+s_i^2},$$

and then

$$x^{(\mathrm{r})}(k)-x^{(r)}(k-1)=-az^{(r)}(k)+c_0+c_1z_1^{(1)}(k)+d_1z_2^{(1)}(k)+\cdots+c_pz_p^{(1)}(k)+d_pz_p^{(1)}(k)$$

$$\frac{x^{(0)}(k)}{v+g\cdot r^k}=-az^{(r)}(k)+c_0+c_1z_1^{(1)}(k)+d_1z_2^{(1)}(k)+\cdots+c_pz_p^{(1)}(k)+d_pz_p^{(1)}(k)\quad .$$

Because $z^{(r)}(k)\approx\alpha_kx^{(r)}(k-1)+(1-\alpha_k)x^{(r)}(k)=\alpha_k\sum_{i=1}^{k-1}\frac{x^{(0)}(i)}{v+\mathrm{g}\cdot r^i}+(1-\alpha_k)\sum_{i=1}^{k}\frac{x^{(0)}(i)}{v+\mathrm{g}\cdot r^i}$, get Theorem 4 using the least square method.

Estimate the parameter, and then calculate the simulation value and prediction value of $x^{(0)}$ with the following Theorem 5.

Theorem 5: $x^{(0)}$'s simulation value is $\hat{x}^{(0)}(k)=(\hat{x}^{(r)}(k)-\hat{x}^{(r)}(k-1))(v+gr^k),(k=2,3,\cdots,m)$ and prediction value is $\hat{x}^{(0)}(k)=(\hat{x}^{(r)}(k)-\hat{x}^{(r)}(k-1))(v+gr^k),(k=m+1,3,\cdots,n)$.

Proof: Record the original time sequence as $x^{(0)}=\left\{x^{(0)}(1),x^{(0)}(2),\cdots,x^{(0)}(n)\right\}$ and the new accumulated generating sequence is $x^{(r)}=\left\{x^{(r)}(1),x^{(r)}(2),\cdots,x^{(r)}(n)\right\}$. Because $x^{(r)}(k)=\sum_{i=1}^{k}\frac{x^{(0)}(i)}{v+g\cdot r^i},\ k=1,2,\cdots,n$, then get

$$\begin{aligned}\hat{x}^{(r)}(k)-\hat{x}^{(r)}(k-1)&=\sum_{i=1}^{k}\frac{x^{(0)}(i)}{v+g\cdot r^{i}}-\sum_{i=1}^{k\text{-}1}\frac{x^{(0)}(i)}{v+g\cdot r^{i}},\\&=\frac{x^{(0)}(k)}{v+g\cdot r^{k}}\end{aligned}$$

and then $\hat{x}^{(0)}(k)=(\hat{x}^{(r)}(k)-\hat{x}^{(r)}(k-1))(v+gr^{k})$.

6. The Nonlinear Parameter Optimization Method of Extended Grey Model GM (1, 1, exp×sin, exp×cos)

6.1 The Nonlinear Parameter Optimization Problem

In fact, the values of v,g,r,α,b_i,s_i in Theorem 4 are required, which can be determined using an optimization method. Suppose there are observation data of n years in which the data from year 1 to year m are used for modeling and the data from year $m+1$ to year n are used for prediction. For $x^{(0)}$, the simulation value and prediction value are recorded as $\hat{x}^{(0)}(k),(k=2,3,\cdots,m)$ and $\hat{x}^{(0)}(k),(k=m+1,3,\cdots,n)$ respectively; the average simulation relative is $MAPE_1=\frac{1}{m-1}\sum_{k=2}^{m}\left|\frac{x^{(0)}(k)-\hat{x}^{(0)}(k)}{x^{(0)}(k)}\right|\times 100\%$; the average prediction relative error is $MAPE_2=\frac{1}{n-m}\sum_{k=m+1}^{n}\left|\frac{x^{(0)}(k)-\hat{x}^{(0)}(k)}{x^{(0)}(k)}\right|\times 100\%$. To avoid big simulation or prediction error, we define the objective function as the minimum of $\max(MAPE_1,MAPE_2)$. Then, have the following optimization problem:

$$\min_{v,g,r,\alpha,b_i,s_i} MAPE=\max(MAPE_1,MAPE_2)$$

$$s.t.\begin{cases}MAPE_1=\dfrac{1}{m-1}\displaystyle\sum_{k=2}^{m}\left|\frac{x^{(0)}(k)-\hat{x}^{(0)}(k)}{x^{(0)}(k)}\right|\times 100\%\\ MAPE_2=\dfrac{1}{n-m}\displaystyle\sum_{k=m+1}^{n}\left|\frac{x^{(0)}(k)-\hat{x}^{(0)}(k)}{x^{(0)}(k)}\right|\times 100\%\\ \hat{x}^{(0)}(t)=(\hat{x}^{(\mathrm{r})}(t)-\hat{x}^{(r)}(t-1))(v+gr^{t})\\ \hat{x}^{(r)}(t)=e^{-a(t-1)}\left\{x^{(r)}(1)+\dfrac{c_0(e^{a(t-1)}-1)}{a}\right.\\ +\displaystyle\sum_{i=1}^{p}\frac{c_ie^{-a}e^{(a+b_i)t}[(a+b_i)\sin(s_it)-s_i\cos(s_it)]}{(a+b_i)^2+s_i^2}-\sum_{i=1}^{p}\frac{c_ie^{-a}e^{(a+b_i)}[(a+b_i)\sin(s_i)-s_i\cos(s_i)]}{(a+b_i)^2+s_i^2}\\ \left.+\displaystyle\sum_{i=1}^{p}\frac{d_ie^{-a}e^{(a+b_i)t}[(a+b_i)\cos(s_it)+s_i\sin(s_it)]}{(a+b_i)^2+s_i^2}-\sum_{i=1}^{p}\frac{d_ie^{-a}e^{(a+b_i)}[(a+b_i)\cos(s_i)+s_i\sin(s_i)]}{(a+b_i)^2+s_i^2}\right\}\\ (a,c_0,c_1,d_1,\cdots,c_p,d_p)^T=(X'X)^{-1}X'Y\\ 0\le b_i\le 1,\ \mathrm{v>0,\ g>0,}\ 0<\mathrm{r}\le 1\end{cases}.$$

It is essentially a nonlinear optimization problem which is solved using the PSO in the paper.

The fitness function is $G(v, g, r, \alpha, b_i, s_i) = \max(MAPE_1, MAPE_2) \to \min$.

6.2 The Basic Idea of the PSO

Initialize a population randomly in the solution space. The population contains several particles of which the positions in the solution space represent the solutions of problems to be solved. A particle determines its flight route according to the optimal position p_i of itself currently and the optimal position p_g of the whole population currently in the solution space, approaching the optimal region by steps.

Suppose $x_i = (x_{i1}, x_{i2}, \cdots, x_{id}, \cdots, x_{iD})$ is the D-dimension position vector of the i^{th} particle, and then calculate the fitness value of x_i currently with the fitness function set in advance to measure the advantage of particle position; $v_i = (v_{i1}, v_{i2}, \cdots, v_{id}, \cdots, v_{iD})$ is the flight speed of particle i, i.e. the moving distance of particle; $p_i = (p_{i1}, p_{i2}, \cdots, p_{id}, \cdots, p_{iD})$ is the optimal position of particle i searched by now; $p_g = (p_{g1}, p_{g2}, \cdots, p_{gd}, \cdots, p_{gD})$ is the optimal position of the whole particle swarm searched by now.

In each iteration, each particle updates speed and position according to the following formulas:

$$v_i^{t+1} = wv_i^t + c_1 r_1 (p_i - x_i^t) + c_2 r_2 (p_g - x_i^t)$$

$$x_i^{t+1} = x_i^t + v_i^{t+1}$$

where $i = 1, 2, \cdots, m$, $d = 1, 2, \cdots, D$, k is the number of iteration and r_1 & r_2 are the random numbers in the range of [0，1].

6.3 Improved PSO

The basic PSO generally has a slow convergence rate and poor precision. The paper proposes an improved PSO combining the adaptive variable weight with the adaptive acceleration constant for the optimization of parameter.

(1) Adaptive Variable Weight

The adaptive variable weight adopts a big w in the early stage of population evolution to realize the big moving speed of particle and a strong global search ability. In the late stage of evolution, it adopts a small w to reduce the moving speed of particle and emphasize the local search, thus improving the precision of optimal particle.

Therefore, the paper uses an inertia weight nonlinear decline strategy, i.e.

$$w = (1 - 0.99(\frac{4}{\pi}\operatorname{atan}(\frac{t}{T_{\max}})))^{\alpha}$$

where w is the adaptive variable weight, t is the number of iteration, $T_{\max}$ is the total number of iteration and $\alpha \geq 0$.

(2) Adaptive Acceleration Constant

The paper uses the dynamic acceleration constant as a new parameter adaptation strategy of PSO. The improvement aims to encourage the particle to move in the whole search space in the early stage of optimization and improve the rate of convergence approaching the optimal solution in the late stage of optimization, i.e.

$$c_1 = c_{\max} - (c_{\max} - c_{\min})[1 - (1 - 0.99(\frac{4}{\pi}\operatorname{atan}(\frac{t}{T_{\max}})))^{\alpha}]$$

$$c_2 = c_{\min} + (c_{\max} - c_{\min})[1 - (1 - 0.99(\frac{4}{\pi}\operatorname{atan}(\frac{t}{T_{\max}})))^{\alpha}]$$

where c_1 and c_2 are adaptive acceleration constants, $c_{\max}$ and $c_{\min}$ are constant parameters set initially, t is the number of iteration, $T_{\max}$ is the total number of iteration and $\alpha \geq 0$.

6.4 The Parameter Optimization Steps Based on Improved PSO

Step 1: Initialize the particle swarm and set parameters. To improve the algorithm's convergence rate and precision, may give the ceiling and floor of particle position of initial population in a small range and the proper initial value.

Step 2：Calculate the fitness of particle according to current particle parameter.

Step 3: Update the historical optimal fitness of particle currently and corresponding position and the optimal fitness of population and corresponding particle position.

Step 4: Calculate the speed and position of particle according to adaptive variable weight w and adaptive acceleration constants c_1 and c_2.

Step 5: Judge whether the algorithm has reached the number or precision of iteration. If yeas, output the optimal particle of population, end; if no, turn to Step 2.

7. Grey Modeling for China's GDP Per Capita

China's GDP has been growing rapidly in the recent 20 years. In 2021, under the impact of covid, China's economy still grew by 8.1% with the gross reaching ￥ 114 trillion and the yearly increasing amount of ￥0.13 million amounting to the total GDP in year 2003, which was a great achievement. With the rapid growth of China's GDP gross, the GDP per capita also reached $ 12, 000 which as only $150 back compared with the GDP per capita of high-income country. The future trend of China's GDP per capita is cared about by everyone, so predicting China's GDP per capita accurately has great significance. The paper builds a grey prediction model for China's GDP per capita using the extended grey model GM (1, 1, exp×sin,exp×cos) proposed. China's GDP per capita is written as $x^{(0)}(t)$ in the unit of RMB 1 Yuan. The data came from the *China Statistical Yearbook.* See table 1 for the specific data.

Table 1 : Calculation Results of Grey Modeling for China's GDP Per Capita

Year	No.	Actual Value $x^{(0)}(t)$	Traditional GM (1, 1) Model		GM (1, 1, exp×sin, exp×cos) Model Proposed	
			Simulation Value	Relative Error %	Simulation Value	Relative Error %
2002	1	9506	-	-	-	-
2003	2	10666	14370.49	34.73	10692.54	0.249
2004	3	12487	15967.42	27.87	12501.0	0.112
2005	4	14368	17741.82	23.48	14315.92	0.362
2006	5	16738	19713.39	17.78	17200.86	2.77
2007	6	20494	21904.06	6.88	20466.89	0.132
2008	7	24100	24338.17	0.9882	23979.27	0.501
2009	8	26180	27042.77	3.296	27658.29	5.65
2010	9	30808	30047.92	2.467	31437.31	2.04
2011	10	36277	33387.02	7.966	35288.75	2.72
2012	11	39771	37097.18	6.723	39204.91	1.42
2013	12	43497	41219.64	5.236	43193.43	0.698
2014	13	46912	45800.2	2.37	47272.98	0.769
2015	14	49922	50889.79	1.939	51469.53	3.1
2016	15	53783	56544.96	5.135	55813.84	3.78
2017	16	59592	62828.57	5.431	60339.62	1.25
2018	17	65534	69810.45	6.526	65082.25	0.689
			Simulation Value	Relative Error %	Simulation Value	Relative Error %
2019	18	70078	77568.2	10.69	70078.0	1.13e-6
2020	19	71828	86188.03	19.99	75363.59	4.92
2021	20	80976	95765.75	18.26	80976.0	1.71e-7
Average Simulation Relative Error (2002-2018)			-	9.93	-	1.64
Average Prediction Relative Error (2019-2021)			-	16.31	-	1.64
Average Overall Relative Error (2002-2021)			-	10.94	-	1.64

First, build a traditional GM (1, 1) model, i.e. the following model

$$\frac{dx^{(1)}(t)}{dt}+ax^{(1)}(t)=b$$

Then, get the parameter estimate through calculation:

$$(a,b)=(-0.105373795,\ 12624.9646)\quad .$$

In this case, the time response equation is

$$\hat{x}^{(1)}(t) = (x^{(1)}(1) - \frac{b}{a})e^{-a(t-1)} + \frac{b}{a}$$
$$= 129317.2357e^{0.1053738(t\text{-}1)}\text{-}119811.2357$$

Calculate and get the simulation values and prediction values of original sequence with $\hat{x}^{(0)}(k) = \hat{x}^{(1)}(k) - \hat{x}^{(1)}(k-1)$. See the results in Table 1. With $RE(t) = \left|\frac{x^{(0)}(t) - \hat{x}^{(0)}(t)}{x^{(0)}(t)}\right| \times 100\%$, calculate and get the relative errors of simulation values and prediction values of original sequence in the periods. With $MAPE_1 = \frac{1}{m-1}\sum_{k=2}^{m}\left|\frac{x^{(0)}(k) - \hat{x}^{(0)}(k)}{x^{(0)}(k)}\right| \times 100\%$, calculate and get the average simulation relative error. With $MAPE_2 = \frac{1}{n-m}\sum_{k=m+1}^{n}\left|\frac{x^{(0)}(k) - \hat{x}^{(0)}(k)}{x^{(0)}(k)}\right| \times 100\%$, calculate and get the average prediction relative error. See Table 1 for the results.

Then, build the extended grey model GM (1, 1, exp×sin, exp×cos) proposed, i.e. the following model

$$\frac{dx^{(r)}(t)}{dt} + ax^{(r)}(t) = c_0 + \sum_{i=1}^{p} e^{b_i t}(c_i \sin(s_i t) + d_i \cos(s_i t))$$

After calculation, get that when $p = 2$, the precision is the highest. Now use the improved PSO proposed to calculate parameters v, g, r, α, b_i and s_i. The values of fixed parameters in the model are as follows: population size $m = 20$, dimension $D = 8$, $T_{\max} = 500$, $c_{\max} = 2.5$, $c_{\min} = 1.5$ and $\alpha = 0.5$. Through calculation, get the values of parameters:

$$(v, g, r, \alpha, b_1, s_1, b_2, s_2)$$
$$= (\ 17.4591,47.5948,0.1636,0.5895,\text{-}1.3456,\text{-}2.3018,\text{-}0.2133,\text{-}0.1429)$$

$$(a, c_0, c_1, d_1, c_2, d_2) = (\text{-}0.07028063,\ 1629.798,\ 376.5244,\ 376.7157,\ 2409.817,\ \text{-}976.4062)$$

In this case, get the time response equation:

$$\hat{x}^{(r)}(t) = 16769.35\mathrm{e}^{(0.07028063\ t)} - 23189.86$$
$$+ \mathrm{e}^{(0.07028063\ t)}\{9681.533\mathrm{e}^{(-0.2835994\ t)}\ [0.2835994\cos(0.142912\ t)\ \ -0.142912\ \sin(0.142912\ t)\ ]$$
$$+ 23894.48\mathrm{e}^{(-0.2835994\ t)}\ [0.142912\cos(0.142912\ t)\ + 0.2835994\sin(0.142912\ t)\ ]$$
$$+51.55732\ \mathrm{e}^{(-1.415898\ t)}\ [2.301794\ \cos(2.301794\ t) +\ 1.415898\sin(2.301794\ t)]$$
$$-51.58352\ \mathrm{e}^{(-1.415898\ t)}\ [1.415898\cos(2.301794\ t)\ \ -2.301794\ \sin(2.301794\ t)]\}$$

From $\hat{x}^{(0)}(k) = (\hat{x}^{(r)}(k) - \hat{x}^{(r)}(k-1))(v + gr^k)$, get the simulation and prediction values of original sequence. See Table 1 for the relative errors and average relative errors in the periods.

To compare the traditional PSO and the improved PSO proposed in terms of convergence rate and precision, the paper makes a calculation. See Table 2 for the results. Figure 1 gives the variation curves of objective functions with the number of iteration of two algorithms. It shows that the improved PSO has the better convergence rate and precision compared with the traditional PSO.

Table 2: The Comparison of Calculation Results of Two PSO Algorithms

Method	Traditional Method	Improved Method
v	18.5242	17.4591
g	40.5133	47.5948
h	0.1101	0.1636
α	0.5903	0.5895
b_1	-1.1616	-1.3456
s_1	-1.8405	-2.3018
b_2	-0.2683	-0.2133
s_2	-0.0505	-0.1429
Number of Iteration	355	185
Objective Function Value G	1.65%	1.64%

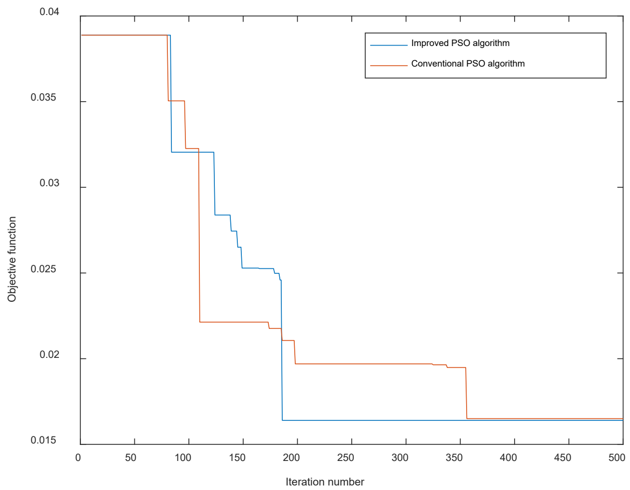

Figure 1: Variation Curves of Objective Function Values with Iteration Number of Two Algorithms

Table 3: Calculation Results of Grey Models for China's GDP Per Capita Built with Other Methods

Year	No.	Actual Value $x^{(0)}(t)$	Model Proposed by Ma and Wang (2019)		Model Proposed by Cheng and Liu (2021)	
			Simulation Value	Relative Error %	Simulation Value	Relative Error %
2002	1	9506	-	-	-	-
2003	2	10666	11677.84	9.49	10546.46	1.12
2004	3	12487	14113.8	13.0	11733.62	6.03
2005	4	14368	16444.15	14.4	14293.92	0.516
2006	5	16738	18814.54	12.4	17335.04	3.57
2007	6	20494	21293.59	3.9	20606.48	0.549
2008	7	24100	23925.13	0.726	24019.0	0.336
2009	8	26180	26743.25	2.15	27541.11	5.2
2010	9	30808	29777.93	3.34	31164.29	1.16
2011	10	36277	33057.82	8.87	34889.76	3.82
2012	11	39771	36611.64	7.94	38723.02	2.64
2013	12	43497	40469.03	6.96	42671.56	1.9
2014	13	46912	44661.25	4.8	46743.83	0.358
2015	14	49922	49221.54	1.4	50948.84	2.06
2016	15	53783	54185.63	0.749	55295.9	2.81
2017	16	59592	59592.02	3.27e-5	59794.63	0.34
2018	17	65534	65482.38	0.0788	64454.93	1.65
			Prediction Value	Relative Error %	Prediction Value	Relative Error %
2019	18	70078	71901.92	2.6	69286.97	1.13
2020	19	71828	78899.7	9.85	74301.26	3.44
2021	20	80976	86529.12	6.86	79508.66	1.81
Average Simulation Relative Error (2002-2018)			-	5.64	-	2.13
Average Prediction Relative Error (2019-2021)			-	6.44	-	2.13
Average Overall Relative Error (2002-2021)			-	5.77	-	2.13

To compare the model proposed with the grey models proposed in other referencing documents in terms of modeling precision, the paper makes calculations. Build a model using the improvement method of grey GM (1, 1) power model given by Ma and Wang (2019) and then get the following estimate of parameter:

$$(a, b, \alpha, c) = (-0.08666978,\ 1815.878,\ 0.1856846,\ 0.9601279).$$

Then, the time response equation is

$$\hat{x}^{(1)}(k) = c\left\{\frac{b}{a} + [x^{(1)}(1)^{(1-\alpha)} - \frac{b}{a}]e^{a(\alpha-1)(k-1)}\right\}^{\frac{1}{\alpha-1}}$$
$$= 0.9601279(\text{-}20951.6807 + 22686.8618e^{0.0706(k\text{-}1)})^{1.2280}$$
.

Calculate and get the simulation value and prediction value of original sequence with $\hat{x}^{(0)}(k) = \hat{x}^{(1)}(k) - \hat{x}^{(1)}(k-1)$, and the results are shown in Table 3. Table 3 shows the relative errors and average relative error in the periods.

Build a model using the improvement method of extended grey GM (1, 1) power model given by Cheng and Liu (2021) and then get the following estimate of parameter:

$$(\alpha, a, b_0, b_1, b_2)$$
$$= (\text{-}0.6463599987,\ \text{-}0.04886932862,\ 3525512.917,\ \text{-}233540.6464,\ 710804.9806).$$

Then, the time response equation is

$$\hat{x}^{(1)}(t) = \left\{e^{-a(1-\alpha)(t-1)}\left[(x^{(1)}(1))^{(1-\alpha)} + (1-\alpha)e^{-a(1-\alpha)}g(t)\right]\right\}^{\frac{1}{1-\alpha}}$$
$$= \left\{e^{0.0805(t-1)}\left[3541738.3101 + 1.7843g(t)\right]\right\}^{0.6074}$$

where

$$g(t) = \frac{b_2 t^2 e^{a(1-\alpha)t}}{a(1-\alpha)} - \frac{2b_2 t e^{a(1-\alpha)t}}{a^2(1-\alpha)^2} + \frac{b_1 t e^{a(1-\alpha)t}}{a(1-\alpha)} + \frac{2b_2 e^{a(1-\alpha)t}}{a^3(1-\alpha)^3} - \frac{b_1 e^{a(1-\alpha)t}}{a^2(1-\alpha)^2} + \frac{b_0 e^{a(1-\alpha)t}}{a(1-\alpha)}$$
$$- \frac{2b_2 e^{a(1-\alpha)}}{a^3(1-\alpha)^3} + \frac{2b_2 e^{a(1-\alpha)}}{a^2(1-\alpha)^2} - \frac{b_2 e^{a(1-\alpha)}}{a(1-\alpha)} + \frac{b_1 e^{a(1-\alpha)}}{a^2(1-\alpha)^2} - \frac{e^{a(1-\alpha)}}{a(1-\alpha)} - \frac{b_0 e^{a(1-\alpha)}}{a(1-\alpha)}$$
$$= 215251872.839\text{-}8834648.6818t^2e^{\text{-}0.0805t}\text{-}216710334.478te^{\text{-}0.0805t}\text{-}2737327947.79e^{0.0805t}.$$

Calculate and get the simulation value and prediction value of original sequence with $\hat{x}^{(0)}(k) = \hat{x}^{(1)}(k) - \hat{x}^{(1)}(k-1)$, and the results are shown in Table 3. Table 3 shows the relative errors and average relative error in the periods.

Figure 2 shows the modeling precision of the models. From Table 1 to Table 3 and Figure 1 to Figure 2 we can see the model built using the grey model GM (1, 1, exp×sin, exp×cos) and modeling method proposed has the simulation precision and prediction precision significantly higher than that of traditional grey GM (1, 1) model and the grey power model proposed by Ma and Wang (2019), and the modeling precision superior to that of the extended grey power model given by Cheng and Liu (2021). The GM (1, 1, exp×sin, exp×cos) model has an average simulation relative error of only 1.64% and an average prediction relative error of only 1.64% with high precision. It shows the model and method proposed have high reliability and effectiveness.

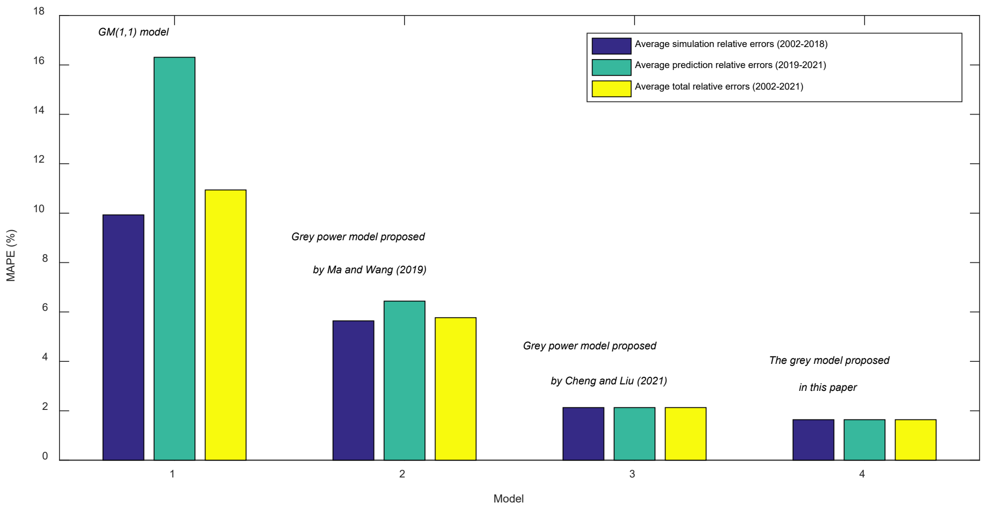

Figure 2: Modeling Precision Histogram of Four Models

7. Conclusions

(1) To increase the data's adaptability to the model, the paper transforms the accumulated generating sequence of original time sequence, i.e. $x^{(0)}=\left\{x^{(0)}(1),x^{(0)}(2),\cdots,x^{(0)}(n)\right\}$. The new accumulated generating sequence is $x^{(r)}=\left\{x^{(r)}(1),x^{(r)}(2),\cdots,x^{(r)}(n)\right\}$ in which $x^{(r)}(k)=\sum_{i=1}^{k}\frac{x^{(0)}(i)}{v+g\cdot r^{i}}$

(2) To further improve the model's modeling precision, the paper, considering the variation characteristic of residual sequence of grey GM (1, 1) model, extends the grey GM (1, 1) model, i.e. building the GM (1, 1, exp×sin, exp×cos) model.

(3) The paper gives the parameter estimation method and time response equation of GM (1, 1, exp×sin, exp×cos) model. Because the grey differential equation for parameter estimation and the whitening equation for prediction have consistent structures, the model has high precision.

(4) The paper optimizes the parameters of transformed accumulated generating sequence and the parameters of model at the same time and the optimization object is $\min\limits_{v,g,r,\alpha,b_i,s_i} MAPE=\max(MAPE_1,MAPE_2)$, thus avoiding high simulation or prediction error. With the improved PSO, the model has high precision and a fast convergence rate.

(5) Using the model and method proposed, the paper builds a GM (1, 1, exp×sin, exp×cos) model for China's GDP per capita. Results show that the model has high precision with an average simulation relative error of only 1.64% and an average prediction relative error of only 1.64%.

(6) To compare the method proposed with other grey models in terms of precision, the paper builds the GM (1, 1) model, the grey GM (1, 1) power model proposed by Ma and Wang (2019) and the extended grey GM (1, 1) power model proposed by Cheng and Liu (2021) for China's GDP per capita. Results show that the model built with the method proposed has the overall precision significantly higher than that of the GM (1, 1) model and the grey GM (1, 1) power model proposed by Ma and Wang (2019), and is superior to the extended grey GM (1, 1) power model proposed by Cheng and Liu (2021).

Authorship contribution statement

Maolin Cheng: Conceptualization, Methodology, Supervision, Project administration, Software, Writing–original draft. Bin Liu: Validation, Formal analysis, Investigation, Resources, Data curation, Visualization.

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Compliance with ethical standards

Conflict of interest

The authors declare that they have no conflict of interest.

Human participants or animals performed

This article does not contain any studies with human participants or animals performed by any of the authors.

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