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The distributivity of extended semi-t-operators over extended S-uninorms on fuzzy truth values

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Abstract

Inspired by the thought of distributivity between semi-t-operator and S-uninorm, this paper primarily explores the distributivity between extended semi-t-operator and extended S-uninorm on fuzzy truth value. Firstly, Zadeh-extended semi-t-operator and S-uninorm are proposed on fuzzy truth value and some results of extended semi-operator are studied under special fuzzy truth values. Then, it concentrates on the sufficient condition about left and right distributivity of extended semi-t-operator over extended S-uninorm under the condition that semi-t-operator is left and right distributive over S-uninorm, respectively. Finally, when parameters satisfy different cases, sufficient conditions for the distributivity between extended semi-t-operator and extended S-uninorm are given under the condition that semi-t-operator satisfies distributivity or conditional distributivity over S-uninorm.

Keywords: Fuzzy truth value, Aggregation function, Semi-t-operator, S-uninorm, Distributivity, Conditional distributivity

1 Introduction

As an extension of fuzzy set, type-2 fuzzy set was defined by Zadeh in 1975 [1], which has been explored comprehensively in theory [2–8] and practical applications [9–13]. Unlike the exact membership function in fuzzy sets, type-2 fuzzy set can be represented as a fuzzy set with fuzzy sets as truth values, that is, the fuzzy membership function (called fuzzy truth value) of type-2 fuzzy set consists of all mappings from $[0, 1]$ into $[0, 1]$. Therefore, type-2 fuzzy set has greater inclusiveness for uncertainty and plays an indispensable role in dealing with fuzzy language uncertainty. Meantime, the operations on type-2 fuzzy set are convolutions of operations on $[0, 1]$ [2]. On the other end of the spectrum, aggregation theory of real numbers has an imperative effect both in actual applications and theory [14–16]. Hence, the aggregation function of real numbers is widely generalized to type-2 fuzzy set [17–19]. More specifically, Gera and Dombi [20] introduced the extended t-norm and t-conorm on fuzzy truth value by pointwise formulas. Further, the extended aggregation function on convex normal fuzzy truth value was proposed by Takáč [17]. On the basis of [17], Torres-Blanc et al. [7] extended the aggregation function of fuzzy set to type-2 fuzzy set with the help of Zadeh's extension principle.

In addition, the distributivity between binary functions was introduced based on the viewpoint of functional equation [21]. Subsequently, the distributivity between many special aggregation functions was explored [22–25]. Accordingly, a natural and important problem is to consider the distributivity between convolution operations on fuzzy truth values, which can be viewed as a link connecting two extended aggregation functions. The distributivity between extended aggregation functions not only enriches the theory of logical algebraic structure for type-2 fuzzy set [26–29], but also enhances its application in approximate reasoning and fuzzy control system [30]. For example, Walker and Walker [2, 31, 32], Harding et al. [5] proposed that the extended minimum was distributive over the extended maximum, meantime, the extended maximum was distributive over the extended minimum. Moreover, the extended t-norm (resp. the extended t-conorm) was distributive over maximum (resp. minimum). In 2014, Hu and Kwong [6] researched the distributivity between the extended t-norm (resp. the extended t-conorm) and maximum (resp. minimum). Further, Xie [33] gave the concept of nullnorm and uninorm on fuzzy truth value and explored the condition that extended uninorm was distributive between minimum or maximum. Afterwards, Zhang and Hu [34] characterized the distributivity of convolution operation over a meet-convolution and a join-convolution. In 2020, Wang and Liu [28] presented the distributivity of extended nullnorm over uninorm under the condition that nullnorm and uninorm were conditionally distributive. Next, when t-norm satisfied conditional distributivity over t-conorm, Liu and Wang [27] investigated the distributivity of extended t-norm over t-conorm. Further, Liu and Wang [29] explored the conditions under which the distributivity between the Z-extended overlap functions and grouping functions holds.

As a further extension of the t-norm and t-conorm, Drygaś [35, 36] defined semi-t-operator through reducing the commutative law of t-operator. By the idea of annihilator in aggregation function, Mas et al. [37] presented the concept of S-uninorm, which can be seen as a significant generalization of uninorm.

Based on the above definitions, Fang and Hu [38] researched several conditional distributivity equations for S-uninorm, where t-norm, T-uninorm and semi-t-operator are conditionally distributive over S-uninorm. Later on, Wang et al. [39] complemented the work of conditional distributivity and distributivity between semi-t-operator and S-uninorm. Furthermore, when parameters satisfied different relations, the sufficient and necessary conditions that semi-t-operator satisfied left (resp. right) distributivity over S-uninorm were studied with the underlying uninorm in $\mathcal{U}_{\min}$. The above conclusions lay a solid foundation for exploring the distributive law between extended aggregation functions on fuzzy truth value.

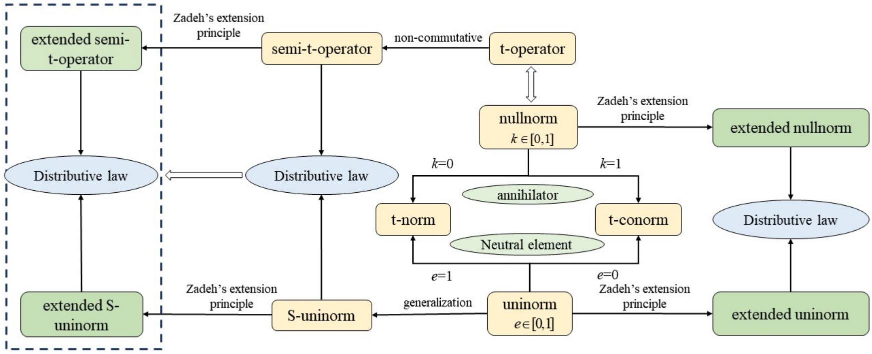

Figure 1 The relationship of distributive law between different logical operators

On the other hand, based on the condition that nullnorm and uninorm satisfy conditional distributivity, Wang and Liu [28] explored the distributive law between extended nullnorm and uninorm on fuzzy truth value. It is well known that semi-t-operator [35, 36] and S-uninorm [37] are generalizations of t-operator (nullnorm) and uninorm, respectively. To further study the extended semi-t-operator and extended S-uninorm and make them easy to apply in application, it is necessary to explore the related properties of semi-t-operator and S-uninorm on fuzzy truth value. The aim of this paper is to generalize the distributivity between extended semi-t-operator and extended S-uninorm, which can be considered as the improvement of the algebraic structure of type-2 fuzzy set in theoretical field. Meanwhile, the specific relationship of distributive law between different fuzzy logical operators is shown in Figure 1.

In this paper, we firstly investigate the extended semi-t-operator and S-uninorm by Zadeh's extension principle, and further explore the distributivity between extended semi-t-operator and S-uninorm based on the results that semi-t-operator satisfies conditional distributivity or distributivity over S-uninorm in [38, 39].

The reminder of this paper is arranged as follows. Section 2 reviews some basic concepts and properties, which are essential for the sequel. In Section 3, the Zadeh-extended semi-t-operator and S-uninorm are proposed, and then some properties of extended semi-t-operator are studied under special fuzzy truth values. Section 4 explores the left and right distributivity of extended semi-t-operator over extended S-uninorm on fuzzy truth values when semi-t-operator is left and right distributive to S-uninorm. In Section 5, the distributivity between extended semi-t-operator and S-uninorm is investigated under the condition that semi-t-operator is (conditionally) distributive to S-uninorm. Section 6 summarizes the full works.

2 Preliminaries

This part is composed of three subsections to review basic knowledge regarding semi-t-operator and S-uninorm, distributivity, conditional distributivity and the extended binary operators on fuzzy truth value.

2.1 Semi-t-operator and S-uninorm

In this subsection, we mainly introduce several types of common binary operators. According to reference [40], the concepts of semi-t-norm, semi-t-conorm, t-norm and t-conorm can be shown below.

- A semi-t-norm $\mathscr{T} : [0,1] \times [0,1] \longrightarrow [0,1]$ is an increasing and associative binary function with neutral element 1.
- A semi-t-conorm $\mathscr{S} : [0,1] \times [0,1] \longrightarrow [0,1]$ is an increasing and associative binary function with neutral element 0.
- A t-norm $T : [0,1] \times [0,1] \longrightarrow [0,1]$ is a commutative semi-t-norm.
- A t-conorm $S : [0,1] \times [0,1] \longrightarrow [0,1]$ is a commutative semi-t-conorm.

Definition 1 [38] A left semi-t-conorm (resp. right semi-t-conorm) is a increasing binary function $\mathbb{S} : [0,1] \times [0,1] \longrightarrow [0,1]$ that has associativity and the left (resp. right) neutral element 0, i.e., $\mathbb{S}(0,u) = u$ (resp. $\mathbb{S}(u,0) = u$) for each $u \in [0,1]$.

In light of the positive t-conorm, Wang et al. [39] defined the positive property of left semi-t-conorm and right semi-t-conorm, that is, if $\mathbb{S}(u,v) = 1$ implies $u = 1$ or $v = 1$, then left (resp. right) semi-t-conorm $\mathbb{S}$ is positive. In a similar way, if $\mathscr{S}(u,v) = 1$ implies $u = 1$ or $v = 1$, then semi-t-conorm $\mathscr{S}$ is positive.

Definition 2 [41] A uninorm $U : [0,1]\times[0,1] \longrightarrow [0,1]$ is an increasing, commutative and associative binary function with neutral element $e \in [0,1]$.

It is worth reminding that uninorm U degenerates into a t-norm T when $e = 1$ and a t-conorm S when $e = 0$. If $U(0,1) = 0$, then it is denoted as a conjunctive uninorm. Further, Fodor et al. [42] proposed the characterization of uninorm U in the following.

Proposition 1 *[42] The uninorm U with neutral element $e \in (0,1)$ can be expressed as follows:*

$$U(u,v) = \begin{cases} eT_U\left(\frac{u}{e}, \frac{v}{e}\right), & \text{if } u,v \in [0,e], \\ e + (1-e)S_U\left(\frac{u-e}{1-e}, \frac{v-e}{1-e}\right), & \text{if } u,v \in [e,1], \\ A(e), & \text{otherwise}, \end{cases} \tag{1}$$

where T_U is a t-norm, S_U is a t-conorm and $A(e)$ meets the condition that $\min(u,v) \leqslant A(e) \leqslant \max(u,v)$ for any $u \in [0,e), v \in (e,1]$ or $u \in (e,1], v \in [0,e)$.

Moreover, the underlying t-norm and t-conorm of uninorm U are denoted as T_U and S_U, respectively. Specially, the family of all uninorms given by Eq. (1) is denoted by $\mathcal{U}_{\min}$, if $A(e) = \min(u,v)$ for any $u \in [0,e), v \in (e,1]$ or $u \in (e,1], v \in [0,e)$.

Definition 3 [37] An S-uninorm $A : [0,1] \times [0,1] \longrightarrow [0,1]$ is an increasing, commutative, associative binary function with $A(0,0) = 0$ and $A(1,1) = 1$, and the following statements hold.

(1) $A(0,u)$ is continuous and $A(1,u)$ is not;
(2) there is an annihilator $\lambda \in [0,1)$ of A, that is, $A(u,\lambda) = \lambda$ for any $u \in [0,1]$;
(3) there exists an IFC element $e \in (0,1)$, which makes e an idempotent element and satisfies the fixed value $A(e,1) = 1$ and continuity, whereby $A(e,u)$ is continuous.

Further, Mas et al. [37] presented the equivalent characterization form of S-uninorm.

Proposition 2 *[37] A binary function $A : [0,1] \times [0,1] \longrightarrow [0,1]$ is an S-uninorm if and only if there exists $\lambda \in [0,1)$, t-conorm S and conjunctive uninorm U with neutral element $e_U \in (0,1)$ such that A is expressed as:*

$$A(u,v) = \begin{cases} \lambda S\left(\frac{u}{\lambda}, \frac{v}{\lambda}\right), & \text{if } u,v \in [0,\lambda], \\ \lambda + (1-\lambda)U\left(\frac{u-\lambda}{1-\lambda}, \frac{v-\lambda}{1-\lambda}\right), & \text{if } u,v \in [\lambda,1], \\ \lambda, & \text{otherwise}, \end{cases} \tag{2}$$

where U and S are called the underlying uninorm and underlying t-conorm, respectively.

All S-uninorms in the form of Eq. (2) consist of a family, called $\mathcal{U}^S_{e,\lambda}$ with IFC element e. Further, unless otherwise stated, the family of $A \in \mathcal{U}^S_{e,\lambda}$ with the underlying uninorm in $\mathcal{U}_{\min}$ is denoted as $(\mathbb{U}^S_{e,\lambda})_{\min}$.

Definition 4 [41] A semi-t-operator $F : [0,1] \times [0,1] \longrightarrow [0,1]$ is an increasing, associative binary function with $F(0,0) = 0, F(1,1) = 1$, meantime, $F(0,u)$, $F(u,0)$, $F(1,u)$ and $F(u,1)$ are continuous for any $u \in [0,1]$.

The family of all semi-t-operators is denoted as $\mathscr{F}_{\varepsilon,\eta}$ with $F(0,1) = \varepsilon$ and $F(1,0) = \eta$. Next, the equivalent characterization of $F \in \mathscr{F}_{\varepsilon,\eta}$ can be given as follows.

Proposition 3 *[35, 36] The semi-t-operator $F \in \mathscr{F}_{\varepsilon,\eta}$ if and only if there are semi-t-norm $\mathscr{T}$ and semi-t-conorm $\mathscr{S}$ such that*

$$F(u,v) = \begin{cases} \varepsilon\mathscr{S}\left(\frac{u}{\varepsilon}, \frac{v}{\varepsilon}\right), & if\ u,v \in [0,\varepsilon], \\ \eta + (1-\eta)\mathscr{T}\left(\frac{u-\eta}{1-\eta}, \frac{v-\eta}{1-\eta}\right), & if\ u,v \in [\eta,1], \\ \varepsilon, & if\ u \leqslant \varepsilon \leqslant v, \\ \eta, & if\ v \leqslant \eta \leqslant u, \\ u, & othersize, \end{cases}$$

when $\varepsilon \leq \eta$ and

$$F(u,v) = \begin{cases} \eta\mathscr{S}\left(\frac{u}{\eta}, \frac{v}{\eta}\right), & if\ u,v \in [0,\eta], \\ \varepsilon + (1-\varepsilon)\mathscr{T}\left(\frac{u-\varepsilon}{1-\varepsilon}, \frac{v-\varepsilon}{1-\varepsilon}\right), & if\ u,v \in [\varepsilon,1], \\ \varepsilon, & if\ u \leqslant \varepsilon \leqslant v, \\ \eta, & if\ v \leqslant \eta \leqslant u, \\ v, & othersize, \end{cases}$$

when $\eta \leq \varepsilon$.

2.2 Distributive law and conditional distributive law

Next, the distributivity and conditional distributivity equations between two binary functions are listed as follows.

Definition 5 [21] Suppose $O_1, O_2 : [0,1] \times [0,1] \longrightarrow [0,1]$ are two binary functions. Then, O_1 is distributive over O_2 when both left distributive law and right distributive law are satisfied:

- left distributive law: $O_1(u, O_2(v,w)) = O_2(O_1(u,v), O_1(u,w))$
- right distributive law: $O_1(O_2(v,w), u) = O_2(O_1(v,u), O_1(w,u))$

for each $u, v, w \in [0,1]$.

Definition 6 [38] Let A be an S-uninorm with underlying uninorm in $\mathcal{U}_{\min}$ and F a semi-t-operator. F satisfies conditional distributivity over A from the left if for any $u, v, w \in [0,1]$ and $A(v,w) < 1$, it holds that

$$F(u, A(v,w)) = A(F(u,v), F(u,w)). \tag{3}$$

In a similar way, F satisfies conditional distributivity over A from the right if for any $u, v, w \in [0,1]$ and $A(v,w) < 1$, it holds that

$$F(A(v,w), u) = A(F(v,u), F(w,u)). \tag{4}$$

If F and A satisfy both Eqs. (3) and (4), then F satisfies the conditional distributivity for A.

2.3 The extended operators on fuzzy truth value

Definition 7 [2, 43] A mapping $h : [0,1] \longrightarrow [0,1]$ is called fuzzy truth value. Further, the family of all fuzzy truth values is represented as $\mathscr{F} = \{h \mid h : [0,1] \longrightarrow [0,1]\}$.

There are some special fuzzy truth values [2, 31], which can be shown as follows.

$$\bar{1}(u) = \begin{cases} 1, u = 1, \\ 0, u \neq 1, \end{cases} \text{ and } \bar{0}(u) = \begin{cases} 1, u = 0, \\ 0, u \neq 0. \end{cases}$$

Further, if $a \in [0,1]$, $\underline{a}(u) = a$ for any $u \in [0,1]$, then $\underline{a}$ is a mapping of fixed value a.

Definition 8 [2, 5] Let $h \in \mathscr{F}$, h^L and h^R are denoted as follows:

$$h^L(u) = \bigvee_{t \leq u} h(t) \text{ and } h^R(u) = \bigvee_{t \geq u} h(t).$$

Definition 9 [2] A fuzzy truth value h is convex if $u \leq v \leq w$, then $h(v) \geq h(u) \wedge h(w)$ for any $u, v, w \in [0,1]$. The family of all convex fuzzy truth values is denoted as $\mathscr{F}_C$. Further, $h \in \mathscr{F}$ is convex if and only if $h = h^L \wedge h^R$.

A fuzzy truth value is normal if $\bigvee\limits_{u \in [0,1]} h(u) = 1$ holds and the family of all normal fuzzy truth values is denoted as $\mathscr{F}_N$.

In light of Zadeh's extension principle [1], the binary function $\diamond : [0,1] \times [0,1] \longrightarrow [0,1]$ can be extended to function $\odot_\diamond : \mathscr{F} \times \mathscr{F} \longrightarrow \mathscr{F}$:

$$(h_1 \odot_\diamond h_2)(w) = \bigvee_{\diamond(u,v)=w} (h_1(u) \wedge h_2(v))$$

where $h_1, h_2 \in \mathscr{F}$.

Definition 10 [2, 5, 6] Let $h_1, h_2 \in \mathscr{F}$, T and S be a t-norm and a t-conorm. The extended t-norm $\odot_T$ and extended t-conorm $\odot_S$ can be showed as follows:

$$(h_1 \odot_T h_2)(w) = \bigvee_{T(u,v)=w} (h_1(u) \wedge h_2(v)),$$

$$(h_1 \odot_S h_2)(w) = \bigvee_{S(u,v)=w} (h_1(u) \wedge h_2(v)).$$

In particular, if T and S take T_M and S_M respectively, then $\sqcap$ and $\sqcup$ are used to denote $\odot_T$ and $\odot_S$, i.e.,

$$(h_1 \sqcap h_2)(w) = \bigvee_{u \wedge v = w} (h_1(u) \wedge h_2(v)),$$

$$(h_1 \sqcup h_2)(w) = \bigvee_{u \vee v = w} (h_1(u) \wedge h_2(v)).$$

Further, Liu [44] proved that the extended continuous binary functions with the properties of monotonically increasing satisfy the distributive law for the extended t-norms $\sqcap$ (resp. the extended t-conorms $\sqcup$), which are listed as follows.

Proposition 4 *[44] Let $\diamond : [0,1] \times [0,1] \longrightarrow [0,1]$ be a binary function which is continuous and monotonically increasing. If $h_1 \in \mathscr{F}$ is convex, then for each $h_2, h_3 \in \mathscr{F}$, the following statements hold.*

(1) $h_1 \odot_\diamond (h_2 \sqcap h_3) = (h_1 \odot_\diamond h_2) \sqcap (h_1 \odot_\diamond h_3)$,
(2) $h_1 \odot_\diamond (h_2 \sqcup h_3) = (h_1 \odot_\diamond h_2) \sqcup (h_1 \odot_\diamond h_3)$.

3 The extended semi-t-operator and extended S-uninorm on fuzzy truth value

First, the definitions of extended semi-t-operator $\odot_F$ and extended S-uninorm $\odot_A$ are constructed below.

Definition 11 Suppose $h_1, h_2 \in \mathscr{F}$, F is an semi-t-operator and A an S-uninorm. The extended semi-t-operator $\odot_F$ is represented as

$$(h_1 \odot_F h_2)(w) = \bigvee_{F(u,v)=w} (h_1(u) \wedge h_2(v)).$$

The extended S-uninorm $\odot_A$ is denoted as

$$(h_1 \odot_A h_2)(w) = \bigvee_{A(u,v)=w} (h_1(u) \wedge h_2(v)).$$

As a special case of [7], Propositions 5 and 6 hold when semi-t-operator F and S-uninorm A are continuous.

Proposition 5 *Let F be a continuous semi-t-operator and $\odot_F$ the extended operator of F. For any $h_1, h_2 \in \mathscr{F}$, it holds that*

$$(h_1 \odot_F h_2)^R = h_1^R \odot_F h_2^R,$$
$$(h_1 \odot_F h_2)^L = h_1^L \odot_F h_2^L.$$

Proposition 5 illustrates that the order of operations between the extended semi-t-operator and the fuzzy truth value h^R (resp. h^L) has no effect on the final results. Further, there is a similar relationship between the extended S-uninorm and the fuzzy truth value h^R (resp. h^L).

Proposition 6 *Let A be a continuous S-uninorm and $\odot_A$ the extended operator of A. For any $h_1, h_2 \in \mathscr{F}$, it holds that*

$$(h_1 \odot_A h_2)^L = h_1^L \odot_A h_2^L,$$
$$(h_1 \odot_A h_2)^R = h_1^R \odot_A h_2^R.$$

Next, the properties of extended semi-t-operator $\odot_F$ on special fuzzy truth value $\underline{1}$ are discussed.

Proposition 7 *Let $F \in \mathscr{F}_{\varepsilon,\eta}$ be a continuous semi-t-operator, $\odot_F$ the extended operator of F. For any $h \in \mathscr{F}$, the following statements can be obtained.*

If $\varepsilon \leqslant \eta$, then

$$(h \odot_F \underline{1})(w) = \begin{cases} h^L(w), & if\ w \leqslant \varepsilon, \\ h^R(w), & if\ w \geqslant \eta, \\ h(w), & if\ \varepsilon < w < \eta, \end{cases}$$

and

$$(\underline{1} \odot_F h)(w) = \begin{cases} h^L(w), & if\ w < \varepsilon, \\ h^R(w), & if\ w > \eta, \\ \bigvee\limits_{t \in [0,1]} h(t), & if\ \varepsilon \leqslant w \leqslant \eta. \end{cases}$$

If $\eta \leqslant \varepsilon$, then

$$(h \odot_F \underline{1})(w) = \begin{cases} h^L(w), & if\ w < \eta, \\ h^R(w), & if\ w > \varepsilon, \\ \bigvee\limits_{t \in [0,1]} h(t), & if\ \eta \leqslant w \leqslant \varepsilon, \end{cases}$$

and

$$(\underline{1} \odot_F h)(w) = \begin{cases} h^L(w), & if\ w \leqslant \eta, \\ h^R(w), & if\ w \geqslant \varepsilon, \\ h(w), & if\ \eta < w < \varepsilon. \end{cases}$$

Proof First, we verify the case of $\varepsilon \leqslant \eta$, it holds that

$$(h \odot_F \underline{1})(w) = \bigvee_{F(u,v)=w} h(u) \wedge \underline{1}(v)$$
$$= \bigvee_{F(u,v)=w} h(u).$$

(1) If $w \leqslant \varepsilon$, then the following two cases are discussed.

- If $(u, v) \in [0, \varepsilon]^2$, then according to Proposition 3 that F is semi-t-conorm with neutral element 0. Hence, one concludes that

$$w = F(u, v) \geq F(u, 0) = u,$$

that is, $w \geq u$.

- If $(u,v) \in [0,\varepsilon] \times [\varepsilon,1]$, then $w = F(u,v) = \varepsilon \geq u$, that is, $w \geq u$.

  Further, for any $u \in [0,w]$,

$$F(u,w) \geq F(0,w) = w,$$
$$F(u,0) \leq F(w,0) = w.$$

Since F is continuous, there is $w' \in [0,w]$ such that $F(u,w') = w$.

In summary, we can obtain that

$$\begin{aligned}(h \odot_F \underline{1})(w) &= \bigvee_{F(u,v)=w} h(u) \\ &= \bigvee_{u \leq w} h(u) \\ &= h^L(w).\end{aligned}$$

(2) If $w \geqslant \eta$, the following two cases are discussed.

- If $(u,v) \in [\eta,1]^2$, then according to Proposition 3 that F is semi-t-norm with neutral element 1. Hence, it holds that

$$w = F(u,v) \leq F(u,1) = u,$$

  that is, $w \leq u$.
- If $(u,v) \in [\eta,1] \times [0,\eta]$, then $w = F(u,v) = \eta \leq u$, that is, $w \leq u$.

  Further, for any $u \in [w,1]$,

$$F(u,w) \leq F(1,w) = w,$$
$$F(u,1) \geq F(w,1) = w.$$

Since F is continuous, there is $w' \in [w,1]$ such that $F(u,w') = w$.

In summary, it can be concluded that

$$\begin{aligned}(h \odot_F \underline{1})(w) &= \bigvee_{F(u,v)=w} h(u) \\ &= \bigvee_{u \geq w} h(u) \\ &= h^R(w).\end{aligned}$$

(3) If $\varepsilon < w < \eta$, then $(u,v) \in (\varepsilon,\eta) \times [0,1]$. According to Proposition 3, $w = F(u,v) = u$, it concludes that

$$\begin{aligned}(h \odot_F \underline{1})(w) &= \bigvee_{F(u,v)=w} h(u)\\ &= \bigvee_{u=w} h(u)\\ &= h(w).\end{aligned}$$

Next, we consider that $(\underline{1} \odot_F h)(w)$ with the case of $\varepsilon \leqslant \eta$.

$$\begin{aligned}(\underline{1} \odot_F h)(w) &= \bigvee_{F(u,v)=w} \underline{1}(u) \wedge h(v)\\ &= \bigvee_{F(u,v)=w} h(v).\end{aligned}$$

(1) If $w < \varepsilon$, then $(u,v) \in [0,\varepsilon]^2$. According to Proposition 3 that F is semi-t-conorm with neutral element 0. Hence, one concludes that

$$w = F(u,v) \geq F(0,v) = v,$$

that is, $w \geq v$.

Further, for any $v \in [0,w]$,

$$\begin{aligned}F(w,v) &\geq F(w,0) = w,\\ F(0,v) &\leq F(0,w) = w.\end{aligned}$$

Since F is continuous, there is $w' \in [0,w]$ such that $F(w',v) = w$.

Therefore, we can obtain that

$$\begin{aligned}(\underline{1} \odot_F h)(w) &= \bigvee_{F(u,v)=w} h(v)\\ &= \bigvee_{w\geq v} h(v)\\ &= h^L(w).\end{aligned}$$

(2) If $w > \eta$, then $(u,v) \in [\eta,1]^2$. According to Proposition 3 that F is semi-t-norm with neutral element 1. Hence, one concludes that

$$w = F(u,v) \leq F(1,v) = v,$$

that is, $w \leq v$.

Further, for any $v \in [w,1]$,

$$F(w,v) \leq F(w,1) = w,$$

$$F(1,v) \geq F(1,w) = w.$$

Since F is continuous, there is $w' \in [w,1]$ such that $F(w',v) = w$.

Therefore, we can obtain that

$$\begin{aligned}(\underline{1} \odot_F h)(w) &= \bigvee_{F(u,v)=w} h(v) \\ &= \bigvee_{w \leq v} h(v) \\ &= h^R(w).\end{aligned}$$

(3) If $\varepsilon \leq w \leq \eta$, then the following five cases are discussed:
(1′) If $w = \varepsilon$ and $(u,v) \in [0,\varepsilon] \times [\varepsilon,1]$, then $F(u,v) = \varepsilon$.
(2′) If $w = \varepsilon$ and $(u,v) \in [0,\varepsilon]^2$, then $v \leq w$.
(3′) If $\varepsilon < w < \eta$ and $(u,v) \in (\varepsilon,\eta) \times [0,1]$, according to Proposition 3, $F(u,v) = u$.
(4′) If $w = \eta$ and $(u,v) \in [\eta,1] \times [0,\eta]$, then $F(u,v) = \eta$.
(5′) If $w = \eta$ and $(u,v) \in [\eta,1]^2$, then $v \geq w$.

In summary, the value of v is not affected by $w = F(u,v)$ in cases (1′), (3′) and (4′), then we can obtain that

$$\begin{aligned}(\underline{1} \odot_F h)(w) &= \bigvee_{F(u,v)=w} h(v) \\ &= \left(\bigvee_{v\in[0,1]} h(v)\right) \vee \left(\bigvee_{v \leq w} h(v)\right) \vee \left(\bigvee_{v \geq w} h(v)\right) \\ &= \bigvee_{t\in[0,1]} h(t).\end{aligned}$$

In a similar way, the corresponding conclusions can be verified when $\eta \leq \varepsilon$. □

Similar to Proposition 7, the properties of extended semi-t-operator $\odot_F$ on the fuzzy truth value $\overline{1}$ are obtained.

Proposition 8 *Suppose $F \in \mathscr{F}_{\varepsilon,\eta}$ is a semi-t-operator, $\odot_F$ the extension of F, then for any $h \in \mathscr{F}$, the following statements can be obtained when $\varepsilon \leq \eta$ or $\varepsilon \geq \eta$.*

$$\left(h \odot_F \overline{1}\right)(w) = \begin{cases} h(w), & if\ w \geqslant \varepsilon, \\ 0, & if\ w < \varepsilon. \end{cases}$$

$$\left(\overline{1} \odot_F h\right)(w) = \begin{cases} h(w), & if\ w \geqslant \eta, \\ 0, & if\ w < \eta. \end{cases}$$

Proof At first, we verify the case of $\varepsilon \leqslant \eta$. If $w \geqslant \varepsilon$, then it holds that

$$\begin{aligned}\left(h \odot_F \overline{1}\right)(w) &= \bigvee_{F(u,v)=w} h(u) \wedge \overline{1}(v)\\ &= \bigvee_{F(u,1)=w} h(u)\\ &= h(w).\end{aligned}$$

On the other hand, if $w < \varepsilon$, then it holds that

$$\left(h \odot_F \overline{1}\right)(w) = \bigvee_{F(u,v)=w} h(u) \wedge \overline{1}(v) = 0.$$

Similarly, the case of $(\overline{1} \odot_F h)(w)$ can be verified as follows.
If $w \geqslant \eta$, then it infers that

$$\begin{aligned}(\overline{1} \odot_F h)(w) &= \bigvee_{F(u,v)=w} \overline{1}(u) \wedge h(v)\\ &= \bigvee_{F(1,v)=w} h(v)\\ &= h(w).\end{aligned}$$

On the other hand, if $w < \eta$, then it infers that

$$\left(\overline{1} \odot_F h\right)(w) = \bigvee_{F(u,v)=w} \overline{1}(u) \wedge h(v) = 0.$$

Similarly, the corresponding conclusions can be verified when $\eta \leq \varepsilon$. □

When ε, η take special values 0 or 1, i.e., $\mathscr{F}_{0,1}$ or $\mathscr{F}_{1,0}$, some properties of the extended semi-t-operator $\odot_F$ can be obtained.

Proposition 9 *Let $F_1 \in \mathscr{F}_{0,1}, F_2 \in \mathscr{F}_{1,0}$ be two semi-t-operators, $\odot_{F_1}$ and $\odot_{F_2}$ be the extended operators of F_1 and F_2, then for any $h_1, h_2 \in \mathscr{F}$, the following statements can be obtained.*

$$\left(h_1 \odot_{F_1} h_2\right)(w) = h_1(w) \wedge \bigvee_{t\in[0,1]} h_2(t),$$

$$\left(h_1 \odot_{F_2} h_2\right)(w) = h_2(w) \wedge \bigvee_{t\in[0,1]} h_1(t).$$

In particular, if $h_1, h_2 \in \mathscr{F}_N$, then

$$\begin{aligned}\left(h_1 \odot_{F_1} h_2\right)(w) &= h_1(w),\\ \left(h_1 \odot_{F_2} h_2\right)(w) &= h_2(w).\end{aligned}$$

Proof Assume that $F_1 \in \mathscr{F}_{0,1}$, that is, $0 = \varepsilon < \eta = 1$, then $F(u,v) = u$ for any $u, v \in [0,1]$. In light of Proposition 3, one concludes that

$$\left(h_1 \odot_{F_1} h_2\right)(w) = \bigvee_{F_1(u,v)=w} h_1(u) \wedge h_2(v)$$

$$= \bigvee_{w=u} h_1(u) \wedge h_2(v)$$
$$= h_1(w) \wedge \bigvee_{v\in[0,1]} h_2(v).$$

Specially, if $h_2 \in \mathscr{F}_N$, then $\bigvee_{v\in[0,1]} h_2(v) = 1$, it follows that

$$(h_1 \odot_{F_1} h_2)(w) = h_1(w) \wedge \bigvee_{t\in[0,1]} h_2(t)$$
$$= h_1(w).$$

In a similar way, $(h_1 \odot_{F_2} h_2)(w) = h_2(w) \wedge \bigvee_{t\in[0,1]} h_1(t)$ can be proven when $F_2 \in \mathscr{F}_{1,0}$ and $(h_1 \odot_{F_2} h_2)(w) = h_2(w)$ with $h_1 \in \mathscr{F}_N$. □

Some properties about the extended semi-t-operator $\odot_F$ and extended S-uninorm $\odot_A$ can be given with the help of literature [6]. Since the semi-t-operator does not satisfy commutativity, the extended semi-t-operator also does not satisfy commutativity.

Proposition 10 *Suppose F is a semi-t-operator, A an S-uninorm, then for any $h, h', h'' \in \mathscr{F}$, the following statements hold.*

(1) If $h \subseteq h'$, then $h \odot_F h'' \subseteq h' \odot_F h''$.
(2) $h \odot_A h' = h' \odot_A h$.
(3) $(h \odot_F h') \odot_F h'' = h \odot_F (h' \odot_F h'')$ and $(h \odot_A h') \odot_A h'' = h \odot_A (h' \odot_A h'')$.
(4) $h \odot_F (h' \cup h'') = (h \odot_F h') \cup (h \odot_F h'')$ and $h \odot_A (h' \cup h'') = (h \odot_A h') \cup (h \odot_A h'')$, where $(h' \cup h'')(x) = h'(x) \vee h''(x)$.
(5) $h \odot_F (h' \cap h'') = (h \odot_F h') \cap (h \odot_F h'')$ and $h \odot_A (h' \cap h'') = (h \odot_A h') \cap (h \odot_A h'')$, where $(h' \cap h'')(x) = h'(x) \wedge h''(x)$.

Proof(1) If $h \subseteq h'$, then it concludes that

$$(h \odot_F h'')(w) = \bigvee_{F(u,v)=w} h(u) \wedge h''(v)$$
$$\leq \bigvee_{F(u,v)=w} h'(u) \wedge h''(v)$$
$$= (h' \odot_F h'')(w).$$

(2) Since the S-uninorm A satisfies the commutativity, it holds that

$$(h \odot_A h')(w) = \bigvee_{A(u,v)=w} h(u) \wedge h'(v)$$
$$= \bigvee_{A(u,v)=w} h'(v) \wedge h(u)$$

$$= (h' \odot_A h)(w).$$

(3) Since the semi-t-operator F satisfies the associativity, it follows that

$$\begin{aligned}
&((h \odot_F h') \odot_F h'')(w) \\
&= \bigvee_{F(s,t)=w} (h \odot_F h')(s) \wedge h''(t) \\
&= \bigvee_{F(s,t)=w} \left(\bigvee_{F(u,v)=s} h(u) \wedge h'(v) \right) \wedge h''(t) \\
&= \bigvee_{F(F(u,v),t)=w} h(u) \wedge h'(v) \wedge h''(t) \\
&= \bigvee_{F(u,F(v,t))=w} h(u) \wedge h'(v) \wedge h''(t) \\
&= \bigvee_{F(u,m)=w} h(u) \wedge \left(\bigvee_{F(v,t)=m} h'(v) \wedge h''(t) \right) \\
&= \bigvee_{F(u,m)=w} h(u) \wedge (h' \odot_F h'')(m) \\
&= (h \odot_F (h' \odot_F h''))(w).
\end{aligned}$$

Hence, $(h \odot_F h') \odot_F h'' = h \odot_F (h' \odot_F h'')$. Similarly, $(h \odot_A h') \odot_A h'' = h \odot_A (h' \odot_A h'')$ can be obtained.

(4) Since $(h' \cup h'')(t) = h'(t) \vee h''(t)$, it concludes that

$$\begin{aligned}
(h \odot_F (h' \cup h''))(w) &= \bigvee_{F(u,v)=w} h(u) \wedge (h' \cup h'')(v) \\
&= \bigvee_{F(u,v)=w} h(u) \wedge (h'(v) \vee h''(v)) \\
&= \bigvee_{F(u,v)=w} (h(u) \wedge h'(v)) \vee (h(u) \wedge h''(v)) \\
&= (h \odot_F h')(w) \vee (h \odot_F h'')(w) \\
&= ((h \odot_F h') \cup (h \odot_F h''))(w).
\end{aligned}$$

(5) Since $(h' \cap h'')(t) = h'(t) \wedge h''(t)$, it concludes that

$$\begin{aligned}
(h \odot_F (h' \cap h''))(w) &= \bigvee_{F(u,v)=w} h(u) \wedge (h' \cap h'')(v) \\
&= \bigvee_{F(u,v)=w} h(u) \wedge (h'(v) \wedge h''(v))
\end{aligned}$$

$$= \bigvee_{F(u,v)=w} (h(u) \wedge h'(v)) \wedge (h(u) \wedge h''(v))$$
$$= (h \odot_F h')(w) \wedge (h \odot_F h'')(w)$$
$$= ((h \odot_F h') \cap (h \odot_F h''))(w).$$

Therefore, we obtain that $h \odot_F (h' \cap h'') = (h \odot_F h') \cap (h \odot_F h'')$. In a similar way, it holds that $h \odot_A (h' \cap h'') = (h \odot_A h') \cap (h \odot_A h'')$.

□

This section mainly explores some properties of the extended semi-t-operator $\odot_F$ and the extended S-uninorm $\odot_A$. Next, the left (resp. right) distributive law between $\odot_F$ and $\odot_A$ will be obtained.

4 Left and right distributivity of extended semi-t-operator over extended S-uninorm on fuzzy truth value

In what follows, the sufficient conditions of the left distributive law between extended semi-t-operator $\odot_F$ and extended S-uninorm $\odot_A$ on fuzzy truth value are given.

4.1 The situation of $e < \min(\varepsilon, \eta) < \max(\varepsilon, \eta) = 1$

In 2021, Wang et al. [39] charactered the equivalent forms about the semi-t-operator and S-uninorm when the left or right distributive law between them is satisfied.

Proposition 11 *[39] Suppose $0 < \lambda < e < \varepsilon < \eta = 1, F \in \mathscr{F}_{\varepsilon,\eta}$, $A \in (\mathbb{U}^S_{e,\lambda})_{\min}$. If the left distributivity between F and A is satisfied, then A and F have the following forms, respectively:*

$$A(u,v) = \begin{cases} \max(u,v), & \text{if } u,v \in [0,\lambda) \text{ or } u,v \in [e,1], \\ \lambda, & \text{if } u \in [0,\lambda], v \in [\lambda,1] \text{ or } u \in [\lambda,1], v \in [0,\lambda], \\ \min(u,v), & \text{if } u \in (\lambda,e], v \in (\lambda,1] \text{ or } u \in [e,1], v \in (\lambda,e), \end{cases} \tag{5}$$

and

$$F(u,v) = \begin{cases} \lambda \mathscr{S}\left(\frac{u}{\lambda}, \frac{v}{\lambda}\right), & \text{if } u,v \in [0,\lambda], \\ A_1(u,v), & \text{if } u \in [0,\lambda), v \in (\lambda,e), \\ B(u,v), & \text{if } u,v \in [\lambda,e), \\ C(u,v), & \text{if } u \in [0,e), v \in [e,\varepsilon), \\ D(u,v), & \text{if } u \in [e,\varepsilon), v \in [0,\lambda), \\ \varepsilon, & \text{if } u \in [e,\varepsilon], v \in [\lambda,\varepsilon] \text{ or } u \in [0,\varepsilon], v \in [\varepsilon,1], \\ u, & \text{if } u \in [\varepsilon,1], v \in [0,1], \\ \max(u,v), & \text{otherwise}, \end{cases} \tag{6}$$

where A is an idempotent S-uninorm; $\mathscr{S}$ is a semi-t-conorm; $A_1 : [0,\lambda) \times (\lambda,e) \longrightarrow [\lambda,\varepsilon]$, $B : [\lambda,e) \times [\lambda,e) \longrightarrow [\lambda,\varepsilon]$, $C : [0,e) \times [e,\varepsilon) \longrightarrow [e,\varepsilon]$ and $D : [e,\varepsilon) \times [0,\lambda) \longrightarrow [e,\varepsilon]$ have associativity, the property of monotonically increasing and the respective *boundary condition* $A_1(0,u) = u$ $(\forall u \in (\lambda, e))$, $B(u, \lambda) = u$ $(\forall u \in [\lambda, e))$, $C(0,u) = u$ $(\forall u \in [e, \varepsilon))$, $D(u, 0) = u$ $(\forall u \in [e, \varepsilon))$ *(A visual image is shown in* Figure. 2*).*

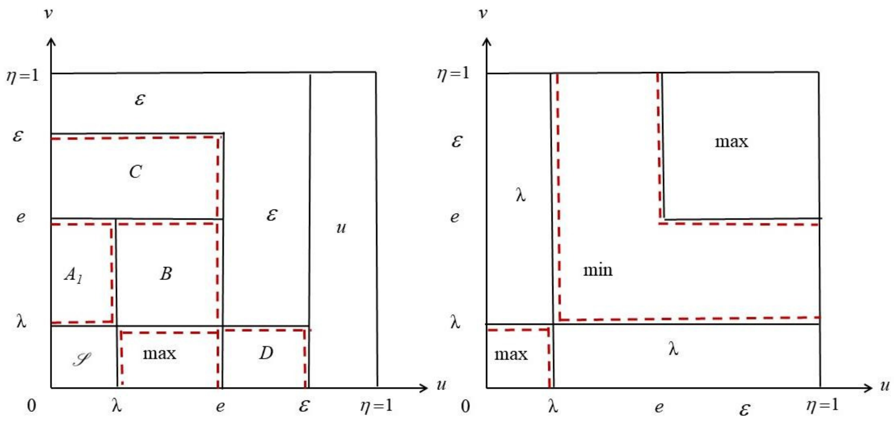

Figure 2 The characterizations of F (left) and A (right) in Proposition 11 [39]

Next, on the basis of left distributivity between F and A, it is natural to consider the left distributive law of the extended operator $\odot_F$ with respect to the extended operator $\odot_A$.

Theorem 12 *Suppose* $0 < \lambda < e < \varepsilon < \eta = 1$, $F \in \mathscr{F}_{\varepsilon,\eta}$ *is continuous,* $A \in (\mathbb{U}^S_{e,\lambda})_{\min}$ *and* F *satisfies the left distributivity over* A*. For any* $h_1 \in \mathscr{F}_C$ *and* $h_2, h_3 \in \mathscr{F}$*, it holds that*

$$h_1 \odot_F (h_2 \odot_A h_3) = (h_1 \odot_F h_2) \odot_A (h_1 \odot_F h_3)\,.$$

Proof First, it follows from Definition 11 that for any $\omega \in [0,1]$,

$$\begin{aligned}
&(h_1 \odot_F (h_2 \odot_A h_3))(\omega) \\
&= \bigvee_{F(y,x)=\omega} h_1(y) \wedge ((h_2 \odot_A h_3)(x)) \\
&= \bigvee_{F(y,x)=\omega} h_1(y) \wedge \left(\bigvee_{A(u,v)=x} h_2(u) \wedge h_3(v) \right) \\
&= \bigvee_{F(y,\,A(u,v))=\omega} h_1(y) \wedge h_2(u) \wedge h_3(v),
\end{aligned}$$

and

$$\begin{aligned}
&((h_1 \odot_F h_2) \odot_A (h_1 \odot_F h_3))\,(\omega) \\
&= \bigvee_{A(y,x)=\omega} ((h_1 \odot_F h_2)(y) \wedge (h_1 \odot_F h_3)(x))
\end{aligned}$$

$$= \bigvee_{A(y,x)=\omega} \left(\left(\bigvee_{F(p,q)=y} h_1(p) \wedge h_2(q) \right) \wedge \left(\bigvee_{F(s,t)=x} h_1(s) \wedge h_3(t) \right) \right)$$
$$= \bigvee_{A(F(p,q),F(s,t))=\omega} h_1(p) \wedge h_2(q) \wedge h_1(s) \wedge h_3(t).$$

Since F is left distributive over A, that is, $F(y, A(u,v)) = A(F(y,u), F(y,v))$. According to Proposition 11, A and F can be expressed as Eqs. (5) and (6), respectively. Next, for any $u, v \in [0,1]$, we divide into four cases to prove.

(1) If $\omega = F(y, A(u,v)) \in [0,\lambda)$, then $A(u,v) = \max(u,v)$. It follows from Proposition 4 that $(h_1 \odot_F (h_2 \odot_A h_3))(\omega) = ((h_1 \odot_F h_2) \odot_A (h_1 \odot_F h_3))(\omega)$.

(2) If $\omega = F(y, A(u,v)) \in [\lambda, e]$, then there exist four cases below.

(i) If $F(y, A(u,v)) = \max(y, A(u,v))$, then $y \in (\lambda, e)$, $A(u,v) \in [0,\lambda)$, that is, $A(u,v) = \max(u,v)$. By Proposition 4 that $(h_1 \odot_F (h_2 \odot_A h_3))(\omega) = ((h_1 \odot_F h_2) \odot_A (h_1 \odot_F h_3))(\omega)$.

(ii) If $F(y, A(u,v)) = A_1(y, A(u,v))$, then $y \in [0,\lambda)$, $A(u,v) \in (\lambda, e)$, that is, $A(u,v) = \min(u,v)$. By Proposition 4 that $(h_1 \odot_F (h_2 \odot_A h_3))(\omega) = ((h_1 \odot_F h_2) \odot_A (h_1 \odot_F h_3))(\omega)$.

(iii) If $F(y, A(u,v)) = B(y, A(u,v))$, then $y \in [\lambda, e)$, $A(u,v) \in [\lambda, e)$.

- If $A(u,v) \in (\lambda, e)$, that is, $A(u,v) = \min(u,v)$, then $(h_1 \odot_F (h_2 \odot_A h_3))(\omega) = ((h_1 \odot_F h_2) \odot_A (h_1 \odot_F h_3))(\omega)$ holds by Proposition 4.
- If $A(u,v) = \lambda$, then $z = F(y, A(u,v)) = B(y,\lambda) = y$ for any $y \in [\lambda, e)$. Since F is left distributive over A, it holds that

$$\begin{aligned}
&((h_1 \odot_F h_2) \odot_A (h_1 \odot_F h_3))(\omega) \\
&= \bigvee_{A(F(p,q),F(s,t))=\omega} h_1(p) \wedge h_2(q) \wedge h_1(s) \wedge h_3(t) \\
&\overset{p=s=y}{\geq} \bigvee_{A(F(y,q),F(y,t))=\omega} h_1(y) \wedge h_2(q) \wedge h_1(y) \wedge h_3(t) \\
&= \bigvee_{F(y,A(q,t))=\omega} h_1(y) \wedge h_2(q) \wedge h_3(t) \\
&= (h_1 \odot_F (h_2 \odot_A h_3))(\omega),
\end{aligned}$$

that is, $(h_1 \odot_F (h_2 \odot_A h_3))(\omega) \leq ((h_1 \odot_F h_2) \odot_A (h_1 \odot_F h_3))(\omega)$ always holds.

Next, we verify that $(h_1 \odot_F (h_2 \odot_A h_3))(\omega) \geq ((h_1 \odot_F h_2) \odot_A (h_1 \odot_F h_3))(\omega)$.

Firstly, we give the statement P.
P. For each $y' \in [\lambda, e)$, $(q,t) \in [0,\lambda] \times [\lambda, 1]$ or $(q,t) \in [\lambda, 1] \times [0,\lambda]$, $p, s \in [0,1]$, if $A(F(p,q), F(s,t)) = A(F(y',q), F(y',t)) = \omega$, then $(h_1 \odot_F (h_2 \odot_A h_3))(\omega) \geq ((h_1 \odot_F h_2) \odot_A (h_1 \odot_F h_3))(\omega)$.

Due to $A(F(p,q),F(s,t)) = A(F(y',q),\, F(y',t)) = \omega$, then $p \le y' \le s$ or $s \le y' \le p$. Otherwise, $y' < p \wedge s$ or $y' > p \vee s$. Assume that $y' < p \wedge s$, there is a contradiction in

$$\begin{aligned}\omega &= A(F(p,q),F(s,t))\\ &\ge A(F(p\wedge s,q),F(p\wedge s,t))\\ &= F(p\wedge s, A(q,t))\\ &= p\wedge s\\ &> y'\\ &= F(y',A(q,t))\\ &= A(F(y',q),F(y',t))\\ &= \omega.\end{aligned}$$

Hence, $y' \ge p \wedge s$. Moreover, it can be proven that $y' \le p \vee s$ in a similar way. Therefore, $p \le y' \le s$ or $s \le y' \le p$. One can conclude that $h_1(y') \ge h_1(p) \wedge h_1(s)$ since h_1 is convex, and then $h_1(y') \wedge h_2(q) \wedge h_3(t) \ge h_1(p) \wedge h_2(q) \wedge h_1(s) \wedge h_3(t)$. In view of the above discussion, P holds.

In light of the statement P, one can obtain that

$$\begin{aligned}&((h_1 \odot_F h_2) \odot_A (h_1 \odot_F h_3))(\omega)\\ &= \bigvee_{A(F(p,q),F(s,t))=\omega} h_1(p) \wedge h_2(q) \wedge h_1(s) \wedge h_3(t)\\ &\le \bigvee_{A(F(y',q),F(y',t))=\omega} h_1(y') \wedge h_2(q) \wedge h_3(t)\\ &= \bigvee_{F(y',\, A(q,t))=\omega} h_1(y') \wedge h_2(q) \wedge h_3(t)\\ &= \bigvee_{F(y,\, A(u,v))=\omega} h_1(y) \wedge h_2(u) \wedge h_3(v)\\ &= (h_1 \odot_F (h_2 \odot_A h_3))(\omega).\end{aligned}$$

(iv) If $F(y,A(u,v)) = S(y,A(u,v))$ (i.e., $\omega = \lambda$), then $y \in [0,\lambda)$, $A(u,v) = \lambda$ or $y = \lambda, A(u,v) \in [0,\lambda]$ or $y \in [0,\lambda), A(u,v) \in [0,\lambda)$. In particular, if $y = \lambda, A(u,v) \in [0,\lambda)$, then $A(u,v) = \max(u,v)$ and $(h_1 \odot_F (h_2 \odot_A h_3))(\lambda) = ((h_1 \odot_F h_2) \odot_A (h_1 \odot_F h_3))(\lambda)$ holds by Proposition 4. Since F is left distributive over A, it holds that

$$(h_1 \odot_F (h_2 \odot_A h_3))\,(\lambda) \le ((h_1 \odot_F h_2) \odot_A (h_1 \odot_F h_3))\,(\lambda).$$

Next, we prove that $(h_1 \odot_F (h_2 \odot_A h_3))(\lambda) \geq ((h_1 \odot_F h_2) \odot_A (h_1 \odot_F h_3))\ (\lambda)$.

$$
\begin{aligned}
&(h_1 \odot_F (h_2 \odot_A h_3))(\lambda)\\
=& \bigvee_{F(y,A(u,v)=\lambda} h_1(y) \wedge h_2(u) \wedge h_3(v)\\
=& \left(\bigvee_{y\in[0,\lambda),\ A(u,v)=\lambda} h_1(y) \wedge h_2(u) \wedge h_3(v) \right) \vee\\
& \left(\bigvee_{y=\lambda,\ A(u,v)\in[0,\lambda]} h_1(y) \wedge h_2(u) \wedge h_3(v) \right) \vee\\
& \left(\bigvee_{y\in[0,\lambda),\ A(u,v)\in[0,\lambda)} h_1(y) \wedge h_2(u) \wedge h_3(v) \right)\\
\geq& \bigvee_{y=\lambda,\ A(u,v)\in[0,\lambda)} h_1(y) \wedge h_2(u) \wedge h_3(v)\\
=& \bigvee_{A(F(p,q),F(s,t))=\lambda} h_1(p) \wedge h_2(q) \wedge h_1(s) \wedge h_3(t)\\
=&((h_1 \odot_F h_2) \odot_A (h_1 \odot_F h_3))(\lambda).
\end{aligned}
$$

(3) If $\omega = F(y, A(u, v)) \in [e, \varepsilon]$, then there exist five cases as follows.

(i) If $F = A_1$, then $y \in [0, \lambda)$, $A(u, v) \in (\lambda, e)$, that is, $A(u, v) = \min(u, v)$.

(ii) If $F = B$, then $y \in [\lambda, e)$, $A(u, v) \in [\lambda, e)$, then it can be proven as case (2)(iii).

(iii) If $F = C$, then $y \in [0, e)$, $A(u, v) \in [e, \varepsilon)$, that is, $A(u, v) = \max(u, v)$.

(iv) If $F = D$, then $y \in [e, \varepsilon)$, $A(u, v) \in [0, \lambda)$, that is, $A(u, v) = \max(u, v)$.

In light of Proposition 4, all of the above cases satisfy that $(h_1 \odot_F (h_2 \odot_A h_3))(\omega) = ((h_1 \odot_F h_2) \odot_A (h_1 \odot_F h_3))(\omega)$.

(v) If $F(y, A(u, v)) = \varepsilon$, then $y \in [0, \varepsilon]$, $A(u, v) \in [\varepsilon, 1]$ or $y \in [e, \varepsilon]$, $A(u, v) \in [\lambda, \varepsilon]$. In particular, if $y \in [0, \varepsilon]$, $A(u, v) \in [\varepsilon, 1]$, then $(h_1 \odot_F (h_2 \odot_A h_3))(\varepsilon) = ((h_1 \odot_F h_2) \odot_A (h_1 \odot_F h_3))(\varepsilon)$ holds by Proposition 4. Since F is left distributive over A, it holds that $(h_1 \odot_F (h_2 \odot_A h_3))(\varepsilon) \leq ((h_1 \odot_F h_2) \odot_A (h_1 \odot_F h_3))(\varepsilon)$.

Next, we prove that $(h_1 \odot_F (h_2 \odot_A h_3))(\varepsilon) \geq ((h_1 \odot_F h_2) \odot_A (h_1 \odot_F h_3))(\varepsilon)$.

$$
\begin{aligned}
&(h_1 \odot_F (h_2 \odot_A h_3))(\varepsilon)\\
=& \bigvee_{F(y,\ A(u,v)=\varepsilon} h_1(y) \wedge h_2(u) \wedge h_3(v)
\end{aligned}
$$

$$\begin{aligned}
&\geq \bigvee_{y\in[0,\varepsilon],\, A(u,v)\in[\varepsilon,\,1]} h_1(y)\wedge h_2(u)\wedge h_3(v)\\
&= \bigvee_{A(F(p,q),F(s,t))=\varepsilon} h_1(p)\wedge h_2(q)\wedge h_1(s)\wedge h_3(t)\\
&= ((h_1\odot_F h_2)\odot_A (h_1\odot_F h_3))(\varepsilon).
\end{aligned}$$

(4) If $\omega = F(y, A(u,v)) \in (\varepsilon,\, 1]$, then $y \in [\varepsilon,\, 1], A(u,v) \in [0,1]$. Four cases are given below.

(i) If $A(u,v) \in [0,\lambda)$, then $A(u,v) = \max(u,v)$;

(ii) If $A(u,v) \in (\lambda, e)$, then $A(u,v) = \min(u,v)$;

(iii) If $A(u,v) \in [e,1]$, then $A(u,v) = \max(u,v)$;

In light of Proposition 4, all of the above cases satisfy that

$$(h_1\odot_F (h_2\odot_A h_3))(\omega) = ((h_1\odot_F h_2)\odot_A (h_1\odot_F h_3))(\omega).$$

(iv) If $A(u,v) = \lambda$, then $F(y, A(u,v)) = y$. Since F is left distributive over A, it holds that

$$(h_1\odot_F (h_2\odot_A h_3))(\omega) \leq ((h_1\odot_F h_2)\odot_A (h_1\odot_F h_3))(\omega).$$

Next, we verify that $(h_1\odot_F (h_2\odot_A h_3))(\omega) \geq ((h_1\odot_F h_2)\odot_A (h_1\odot_F h_3))(\omega)$.

Firstly, we give the statement Q.

Q. For each $y' \in [\varepsilon,\, 1]$, $(q,t) \in [0,\lambda] \times [\lambda,\, 1]$ or $(q,t) \in [\lambda,\, 1] \times [0,\lambda]$, $p, s \in [0,1]$, if $A(F(p,q), F(s,t)) = A(F(y',q), F(y',t)) = \omega$, then $(h_1\odot_F (h_2\odot_A h_3))(\omega) \geq ((h_1\odot_F h_2)\odot_A (h_1\odot_F h_3))(\omega)$.

Due to $A(F(p,q), F(s,t)) = A(F(y',q),\, F(y',t)) = \omega$, then $p \leq y' \leq s$ or $s \leq y' \leq p$. If not, $y' < p\wedge s$ or $y' > p\vee s$. Assume that $y' < p\wedge s$, there is a contradiction in

$$\begin{aligned}
\omega &= A(F(p,q), F(s,t))\\
&\geq A(F(p\wedge s,q), F(p\wedge s,t))\\
&= F(p\wedge s, A(q,t))\\
&= p\wedge s\\
&> y'\\
&= F(y', A(q,t))\\
&= A(F(y',q), F(y',t))\\
&= \omega.
\end{aligned}$$

Hence, $y' \geq p \wedge s$. Moreover, $y' \leq p \vee s$ can be proven in a similar way. Therefore, $p \leq y' \leq s$ or $s \leq y' \leq p$. One can conclude that $h_1(y') \geq h_1(p) \wedge h_1(s)$ since h_1 is convex, and then $h_1(y') \wedge h_2(q) \wedge h_3(t) \geq h_1(p) \wedge h_2(q) \wedge h_1(s) \wedge h_3(t)$. In view of the above discussion, Q holds.

In light of the statement Q, one can obtain that

$$\begin{aligned}
&((h_1 \odot_F h_2) \odot_A (h_1 \odot_F h_3))(\omega) \\
&= \bigvee_{A(F(p,q),\, F(s,t))=\omega} h_1(p) \wedge h_2(q) \wedge h_1(s) \wedge h_3(t) \\
&\leq \bigvee_{A(F(y',q),F(y',t))=\omega} h_1(y') \wedge h_2(q) \wedge h_3(t) \\
&= \bigvee_{F(y',\, A(q,t))=\omega} h_1(y') \wedge h_2(q) \wedge h_3(t) \\
&= \bigvee_{F(y,\, A(u,v))=\omega} h_1(y) \wedge h_2(u) \wedge h_3(v) \\
&= (h_1 \odot_F (h_2 \odot_A h_3))(\omega).
\end{aligned}$$

Finally, Cases (1), (2), (3) and (4) satisfy that $(h_1 \odot_F (h_2 \odot_A h_3))(\omega) = ((h_1 \odot_F h_2) \odot_A (h_1 \odot_F h_3))(\omega)$ for any $\omega \in [0,1]$. □

Lemma 1 [39] Suppose $0 < \lambda < e < \varepsilon < \eta = 1, F \in \mathscr{F}_{\varepsilon,\eta}, A \in (\mathbb{U}^S_{e,\lambda})_{\min}$. F satisfies left distributivity over A, if A is idempotent and F is constructed as Eq. (6) and B can be represented as

$$B(u,v) = \lambda + (e-\lambda)\mathbb{S}'\left(\frac{u-\lambda}{e-\lambda}, \frac{v-\lambda}{e-\lambda}\right),$$

where $u, v \in [\lambda, e)$ and $\mathbb{S}'$ is a positive right semi-t-conorm.

Lemma 1 gives the sufficient condition that F satisfies left distributivity over A. Furthermore, combined with Lemma 1 and Theorem 12, a sufficient condition about left distributivity between extended semi-t-operator $\odot_F$ and extended S-uninorm $\odot_A$ can be shown.

Proposition 13 *Suppose* $0 < \lambda < e < \varepsilon < \eta = 1, F \in \mathscr{F}_{\varepsilon,\eta}$ *is continuous,* $A \in (\mathbb{U}^S_{e,\lambda})_{\min}$. *For any* $h_1 \in \mathscr{F}_C$ *and* $h_2, h_3 \in \mathscr{F}$, *then it holds that*

$$h_1 \odot_F (h_2 \odot_A h_3) = (h_1 \odot_F h_2) \odot_A (h_1 \odot_F h_3),$$

if A *is idempotent and* F *is constructed as Eq. (6) and* B *can be represented as*

$$B(u,v) = \lambda + (e-\lambda)\mathbb{S}'\left(\frac{u-\lambda}{e-\lambda}, \frac{v-\lambda}{e-\lambda}\right),$$

where $u, v \in [\lambda, e)$ *and* $\mathbb{S}'$ *is positive right semi-t-conorm.*

Proof It can be immediately verified by Theorem 12 and Lemma 1. □

Similarly, when semi-t-operator F satisfies the right distributive law for S-uninorm A, then the extended semi-t-operator $\odot_F$ satisfies distributive law over extended S-uninorm $\odot_A$.

Proposition 14 *[39] Suppose $0 < \lambda < e < \eta < \varepsilon = 1, F \in \mathscr{F}_{\varepsilon,\eta}, A \in (\mathbb{U}^S_{e,\lambda})_{\min}$. If the right distributivity between F and A is satisfied, then A is an idempotent S-uninorm and F has the following form:*

$$F(u,v) = \begin{cases} \lambda\mathscr{S}\left(\frac{u}{\lambda},\frac{v}{\lambda}\right), & \text{if } u,v \in [0,\lambda]; \\ A_1(u,v), & \text{if } u \in (\lambda,e), v \in [0,\lambda); \\ B(u,v), & \text{if } u,v \in [\lambda,e); \\ C(u,v), & \text{if } u \in [e,b), v \in [0,e); \\ D(u,v), & \text{if } u \in [0,\lambda), v \in [e,\eta); \\ \eta, & \text{if } u \in [\lambda,\eta], v \in [e,\eta] \text{ or } u \in [\eta,1], v \in [0,\eta]; \\ v, & \text{if } u \in [0,1], v \in [\eta,1]; \\ \max(u,v), & \text{otherwise}, \end{cases} \tag{7}$$

where $\mathscr{S}$ is a semi-t-conorm, $A_1 : (\lambda,e)\times[0,\lambda) \longrightarrow [\lambda,\eta]$, $B : [\lambda,e)\times[\lambda,e) \longrightarrow [\lambda,\eta]$, $C : [e,\eta)\times[0,e) \longrightarrow [e,\eta]$, $D : [0,\lambda)\times[e,\eta) \longrightarrow [e,\eta]$ have associativity, the property of monotonically increasing and the respective boundary condition $A_1(u,0) = u$ $(\forall u \in (\lambda,e))$, $B(\lambda,u) = u$ $(\forall u \in [\lambda,e))$, $C(u,0) = u$ $(\forall u \in [e,\eta))$, $D(0,u) = u$ $(\forall u \in [e,\eta))$ (A visual image is shown in Figure. 3*)*

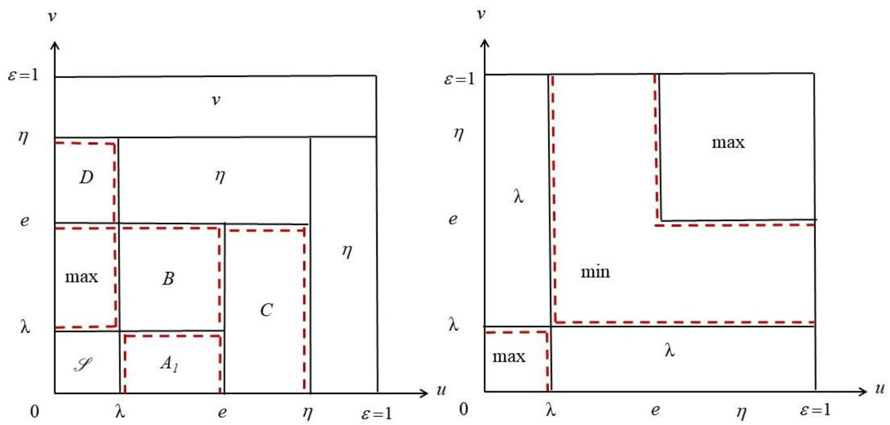

Figure 3 The characterizations of F (left) and A (right) in Proposition 14 [39]

Theorem 15 *Suppose $0 < \lambda < e < \eta < \varepsilon = 1$, $F \in \mathscr{F}_{\varepsilon,\eta}$ is continuous, $A \in (\mathbb{U}^S_{e,\lambda})_{\min}$ and F satisfies the right distributivity over A. For any $h_1 \in \mathscr{F}_C$ and $h_2, h_3 \in \mathscr{F}$, then it holds that*

$$(h_2 \odot_A h_3) \odot_F h_1 = (h_2 \odot_F h_1) \odot_A (h_3 \odot_F h_1).$$

Proof It can be verified in an analogical method as Theorem 12. □

Lemma 2 [39] Suppose $0 < \lambda < e < \eta < \varepsilon = 1, F \in \mathscr{F}_{\varepsilon,\eta}, A \in (\mathbb{U}^S_{e,\lambda})_{\min}$. F satisfies right distributivity over A, if A is idempotent and F is constructed as Eq. (7) and B can be represented as

$$B(u,v) = \lambda + (e-\lambda)\mathbb{S}'\left(\frac{u-\lambda}{e-\lambda}, \frac{v-\lambda}{e-\lambda}\right),$$

where $u, v \in [\lambda, e)$ and $\mathbb{S}'$ is a positive left semi-t-conorm.

Furthermore, a sufficient condition of right distributivity between extended semi-t-operator $\odot_F$ and S-uninorm $\odot_A$ can be shown below.

Proposition 16 *Suppose* $0 < \lambda < e < \eta < \varepsilon = 1, F \in \mathscr{F}_{\varepsilon,\eta}$ *is continuous,* $A \in (\mathbb{U}^S_{e,\lambda})_{\min}$. *For any* $h_1 \in \mathscr{F}_C, h_2, h_3 \in \mathscr{F}$, *then*

$$(h_2 \odot_A h_3) \odot_F h_1 = (h_2 \odot_F h_1) \odot_A (h_3 \odot_F h_1)$$

holds, if A is idempotent and F is constructed as (7) where B can be represented as:

$$B(u,v) = \lambda + (e-\lambda)\mathbb{S}'\left(\frac{u-\lambda}{e-\lambda}, \frac{v-\lambda}{e-\lambda}\right),$$

where $u, v \in [\lambda, e)$ *and* $\mathbb{S}'$ *is positive left semi-t-conorm.*

Proof It follows immediately from Lemma 2 and Theorem 15. □

4.2 The situation of $0 = \min(\varepsilon, \eta) < e < \max(\varepsilon, \eta) = 1$

Lemma 3 [39] Suppose $0 = \varepsilon \leqslant \lambda < e < \eta = 1, F \in \mathscr{F}_{\varepsilon,\eta}, A \in (\mathbb{U}^S_{e,\lambda})_{\min}$. Then the following statements hold.

(1) F satisfies the left distributivity over $A \iff A$ is idempotent.
(2) F is right distributive over A.

Theorem 17 *Suppose* $0 = \varepsilon \leqslant \lambda < e < \eta = 1, F \in \mathscr{F}_{\varepsilon,\eta}, A \in (\mathbb{U}^S_{e,\lambda})_{\min}$. *If F satisfies left distributivity over A, then for all* $h_1 \in \mathscr{F}_C$ *and* $h_2, h_3 \in \mathscr{F}$, *it holds that*

$$h_1 \odot_F (h_2 \odot_A h_3) = (h_1 \odot_F h_2) \odot_A (h_1 \odot_F h_3).$$

Proof Consider that $\varepsilon = 0$ and $\eta = 1$. According to Proposition 3, $F(u,v) = u$ for all $u, v \in [0,1]$. Meanwhile, since F is left distributive over A, we obtain that A is idempotent by Lemma 3, that is, $A(t,t) = t$ for any $t \in [0,1]$. Hence, it holds that

$$\begin{aligned} &(h_1 \odot_F (h_2 \odot_A h_3))(\omega) \\ &= \bigvee_{F(y,\, A(u,v)) = y = \omega} h_1(y) \wedge h_2(u) \wedge h_3(v) \end{aligned}$$

$$= \bigvee_{u,v\in[0,1]} h_1(\omega) \wedge h_2(u) \wedge h_3(v)$$
$$= h_1(\omega) \wedge \left(\bigvee_{u\in[0,1]} h_2(u)\right) \wedge \left(\bigvee_{v\in[0,1]} h_3(v)\right),$$

and

$$((h_1 \odot_F h_2) \odot_A (h_1 \odot_F h_3))(\omega)$$
$$= \bigvee_{A(F(p,q),F(s,t))=\omega} h_1(p) \wedge h_2(q) \wedge h_1(s) \wedge h_3(t)$$
$$= \bigvee_{A(p,s)=\omega,q,t\in[0,1]} h_1(p) \wedge h_2(q) \wedge h_1(s) \wedge h_3(t)$$
$$= \left(\bigvee_{A(p,s)=\omega} h_1(p) \wedge h_1(s)\right) \wedge \left(\bigvee_{q\in[0,1]} h_2(q)\right) \wedge \left(\bigvee_{t\in[0,1]} h_3(t)\right).$$

To illustrate the left distributivity of $\odot_F$ to $\odot_A$, only $h_1(\omega) = \bigvee_{A(p,s)=\omega} h_1(p) \wedge h_1(s)$ needs to be stated.

On the one hand, it can be obtained that

$$h_1(\omega) = \bigvee_{A(\omega,\omega)=\omega} h_1(\omega) \wedge h_1(\omega) \leq \bigvee_{A(p,s)=\omega} h_1(p) \wedge h_1(s).$$

On the other hand, if $A(p, s) = \omega$, then $p \wedge s \leq \omega \leq p \vee s$. In fact, assume that $p \wedge s > \omega$, then

$$\omega = A(p, s) \geq A(p \wedge s, p \wedge s) = p \wedge s > \omega,$$

which is a contradiction. Similarly, it can be shown that $\omega > p \vee s$ does not hold. Hence, we obtain that $p\wedge s \leq \omega \leq p\vee s$. Since $h_1 \in \mathscr{F}_C$, that is, $h_1(\omega) \geq h_1(p)\wedge h_1(s)$, further, $h_1(\omega) \geq \bigvee_{A(p,s)=\omega} h_1(p) \wedge h_1(s)$.

Therefore, $h_1(\omega) = \bigvee_{A(p,s)=\omega} h_1(p) \wedge h_1(s)$.

In summary, $(h_1 \odot_F (h_2 \odot_A h_3))(\omega) = ((h_1 \odot_F h_2) \odot_A (h_1 \odot_F h_3))(\omega)$ holds for any $\omega \in [0, 1]$. □

Further, we verify that the extended semi-t-operator $\odot_F$ satisfies the right distributive law over extended S-uninorm $\odot_A$ under the condition $\varepsilon = 0, \eta = 1$.

Theorem 18 *Suppose* $0 = \varepsilon \leqslant \lambda < e < \eta = 1, F \in \mathscr{F}_{\varepsilon,\eta}, A \in (\mathbb{U}^S_{e,\lambda})_{\min}$. *Then the right distributivity between* F *and* A *is satisfied and for any* $h_1, h_2, h_3 \in \mathscr{F}$, *it concludes that*

$$(h_2 \odot_A h_3) \odot_F h_1 = (h_2 \odot_F h_1) \odot_A (h_3 \odot_F h_1).$$

Proof According to Proposition 3, $F(u, v) = u$ for any $u, v \in [0, 1]$ when $\varepsilon = 0$ and $\eta = 1$. One concludes that

$$((h_2 \odot_A h_3) \odot_F h_1)(\omega)$$

$$= \bigvee_{F(A(p,q),y)=\omega} h_2(p) \wedge h_3(q) \wedge h_1(y)$$
$$= \left(\bigvee_{A(p,q)=\omega} h_2(p) \wedge h_3(q) \right) \wedge \left(\bigvee_{y\in[0,1]} h_1(y) \right),$$

and

$$((h_2 \odot_F h_1) \odot_A (h_3 \odot_F h_1))(\omega)$$
$$= \bigvee_{A(F(p,q),F(s,t))=\omega} h_2(p) \wedge h_1(q) \wedge h_3(s) \wedge h_1(t)$$
$$= \left(\bigvee_{A(p,s)=\omega} h_2(p) \wedge h_3(s) \right) \wedge \left(\bigvee_{q,t\in[0,1]} (h_1(q) \wedge h_1(t)) \right)$$
$$= \left(\bigvee_{A(p,s)=\omega} h_2(p) \wedge h_3(s) \right) \wedge \left(\bigvee_{y\in[0,1]} h_1(y) \right).$$

Hence, $((h_2 \odot_A h_3) \odot_F h_1)(\omega) = ((h_2 \odot_F h_1) \odot_A (h_3 \odot_F h_1))(\omega)$ holds for all $\omega \in [0,1]$. □

Notice that the right distributivity between extended semi-t-operator $\odot_F$ and extended S-uninorm $\odot_A$ does not need the condition of $f \in \mathscr{F}_C$.

Further, the similar conclusions can be obtained under condition $\varepsilon = 1, \eta = 0$.

Lemma 4 [39] Suppose $0 = \eta \leqslant \lambda < e < \varepsilon = 1, F \in \mathscr{F}_{\varepsilon,\eta}, A \in (\mathbb{U}^S_{e,\lambda})_{\min}$. Then the following statements hold.

(1) F satisfies left distributive law over A.
(2) F satisfies right distributive law over $A \iff A$ is idempotent.

Theorem 19 *Suppose $0 = \eta \leqslant \lambda < e < \varepsilon = 1, F \in \mathscr{F}_{\varepsilon,\eta}, A \in (\mathbb{U}^S_{e,\lambda})_{\min}$. Then the following statements hold.*

(1) For any $h_1, h_2, h_3 \in \mathscr{F}$, it holds that

$$h_1 \odot_F (h_2 \odot_A h_3) = (h_1 \odot_F h_2) \odot_A (h_1 \odot_F h_3).$$

(2) If F is right distributivity over A, then for any $h_1 \in \mathscr{F}_C$, $h_2, h_3 \in \mathscr{F}$, we have that

$$(h_2 \odot_A h_3) \odot_F h_1 = (h_2 \odot_F h_1) \odot_A (h_3 \odot_F h_1).$$

Proof It can be verified in an analogical method as Theorems 17 and 18. □

5 Distributivity about extended semi-t-operator over extended S-uninorm on fuzzy truth value

In what follows, under the condition that semi-t-operator F is distributive and conditionally distributive over S-uninorm A [39], the distributivity between extended semi-t-operator $\odot_F$ and S-uninorm $\odot_A$ is explored on the situation of $e < \min(\varepsilon, \eta) < \max(\varepsilon, \eta) = 1$ and $0 = \min(\varepsilon, \eta) < e < \max(\varepsilon, \eta) = 1$, respectively.

5.1 The situation of $e < \min(\varepsilon, \eta) < \max(\varepsilon, \eta) = 1$

Next, the distributivity between extended semi-t-operator $\odot_F$ and S-uninorm $\odot_A$ is discussed when $e < \varepsilon < \eta = 1$ and $e < \eta < \varepsilon = 1$.

Proposition 20 *[39] Suppose* $0 < \lambda < e < \varepsilon < \eta = 1, F \in \mathscr{F}_{\varepsilon,\eta}, A \in (\mathbb{U}^S_{e,\lambda})_{\min}$. *If the distributive law between* F *and* A *is satisfied, then* A *is an idempotent S-uninorm and* F *can be characterized as:*

$$F(u,v) = \begin{cases} \lambda \mathscr{S}\left(\frac{u}{\lambda}, \frac{v}{\lambda}\right), & \text{if } u.v \in [0,\lambda]; \\ B(u,v), & \text{if } u,v \in [\lambda, e); \\ C(u,v), & \text{if } u \in [0,\lambda), v \in [e,\varepsilon); \\ D(u,v), & \text{if } u \in [e,\varepsilon), v \in [0,\lambda); \\ \varepsilon, & \text{if } u \in [\lambda,\varepsilon], v \in [e,\varepsilon] \text{ or } u \in [e,\varepsilon], v \in [\lambda,\varepsilon] \text{ or } u \in [0,\varepsilon], v \in [\varepsilon,1]; \\ u, & \text{if } u \in [\varepsilon,1], v \in [0,1]; \\ \max(u,v), & \text{otherwise}, \end{cases} \tag{8}$$

where $\mathscr{S}$ *is a semi-t-conorm,* $B : [\lambda, e) \times [\lambda, e) \longrightarrow [\lambda, \varepsilon]$, $C : [0, \lambda) \times [e, \varepsilon) \longrightarrow [e, \varepsilon]$, $D : [e, \varepsilon) \times [0, \lambda) \longrightarrow [e, \varepsilon]$ *have associativity, the property of monotonically increasing and the respective boundary condition* $B(u, \lambda) = u$ *and* $B(\lambda, v) = v$ $(\forall u, v \in [\lambda, e))$, $C(0, v) = v$ $(\forall v \in [e, \varepsilon))$, $D(u, 0) = u(\forall u \in [e, \varepsilon))$ *(A visual image is shown in* Figure 4*).*

Lemma 5 [39] Suppose $0 < \lambda < e < \varepsilon < \eta = 1, F \in \mathscr{F}_{\varepsilon,\eta}, A \in (\mathbb{U}^S_{e,\lambda})_{\min}$. F satisfies distributivity over A, if A is idempotent and F is constructed as Eq. (8) and B can be represented as

$$B(u,v) = \lambda + (e - \lambda)\mathscr{S}'\left(\frac{u-\lambda}{e-\lambda}, \frac{v-\lambda}{e-\lambda}\right),$$

where $u, v \in [\lambda, e)$ and $\mathscr{S}'$ is a positive semi-t-conorm.

In light of Proposition 20, the following conclusions can be obtained similarly as Theorem 12.

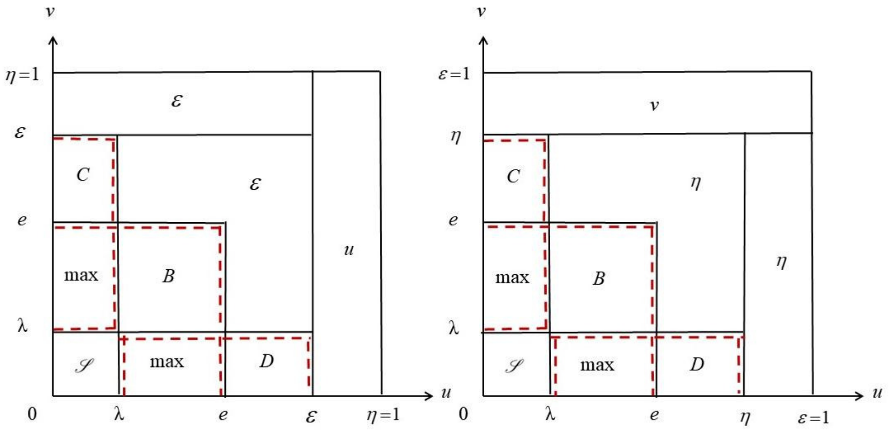

Figure 4 The characterizations of F in Proposition 20 (left) and Proposition 23 (right) [39]

Theorem 21 *Suppose* $0 < \lambda < e < \varepsilon < \eta = 1$, $F \in \mathscr{F}_{\varepsilon,\eta}$ *is continuous,* $A \in (\mathbb{U}^S_{e,\lambda})_{\min}$ *and* F *satisfies the distributivity over* A*. For any* $h_1 \in \mathscr{F}_C$*,* $h_2, h_3 \in \mathscr{F}$*, then we have that*

$$h_1 \odot_F (h_2 \odot_A h_3) = (h_1 \odot_F h_2) \odot_A (h_1 \odot_F h_3)\,,$$

and

$$(h_2 \odot_A h_3) \odot_F h_1 = (h_2 \odot_F h_1) \odot_A (h_3 \odot_F h_1)\,.$$

Proof It can be verified in an analogical method as Theorem 12. □

Further, combined with Lemma 5, the sufficient condition that $\odot_F$ is distributive over $\odot_A$ can be given below.

Proposition 22 *Suppose* $0 < \lambda < e < \varepsilon < \eta = 1, F \in \mathscr{F}_{\varepsilon,\eta}$ *is continuous,* $A \in (\mathbb{U}^S_{e,\lambda})_{\min}$*. For any* $h_1 \in \mathscr{F}_C$ *and* $h_2, h_3 \in \mathscr{F}$*, then*

$$h_1 \odot_F (h_2 \odot_A h_3) = (h_1 \odot_F h_2) \odot_A (h_1 \odot_F h_3)$$

and

$$(h_2 \odot_A h_3) \odot_F h_1 = (h_2 \odot_F h_1) \odot_A (h_3 \odot_F h_1)$$

hold, if A *is idempotent,* F *is constucted as (8) and* B *can be represented as:*

$$B(u,v) = \lambda + (e-\lambda)\mathscr{S}'\left(\frac{u-\lambda}{e-\lambda}, \frac{v-\lambda}{e-\lambda}\right),$$

where $u, v \in [\lambda, e)$ *and* $\mathscr{S}'$ *is positive semi-t-conorm.*

Proof It follows from Lemma 5 that F satisfies the distributive law for A. Further, the distributivity between $\odot_F$ and $\odot_A$ can be proved similar to Theorem 12. □

The following conclusions hold when $e < \eta < \varepsilon = 1$.

Proposition 23 *[39] Suppose* $0 < \lambda < e < \eta < \varepsilon = 1, F \in \mathscr{F}_{\varepsilon,\eta}, A \in (\mathbb{U}^S_{e,\lambda})_{\min}$. *If the distributivity between* F *and* A *is satisfied, then* A *is idempotent S-uninorm and* F *can be characterized as:*

$$F(u,v) = \begin{cases} \lambda\mathscr{S}\left(\frac{u}{\lambda},\frac{v}{\lambda}\right), & \text{if } u,v \in [0,\lambda]; \\ B(u,v), & \text{if } u,v \in [\lambda,e); \\ C(u,v), & \text{if } u \in [0,\lambda), v \in [e,\eta); \\ D(u,v), & \text{if } u \in [e,\eta), v \in [0,\lambda); \\ \eta, & \text{if } u \in [\lambda,\eta], v \in [e,\eta] \text{ or } u \in [e,\eta], v \in [\lambda,\eta] \text{ or } u \in [\eta,1], v \in [0,\eta]; \\ v, & \text{if } u \in [0,1], v \in [\eta,1]; \\ \max(u,v), & \text{otherwise}, \end{cases} \tag{9}$$

where $\mathscr{S}$ *is a semi-t-conorm,* $B : [\lambda,e) \times [\lambda,e) \longrightarrow [\lambda,\eta]$, $C : [0,\lambda) \times [e,\eta) \longrightarrow [e,\eta]$, $D : [e,\eta) \times [0,\lambda) \longrightarrow [e,\eta]$ *have associativity, the property of monotonically increasing and the respective boundary condition* $B(u,\lambda) = u$ *and* $B(\lambda,v) = v$ $(\forall u,v \in [\lambda,e))$, $C(0,v) = v$ $(\forall v \in [e,\eta))$, $D(u,0) = u$ $(\forall u \in [e,\eta))$ *(A visual image is shown in* Figure. 4*).*

Lemma 6 [39] Suppose $0 < \lambda < e < \eta < \varepsilon = 1, F \in \mathscr{F}_{\varepsilon,\eta}, A \in (\mathbb{U}^S_{e,\lambda})_{\min}$. F satisfies distributivity over A, if A is idempotent and F is constructed as Eq. (9) and B can be represented as

$$B(u,v) = \lambda + (e-\lambda)\mathscr{S}'\left(\frac{u-\lambda}{e-\lambda}, \frac{v-\lambda}{e-\lambda}\right),$$

where $u,v \in [\lambda,e)$ and $\mathscr{S}'$ is a positive semi-t-conorm.

According to Proposition 23, we can obtain the following conclusions similarly as Theorem 12.

Theorem 24 *Suppose* $0 < \lambda < e < \eta < \varepsilon = 1$, $F \in \mathscr{F}_{\varepsilon,\eta}$ *is continuous,* $A \in (\mathbb{U}^S_{e,\lambda})_{\min}$ *and* F *satisfies the distributivity over* A. *For any* $h_1 \in \mathscr{F}_C$ *and* $h_2, h_3 \in \mathscr{F}$, *it holds that*

$$h_1 \odot_F (h_2 \odot_A h_3) = (h_1 \odot_F h_2) \odot_A (h_1 \odot_F h_3),$$

and

$$(h_2 \odot_A h_3) \odot_F h_1 = (h_2 \odot_F h_1) \odot_A (h_3 \odot_F h_1).$$

Proof It can be verified in an analogical method as Theorem 15. □

Further, combined with Lemma 6, the sufficient condition of the distributivity between $\odot_F$ and $\odot_A$ can be given below.

Proposition 25 *Suppose* $0 < \lambda < e < \eta < \varepsilon = 1, F \in \mathscr{F}_{\varepsilon,\eta}$ *is continuous,* $A \in (\mathbb{U}_{e,\lambda}^{S})_{\min}$. *For any* $h_1 \in \mathscr{F}_C$ *and* $h_2, h_3 \in \mathscr{F}$, *then we conclude that*

$$h_1 \odot_F (h_2 \odot_A h_3) = (h_1 \odot_F h_2) \odot_A (h_1 \odot_F h_3)$$

and

$$(h_2 \odot_A h_3) \odot_F h_1 = (h_2 \odot_F h_1) \odot_A (h_3 \odot_F h_1)$$

hold, if A *is idempotent,* F *is constructed as (9) and* B *can be represented as:*

$$B(u, v) = \lambda + (e - \lambda)\mathscr{S}'\left(\frac{u - \lambda}{e - \lambda}, \frac{v - \lambda}{e - \lambda}\right),$$

where $u, v \in [\lambda, e)$ *and* $\mathscr{S}'$ *is positive semi-t-conorm.*

Proof It is a direct consequence according to Lemma 6 and Theorem 24. □

Lemma 7 [39] Suppose $0 < \lambda < e < \min(\varepsilon, \eta) < \max(\varepsilon, \eta) = 1, F \in \mathscr{F}_{\varepsilon,\eta}, A \in (\mathbb{U}_{e,\lambda}^{S})_{\min}$. Then the following statements hold.

(1) F satisfies conditional distributivity for A from the left $\Longleftrightarrow$ F satisfies left distributivity for A when $e < \varepsilon < \eta = 1$.
(2) F satisfies conditional distributivity for A from the right $\Longleftrightarrow$ F satisfies right distributivity for A when $e < \eta < \varepsilon = 1$.

Next, based on the condition that F has conditional distributivity over A form left (resp. right), the left (resp. right) distributive law between $\odot_F$ and $\odot_A$ is given in the following theorems.

Theorem 26 *Suppose* $0 < \lambda < e < \varepsilon < \eta = 1$, $F \in \mathscr{F}_{\varepsilon,\eta}$ *is continuous,* $A \in (\mathbb{U}_{e,\lambda}^{S})_{\min}$ *and* F *satisfies the conditional distributivity over* A *from the left. For any* $h_1 \in \mathscr{F}_C$ *and* $h_2, h_3 \in \mathscr{F}$, *it holds that*

$$h_1 \odot_F (h_2 \odot_A h_3) = (h_1 \odot_F h_2) \odot_A (h_1 \odot_F h_3).$$

Proof In light of Theorem 12 and Lemma 7, it can be verified immediately. □

Theorem 27 *Suppose* $0 < \lambda < e < \eta < \varepsilon = 1$, $F \in \mathscr{F}_{\varepsilon,\eta}$ *is continuous,* $A \in (\mathbb{U}_{e,\lambda}^{S})_{\min}$ *and* F *satisfies the conditional distributivity over* A *from the right. For any* $h_1 \in \mathscr{F}_C$ *and* $h_2, h_3 \in \mathscr{F}$, *then we have that*

$$(h_2 \odot_A h_3) \odot_F h_1 = (h_2 \odot_F h_1) \odot_A (h_3 \odot_F h_1).$$

Proof In light of Theorem 15 and Lemma 7, it can be verified immediately. □

5.2 The situation of $0 = \min(\varepsilon, \eta) < e < \max(\varepsilon, \eta) = 1$

Lemma 8 [39] Suppose $0 = \min(\varepsilon, \eta) \leq \lambda < e < \max(\varepsilon, \eta) = 1, F \in \mathscr{F}_{\varepsilon,\eta}, A \in (\mathbb{U}^S_{e,\lambda})_{\min}$. Then the following statements hold.

(1) F satisfies conditional distributivity over $A \Longleftrightarrow F$ satisfies distributivity over A.
(2) F satisfies conditional distributivity over $A \Longleftrightarrow F$ satisfies left distributive over A when $\varepsilon = 0$ and $\eta = 1$.
(3) F satisfies conditional distributivity over $A \Longleftrightarrow F$ satisfies right distributivity over A when $\varepsilon = 1$ and $\eta = 0$.

Theorem 28 *Suppose $0 = \varepsilon \leq \lambda < e < \eta = 1, F \in \mathscr{F}_{\varepsilon,\eta},\ A \in (\mathbb{U}^S_{e,\lambda})_{\min}$ and F satisfies left distributivity (resp. conditionally distributivity or distributivity) for A. For any $h_1 \in \mathscr{F}_C$ and $h_2, h_3 \in \mathscr{F}$, then we can obtain that*

$$h_1 \odot_F (h_2 \odot_A h_3) = (h_1 \odot_F h_2) \odot_A (h_1 \odot_F h_3),$$

and

$$(h_2 \odot_A h_3) \odot_F h_1 = (h_2 \odot_F h_1) \odot_A (h_3 \odot_F h_1).$$

Proof In light of Theorem 17 and Lemma 8, it can be verified immediately. □

Theorem 29 *Suppose $0 = \eta \leq \lambda < e < \varepsilon = 1, F \in \mathscr{F}_{\varepsilon,\eta},\ A \in (\mathbb{U}^S_{e,\lambda})_{\min}$ and F satisfies right distributivity (resp. conditional distributivity or distributivity) for A. For any $h_1 \in \mathscr{F}_C$ and $h_2, h_3 \in \mathscr{F}$, then we get that*

$$h_1 \odot_F (h_2 \odot_A h_3) = (h_1 \odot_F h_2) \odot_A (h_1 \odot_F h_3),$$

and

$$(h_2 \odot_A h_3) \odot_F h_1 = (h_2 \odot_F h_1) \odot_A (h_3 \odot_F h_1).$$

Proof In light of Theorem 19 and Lemma 8, it can be verified immediately. □

6 Conclusion

This paper mainly researches the distributive law between extended semi-t-operator and S-uninorm on fuzzy truth value where the underlying uninorm of S-uninorm is $\mathcal{U}_{\min}$, which further generalizes the corresponding results of [38, 39] to fuzzy truth value. The core contents of this paper can be summarized as follows:

(1) Applying Zadeh's extension principle to give the extended semi-t-operator and extended S-uninorm, and then some results of extended semi-t-operator are shown, such as the operations of extended semi-t-operator and special fuzzy truth values $\underline{1}$, $\overline{1}$, commutativity, associativity and so on.

(2) Based on the two situations that $e < \min(\varepsilon, \eta) < \max(\varepsilon, \eta) = 1$ and $0 = \min(\varepsilon, \eta) < e < \max(\varepsilon, \eta) = 1$, the sufficient condition of left (resp. right) distributive law between extended semi-t-operator and extended S-uninorm is investigated. Later on, the distributive law between extended semi-t-operator and extended S-uninorm is studied when the semi-t-operator satisfies distributivity or conditional distributivity for S-uninorm.

In future work, a meaningful question is to characterize the distributivity of extended semi-t-operator over extended S-uninorm when the underlying uninorm of S-uninorm is continuous in $(0,1)^2$, idempotent and locally internal [42, 45].

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Compliance with Ethical Standards

Conflict of interest. The authors declare that they have no conflict of interest.
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Author contributions

Bin Yang: Analysis, Preparation, Writing-review & editing. **Wei Li:** Methodology, Writing-review & editing. **Yuanhao Liu:** Writing-original draft. **Jing Xu:** Methodology.

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