

# FINANCIAL EQUILIBRIA IN THE SEMIMARTINGALE SETTING: COMPLETE MARKETS AND MARKETS WITH WITHDRAWAL CONSTRAINTS

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**ABSTRACT.** We establish existence of stochastic financial equilibria on filtered spaces more general than the ones generated by finite-dimensional Brownian motions. These equilibria are expressed in real terms and span complete markets or markets with withdrawal constraints. We deal with random endowment density streams which admit jumps and general time-dependent utility functions on which only regularity assumptions are imposed. As a side-product of the proof of the main result, we establish a novel characterization of semimartingale functions.

## 1. INTRODUCTION

*Existing results and history of the problem.* The existence of financial equilibria in continuous-time financial markets is one of the central problems in financial theory and mathematical finance. Unlike the problems of utility maximization and asset pricing where the price dynamics are given, the equilibrium problem is concerned with the origin of security prices themselves. More precisely, our goal is to construct a stochastic market with the property that when the price-taking agents act rationally, supply equals demand. Of course, there are many ways to interpret the previous sentence, even in the setting of continuous-time stochastic finance - let alone broader financial theory or economics as a whole. We are, therefore, really talking about a whole class of problems.

Before delving into the specifics of our formulation, let us briefly touch upon the history of the problem. Given the amount of research published on the various facets of the financial equilibrium, we can only mention a tiny fraction of the work leading directly to the present paper. Many seminal contributions not directly related to our research are left out. The notion of competitive equilibrium prices as an expression of the basic idea that the laws of supply and demand determine prices was introduced by Leon Walras (see [Wal74]) 130 years ago. Rigorous mathematical theory starts with [AD54]. Continuous-time stochastic models have been investigated by [DH85]

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and [Duf86], among many others. The direct predecessor of this paper is the work of Karatzas, Lakner, Lehoczky and Shreve in [KLLS91], [KLS90] and [KLS91]. A convenient exposition of the results of these papers can be found in Chapter 4. of [KS98]. Recently, existence of an equilibrium functional when utilities exhibit intertemporal substitution properties has been established in [BR01].

*Our contributions.* The motive leading our research was to investigate how the relaxation of the assumption that the filtration is generated by a Brownian motion affects the existence theory for the financial equilibrium, and how stringent conditions on the primitives (utilities, endowments, filtration) one needs to assume in this case. We were particularly keen to impose minimal conditions on utility functions and to allow endowment density processes to admit jumps. As we are primarily concerned with the *existence* of an equilibrium market, we stress that we have not pursued in any detail the questions of uniqueness or the financial consequences of our setup. We leave this interesting line of research for the future, and direct the reader to [Dan93] and [DP92]. In the following paragraphs we describe several directions in which this work extends existing theory.

First, we start from a right-continuous and complete filtration which we *do not* require to be generated by a Brownian motion. Consequently, we look for the price processes in the set of all finite-dimensional semimartingales, thus allowing for the equilibrium prices with jumps. The conditions we impose on the filtration are directly related with the possibility of obtaining a *finite* number of assets spanning all uncertainty. In this way, virtually any complete arbitrage-free market known in the financial literature can arise as an equilibrium in our setting.

Second, we introduce a simple constraint in our model by limiting the amounts the agents can withdraw from the trading account in order to finance a consumption plan. This constraint is phrased in terms of a withdrawal-cap process, which we allow to take infinite values - effectively including the possibility of a fully complete market, with no withdrawal cap whatsoever.

Third, we relax regularity requirements imposed on the utility functions. While these are still stronger than the typical conditions found in the utility-maximization literature, we show that one can develop the theory with assumptions less stringent than, e.g. those in Chapter 4., [KS98]. We also deal with utility functionals which are not necessarily Mackey-continuous due to unboundedness of the utility functions in the neighborhood of zero. Moreover, there is no need for fine growth conditions such as *asymptotic elasticity* (see [KS99]) in our setting. A principal feature of our model - jumps in the endowment density processes - warrants the use and development of

tools from the general theory of stochastic processes. It is in this spirit that we provide a novel characterization of semimartingale functions (the functions of both time and space arguments, that yield semimartingales when applied to semimartingales). Finally, a result due to Mémin and Shiryaev ([MS79]) is used as the most important ingredient in establishing a sufficient condition on a positive semimartingale for the local martingale part in its multiplicative decomposition to be a true martingale.

Another feature in which this paper differs from the classical work (e.g. [KLS90], [KLLS91]) is in that we do not introduce the representative agent's utility function (which is impossible due to withdrawal constraints). Instead we use Negishi's approach (see [Neg60]) in the version described in [MCZ91]. This way the proof the existence of a financial equilibrium is divided into two steps. In the first step we establish the existence of an equilibrium pricing functional (an *abstract equilibrium*). Next, we implement this pricing functional through a stochastic market consisting of a finite number of semimartingale-modeled assets.

*Organization of the paper and some remarks on the notation.* After the Introduction, in Section 2 we describe the model, state the assumptions on its ingredients and pose the central problem of this work. Section 3 introduces an abstract setup and establishes the existence of a financial equilibrium there. In Section 4, we transform the abstract equilibrium into a stochastic equilibrium as defined in Section 2. Finally, in Appendix A we develop the semimartingale results used in Section 4: characterization of semimartingale functions, and regularity of multiplicative decompositions. Apart from being indispensable for the main result of our work, we hope they will be of independent interest, as well.

Throughout this paper, all stochastic processes will be defined on the time horizon  $[0, T]$ , where  $T$  is a positive constant. To relieve the notation, the stochastic process  $(X_t)_{t \in [0, T]}$  will be simply denoted by  $X$ , and its left-limit process  $(X_{t-})_{t \in [0, T]}$ , by  $X_-$ . Unless specified otherwise, (in)equalities between càdlàg processes will be understood pointwise, modulo indistinguishability, i.e.,  $X \leq Y$  will mean  $X_t \leq Y_t$ , for all  $t \in [0, T]$ , a.s. Finally, we use both notations “ $X(t)$ ” and “ $X_t$ ” interchangeably, the choice depending on typographical circumstances.

## 2. THE MODEL

*The information structure.* We consider a stochastic economy on a finite time horizon  $[0, T]$ . The uncertainty reveals itself gradually and is modeled by a right-continuous and complete filtration  $(\mathcal{F}_t)_{t \in [0, T]}$  on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , where we assume that  $\mathcal{F}_0 = \{\emptyset, \Omega\} \bmod \mathbb{P}$  and  $\mathcal{F} = \mathcal{F}_T$ . In order for the finite-dimensional stochastic

process spanning all the uncertainty to exist, the size of the filtration  $(\mathcal{F}_t)_{t \in [0, T]}$  must be restricted:

**Definition 2.1.** A filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$ , with  $(\mathcal{F}_t)_{t \in [0, T]}$  satisfying the usual conditions, is said to have the **finite representation property** if for any probability  $\mathbb{Q}$ , equivalent to  $\mathbb{P}$ , there exist a finite number  $n$  of  $\mathbb{Q}$ -martingales  $Y^1, \dots, Y^n$  such that

- (1)  $Y^i$  and  $Y^j$  are orthogonal for  $i \neq j$ , i.e., the quadratic covariation  $[Y^i, Y^j]_t$  vanishes for all  $t \in [0, T]$ , a.s.
- (2) for every bounded  $\mathbb{Q}$ -martingale  $M$  there exists an  $n$ -dimensional predictable,  $(Y^1, \dots, Y^n)$ -integrable process  $(H^1, \dots, H^n)$  such that

$$M_t = \mathbb{E}^{\mathbb{Q}}[M_T] + \sum_{i=1}^n \int_0^t H_u^i dY_u^i, \text{ for all } t \in [0, T], \text{ a.s.}$$

The smallest such number  $n$  is called the **martingale multiplicity** of  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$ .

**Example 2.2.** The filtered probability spaces with finite representation property include  $n$ -dimensional Brownian filtration, filtrations generated by Poisson processes, filtrations generated by Dritschel-Protter semimartingales (see [PD99]), or combinations of the above.

*Remark 2.3.* The notion of martingale multiplicity and the related notion of the *spanning number of a filtration* have been introduced by Duffie in [Duf86]. Definition 2.1 differs from Duffie's in that we explicitly require the existence of martingales  $(Y^1, \dots, Y^n)$ , for *each* probability measure  $\mathbb{Q} \sim \mathbb{P}$ . In [Duf85], Duffie proves that if we only considered probability measures with  $\frac{d\mathbb{Q}}{d\mathbb{P}} \in \mathbb{L}^\infty$  in Definition 2.1, it would be enough to postulate the existence of the processes  $(Y^1, \dots, Y^n)$  under  $\mathbb{P}$ . It is an open question whether one can achieve such a simplification under less stringent conditions on  $\mathbb{Q}$ .

**Assumption 2.4** (Finite representation property). The filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$  has the finite representation property.

*Remark 2.5.* The finite representation property is used to ensure that the existence of a stochastic implementation of an abstract financial equilibrium with only a finite number of assets. Without this property one could still build a financial equilibrium, but the number of assets needed to span all the uncertainty might be infinite.

*Random endowments.* There are  $d \in \mathbb{N}$  agents in our economy each of whom is receiving a **random endowment** - a bounded and strictly positive income stream, modeled by a semimartingale  $e^i$ . We interpret the random variable  $\int_0^t e_u^i du$  as the

total income received by agent  $i$  on the interval  $[0, t]$ , for  $t < T$ . At time  $t = T$  there is a lump endowment of  $e^i(T)$ . To simplify the notation, we introduce the measure  $\kappa$  on  $[0, T]$  by  $d\kappa_t = dt$  on  $[0, T)$  and  $\kappa(\{T\}) = 1$ . The cumulative random endowment on  $[0, t]$  can now be represented as  $\int_0^t e_t^i d\kappa_t$ , for all  $t \in [0, T]$ .

*Remark 2.6.* The results in this paper can be extended to the case where  $\kappa$  is an optional random measure with  $\kappa(\{T\}) > 0$ , a.s. We do not pursue such an extension, as it would not add to the content in any interesting way.

In order for certain stochastic exponentials to be uniformly integrable martingales, we need to impose a regularity requirement on  $e^i$ ,  $i = 1, \dots, d$ , described in detail in Appendix A.

**Definition 2.7.** For a special semimartingale  $X$ , let  $\mathcal{N}(X) = \langle M, M \rangle_T$ , where  $X = M + A$  is a decomposition of  $X$  into a local martingale  $M$  and a predictable process  $A$  of finite variation, and  $\langle M, M \rangle$  denotes the compensator of the quadratic variation  $[M, M]$ .

*Remark 2.8.* The random variable  $\mathcal{N}(X)$  from Definition 2.7 will usually be used in requirements of the form  $\mathcal{N}(X) \in \mathbb{L}^\infty$ . Existence of the compensator  $\langle M, M \rangle$  and the special semimartingale property of  $X$  are tacitly assumed as parts of such requirements.

The full strength of the following assumption on random endowment processes  $e^i$ ,  $i = 1, \dots, d$ , is needed for the existence of a stochastic equilibrium (Theorem 4.6), and only part 1. for the abstract equilibrium (Theorem 3.7).

**Assumption 2.9** (Regularity of random endowments). For  $i = 1, \dots, d$ ,

- (1)  $e^i$  is an optional process, with  $\varepsilon \leq e^i \leq 1/\varepsilon$ , for some  $\varepsilon > 0$ ,
- (2)  $e^i$  is a (special) semimartingale and  $\mathcal{N}(e^i) \in \mathbb{L}^\infty$ .

**Example 2.10.** Processes  $e^i$  satisfying conditions of Assumption 2.9 include linear combinations of processes of the form  $Y_t = h(t, X_t)$  where  $1/\varepsilon \geq h \geq \varepsilon > 0$  is a  $C^{1,2}$ -function, with  $h_x$ , and  $h_{xx}$  uniformly bounded, and  $X$  is a diffusion process with a bounded diffusion coefficient, or a Lévy process whose jump measure  $\nu$  satisfies  $\int_{\mathbb{R}} x^2 \nu(dx) < \infty$ . Homogeneous and inhomogeneous Poisson processes and non-exploding continuous-time Markov chains are examples of allowable processes  $X$ .

*Utility functions.* Apart from being characterized by the random endowment process, each agent represents her attitude towards risk by a von Neumann-Morgenstern utility function  $U^i$ . Before we list the exact regularity assumptions placed on  $U^i$ , we need the following definition:

**Definition 2.11.** For a continuously differentiable function  $f : [x_1, x_2] \rightarrow \mathbb{R}$  we define the **total convexity norm**  $\|f\| = \|f\|_{[x_1, x_2]}$  by

$$\|f\|_{[x_1, x_2]} = |f(x_1)| + |f'(x_1)| + TV(f'; [x_1, x_2]),$$

where  $TV(f'; [x_1, x_2])$  denotes the total variation of the derivative  $f'$  of  $f$  on  $[x_1, x_2]$ . A function  $f : [0, T] \times [x_1, x_2] \rightarrow \mathbb{R}$ , continuously differentiable in the second variable, is said to be **convexity-Lipschitz** if there exists a constant  $C$  such that, for all  $t, s \in [0, T]$ , we have  $\|f(t, \cdot) - f(s, \cdot)\| \leq C|t - s|$ . A function  $f : [0, T] \times I \rightarrow \mathbb{R}$  (where  $I$  is a subset of  $\mathbb{R}$ ) is called **locally convexity-Lipschitz** if its restriction  $f|_{[0, T] \times [x_1, x_2]}$  is convexity-Lipschitz, for any compact interval  $[x_1, x_2]$ .

*Remark 2.12.* A sufficient condition for a function  $f : [0, T] \times I \rightarrow \mathbb{R}$  to be (locally) convexity-Lipschitz is that  $f(t, \cdot) \in C^2(I)$ , for all  $t \in [0, T]$ , and  $f_{xx}(x, \cdot)$  is Lipschitz, (locally) uniformly in  $x$ .

**Assumption 2.13** (Regularity of utilities). For each  $i = 1, \dots, d$ , the utility function  $U^i : [0, T] \times (0, \infty) \rightarrow \mathbb{R}$  has the following properties

- (1)  $U^i(t, \cdot)$  is strictly concave, continuously differentiable and strictly increasing for each  $t \in [0, T]$ . Moreover, the function  $U(\cdot, x)$ , is bounded for any  $x \in (0, \infty)$ .
- (2) The *inverse-marginal-utility* functions  $I^i : [0, T] \times (0, \infty) \rightarrow (0, \infty)$ ,  $I^i(t, y) = U_x(t, \cdot)^{-1}(t, y)$  are locally convexity-Lipschitz and satisfy

$$\lim_{y \rightarrow \infty} I^i(t, y) = 0, \quad \lim_{y \rightarrow 0} I^i(t, y) = \infty, \quad \text{uniformly in } t \in [0, T]. \quad (2.1)$$

**Example 2.14.** The most important example of a utility function satisfying Assumption 2.13 is so-called *discounted utility*  $U(t, x) = \exp(-\beta t)\hat{U}(x)$ , where  $\beta > 0$  is the impatience factor, and  $\hat{U} \in C^2(\mathbb{R}_+)$  satisfies  $\hat{U}' > 0$  and  $\hat{U}''$  is a strictly negative function of finite variation on compacts. A sufficient (but not necessary) condition for this is  $\hat{U} \in C^3(\mathbb{R}_+)$ . Power utilities  $\hat{U}(x) = x^p/p$ , for  $p \in (-\infty, 1) \setminus \{0\}$  and  $\hat{U}(x) = \log(x)$  belong to this class.

*Remark 2.15.* Unlike the problems of utility maximization (see [KS99], e.g.) where the utility function is only required to be strictly concave and continuously differentiable, existence of financial equilibria requires a higher degree of smoothness (compare to Chapter 4., [KS98], where the existence of three continuous derivatives is postulated in the Brownian setting).

Total utility accrued by an agent whose consumption equals  $c_t(\omega)$  at time  $t \in [0, T]$  in the state of the world  $\omega \in \Omega$ , will be modeled as the aggregate of *instantaneous utilities*  $U^1(t, c_t(\omega))$  in an additive way. More precisely, for each agent  $i = 1, \dots, d$ , we

define the **utility functional**  $\mathbb{U}^i$ , taking values in  $[-\infty, \infty]$ . Its action on an optional process  $c$  is given by  $\mathbb{U}^i(c) \triangleq \mathbb{E}[\int_0^T U^i(t, c(t)) d\kappa_t]$  when  $\mathbb{E}[\int_0^T \min(0, U^i(t, c(t))) d\kappa_t] > -\infty$  and  $\mathbb{U}^i(c) = -\infty$ , otherwise.

*Remark 2.16.* Due to the fact that the final time-point  $t = T$  plays a special role in the definition of the endowment processes  $e^i$ , one would like to be able to redefine the agent's utility quite freely there. Utility functions with virtually no continuity requirements at  $t = T$  are indeed possible to include in our framework, but we decided not to go through with this in order to keep the exposition as simple as possible. It will suffice to note that most of the restrictions involving the time variable placed on the utility functions in Assumption 2.13 are there to ensure that the pricing processes obtained in Theorem 3.7 are semimartingales and not merely optional processes. All of them superfluous at  $t = T$ , since the semimartingale property of a process  $(X_t)_{t \in [0, T]}$  is preserved if we replace  $X_T$  by another  $\mathcal{F}_T$ -measurable random variable.

*Investment and consumption.* The basic premise of equilibrium analysis is that agents engage in trade with each other in order to improve their utilities. To facilitate this exchange, a stock market consisting of a finite number of risky assets  $S$ , and one riskless asset  $B$  is to be set up. In order to have a meaningful mathematical theory, we shall require these processes to be semimartingales with respect to  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$ . Moreover, both the riskless asset  $B$  and its left-limit process  $B_-$  will be required to be strictly positive càdlàg predictable processes of finite variation.

An agent trades in the market by dynamically readjusting the portion of her wealth kept in various risky, or the riskless asset. This is achieved by a choice of a portfolio process  $H$  (in an adequate admissibility class to be specified shortly) with the same number of components as  $S$ . At the same time, the agent will accrue utility by choosing the consumption rate according to an optional consumption process  $c$ . The components of the process  $H$  stand for the number of shares of each risky asset held in the portfolio. The trading is financed by borrowing from (or depositing in) the riskless asset. With that in mind, the equation governing the dynamics of the wealth  $X^{H, c, e}$  of an agent becomes

$$dX_t^{H, c, e} = H_t dS_t + \frac{(X_{t-}^{H, c, e} - H_t S_{t-})}{B_{t-}} dB_t - c(t) d\kappa_t + e(t) d\kappa_t. \quad (2.2)$$

We assume that the agent has no initial wealth, i.e.,  $X_0^{H, c, e} = 0$  (this assumption is in place only to simplify exposition). The net effect of market involvement of the agent is a redistribution of wealth across times and states of the world. The income stream  $e$  (which would have been the only possibility without the market) gets swapped for another stream - the consumption process  $c$ .



There are, invariably, exogenous factors which limit the scope of the market activity. In this paper we deal with one of the simplest such limitations - withdrawal constraints. After having traded for the day (with the net gain of  $H_t dS_t + (X_{t-}^{H,c,e} - H_t S_{t-})/B_{t-} dB_t$ ), and having received the endowment  $e_t d\kappa_t$ , the agent decides to consume  $c_t d\kappa_t$ . If this amount is too large, it is likely to be unavailable for withdrawal from the trading account on a short notice. Therefore, a cap of  $\Gamma^i$  is placed on the amount agent  $i$  can consume at time  $t$ . We assume that  $\Gamma^i, i = 1, \dots, d$  are  $(0, \infty]$ -valued càdlàg adapted process satisfying  $\Gamma^i > e^i$ . We impose no withdrawal restrictions for  $t = T$ , effectively requiring  $\Gamma_T^i = \infty$  a.s. Moreover, an assumption analogous to Assumption 2.9 is placed on  $\Gamma^i$ :

**Assumption 2.17.** For each  $C > 0$ , the stochastic process  $\min(\Gamma^i, C)$  is a semimartingale satisfying  $\mathcal{N}(\min(\Gamma^i, C)) \in \mathbb{L}^\infty$ .

In addition to an abstract, exogenously given withdrawal-cap processes, in the following example we describe several other possibilities.

**Example 2.18.** In all of the following examples, we set  $\Gamma_T^i = \infty$ :

- (1) *Complete markets:*  $\Gamma_t^i = \infty, t \in [0, T]$ .
- (2) *Proportional constraints:* For a constant  $\gamma > 1$ ,  $\Gamma_t^i = \gamma e_t^i, t \in [0, T]$ .
- (3) *Constant overdraft limit:* for  $\delta > 0$  we set  $\Gamma_t^i = e_t^i + \delta, t \in [0, T]$ .

*Market Equilibrium.* Before giving a rigorous definition of an equilibrium market, we introduce the notion of affordability for a consumption process  $c$ . Here we assume that the market structure (in the form of the withdrawal-cap process  $\Gamma$ , a finite-dimensional semimartingale  $S$  (risky assets), and a positive predictable càdlàg process  $B$  of finite variation (riskless asset)) and the random endowment process  $e$  are given.

**Definition 2.19.** An  $(S, B, e, \Gamma)$ -**affordable consumption-investment strategy** is a pair  $(H, c)$  of an  $S$ -integrable predictable **portfolio process**  $H$ , and an optional **consumption process**  $c \geq 0$  such that

- (1) There exists  $a \in \mathbb{R}$  such that  $a + \int_0^t H_u dS_u \geq 0$ , for all  $t \in [0, T]$ , a.s.
- (2) The wealth process  $(X_t)_{t \in [0, T]}$ , as defined in (2.2), satisfies  $X_T \geq 0$ , a.s.
- (3) The consumption process  $c$  satisfies  $c_t \leq \Gamma_t$  for all  $t \in [0, T]$ , a.s.

**Definition 2.20.** A pair  $(S, B)$  of a finite-dimensional semimartingale  $S$  and a positive predictable càdlàg process  $B$  of finite variation is said to form an **equilibrium market** if for each agent  $i = 1, \dots, d$  here exists an  $(S, B, e^i, \Gamma^i)$ -affordable consumption-investment strategy  $(H^i, c^i)$  satisfying the following two conditions:

- (1)  $\sum_i c_t^i = \sum_i e_t^i$  and  $\sum_i H_t^i = 0$ , for all  $t \in [0, T]$ , a.s.



- (2) For each  $i$ ,  $c^i$  maximizes the utility functional  $\mathbb{U}^i(\cdot)$  over all  $(S, B, e^i, \Gamma^i)$ -affordable consumption-investment strategies  $(H, c)$ .

### 3. EXISTENCE OF AN ABSTRACT EQUILIBRIUM

In this section we establish the existence of an abstract version of a market equilibrium. The notion of an abstract equilibrium encapsulates the tenet that markets in equilibrium should clear when all agents act rationally. The full-fledged stochastic market has been abstracted away in favor of a pricing functional  $\mathbb{Q}$ .  $\mathbb{Q}$  will be an element of the topological dual  $(\mathbb{L}^\infty)^*$  of the *consumption space*  $\mathbb{L}^\infty$ , so that the action  $\langle \mathbb{Q}, c \rangle$  of  $\mathbb{Q}$  onto a consumption process  $c$  has the natural interpretation of the price of the consumption stream  $c$ . Our setup allows for utility functions unbounded in the neighborhood of  $x = 0$  (in order to be able to deal with the important examples from financial theory). Even though these utilities follow the philosophy of the von Neumann - Morgenstern theory, they are *not* von Neumann - Morgenstern utilities in the sense of [Bew72]. In fact, the corresponding utility functionals are not necessarily Mackey-continuous and thus the abstract theory pioneered by Truman Bewley and others does not apply directly to our setting. The structure of our proof of the existence of an abstract equilibrium follows the skeleton laid out in [MCZ91]. For that reason we focus on the substantially novel parts of the proof and only outline the rest. In particular, we present a detailed proof of closedness of the set of utility vectors in Lemma 3.3, but merely refer to the corresponding parts of [MCZ91] for the results whose derivation is a more-or-less straightforward modification of existing results.

*Functional-analytic setup.* In what follows,  $\mathbb{L}^\infty$  will denote the Banach space of  $(\kappa \otimes \mathbb{P})$ -essentially bounded processes, measurable with respect to the  $\sigma$ -algebra  $\mathcal{O}$  of  $(\mathcal{F}_t)_{t \in [0, T]}$ -optional sets.  $\mathbb{L}_+^\infty$  will denote the positive orthant of  $\mathbb{L}^\infty$ , i.e., the set of all  $(\kappa \otimes \mathbb{P})$ -a.e. nonnegative elements in  $\mathbb{L}^\infty$ . All  $(\mathcal{F}_t)_{t \in [0, T]}$ -optional processes will be identified with the corresponding  $\mathcal{O}$ -measurable random variables without explicit mention, and the equalities and inequalities will always be understood in  $(\kappa \otimes \mathbb{P})$ -a.e. sense.

The set of all bounded consumption processes  $c$  satisfying the consumption constraints introduced via cap processes  $\Gamma^i$ , will be denoted by  $\mathcal{A}^i$ , i.e.,  $\mathcal{A}^i = \{c \in \mathbb{L}_+^\infty : c \leq \Gamma^i\}$ . Also, define  $\mathcal{A} = \{(c^i)_{i=1, \dots, d} : c^i \in \mathcal{A}^i\}$ , and its subset  $\mathcal{A}^f$  consisting of only those allocations which can be produced by redistributing the aggregate endowment  $e = \sum_i e^i$ , i.e.,  $\mathcal{A}^f = \{(c^i)_{i=1, \dots, d} \in \mathcal{A} : \sum_i c^i = e\}$

The topological dual  $(\mathbb{L}^\infty)^*$  of  $\mathbb{L}^\infty$  can be identified with the set of all finitely-additive measures  $\mathbb{Q}$  on the  $\sigma$ -algebra  $\mathcal{O}$ , weakly-absolutely continuous with respect to  $\kappa \otimes \mathbb{P}$ , i.e. for  $A \in \mathcal{O}$ ,  $\mathbb{Q}[A] = 0$  whenever  $(\kappa \otimes \mathbb{P})[A] = 0$ .

*Remark 3.1.* We will consider the set of finitely-additive probabilities as a subset of  $(\mathbb{L}^\infty)^*$ , supplied with the weak \* topology  $\sigma((\mathbb{L}^\infty)^*, \mathbb{L}^\infty)$ . It is a consequence of Alaoglu's theorem that any collection of finitely-additive probabilities is relatively  $\sigma((\mathbb{L}^\infty)^*, \mathbb{L}^\infty)$ -compact. Furthermore, the closedness of the set of finitely-additive probabilities (in the space of all finite-additive measures, and w.r.t the  $\sigma((\mathbb{L}^\infty)^*, \mathbb{L}^\infty)$ -topology) implies that the cluster-points of nets of finitely-additive probabilities are finitely-additive probabilities themselves. In the sequel, weak \* topology will always refer to the  $\sigma((\mathbb{L}^\infty)^*, \mathbb{L}^\infty)$  topology of the pair  $((\mathbb{L}^\infty)^*, \mathbb{L}^\infty)$ .

We can now define the concept of an abstract equilibrium. Instead of a semimartingale price process, an abstract equilibrium requires the existence of a finitely-additive probability  $\mathbb{Q} \in (\mathbb{L}^\infty)^*$  which takes the role of a pricing functional acting directly on consumption processes. Given such a finitely-additive probability  $\mathbb{Q}$ , the **budget set**  $B^i(\mathbb{Q})$  of agent  $i$  is defined by  $B^i(\mathbb{Q}) = \{c \in \mathbb{L}_+^\infty : c \in \mathcal{A}^i \text{ and } \langle \mathbb{Q}, c \rangle \leq \langle \mathbb{Q}, e^i \rangle\}$ .

**Definition 3.2.** A pair  $(\mathbb{Q}, (c^i)_{i=1,\dots,d})$  of a finitely-additive probability  $\mathbb{Q}$  and an allocation  $(c^i)_{i=1,\dots,d} \in \mathcal{A}$  is called an **abstract equilibrium** if

- (1)  $\sum_i c^i = \sum_i e^i$ , i.e.,  $(c^i)_{i=1,\dots,d} \in \mathcal{A}^f$ .
- (2) For any  $i = 1, \dots, d$ ,  $c^i \in B^i(\mathbb{Q})$  and  $\mathbb{U}^i(c^i) \geq \mathbb{U}^i(c)$  for all  $c \in B^i(\mathbb{Q})$ .

*Existence of an abstract equilibrium.* To simplify notation in some proofs and statements we assume that the utility functionals  $\mathbb{U}^i$  are normalized so that  $\mathbb{U}^i(e^i) = 0$  for all  $i = 1, \dots, d$ .

We start by introducing  $\mathcal{U}^f$  - the set of all  $d$ -tuples of utilities which can be achieved by different allocations  $(c^i)_{i=1,\dots,d} \in \mathcal{A}^f$ , i.e.,

$$\mathcal{U}^f = \{(\mathbb{U}^1(c^1), \dots, \mathbb{U}^d(c^d)) : (c^i)_{i=1,\dots,d} \in \mathcal{A}^f\}, \quad (3.1)$$

and  $\mathcal{U}_-^f = \mathcal{U}^f - [0, \infty)^d$  - the set of all vectors in  $\mathbb{R}^d$  dominated by some element in  $\mathcal{U}^f$ . The elements in  $\mathcal{U}_-^f$  will be called **utility vectors**. Our first lemma identifies several properties of  $\mathcal{U}_-^f$ , the most important of which is closedness.

**Lemma 3.3.** *The set  $\mathcal{U}_-^f$  is non-empty, convex and closed.*

*Proof.*  $\mathcal{U}_-^f$  is obviously non-empty, and its convexity follows easily from convexity of  $\mathcal{A}^f$ . It remains to show that it is closed. Let  $\{\mathbf{u}_n\}_{n \in \mathbb{N}}$ ,  $\mathbf{u}_n = (u_n^1, u_n^2, \dots, u_n^d)$ , be a sequence in  $\mathcal{U}_-^f$  converging to  $\mathbf{u} = (u^1, u^2, \dots, u^d) \in \mathbb{R}^d$ . By the definition of the set  $\mathcal{U}_-^f$ , there exist two sequences  $\mathbf{c}_n = (c_n^1, c_n^2, \dots, c_n^d) \in \mathcal{A}^f$  and  $\mathbf{r}_n = (r_n^1, \dots, r_n^d) \in \mathbb{R}_+^d$

such that  $\mathbb{U}^i(c_n^i) = u_n^i + r_n^i$ . Since  $u_n^i \leq \mathbb{U}^i(c_n^i) \leq \mathbb{U}^i(e) < \infty$ , we can assume - passing to a subsequence if necessary - that there exists a vector  $\hat{\mathbf{u}} = (\hat{u}^1, \dots, \hat{u}^d)$  such that  $\mathbb{U}^i(c_n^i) \rightarrow \hat{u}^i \geq u^i$ .

For any  $i = 1 \dots d$ , the sequence  $\{c_n^i\}_{n \in \mathbb{N}}$  is bounded in  $\mathbb{L}^\infty$ , and therefore also in  $\mathbb{L}^1(\kappa \otimes \mathbb{P})$ . By a simple extension of the classical Komlos' theorem (see [Sch86]) to the case of  $\mathbb{R}^d$ -valued random variables, there exists an infinite array of nonnegative weights  $(\alpha_k^n)_{k=n, \dots, k_n}^{n \in \mathbb{N}}$  and a  $d$ -tuple  $(c^i)_{i=1, \dots, d}$  of nonnegative optional processes with the following properties:  $\sum_{k=n}^{k_n} \alpha_k^n = 1$  and  $\tilde{c}_n^i = \sum_{k=n}^{k_n} \alpha_k^n c_k^i \rightarrow c^i$ ,  $(\kappa \otimes \mathbb{P})$ -a.e. Consequently,  $\sum_i c^i = e$  and  $c^i \leq \Gamma^i$ , so that  $(c^i)_{i=1, \dots, d} \in \mathcal{A}^f$ .

To show that  $\mathbf{u} \in \mathcal{U}_-^f$ , we use concavity and right-continuity of the utility functions and the Fatou Lemma (the use of which is justified by the fact that  $\tilde{c}_n^i \leq e$ , for all  $i$  and all  $n \in \mathbb{N}$ ) in the following chain of inequalities:

$$\mathbb{U}^i(c^i) = \mathbb{U}^i(\lim_n \tilde{c}_n^i) \geq \overline{\lim}_n \mathbb{U}^i(\tilde{c}_n^i) \geq \overline{\lim}_n \sum_{k=n}^{k_n} \alpha_k^n \mathbb{U}^i(c_k^i) = \lim_n \mathbb{U}^i(c_n^i) = \hat{u}^i \geq u^i,$$

□

The next task is to establish the existence of *supporting measures* for *weakly optimal* utility vectors. We start with definitions of these two concepts.

**Definition 3.4.** A finitely-additive probability  $\mathbb{Q}$  is said to **support** a vector  $\mathbf{u} = (u^1, \dots, u^d) \in \mathbb{R}^d$  if for any allocation  $\mathbf{c} = (c^i)_{i=1, \dots, d} \in \mathcal{A}$  with the property that  $\mathbb{U}^i(c^i) \geq u^i$  for all  $i = 1, \dots, d$ , we have  $\langle \mathbb{Q}, \sum_i c^i \rangle \geq \langle \mathbb{Q}, \sum_i e^i \rangle$ . The set of all finitely-additive probability measures supporting a vector  $\mathbf{u} \in \mathbb{R}^d$  is denoted by  $P(\mathbf{u})$ .

**Definition 3.5.** A vector  $\mathbf{u} = (u^1, \dots, u^d)$  in  $\mathcal{U}_-^f$  is said to be *weakly optimal* if there is no allocation  $(c^i)_{i=1, \dots, d} \in \mathcal{A}^f$  with the property that  $\mathbb{U}^i(c^i) > u^i$  for all  $i = 1, \dots, d$ .

**Lemma 3.6** (Second Fundamental Theorem of Welfare Economics). *For a weakly optimal utility vector  $\mathbf{u} \in \mathcal{U}_-^f$ , the set  $P(\mathbf{u})$  of finitely-additive probabilities supporting  $\mathbf{u}$  is non-empty, convex and weak \* compact*

*Proof.* The proof relies on a well-know separating-hyperplane-type argument. See [MCZ91], Section 8., pp. 1859-1860 for more details. □

Having established the closedness and convexity of the set  $\mathcal{U}_-^f$  in Lemma 3.3, and the existence of supporting functionals for weakly optimal utility vectors in Lemma 3.6, it suffices to use the proof of Theorem 7.1, p. 1856 in [MCZ91] to establish the following abstract existence theorem:

**Theorem 3.7.** *Under Assumptions 2.9.1, 2.13.1 and 2.13.2, there exists an abstract equilibrium  $(\mathbb{Q}, (c^i)_{i=1, \dots, d})$ .*

## 4. FROM ABSTRACT TO STOCHASTIC EQUILIBRIA

Our next task is to show that the abstract equilibrium obtained in the previous section can be implemented as a stochastic equilibrium. We first note that the equilibrium functional  $\mathbb{Q}$  must be countably-additive and equivalent to  $\kappa \otimes \mathbb{P}$ . We omit the proof as it follows the argument from Theorem 8.2, p. 1863 in [MCZ91], using the fact that  $\Gamma^i > e^i$  and  $\Gamma_T^i = \infty$  for all  $i = 1, \dots, d$ .

**Lemma 4.1.** *Let  $(\mathbb{Q}, (c^i)_{i=1, \dots, d})$  be an abstract equilibrium. Then  $\mathbb{Q}$  is countably additive and equivalent to  $\kappa \otimes \mathbb{P}$ .*

In Lemma 4.2 we use convex duality to describe the solutions of agents' utility-maximization problems in an equilibrium:

**Lemma 4.2.** *Suppose that  $(\mathbb{Q}, (c^i)_{i=1, \dots, d})$  is an abstract equilibrium. Then there exist constants  $\lambda^i > 0$ ,  $i = 1, \dots, d$ , such that the consumption processes  $c^i$ ,  $i = 1, \dots, d$  are of the form*

$$c_t^i = \min(\Gamma_t^i, I^i(t, \lambda^i Q_t)), \quad (4.1)$$

where  $Q = (Q_t)_{t \in [0, T]}$  is the optional version of the Radon-Nikodym derivative of  $\mathbb{Q}$  with respect to  $\kappa \otimes \mathbb{P}$ .

*Proof.* We prove the lemma for  $i = 1$ . Let  $N(c^1)$  be the set of all  $c \in \mathbb{L}_+^\infty$  such that  $c \leq \min(\Gamma^1, \|c^1\|_{\mathbb{L}^\infty})$ .  $N(c^1)$  is a  $\sigma(\mathbb{L}^\infty, \mathbb{L}^1)$ -compact subset in  $\mathbb{L}_+^\infty$ , and by Komlos' Lemma the restriction of  $\mathbb{U}^1$  to  $N(c^1)$  is  $\sigma(\mathbb{L}^\infty, \mathbb{L}^1)$ -upper-semicontinuous and concave. By Lemma 4.1, the finitely-additive measure  $\mathbb{Q}$  is countably-additive so the Lagrangean function  $L : N(c^1) \times [0, \infty) \rightarrow [-\infty, \infty)$ ,  $L(c, \lambda) = \mathbb{U}^1(c) - \lambda \langle \mathbb{Q}, c - e^1 \rangle$  satisfies the conditions of the Minimax theorem (see [Sio58]). We know that the maximizer  $c^1$  of the functional  $\mathbb{U}^1$  over  $B^1(\mathbb{Q})$  trivially satisfies  $c^1 \leq \|c^1\|_{\mathbb{L}^\infty}$ , so

$$\begin{aligned} \mathbb{U}^1(c^1) &= \sup_{c \in B^1(\mathbb{Q}) \cap N(c^1)} \mathbb{U}^1(c) = \sup_{c \in N(c^1)} \inf_{\lambda \geq 0} L(c, \lambda) = \inf_{\lambda \geq 0} \sup_{c \in N(c^1)} L(c, \lambda) \\ &= \inf_{\lambda \geq 0} \left( \lambda \langle \mathbb{Q}, e^1 \rangle + \mathbb{E} \int_0^T V(t, \lambda Q_t; m_t^1) d\kappa_t \right), \end{aligned}$$

where  $m_t^1 = \min(\Gamma_t^1, \|c^1\|_{\mathbb{L}^\infty})$ , and the function  $V : [0, T] \times [0, \infty) \times (0, \infty) \rightarrow \mathbb{R}$  is given by

$$V(t, \lambda; \xi) \triangleq \sup_{x \in [0, \xi]} (U^1(t, x) - x\lambda) = \begin{cases} V(t, \lambda; \infty), & \lambda > U_x^1(t, \xi), \\ U^1(t, \xi) - \lambda\xi, & \lambda \leq U_x^1(t, \xi). \end{cases}$$

$V$  is convex and nonincreasing in  $\lambda$ , and nondecreasing in  $\xi$ . The function  $v : [0, \infty) \rightarrow [-\infty, \infty]$ , where  $v(\lambda) = \lambda \langle \mathbb{Q}, e^1 \rangle + \mathbb{E} \int_0^T V(t, \lambda Q_t; m_t^1) d\kappa_t$ , is convex and proper,

since  $\inf_{\lambda \geq 0} v(\lambda) = \mathbb{U}^1(c^1) \in (-\infty, \infty)$ . Furthermore, Assumption 2.13.1 implies the inequality  $V(t, \lambda; m_t^1) \leq U^1(t, \|c^1\|_{\mathbb{L}^\infty})$  and the existence of a constant  $D > 0$  such that  $\mathbb{U}^1(c^1) \leq v(\lambda) \leq \lambda \langle \mathbb{Q}, e^1 \rangle + D$ , for all  $\lambda > 0$ .

Assumption 2.13.2 ensures the existence of a constant  $C > \|c^1\|_{\mathbb{L}^\infty}$  such that  $I^1(t, C) < \frac{1}{2} \langle \mathbb{Q}, e^1 \rangle$  for all  $t \in [0, T]$ . Then, for all  $\xi, \lambda, \lambda_0 > 0$  with the property that  $\lambda > \lambda_0 > \max(C, U_x^1(t, \xi))$ , we have

$$\begin{aligned} V(t, \lambda; \xi) &\geq V(t, \lambda_0; \xi) + (\lambda - \lambda_0) V_\lambda(t, \lambda_0; \xi) \\ &= V(t, \lambda_0; \infty) - (\lambda - \lambda_0) I^1(t, \lambda_0) \geq V(t, \lambda_0; \infty) - \frac{1}{2} \langle \mathbb{Q}, e^1 \rangle (\lambda - \lambda_0). \end{aligned}$$

Therefore, if we let  $L = \underline{\lim}_{\lambda \rightarrow \infty} \left( \frac{v(\lambda)}{\lambda} - \langle \mathbb{Q}, e^1 \rangle \right) \in [-\infty, \infty]$ , we have

$$\begin{aligned} L &= \underline{\lim}_{\lambda \rightarrow \infty} \frac{1}{\lambda} \mathbb{E} \int_0^T V(t, \lambda Q_t; m_t^1) d\kappa_t \geq \underline{\lim}_{\lambda \rightarrow \infty} \frac{1}{\lambda} \mathbb{E} \int_0^T V(t, \lambda Q_t; m_t^1) \mathbf{1}_{\{Q_t > \frac{C}{\lambda_0}\}} d\kappa_t \\ &\quad + \underline{\lim}_{\lambda \rightarrow \infty} \frac{1}{\lambda} \mathbb{E} \int_0^T V(t, \lambda Q_t; m_t^1) \mathbf{1}_{\{Q_t \leq \frac{C}{\lambda_0}\}} d\kappa_t \\ &\geq \underline{\lim}_{\lambda \rightarrow \infty} \left( \frac{1}{\lambda} \mathbb{E} \int_0^T V(t, \lambda_0 Q_t; m_t^1) \mathbf{1}_{\{Q_t > \frac{C}{\lambda_0}\}} d\kappa_t - \frac{1}{2\lambda} \langle \mathbb{Q}, e^1 \rangle (\lambda - \lambda_0) \right) \\ &\quad + \underline{\lim}_{\lambda \rightarrow \infty} \frac{1}{\lambda} \mathbb{E} \int_0^T V(t, \lambda Q_t; m_t^1) \mathbf{1}_{\{Q_t \leq \frac{C}{\lambda_0}\}} d\kappa_t \\ &\geq -\frac{1}{2} \langle \mathbb{Q}, e^1 \rangle + \underline{\lim}_{\lambda \rightarrow \infty} \frac{1}{\lambda} \mathbb{E} \int_0^T V(t, \lambda Q_t; m_t^1) \mathbf{1}_{\{Q_t \leq \frac{C}{\lambda_0}\}} d\kappa_t \geq -\frac{1}{2} \langle \mathbb{Q}, e^1 \rangle. \end{aligned}$$

Hence,  $\lim_{\lambda \rightarrow \infty} v(\lambda) = \infty$  and there exists a constant  $\lambda^1 \in [0, \infty)$  such that  $v(\lambda^1) = \mathbb{U}^1(c^1)$ , i.e.,

$$\begin{aligned} \mathbb{E} \int_0^T U^1(t, c^1(t)) d\kappa_t &= \mathbb{E} \int_0^T \lambda^1 Q_t e_t^1 d\kappa_t + \mathbb{E} \int_0^T V(t, \lambda^1 Q_t; m_t^1) d\kappa_t \\ &\geq \mathbb{E} \int_0^T \lambda^1 Q_t c_t^1 d\kappa_t + \mathbb{E} \int_0^T V(t, \lambda^1 Q_t; m_t^1) d\kappa_t. \end{aligned}$$

On the other hand,  $U^1(t, x) \leq \lambda^1 Q_t x + V(t, \lambda^1 Q_t; m_t^1)$  for all  $t \in [0, T]$  and  $x \in [0, m_t^1]$  (with equality only for  $x = \min(m_t^1, I^1(t, \lambda^1 Q_t))$ ), so  $c^1$  must be of the form (4.1). To rule out the possibility  $\lambda^1 = 0$ , note that it would force  $c^1 = \Gamma^1$  and violate the budget constraint since  $\Gamma^1 > e^1$ .  $\square$

**Proposition 4.3.** *The process  $Q$  has a modification which is a semimartingale, and there exists a constant  $\varepsilon > 0$  such that  $\varepsilon \leq Q \leq 1/\varepsilon$ .*

*Proof.* By Lemma 4.2 there exists constants  $\lambda^i > 0$  such that  $e_t = \sum_i c_t^i = \sum_i \min(\Gamma_t^i, I^i(t, \lambda^i Q_t))$ ,  $\kappa \otimes \mathbb{P}$ -a.s. Since  $(\kappa \otimes \mathbb{P})[\sum_i \Gamma_t^i > e] = 1$ , we have  $e_t = \min_{\mathbf{b} \in B} (\sum_i b_i I^i(t, \lambda^i Q_t) + \sum_i (1 - b_i) \Gamma_t^i)$ , where  $B = \{0, 1\}^d \setminus \{0, \dots, 0\}$ .

For  $\mathbf{b} \in B$ , the function  $I^{\mathbf{b}}$ , defined by  $I^{\mathbf{b}}(t, y) = \sum_i b_i I^i(t, \lambda^i y)$ , is strictly decreasing in its second argument and shares the properties in Assumption 2.13 with each  $I^i$ . Therefore, there exists a function  $J^{\mathbf{b}} : [0, T] \times (0, \infty) \rightarrow (0, \infty)$  such that  $I^{\mathbf{b}}(t, J^{\mathbf{b}}(t, x)) = x$ , for all  $(t, x) \in [0, T] \times (0, \infty)$ . Thus, with  $\Gamma_t^{\mathbf{b}} = \sum_i (1 - b_i) \Gamma_t^i$ , we have

$$e_t \geq x \Leftrightarrow I^{\mathbf{b}}(t, Q_t) + \Gamma_t^{\mathbf{b}} \geq x, \forall \mathbf{b} \in B \Leftrightarrow Q_t \leq J^{\mathbf{b}}(t, x - \Gamma_t^{\mathbf{b}}), \forall \mathbf{b} \in B,$$

with  $J(t, x - \Gamma_t^{\mathbf{b}}) = \infty$  for  $x \leq \Gamma_t^{\mathbf{b}}$ . Consequently,  $Q_t = \min_{\mathbf{b} \in B} J^{\mathbf{b}}(t, e_t - \Gamma_t^{\mathbf{b}})$ . Knowing that the semimartingale property is preserved under maximization, it will be enough to prove that for each  $\mathbf{b} \in B$ ,  $J^{\mathbf{b}}$  is a semimartingale function (see Definition A.1). By Inada conditions (2.1) - holding uniformly in  $t \in [0, T]$  -  $I^{\mathbf{b}}$  maps compact sets of the form  $[0, T] \times [y_1, y_2]$  into compact intervals. The function  $I^{\mathbf{b}}$  is locally convexity-Lipschitz, so the conclusion that  $Q$  is a semimartingale follows from Proposition A.6.

To show boundedness, we first set  $\mathbf{b}_1 = (1, \dots, 1)$  to conclude that  $Q_t \leq J^{\mathbf{b}_1}(t, e_t - \Gamma_t^{\mathbf{b}_1}) = J^{\mathbf{b}_1}(t, e_t) \in \mathbb{L}^\infty$ . On the other hand,  $Q_t = \min_{\mathbf{b} \in B} J^{\mathbf{b}}(t, e_t - \Gamma_t^{\mathbf{b}}) \geq \min_{\mathbf{b} \in B} J^{\mathbf{b}}(t, e_t)$  - a positive quantity, uniformly bounded from below. Therefore, the semimartingale  $Q_t$  is positive and uniformly bounded from above and away from zero.  $\square$

**Proposition 4.4.** *The process  $Q$  admits a multiplicative decomposition  $Q = \hat{Q}\beta$  where  $\hat{Q}$  is a strictly positive uniformly integrable martingale, and  $\beta$  is a strictly positive càdlàg predictable process of finite variation.*

*Proof.* By the representation  $Q_t = \min_{\mathbf{b} \in B} J^{\mathbf{b}}(t, e_t - \Gamma_t^{\mathbf{b}})$ , and boundedness of  $Q$  from above, there exists a constant  $C > 0$  such that  $Q_t = \min_{\mathbf{b} \in B} J^{\mathbf{b}}(t, \max(C, e_t - \Gamma_t^{\mathbf{b}}))$ . Propositions A.5, A.7 and A.8 complete the proof.  $\square$

*Construction of the equilibrium market.* Thanks to Proposition 4.4, there exists a measure  $\hat{\mathbb{Q}}$  (with  $\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}} = \frac{\hat{Q}_T}{\mathbb{E}[\hat{Q}_T]}$ ) equivalent to  $\mathbb{P}$  such that

$$Q_t = \mathbb{E}[\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}} | \mathcal{F}_t] \beta_t, \text{ and } \langle \mathbb{Q}, c \rangle = \mathbb{E} \int_0^T Q_u c_u d\kappa_u = \mathbb{E}^{\hat{\mathbb{Q}}} \int_0^T c_u \beta_u d\kappa_u. \quad (4.2)$$

In words, the action of the pricing functional  $Q$  on a consumption stream  $c$  can be represented as a  $\hat{\mathbb{Q}}$ -expectation of a discounted version  $c(u)\beta_u$  of  $c$ .

Let  $n \in \mathbb{N}$  be the martingale multiplicity of the filtration  $(\mathcal{F}_t)_{t \in [0, T]}$  under  $\hat{\mathbb{Q}}$ , and let  $(Y_1, \dots, Y_n)$  be an  $n$ -dimensional positive  $\hat{\mathbb{Q}}$ -martingale described in Definition 2.1.

Define the riskless asset  $B$  and the stock price process  $S = (S_1, \dots, S_n)$  as follows

$$B(t) = 1/\beta(t), S_j(t) = B(t)Y_j(t), \quad t \in [0, T], \quad j = 1, \dots, n. \quad (4.3)$$

**Lemma 4.5.** *The pair  $(S, B)$ , defined in (4.3) is an equilibrium market.*

*Proof.* Let  $(\mathbb{Q}, (c^i)_{i=1, \dots, d})$  be the abstract equilibrium which produced  $(S, B)$ , and let the measure  $\hat{\mathbb{Q}}$  be as in (4.2). For  $i = 1, \dots, d$ , define the  $\hat{\mathbb{Q}}$ -martingale  $\tilde{X}^i$  by  $\tilde{X}_t^i = \mathbb{E}^{\hat{\mathbb{Q}}}[\int_0^T (c^i(u) - e^i(u))\beta_u du | \mathcal{F}_t]$ . By the finite representation property (Assumption 2.4), for each  $i = 1, \dots, d$  there exists an  $S$ -integrable portfolio process  $H^i$  such that  $\tilde{X}_t^i = \tilde{X}_0^i + \int_0^t \tilde{H}^i dS_u$ . Moreover, the boundedness of processes  $c^i$  and  $e^i$  guarantees that  $\tilde{H}^i$  satisfies part 1. of Definition 2.19. Standard calculations involving integration by parts and using the fact that  $B$  is a predictable process of finite variation imply that the wealth process  $X^{\tilde{H}^i, c^i, e^i}$  defined as in (2.2) is bounded and satisfies  $X_T^{\tilde{H}^i, c^i, e^i} \geq 0$ . Therefore,  $(\tilde{H}^i, c^i)$  is an affordable consumption-investment strategy (as described in Definition 2.19).

Since  $\sum_i \tilde{X}^i = 0$ , the mutual orthogonality of the  $\hat{\mathbb{Q}}$ -martingales  $Y_1, \dots, Y_n$  implies that  $\sum_{i=1}^d \tilde{H}_j^i(t) = 0$ ,  $d[Y_j, Y_j]_t - \text{a.e.}$ , for all  $j$ . In order to have markets clear for every  $t \in [0, T]$ , we define the portfolio process  $H^i = (H_1^i, \dots, H_n^i)$  by  $H_j^i(t) = \tilde{H}_j^i(t) \mathbf{1}_{\{\sum_i H_j^i(t)=0\}}$ , for each  $i = 1, \dots, d$ , so that

- (1)  $H_j^i(t) = \tilde{H}_j^i(t)$ ,  $d[Y_j, Y_j]$ -a.e. (implying indistinguishability of the wealth processes  $X^{H^i, c^i, e^i}$  and  $X^{\tilde{H}^i, c^i, e^i}$ ) and
- (2)  $\sum_i H_j^i(t) = 0$ , for all  $t$ , a.s. and all  $j = 1, \dots, n$ .

Therefore, the  $d$ -tuple  $(H^i, c^i)$  satisfies the part 1. of Definition 2.20.

It remains to show that  $c^i$  maximizes  $\mathbb{U}^i$  over all consumption process  $c'$  with  $\mathbb{U}^i(c') \in (-\infty, \infty)$  for which there exists a portfolio process  $H'$  such that  $(H', c')$  is  $(S, B, e^i, \Gamma^i)$ -affordable. We first note that each such  $c'$  satisfies  $\langle \mathbb{Q}, c' \rangle \leq \langle \mathbb{Q}, e^i \rangle$ . This is due to (4.2) and the fact that the discounted wealth  $X' = \beta X^{H', c', e^i}$  (which satisfies  $X'_T \geq 0$ ) can be represented as a sum of a  $\hat{\mathbb{Q}}$ -martingale and a term of the form  $\int_0^t \beta_u (c'(u) - e^i(u)) d\kappa_u$ . Finally, because  $c' \wedge k \in B^i(\mathbb{Q})$ , for any  $k \in \mathbb{N}$ , the properties of the abstract equilibrium imply that  $\mathbb{U}^i(c^i) \geq \mathbb{U}^i(c' \wedge k)$  and the Monotone Convergence Theorem yields  $\mathbb{U}^i(c^i) \geq \mathbb{U}^i(c')$ .  $\square$

**Theorem 4.6.** *Suppose that*

- (1)  $(\Omega, \mathcal{F}, (\mathcal{F})_{t \in [0, T]}, \mathbb{P})$  is a filtered probability space satisfying Assumption 2.4,
- (2)  $(e^i)_{i=1, \dots, d}$  are random endowment processes verifying Assumption 2.9,
- (3)  $(U^i)_{i=1, \dots, d}$  are utility functions for which Assumption 2.13 is valid, and
- (4)  $(\Gamma^i)_{i=1, \dots, d}$  are withdrawal cap processes satisfying Assumption 2.17.



Then there exist an equilibrium market  $(S, B)$  consisting of a finite-dimensional semimartingale risky-asset process  $S$  and a positive predictable riskless-asset process  $B$  of finite variation for which the following additional properties hold

- (1) The market  $(S, B)$  is arbitrage free, i.e., there exists a unique measure  $\hat{\mathbb{Q}}$  equivalent to  $\mathbb{P}$ , such that the discounted prices  $S/B$  of risky assets are  $\hat{\mathbb{Q}}$ -martingales.
- (2) The optimal consumption densities  $c^i$  in the market  $(S, B)$  are uniformly bounded from above.

#### APPENDIX A. SEMIMARTINGALE FUNCTIONS AND MULTIPLICATIVE DECOMPOSITIONS

In this section we provide several results which give sufficient conditions for 1) a process obtained by applying a function to a semimartingale to be a semimartingale, and 2) for a local martingale part in a multiplicative decomposition of a positive process to be a uniformly integrable martingale. These results can be improved in several directions; we are aiming for conditions easily verifiable in practice. In what follows,  $I$  and  $J$  will denote generic open intervals in  $\mathbb{R}$ . For a process  $A$  of finite variation,  $|A| = (|A|_t)_{t \in [0, T]}$  will denote its total variation process.

*Semimartingale functions.*

**Definition A.1.** A function  $f : [0, T] \times I \rightarrow \mathbb{R}$  is called a **semimartingale function** if the process  $Y$  defined by  $Y_t = f(t, X_t)$ ,  $t \in [0, T]$  is a càdlàg semimartingale for each semimartingale  $X$  taking values in  $I$  and defined on an arbitrary filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$ .

In this section we provide a set of sufficient conditions for a function  $f : [0, T] \times I \rightarrow \mathbb{R}$  to be a semimartingale function. We go beyond basic  $C^{1,2}$ -differentiability required by the Itô formula and place much less restrictive assumptions on  $f$ . Apart from being indispensable in Section 4, we hope that the obtained result holds some independent probabilistic interest.

**Theorem A.2.** Suppose that a function  $f : [0, T] \times I \rightarrow \mathbb{R}$  can be represented as  $f(t, x) = f^1(t, x) - f^2(t, x)$ , where for  $i = 1, 2$ ,

- (1)  $f^i$  is Lipschitz in the time variable, uniformly for  $x$  in compact intervals.
- (2)  $f^i$  is convex in the second variable.
- (3) The right derivative  $f_{x+}^i$  is bounded on compact subsets of  $[0, T] \times I$  and satisfies  $f_{x+}^i(t, x) = \lim_{(s, x') \rightarrow (t, x)} f_{x+}^i(s, x')$ , when  $(s, x') \rightarrow (t, x)$  and  $x' \geq x$ .

Then  $f$  is a semimartingale function. Moreover, for a semimartingale  $X$  the local martingale part  $\tilde{M}$  in the semimartingale decomposition of  $f(t, X_t) = f(0, X_0) +$

$\tilde{M}_t + \tilde{A}_t$  is given by  $\tilde{M}_t = \int_0^t f_{x+}(s, X_{s-}) dM_s$ , where  $M$  is the local martingale part in the semimartingale decomposition  $X_t = X_0 + M_t + A_t$ .

Before delving into the proof of Theorem A.2, we recall the concept of Fatou-convergence and some useful compactness-type results related to it.

**Definition A.3.** A sequence  $(X^n)_{n \in \mathbb{N}}$  of càdlàg adapted processes is said to **Fatou-converge** towards a càdlàg adapted process  $X$  if

$$X_t = \lim_{q \searrow t} \lim_n X_q^n, \text{ a.s., for all } t \in [0, T), \text{ and } X_T = \lim_n X_T^n, \text{ a.s.,}$$

where the first limit is taken over rational numbers  $q > t$ .

**Lemma A.4.**

- (1) Let  $(A^n)_{n \in \mathbb{N}}$  be a sequence of non-decreasing adapted càdlàg processes taking values in  $[0, \infty)$ . Then there exists a sequence  $(\tilde{A}^n)_{n \in \mathbb{N}}$  of convex combinations  $\tilde{A}^n \in \text{conv}(A^n, A^{n+1}, \dots)$  and a non-decreasing càdlàg process  $\tilde{A}$  taking values in  $[0, \infty]$  such that  $\tilde{A}^n$  Fatou-converges to  $\tilde{A}$ .
- (2) Let  $(A^n)_{n \in \mathbb{N}}$  be a sequence of finite-variation càdlàg processes on  $[0, T]$ , with uniformly bounded total variations, i.e.,

$$|A^n|_T \leq C \text{ a.s., for some constant } C > 0 \text{ and all } n \in \mathbb{N}.$$

Then there exists a sequence  $(\tilde{A}^n)_{n \in \mathbb{N}}$  of convex combinations  $\tilde{A}^n \in \text{conv}(A^n, A^{n+1}, \dots)$  and a càdlàg process  $\tilde{A}$  of finite variation with  $|\tilde{A}|_T \leq C$  such that  $\tilde{A}^n$  Fatou-converge towards  $\tilde{A}$ .

*Proof.* Part 1. is a restatement of Theorem 4.2 in [Kra96]. To prove part 2., note that the boundedness of total variations of processes  $A^n$  implies that the increasing and decreasing parts  $A^{\uparrow, n}$  and  $A^{\downarrow, n}$  of  $A^n$  satisfy  $A_T^{\uparrow, n} + A_T^{\downarrow, n} \leq C$  a.s. for all  $n$ . Applying part 1. to increasing and decreasing parts and noting that the limiting processes  $\tilde{A}^{\uparrow}$  and  $\tilde{A}^{\downarrow}$  satisfy  $\tilde{A}^{\uparrow} + \tilde{A}^{\downarrow} \leq C$  a.s., leads to the desired conclusion.  $\square$

*Of Theorem A.2.* Let  $X$  be a semimartingale taking values in the open interval  $I$ . Our goal is to prove that the process  $Y$  defined by  $Y_t = f(t, X_t)$  is a semimartingale. We first extend the time-domain of  $X$  and  $Y$  by setting  $X_t = X_T$  and  $Y_t = f(t, X_T)$  for  $t \in (T, \infty)$ . By Theorem 6, p. 54 in [Pro04], it will be enough to find an increasing sequence  $(T_n)_{n \in \mathbb{N}}$  of stopping times with  $T_n \nearrow \infty$ , a.s., such that the *pre-stopped* processes  $Y^{T_n-}$  defined by

$$Y_t^{T_n-} = Y_t \mathbf{1}_{\{0 \leq t < T_n\}} + Y_{T_n-} \mathbf{1}_{\{t \geq T_n\}} = f(t \wedge T_n, X_t^{T_n-})$$

are semimartingales. Taking  $T_n = \inf \{t \geq 0 : X_t \geq n\} \wedge n$ , we reduce the problem to the case where the semimartingale  $X$  takes values in a compact interval  $[x_1, x_2]$ , for  $t \in [0, S)$ , where  $S = T \wedge T_n$ .

Let  $\eta^n : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  be a sequence of standard mollifier functions with supports lying in the lower half-plane and shrinking to a point, i.e.,

- (1)  $\eta^n \in C^\infty(\mathbb{R} \times \mathbb{R})$ .
- (2)  $\eta^n(t, x) \geq 0$ , for all  $t, x$  and  $\int_{\mathbb{R} \times \mathbb{R}} \eta^n(t, x) dt dx = 1$ .
- (3) The supports  $SS_n$  of  $\eta^n$  satisfy  $SS_n \subseteq \mathbb{R} \times (-\infty, 0]$  and  $|t| + |x| \leq 1/n$  for all  $(t, x) \in SS_n$ .

Let the functions  $f^n : [0, T] \times I_n \rightarrow \mathbb{R}$ , where  $I_n = \{x \in I : d(x, I^c) > 1/n\}$ , be the mollified versions of  $f$ , i.e.,

$$f^n(t, x) = (\eta^n * f)(t, x) = \int_{\mathbb{R} \times \mathbb{R}} \eta^n(s, y) f(t - s, x - y) ds dy,$$

where we set  $f(t, x) = f(T, x)$  for  $t > T$  and  $f(t, x) = f(0, x)$  for  $t < 0$ . By standard arguments, the functions  $f^n(t, x)$  have the following properties

- (1)  $f^n(t, x) \rightarrow f(t, x)$  for all  $(t, x) \in [0, T] \times I$ , uniformly on compacts.
- (2)  $f^n(t, x) \in C^\infty([0, T] \times I_n)$ .
- (3) Let  $C > 0$  be a constant such that  $|f(t_2, x) - f(t_1, x)| \leq C |t_2 - t_1|$ , for all  $t_1, t_2 \in [0, T]$  and  $x \in [x_1, x_2]$ . Then the absolute value  $|f_t^n|$  of the time derivative  $f_t^n$  is bounded by the constant  $C$ , uniformly over  $n \in \mathbb{N}$  and  $(t, x) \in [0, T] \times [x_1, x_2]$ .
- (4) By condition 3. in the statement of the theorem and the fact that the support  $SS_n$  lies in the lower half-plane, we have  $f_x^n(t, x) \rightarrow f_{x+}(t, x)$ , for all  $(t, x) \in [0, T] \times I$ .

For  $n \in \mathbb{N}$  such that  $[x_1, x_2] \subseteq I_n$ , the Itô formula applied to  $f^n$  implies that  $f^n(t, X_t) = f^n(0, X_0) + M_t^n + A_t^n + B_t^n$ , where

$$\begin{aligned} M_t^n &= \int_0^t f_x^n(s, X_{s-}) dX_s, \quad A_t^n = \int_0^t f_t^n(s, X_s) ds, \text{ and} \\ B_t^n &= \frac{1}{2} \int_0^t f_{xx}^n(x, X_{s-}) d[X, X]_s^c \\ &\quad + \sum_{0 < s \leq t} (f^n(s, X_s) - f^n(s, X_{s-}) - f_x^n(s, X_{s-}) \Delta X_s). \end{aligned}$$

Note that:

- (1) Using properties 3. and 4. (above) of  $f^n$  and the Dominated Convergence Theorem for stochastic integrals (see [Pro04], Theorem 32, p. 174), we have

$M_t^n \rightarrow M_t = \int_0^t f_{x+}(s, X_{s-}) dX_s$ , uniformly in  $t \in [0, T]$ , in probability. It suffices to take a subsequence to obtain convergence in the Fatou sense.

- (2) By convexity of  $f^n$  in the second variable, the processes  $B_t^n$  are non-decreasing. Thus, by Lemma A.4, after a passage to a sequence of convex combinations they Fatou-converge towards a non-decreasing càdlàg adapted process  $B$  taking values in  $[0, \infty]$ .
- (3) The processes in the sequence  $A_t^n$  have total variation uniformly bounded by  $CT$ , so by part 2. of Lemma A.4, there exists a sequence of their convex combinations Fatou-converging towards a process  $A$  of finite variation with the total variation bounded by the same constant  $CT$ .

Compounding all subsequences and sequences of convex combinations above, we obtain that  $f(t, X_t) - f(0, X_0) - M_t - A_t = B_t$ . We can conclude that  $B_T < \infty$ , a.s. and  $f(t, X_t) = f(0, X_0) + M_t + A_t + B_t$  is a semimartingale.  $\square$

**Proposition A.5.** *Every locally convexity-Lipschitz function  $f : [0, T] \times I \rightarrow \mathbb{R}$  admits a decomposition  $f = f^1 - f^2$ , where  $f^1$  and  $f^2$  satisfy conditions 1.-3. of Theorem A.2. In particular,  $f$  is a semimartingale function.*

*Proof.* We shall construct the desired decomposition only on a compact interval  $[x_1, x_2]$  in  $I$ , as the general case follows immediately.

For a fixed  $t \in [0, T]$ , the finite-variation function  $f_x(t, \cdot)$  admits a decomposition into a difference of a pair  $f^\uparrow(t, \cdot)$  and  $f^\downarrow(t, \cdot)$  of non-increasing and non-negative functions. Lipschitz continuity of the total variation of the derivative  $f_x$  implies that the functions  $f^\uparrow$  and  $f^\downarrow$  are Lipschitz continuous in  $t$ , uniformly in  $x \in [x_1, x_2]$ . It is now easy to check that the sought-for decomposition is  $f = f^1 - f^2$ , where  $f^1(t, x) = f(t, x_1) + \int_{x_1}^x f^\uparrow(t, \xi) d\xi$ , and  $f^2(t, x) = \int_{x_1}^x f^\downarrow(t, \xi) d\xi$ .  $\square$

**Proposition A.6.** *Let  $f : [0, T] \times I \rightarrow \mathbb{R}$  be locally convexity-Lipschitz, with the derivative  $f_x$  positive and bounded away from 0 on compact subsets of  $[0, T] \times I$ . If the function  $g : [0, T] \times J \rightarrow \mathbb{R}$  satisfies  $f(t, g(t, y)) = y$ , for all  $(t, y) \in [0, T] \times J$ , then  $g$  is a semimartingale function.*

*Proof.* We note first that the assumptions of the proposition imply that both  $f$  and  $g$  are continuous and strictly increasing in the second argument. To simplify the proof, we shall restrict the domain of  $g$  to a compact set of the form  $[0, T] \times [y_1, y_2]$ , so that the range of  $g$  is contained in a compact set  $[x_1, x_2] \subseteq I$ . The general case will follow by *pre-stopping* - the technique used in the proof of Theorem A.2. Using the relationships  $0 = f(t, g(t, y)) - f(s, g(s, y))$  and  $g_y(t, y) f_x(t, g(t, y)) = 1$  together with the properties of function  $f$  postulated in the statement, it is tedious but straightforward to prove that both  $g$  and  $g_y$  are Lipschitz continuous in both variables.

Our next task is to decompose the function  $g$  into a difference of two functions satisfying conditions 1.-3. in Theorem A.2. By Proposition A.5,  $f$  has a decomposition  $f(t, x) = f^1(t, x) - f^2(t, x)$  with properties 1.-3. from Theorem A.2. Let  $h^i(t, y)$  denote the compositions  $f_x^i(t, g(t, y))$ ,  $i = 1, 2$ , and let  $h(t, x) = h^1(t, x) - h^2(t, x)$  so that  $f_x(t, g(t, y)) = h(t, y)$ . Then, for  $i = 1, 2$ ,  $h^i(t, \cdot)$  is a non-decreasing function and for  $y \in [y_1, y_2]$ ,

$$\begin{aligned} g_y(t, y) - g_y(t, y_1) &= - \int_{y_1}^y \frac{h(t, d\eta)}{(f_x(t, g(t, \eta)))^2} \\ &= - \int_{y_1}^y g_y(t, \eta)^2 (h^1(t, d\eta) - h^2(t, d\eta)), \end{aligned} \quad (\text{A.1})$$

where  $h^i(t, d\eta)$  stands for the Lebesgue-Stieltjes measure induced by  $h^i(t, \cdot)$ . With  $g^1$  and  $g^2$  defined as

$$\begin{aligned} g^1(t, y) &= g(t, y_1) + g_y(t, y_1)(y - y_1) + \int_{y_1}^y \int_{y_1}^z g_y(t, \eta)^2 h^2(t, d\eta) dz, \\ g^2(t, y) &= \int_{y_1}^y \int_{y_1}^z g_y(t, \eta)^2 h^1(t, d\eta) dz, \end{aligned}$$

(A.1) implies that  $g(t, y) = g^1(t, y) - g^2(t, y)$ .

What follows is the proof of Lipschitz-continuity of  $g_y^2(\cdot, y)$ . A simple change of variables - valid due to the continuity of the function  $g$  - yields  $g_y^2(t, y) = \int_{g(t, y_1)}^{g(t, y)} g_y(t, f(t, \xi))^2 f_x^1(t, d\xi)$ , so, for  $t, s \in [0, T]$  the difference  $g_y^2(t, y) - g_y^2(s, y)$  can be decomposed into the sum  $I_1 + I_2 + I_3 + I_4$  where

$$\begin{aligned} I_1 &= \int_{g(t, y_1)}^{g(s, y_1)} g_y(s, f(s, \xi))^2 f_x^1(s, d\xi), \quad I_2 = \int_{g(s, y)}^{g(t, y)} g_y(s, f(s, \xi))^2 f_x^1(s, d\xi), \\ I_3 &= \int_{g(t, y_1)}^{g(t, y)} (g_y(t, f(t, \xi))^2 - g_y(s, f(s, \xi))^2) f_x^1(t, d\xi), \quad \text{and} \\ I_4 &= \int_{g(t, y_1)}^{g(t, y)} g_y(s, f(s, \xi))^2 (f_x^1(t, d\xi) - f_x^1(s, d\xi)) \end{aligned}$$

Due to boundedness of  $g$  and  $g_y$  and Lipschitz continuity of  $g, g_y$  and  $f_x$ , the absolute values of the expressions  $I_1, I_2$  and  $I_3$  are easily seen to be bounded by a constant multiple of  $|t - s|$ . Additionally, the Lipschitz property of the total-variation functional allows us to conclude the same for  $I_4$ . Consequently, there exists a constant  $C$  such that  $|g_y^2(t, y) - g_y^2(s, y)| \leq C |t - s|$ , for all  $y \in [y_1, y_2]$ .

Finally, to show that  $g$  is a semimartingale function, it suffices to check that both  $g^1$  and  $g^2$  satisfy conditions 1.-3. of Theorem A.2. The increase of the functions  $h^1(t, \cdot)$

and  $h^2(t, \cdot)$  implies that  $g^1(t, \cdot)$  and  $g^2(t, \cdot)$  are convex. Lipschitz-continuity of  $g$  and  $g^2$  in the time variable implies the same for  $g^1 = g + g^2$ . Finally, the derivatives  $g_y^1$  and  $g_y^2$  are continuous due to the continuity of functions  $(f^1)_x(t, \cdot)$  and  $(f^2)_x(t, \cdot)$ .  $\square$

*The multiplicative decomposition of positive semimartingales.* A key step in the transition from abstract to stochastic equilibria is the multiplicative decomposition of the pricing functional which enforces the abstract equilibrium. In this paragraph we give sufficient conditions on a positive semimartingale in order for the local martingale part in its multiplicative decomposition to be, in fact, a uniformly integrable martingale.

The following proposition establishes some useful stability properties of the condition  $\mathcal{N}(X) \in \mathbb{L}^\infty$ .

**Proposition A.7.**

- (1) Let  $X^1$  and  $X^2$  be semimartingales, and let  $X = \min(X^1, X^2)$ . If  $\mathcal{N}(X^1) \in \mathbb{L}^\infty$  and  $\mathcal{N}(X^2) \in \mathbb{L}^\infty$ , then  $\mathcal{N}(X) \in \mathbb{L}^\infty$ .
- (2) Suppose  $f : [0, T] \times I \rightarrow \mathbb{R}$  is a function verifying the conditions of Theorem A.2, and  $X$  is a bounded positive semimartingale, bounded away from 0, such that  $\mathcal{N}(X) \in \mathbb{L}^\infty$ . Then the process  $Y$ , defined by  $Y_t = f(t, X_t)$ , satisfies  $\mathcal{N}(Y) \in \mathbb{L}^\infty$ .

*Proof.*

- (1) Let  $X = M + A$ ,  $X^i = M^i + A^i$ ,  $i = 1, 2$  be the semimartingale decompositions of  $X$ ,  $X^1$  and  $X^2$ . The Meyer-Itô formula (see Theorem 70, p. 214 in [Pro04]) states that  $M_t = \int_0^t \mathbf{1}_{\{X_{s-}^1 \leq X_{s-}^2\}} dM_s^1 + \int_0^t \mathbf{1}_{\{X_{s-}^1 > X_{s-}^2\}} dM_s^2$ , so  $\langle M, M \rangle_T \leq \langle M^1, M^1 \rangle_T + \langle M^2, M^2 \rangle_T \in \mathbb{L}^\infty$ .
- (2) Assume that  $X$  takes values in  $[\varepsilon, 1/\varepsilon]$ , for some  $\varepsilon > 0$ . It suffices to note that the right-continuous function  $f_{x+}(t, x)$  is bounded on the compact set  $[\varepsilon, 1/\varepsilon] \times [0, T]$ , and that Proposition A.2 implies that the local martingale part of the semimartingale  $Y_t$  is given by  $\int_0^t f_{x+}(s, X_{s-}) dM_s$ .

$\square$

**Proposition A.8.** Let  $X$  be a positive semimartingale bounded from above and away from zero, such that  $\mathcal{N}(X) \in \mathbb{L}^\infty$ . Then  $X$  admits a multiplicative decomposition  $X = \hat{Q}\beta$  where  $\beta$  is a positive predictable process of finite variation, and  $\hat{Q}$  is a positive uniformly integrable martingale.

*Proof.* Without loss of generality we assume  $X_0 = 1$ . By Theorem 8.21, p. 138 in [JS03], along with the semimartingale decomposition  $X = M + A$ ,  $X$  also admits a multiplicative decomposition of the form  $X = \hat{Q}\beta$ . The same theorem states that  $\hat{Q} =$

$\mathcal{E}(\hat{M})$  and  $1/\beta = \mathcal{E}(\hat{A})$ , where  $\hat{M}_t = \int_0^t H_s dM_s$ ,  $\hat{A}_t = -\int_0^t H_s dA_s$ , and  $H = \frac{1}{X_- + \Delta A}$  is the reciprocal of the predictable projection  ${}^p(X)$  of  $X$ . For a constant  $\varepsilon > 0$  such that  $\varepsilon \leq X \leq 1/\varepsilon$  we obviously have  $\varepsilon \leq H \leq 1/\varepsilon$ , a.s. Thanks to the boundedness of  $H$ , the compensator  $\langle \hat{M}, \hat{M} \rangle$  of the quadratic variation  $[\hat{M}, \hat{M}]$  satisfies  $\langle \hat{M}, \hat{M} \rangle_T = \int_0^T H_u^2 d\langle M, M \rangle_u \leq 1/\varepsilon^2 \langle M, M \rangle_T = \mathcal{N}(X) \in \mathbb{L}^\infty$ . The conclusion now follows from Théorème 1, p. 147 in [MS79], aided by the fact that absolute values  $|\Delta M|$  of the jumps of the local martingale  $M$  are uniformly bounded (see Lemma 4.24, p. 44 in [JS03]).  $\square$

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