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Abstract We propose a valuation method for financial assets subject to default risk, where investors cannot observe the state variable triggering the default but observe a correlated price process. The model is sufficiently general to encompass a large class of structural models and can be seen as a generalization of the model of Duffie and Lando (Econometrica 69:633–664, 2001). In this setting we prove that the default time is totally inaccessible in the market's filtration and derive the conditional default probabilities and the intensity process. Finally, we provide pricing formulas for default-sensitive claims and illustrate in particular examples the shapes of the credit spreads.

Keywords Imperfect information · Default time · Hazard process

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1 Introduction

Explaining the components of credit risk reflected in corporate bond yield spreads is certainly one of the most important questions in the credit world. An important direction of research was provided by the seminal work of Merton [21] that pioneered the "structural" representation in credit risk modeling. Its crucial assumption is that both equity and bond markets react to a same underlying indicator of the financial health, usually the ratio between the market value of the firm's assets and the obligations payable at maturity. The default event is defined as a stopping time for this indicator process and defaultable securities are seen as derivatives written on the firm's value. Then, classical option pricing theory is used in order to price debt and derive spreads. Hence, the market value of the firm is a key input of these structural models.

However, in most practical situations, investors face incomplete information regarding the firm's assets which become the object of estimations based on the accounting reports and the available market information. For instance, Moody's KMV commercial software, which uses a Merton-derived framework for computing the expected default frequencies, proposes the determination of the firm's asset value from the observed equity price and equity volatility in the context of the existing balance sheet.

Using an estimation of the firm's value within the Black–Scholes framework for the pricing of options embedded in the defaultable claims may represent an important drawback attached to this type of approach. The Black–Scholes framework assumes that the underlying asset is traded. Hence, when applied to the defaultable claims, it ignores the risks attached to the estimation of the firm's value, risks that in practice should bear some premium for investors, since it is not hedgeable.

Moreover and probably in relation to this problem, most structural models fail to produce spreads consistent with empirical observations: they predict lower spreads that decrease to zero for short maturities as documented by [17]. From a modeling point of view, this drawback comes from the fact that the valuation takes place in the filtration of the firm's assets, where the default event is a predictable stopping time. Being predictable, short-term default risk is not priced by the models; nevertheless, in the real world, investors price the risk of unexpected defaults.

Motivated by these observations, an emerging class of literature aims to explicitly model imperfect information, while keeping the structural economic explanation of the default event. Representatives of this approach are Duffie and Lando [8], where investors are supposed to observe at discrete time intervals the assets value plus a noise; Giesecke and Goldberg [12], where the asset's default point is not observed; Çetin et al. [5], where only the sign of the fundamental process is supposed to be publicly known. In Jeanblanc and Valchev [15], a study of the credit spreads for different types of discrete information is proposed, while Guo et al. [13] model a delayed information arrival, which can be either discrete or continuous. See also Giesecke [11] for an overview of the different models of information imperfections within the structural modeling.

In most of these models, the default time is totally inaccessible in the market filtration and has an intensity determined endogenously as a function of the firm's characteristics and the type of information available in the credit risk market. Hence, the models can account for the short-term uncertainty inherent to the credit market and predict higher spreads for short maturities than the original structural ones.

In this paper, we propose an alternative model with noisy information, where the market continuously observes a process correlated with the unobserved indicator of the credit quality. One may think of the indicator process as the asset's value and the observation process as the equity market price or an estimation of the firm's value using available market and accounting information. Other interpretations remain possible.

We believe that the continuous-time framework is more suitable to model public information about firms. On the one hand, we can observe in practice that crucial pieces of news regarding firms, such as earnings figures, do not only appear in a discrete manner, but also in a continuous way, e.g., LBOs, restructurations, or drug approval/rejection in the case of a biotech company. On the other hand, the continuous-time framework makes it possible to directly use a price process as synthesizing public information about a firm, the way KMV uses equity prices to obtain the estimation of the firm's value.

Note that the idea of modeling a correlated diffusion as an observation process for investors also appears in Kusuoka [19], in a filtering model where the drift of the observation process contains information about the state variable triggering the default. However, as the author points out, a pricing method in that framework is difficult since the so-called (H)-hypothesis is not valid.

Our final goal being to price general credit-related claims, we propose a construction making the (H)-hypothesis valid. Our model encompasses a large class of continuous diffusions representing the fundamental process triggering default. In this general framework, the default time is proved to be a totally inaccessible stopping time. We give explicit formulas for the conditional default probabilities and the intensity process and point out that they are non-Markovian. This feature may shed some light on the role of imperfect information on observed phenomena such as the "rating momentum," i.e., the dependence of the new rating on the last few ones.

Finally, we propose a pricing method in the spirit of Elliott et al. [9] but compatible with the imperfect information model and relying on the loss-given-default process. This may be convenient in practice, since the loss-given-default is generally subject to separate estimations by the financial institutions, as recommended by the Basel Committee. The proposed formulas also have a simple financial interpretation: an asset affected by default risk acts like an otherwise similar default-free asset but paying a flow of "negative dividends," which are proportions of the loss-given-default process.

The remainder of the paper is organized as follows. Section 2 presents the modeling assumptions in the general case. Section 3 exhibits the characterization of the conditional default probabilities. In Sect. 4, we provide pricing formulas of defaultable bonds and other default-sensitive claims which are compatible with the imperfect information model. Finally, Sect. 5 is dedicated to the analysis of some particular examples of credit spreads.

2 The model

We suppose that the randomness of the economy is represented by a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. All the filtrations introduced through the paper will be completed by the inclusion of the $\mathbb{P}$-null sets. The probability $\mathbb{P}$ is supposed to be the "real" probability measure as opposed to the "risk-adjusted" probability measure, since firm's fundamentals such as accounting indicators as well as events affecting the whole credit and equity markets are observed in the "real world." We do not introduce any change of measure before Sect. 4.

Our goal is to explain the credit spreads by two factors: (i) a fundamental process of the credit quality (for instance, the firm's assets or cash flows), (ii) the characteristics of the information of market participants with regard to the fundamental process. Since they are not our first focus, default-free interest rates are supposed to be deterministic.

2.1 The structural assumptions

As in common structural models, we consider an economy where the corporate default risk is measured by the distance of some fundamental process to a default threshold. Typically, such a fundamental process is the total value of assets or, alternatively, total cash flows, and the default barrier represents a debt-covenant violation depending on the liability structure of the firm. We concentrate our analysis on a single firm in this economy and denote by $X = (X_t)_{t\geq 0}$ its fundamental process and by $b(t)$, $t \geq 0$, the default threshold supposed to be a continuous function of time with $b(0) < X_0$. The default time is

$$\tau = \inf\{t : X_t \leq b(t)\}.$$

In addition, we state the following fundamental hypotheses:

(**A**) The fundamental process which triggers the default is assumed to be of the form $X_t = F(B_t, t)$, where B is a Brownian motion, the function F is continuous, and $x \to F(x, t)$ can be inverted. Without any loss of generality, we will assume that $x \to F(x, t)$ is increasing.

For simplicity, we choose to work under condition (**A**). However, via a deterministic time-change, our results apply to the class of diffusions characterized as follows:

(**A**$'$) The process X is of the form $X_t = F(m_t, t)$, where $(m_t)_{t\geq 0}$ is a Gaussian martingale and F a continuous function invertible with respect to the first argument.

2.2 The market information

We define the observation process $Y = (Y_t)_{t\geq 0}$ as the observation of the agent, e.g., a price process correlated with the fundamental value X, this latter being unobservable. We suppose that the process Y follows a diffusion of the type

$$dY_t = \mu(Y_t, t)\,dt + \sigma(Y_t, t)\,dB_t + s(Y_t, t)\,dB'_t \tag{2.1}$$

$$= \mu(Y_t, t)\,dt + \sigma_1(Y_t, t)\,dB_t, \tag{2.2}$$

$$Y_0 = y_0,$$

where B and B' are two independent Brownian motions.

We denote by $(\mathcal{G}_t)_{t\geq 0}$ the natural filtration of the pair (B, B'). The process β defined as

$$\beta_t = \int_0^t \frac{\sigma(Y_u, u)\,dB_u + s(Y_u, u)\,dB'_u}{\sigma_1(Y_u, u)}$$

with $\sigma_1(y, t) = \sqrt{\sigma(y, t)^2 + s(y, t)^2}$ is a $(\mathcal{G}_t)$-Brownian motion, since it is a $(\mathcal{G}_t)$-martingale with bracket t.

In (2.1) and (2.2), we suppose that the functions $s(y, t)$ and $\sigma_1(y, t)$ are strictly positive on $\mathcal{Y} \times [0, \infty)$ with $\mathcal{Y}$ being the domain of the process Y. Note that $\sigma(y, t)$ is allowed to be negative in order to capture negative correlation between X and Y, a phenomenon documented for instance in [14]. Also, (2.2) is supposed to have a strong solution, i.e., adapted to the filtration of β completed with respect to $\mathbb{P}$.

We require that market investors be able to observe both the process β, whose filtration is denoted by

$$(\mathcal{F}_t)_{t\geq 0} := \sigma(\beta_s, s \leq t)_{t\geq 0},$$

and the default state, so that the market information filtration $(\mathcal{F}^\tau_t)_{t\geq 0}$ is such that, for every $t \geq 0$,

$$\mathcal{F}^\tau_t := \mathcal{F}_t \vee \sigma(s \wedge \tau, s \leq t).$$

In short, we have constructed three different nested filtrations, namely,

$$\mathcal{F}_t \subset \mathcal{F}^\tau_t \subset \mathcal{G}_t$$

for $t \geq 0$. In addition, the default time satisfies the following property.

Lemma 2.1 *The default time τ is a $(\mathcal{G}_t)$-predictable stopping time; it is an $(\mathcal{F}^\tau_t)$-totally inaccessible stopping time for ordinary market investors and is not an $(\mathcal{F}_t)$-stopping time.*

Note that by construction $(\mathcal{F}^\tau_t)$ is the smallest filtration containing $(\mathcal{F}_t)$ and making τ a stopping time. We postpone to the Appendix the proof of this lemma, as it makes use of some results to be developed in the next section.

3 Immersion of filtrations and the $(\mathcal{F}_t)$-conditional default probabilities

A filtration $(\mathcal{F}_t)$ is said to be *immersed* in some larger filtration when all $(\mathcal{F}_t)$- martingales keep this property in the larger filtration. In default models, one often encounters the immersion of $(\mathcal{F}_t)$ in $(\mathcal{F}^\tau_t)$, a property also known as the (H)-*hypothesis*. This property is essential for the valuation of securities and also has important mathematical consequences, which were first studied in [3]. Let us introduce U as the optional projection of the process $1_{(\tau\le t)}$ onto the filtration $(\mathcal{F}_t)$. Then

$$U_t = \mathbb{P}(\tau \le t|\mathcal{F}_t) \quad \text{a.s.}, \tag{3.1}$$

and an equivalent formulation of the (H)-hypothesis is that

$$\mathbb{P}(\tau \le t|\mathcal{F}_t) = \mathbb{P}(\tau \le t|\mathcal{F}_\infty) \quad \text{for all } t, \tag{3.2}$$

meaning that the process U is increasing.

The following proposition shows that, in our framework, $(\mathcal{F}_t)$ is not only immersed in $(\mathcal{F}_t^\tau)$ but also in the larger filtration $(\mathcal{G}_t)$.

Proposition 3.1 *Any $(\mathcal{F}_t)$-local martingale is also a $(\mathcal{G}_t)$-local martingale, hence an $(\mathcal{F}_t^\tau)$-local martingale.*

Proof If M is an $(\mathcal{F}_t)$-local martingale, there exist an $(\mathcal{F}_t)$-predictable process and a constant m such that $M_t = m + \int_0^t h_u\, dB_u$. Since the process β is a $(\mathcal{G}_t)$-Brownian motion, M is a $(\mathcal{G}_t)$-local martingale. □

In order to estimate the process U, we will show that conditionally on $\mathcal{F}_\infty$, the random time τ can be viewed as a first passage time of a Gaussian process. We begin by introducing some useful processes. First, the $(\mathcal{G}_t)$-martingale D is defined by

$$D_t = \int_0^t \frac{\eta(Y_u,u)\, dB_u - s(Y_u,u)\, dB'_u}{\sigma_1(Y_u,u)} \tag{3.3}$$

with

$$\eta(y,t) = \frac{s(y,t)^2}{\sigma(y,t)}.$$

It can be checked that $d\langle\beta, D\rangle_t = 0$, which means that D and β are orthogonal. As a consequence, the $(\mathcal{G}_t)$-martingale D is not $(\mathcal{F}_t)$-adapted; we introduce $(\mathcal{D}_t)_{t\ge 0} = \sigma(D_u, u\le t)_{t\ge 0}$.

We also define two orthogonal $(\mathcal{G}_t)$-martingales M and N for $t \ge 0$ by

$$M_t = \int_0^t \frac{\sigma_1(Y_u,u)}{\sigma(Y_u,u)+\eta(Y_u,u)}\, dB_u, \tag{3.4}$$

$$N_t = \int_0^t \frac{\sigma_1(Y_u,u)}{\sigma(Y_u,u)+\eta(Y_u,u)}\, dD_u. \tag{3.5}$$

Note that M is $(\mathcal{F}_t)$-adapted. From $B_t = M_t + N_t$ and

$$B'_t = \int_0^t \frac{\eta(Y_u,u)}{s(Y_u,u)}\, dM_u - \int_0^t \frac{\sigma(Y_u,u)}{s(Y_u,u)} dN_u$$

we deduce that

$$\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{D}_t, \quad t \geq 0.$$

We find it convenient to introduce for the conditional probability the notation

$$\forall A \in \mathcal{F}, \quad \tilde{\mathbf{P}}(A) = \mathbb{P}(A|\mathcal{F}_\infty).$$

Lemma 3.2 *The process $(N_t)_{t\geq 0}$ is a $\tilde{\mathbf{P}}$-Gaussian martingale.*

Proof It follows from $\langle \beta, D\rangle = 0$ that $\langle \beta, N\rangle = 0$, hence N and β are orthogonal. From Knight's theorem we know that there exists a $(\mathcal{G}_t)$-Brownian motion W independent from β such that $N_t = W_{\langle N\rangle_t}$, $t \geq 0$. Since all $\langle N\rangle_t, t \geq 0$, are $\mathcal{F}_\infty$-measurable, we conclude that conditionally on $\mathcal{F}_\infty$, the process $(N_t)_{t\geq 0}$ is a deterministic time-changed Brownian motion, i.e., a Gaussian martingale. □

We are now able to state one of our main results derived in particular from assumption (**A**) introduced in Sect. 1.

Proposition 3.3 *Under assumption* (**A**), *the* $(\mathcal{F}_t)$*-conditional default probability at time* t *can be approximated via*

$$U_t = \mathbb{P}(\tau \leq t|\mathcal{F}_t) = \lim_{k\to\infty} \sum_{i=1}^{k} q_i, \tag{3.6}$$

where, for every fixed k, *we set* $\Delta t = t/k$ *and, for* $i = 2, \dots, k$ *and* $j = 1, \dots, i-1$,

$$q_1 = \Phi(a_1), \qquad q_i = \Phi(a_i) - \sum_{j=1}^{i-1} \Phi(b_{i,j}) q_j,$$

$$a_i = \frac{c_{i\Delta t}}{\sqrt{\langle N\rangle_{i\Delta t}}}, \qquad b_{i,j} = \frac{c_{i\Delta t} - c_{j\Delta t}}{\sqrt{\langle N\rangle_{i\Delta t} - \langle N\rangle_{j\Delta t}}},$$

with

$$c_t = F^{-1}\big(b(t), t\big) - M_t,$$

and where Φ *denotes the cumulative function of the standard normal law.*

Proof We begin by recalling an important result involving the first passage time of a Brownian motion through a continuous barrier (for developments on the subject, see [4, 10] and, more recently, [22]). Let $(W_t)_{t\geq 0}$ be a $\mathbb{P}$-Brownian motion and h a continuous function with $h(0) < 0$. We introduce the hitting time

$$T_h = \inf\big\{t : W_t \leq h(t)\big\}$$

and its distribution function

$$\pi^{\mathbb{P},h}(t) := \mathbb{P}(T_h \leq t), \quad t \geq 0.$$

According to Fortet [10], Sect. II.3, the function $\pi^{\mathbb{P},h}(t)$ satisfies the integral equation

$$\Phi\big(h(t)/\sqrt{t}\,\big) = \int_0^t \Phi\left(\frac{h(t)-h(u)}{\sqrt{t-u}}\right) d\pi^{\mathbb{P},h}(u). \tag{3.7}$$

As a consequence of Lemma 3.2, there exists a $\tilde{\mathbf{P}}$-Brownian motion W such that $N_t = W_{\langle N\rangle_t}$ for any $t \geq 0$. Define $c_t := F^{-1}(b(t), t) - M_t$. Finally, note that both (c_t) and the quadratic variation $\langle N\rangle$ are $\tilde{\mathbf{P}}$-a.s. deterministic. Hence, there exists a process $h(t)$ which also is $\tilde{\mathbf{P}}$-a.s. deterministic and such that $c_t = h(\langle N\rangle_t)$.

Now note that

$$\big\{X_t < b(t)\big\} = \big\{B_t < F^{-1}\big(b(t), t\big)\big\} = \{N_t < c_t\} = \big\{W_{\langle N\rangle_t} < h\big(\langle N\rangle_t\big)\big\}.$$

Hence, using first equality (3.2), we may write that

$$U_t = \mathbb{P}(\tau \leq t|\mathcal{F}_\infty) = \tilde{\mathbf{P}}\big(\exists u \in [0, t] : X_u < b(u)\big) = \tilde{\mathbf{P}}\big(\exists u \in \big[0, \langle N\rangle_t\big] : W_u < h(u)\big).$$

Consequently,

$$U_t = \pi^{\tilde{\mathbf{P}},h}\big(\langle N\rangle_t\big). \tag{3.8}$$

We may now apply Fortet's integral equation (3.7) and obtain

$$\Phi\left(\frac{h(\langle N\rangle_t)}{\sqrt{\langle N\rangle_t}}\right) = \int_0^{\langle N\rangle_t} \Phi\left(\frac{h(\langle N\rangle_t) - h(s)}{\sqrt{\langle N\rangle_t - s}}\right) d\pi^{\tilde{\mathbf{P}},h}(s).$$

Via an obvious change of variable in the above integral, the nondecreasing process $(U_t)_{t\geq 0}$ satisfies

$$\Phi\left(\frac{c_t}{\sqrt{\langle N\rangle_t}}\right) = \int_0^t \Phi\left(\frac{c_t - c_s}{\sqrt{\langle N\rangle_t - \langle N\rangle_s}}\right) dU_s. \tag{3.9}$$

A more intuitive form of the above integral equation, useful later on, is

$$\tilde{\mathbf{P}}(N_t \leq c_t) = \int_0^t \tilde{\mathbf{P}}(N_t - N_s \leq c_t - c_s)\, dU_s. \tag{3.10}$$

The last step to approximate U_t is to discretize the time interval $[0, t]$ in (3.9) into n equal subintervals of length Δt. This kind of approximation of the first passage time density was already employed in default models (see [6]) as well as insurance models [1].[1] We define $a_i = \frac{c_{i\Delta t}}{\sqrt{\langle N\rangle_{i\Delta t}}}$ and $b_{i,j} = \frac{c_{i\Delta t} - c_{j\Delta t}}{\sqrt{\langle N\rangle_{i\Delta t} - \langle N\rangle_{j\Delta t}}}$ for $j < i$ and $b_{i,i} = \infty$. We approximate (3.9) by considering that default may only occur at the ends of the subperiods. Thus, we write

$$\Phi(a_n) = \sum_{i=1}^n \Phi(b_{n,i})\mathbb{P}\big(\tau \in \big((i-1)\Delta t, i\Delta t\big]\big|\mathcal{F}_\infty\big). \tag{3.11}$$

[1] See [2] and the references therein for a study of the convergence of this type of approximations.

From the above equality we derive a recursive system of n equations with n unknowns

$$q_i := \mathbb{P}\big(\tau \in \big((i-1)\Delta t, i\Delta t\big]\big|\mathcal{F}_\infty\big), \quad i = 1, \ldots, n.$$

Thus, for the first time interval, we have

$$\Phi(a_1) = \mathbb{P}\big(\tau \in (0, \Delta t]\big|\mathcal{F}_\infty\big) = q_1.$$

For two intervals, we have

$$\begin{aligned}\Phi(a_2) &= \Phi(b_{2,1})\mathbb{P}\big(\tau \in (0, \Delta t]\big|\mathcal{F}_\infty\big) + \mathbb{P}\big(\tau \in (\Delta t, 2\Delta t]\big|\mathcal{F}_\infty\big)\\ &= \Phi(b_{2,1})q_1 + q_2.\end{aligned}$$

Solving this system by iteration leads to the stated solution in (3.6). □

For pricing purposes, one also needs an estimate of the $(\mathcal{F}_t)$-conditional probability that default occurs prior to a fixed time $T > t$, T being the maturity of a claim. Indeed, the following formula (see [7], III.41, p. 58) will allow us to go from $(\mathcal{F}_t)$-conditional probabilities to $(\mathcal{F}_t^\tau)$-conditional probability and vice versa:

$$\mathbb{P}(t < \tau \le T|\mathcal{F}_t^\tau) = \mathbf{1}_{(\tau>t)} \frac{\mathbb{P}(t < \tau \le T|\mathcal{F}_t)}{\mathbb{P}(t > \tau|\mathcal{F}_t)}.$$

Proposition 3.4 *Under assumption* (**A**), *the* $(\mathcal{F}_t)$*-conditional default probability on the interval* $(t, T]$ *is given by the formula*

$$\mathbb{P}(t < \tau \le T|\mathcal{F}_t) = \lim_{n\to\infty} \sum_{j=1}^{n} p_j, \tag{3.12}$$

where, for every fixed n, *we set* $\Delta t = (T-t)/n$, $k = [t/\Delta t]$, *and*

$$p_1 = \int_{-\infty}^{\infty} \left[\Phi(A_1) - \sum_{i=1}^{k} \Phi(C_{i,1})q_i\right]\varphi(x)\,dx,$$

$$p_j = \int_{-\infty}^{\infty} \left[\Phi(A_j) - \sum_{i=1}^{k} \Phi(C_{i,j})q_i\right]\varphi(x)\,dx - \sum_{i=1}^{j-1} p_i\Phi(B_{i,j}), \quad j = 2, \ldots, n,$$

with

$$A_i = \frac{\tilde{F}(t+i\Delta t) - M_t - x\sqrt{i\Delta t}}{\sqrt{\langle N\rangle_t}}, \quad 1 \le i \le n,$$

$$B_{i,j} = \frac{\tilde{F}(t+j\Delta t) - \tilde{F}(t+i\Delta t)}{\sqrt{(j-i)\Delta t}}, \quad 0 \le i < j \le n,$$

$$C_{i,j} = \frac{\tilde{F}(t+j\Delta t) - M_t - c_{i\Delta t} - x\sqrt{j\Delta t}}{\sqrt{\langle N\rangle_t - \langle N\rangle_{i\Delta t}}}, \quad 1 \le i \le k-1,\ 1 \le j \le n,$$

$$C_{k,j} = B_{0,j} \quad \textit{and} \quad \tilde{F}(t) = F^{-1}\big(b(t), t\big).$$

In the above, φ *denotes the density function of a standard normal distribution, and* $[x]$ *is the integer part of* x. *The variables* q_i, $i = 1, \dots, k$, *are defined in Proposition* 3.3.

We begin with an auxiliary result.

Lemma 3.5 *Set*

$$\Theta(T,s) := \tilde{\mathbf{P}}(N_T - N_s \le c_T - c_s).$$

For $t < T$, *we then have*

$$\mathbb{P}(N_T \le c_T | \mathcal{F}_t) = \int_0^t \mathbb{E}\big[\Theta(T,s)\big|\mathcal{F}_t\big]\, dU_s + \int_t^T \mathbb{E}\big[\Theta(T,s)\big]\, d_s\mathbb{P}(\tau \le s|\mathcal{F}_t). \quad (3.13)$$

Proof Using (3.10) for the interval $[0,T]$ and conditioning with respect to $\mathcal{F}_t$, we obtain

$$\begin{aligned}\mathbb{P}(N_T \le c_T | \mathcal{F}_t) &= \mathbb{E}\bigg(\int_0^T \Theta(T,s)\, dU_s \bigg| \mathcal{F}_t\bigg)\\ &= \int_0^t \mathbb{E}\big(\Theta(T,s)\big|\mathcal{F}_t\big)\, dU_s + \mathbb{E}\bigg(\int_t^T \Theta(T,s)\, dU_s \bigg| \mathcal{F}_t\bigg).\end{aligned}$$

The result follows by writing the integral in the last term as the limit of a sum, applying dominated convergence, and using the fact that, for $s < T$, $\mathbb{E}[\Theta(T,s)|\mathcal{F}_s] = \Phi(\frac{\tilde{F}(T) - \tilde{F}(s)}{\sqrt{T-s}}) = \mathbb{E}[\Theta(T,s)]$. □

Proof of Proposition 3.4 We shall use the results and notation of the proof of Proposition 3.3. First, we find an expression for the expectations to be computed in the integral equation (3.13). The left-hand side equals

$$\begin{aligned}\mathbb{P}\big(B_T \le \tilde{F}(T)|\mathcal{F}_t\big) &= \mathbb{E}\big(\mathbb{P}\big(B_T - B_t \le \tilde{F}(T) - B_t|\mathcal{G}_t\big)\big|\mathcal{F}_t\big)\\ &= \mathbb{E}\bigg(\Phi\bigg(\frac{\tilde{F}(T) - B_t}{\sqrt{T-t}}\bigg)\bigg|\mathcal{F}_t\bigg)\\ &= \int_{-\infty}^{\infty} \mathbb{P}\bigg(x \le \frac{\tilde{F}(T) - B_t}{\sqrt{T-t}}\bigg|\mathcal{F}_t\bigg)\varphi(x)\, dx\\ &= \int_{-\infty}^{\infty} \mathbb{P}\big(N_t \le \tilde{F}(T) - M_t - x\sqrt{T-t}\big|\mathcal{F}_t\big)\varphi(x)\, dx\\ &= \int_{-\infty}^{\infty} \Phi\bigg(\frac{\tilde{F}(T) - M_t - x\sqrt{T-t}}{\sqrt{\langle N\rangle_t}}\bigg)\varphi(x)\, dx.\end{aligned}$$

Let us now turn to the right-hand side of (3.13). In the case $s < t$, we write, mimicking the above computation,

$$\begin{aligned}\mathbb{E}\big[\Theta(T,s)\big|\mathcal{F}_t\big] &= \mathbb{E}\bigg[\Phi\bigg(\frac{\tilde{F}(T) - \tilde{F}(s) + B_s - B_t}{\sqrt{T-t}}\bigg)\bigg|\mathcal{F}_t\bigg]\\ &= \int_{-\infty}^{\infty} \Phi\bigg(\frac{\tilde{F}(T) - M_t - c_s - x\sqrt{T-t}}{\sqrt{\langle N\rangle_t - \langle N\rangle_s}}\bigg)\varphi(x)\, dx.\end{aligned}$$

Plugging all these results into (3.13) and rearranging terms leads to

$$\begin{aligned}
&\int_t^T \Phi\left(\frac{\tilde{F}(T)-\tilde{F}(s)}{\sqrt{T-s}}\right) d_s\mathbb{P}(\tau \le s|\mathcal{F}_t) \\
&\quad = \int_{-\infty}^{\infty}\left\{\Phi\left(\frac{\tilde{F}(T)-M_t-x\sqrt{T-t}}{\sqrt{\langle N\rangle_t}}\right)\right. \\
&\qquad \left.-\int_0^t \Phi\left(\frac{\tilde{F}(T)-M_t-c_s-x\sqrt{T-t}}{\sqrt{\langle N\rangle_t-\langle N\rangle_s}}\right) dU_s\right\}\varphi(x)\,dx. \qquad (3.14)
\end{aligned}$$

The integrals above may be approximated by sums if we partition the interval $[0,t]$ into k subperiods and the interval $[t,T]$ into n subperiods and then let both k and n go to infinity. To keep notation simple, we choose to introduce for both intervals $[0,t]$ and $[t,T]$ subperiods of equal length $\Delta t=(T-t)/n$ by setting $k=[t/\Delta t]$. We mention however that, in a practical implementation, k can of course be chosen independently from n.

As before, we suppose that default may only occur at the ends of the subperiods. Let

$$p_j=\mathbb{P}\big(\tau\in\big(t+(j-1)\Delta t, t+j\Delta t\big]\big|\mathcal{F}_t\big), \quad j=1,\dots,n,$$

and, as in Proposition 3.3,

$$q_i=\mathbb{P}\big(\tau\in(i-1)\Delta t, i\Delta t\big|\mathcal{F}_\infty\big), \quad i=1,\dots,k.$$

As an approximation of (3.14), we obtain

$$\begin{aligned}
&\sum_{i=1}^n \Phi\left(\frac{\tilde{F}(t+n\Delta t)-\tilde{F}(t+i\Delta t)}{\sqrt{(n-i)\Delta t}}\right)p_i \\
&\quad = \int_{-\infty}^{\infty}\left\{\Phi\left(\frac{\tilde{F}(t+n\Delta t)-M_t-x\sqrt{n\Delta t}}{\sqrt{\langle N\rangle_t}}\right)\right. \\
&\qquad \left.-\sum_{i=1}^k \Phi\left(\frac{\tilde{F}(t+n\Delta t)-M_t-c_{i\Delta t}-x\sqrt{n\Delta t}}{\sqrt{\langle N\rangle_t-\langle N\rangle_{i\Delta t}}}\right)q_i\right\}\varphi(x)\,dx,
\end{aligned}$$

or, in a reduced form,

$$\sum_{i=1}^n \Phi(B_{i,n})p_i=\int_{-\infty}^{\infty}\left\{\Phi(A_n)-\sum_{i=1}^k \Phi(C_{i,n})q_i\right\}\varphi(x)\,dx.$$

This leads us to a recursive system of n equations where the unknowns are $p_i, i=1,\dots,n$, and the quantities $q_i, i=1,\dots,k$, computed above. We obtain

$$p_1=\int_{-\infty}^{\infty}\left[\Phi(A_1)-\sum_{i=1}^k \Phi(C_{i,1})q_i\right]\varphi(x)\,dx,$$

$$p_2 = \int_{-\infty}^{\infty} \left[\Phi(A_2) - \sum_{i=1}^{k} \Phi(C_{i,2}) q_i \right] \varphi(x)\, dx - p_1 \Phi(B_{1,2}),$$

$$p_n = \int_{-\infty}^{\infty} \left[\Phi(A_n) - \sum_{i=1}^{k} \Phi(C_{i,n}) q_i \right] \varphi(x)\, dx - \sum_{i=1}^{n-1} p_i \Phi(B_{i,n}). \qquad \square$$

Remark 3.6 (Condition ($\mathbf{A}'$)) As announced in the previous section, similar formulas can be obtained for the more general class of diffusions defined via condition ($\mathbf{A}'$), namely $X_t = F(m_t, t)$, where m is a $(\mathcal{G}_t)$-martingale of the form $m_t = \int_0^t h(s)\, dB_s$ with h being a Borel function. We remark that we can recover condition (**A**) via a deterministic time-change, since m is Gaussian. The corresponding default probabilities are simply obtained by replacing in the above formulas N_t by $N'_t = \int_0^t h(s)\, dM_s$ and M_t by $M'_t = \int_0^t h(s)\, dM_s$. Moreover, we need to replace in Proposition 3.4 $\sqrt{i\,\Delta t}$ with $\sqrt{\langle m\rangle_{i\,\Delta t}}$ and similarly $\sqrt{(j-i)\Delta t}$ with $\sqrt{\langle m\rangle_{(j-i)\Delta t}}$. An illustration is provided in Sect. 5 for the Ornstein–Uhlenbeck process.

At this point, a very natural question is whether in our framework an intensity process exists for the default time. This would provide a link with the reduced form default modeling and with other imperfect information models. The answer is given in the next lemma. Note however that the valuation method we introduce in the next section is not "intensity-based," but it is "intensity-compatible."

Lemma 3.7 *Suppose that the function $\tilde{F}(t) = F^{-1}(b(t), t)$ is differentiable and set $\tilde{F}'(t) = \frac{\partial \tilde{F}}{\partial t}(t)$. Then there exists an intensity process $(\lambda_t)_{t\geq 0}$ for the default time given by*

$$\lambda_t = \frac{1_{(\tau>t)}}{(1-U_t)} \tilde{\lambda}_t,$$

where

$$\tilde{\lambda}_t = \frac{\tilde{F}'(t)}{\sqrt{\langle N\rangle_t}} \varphi\left(\frac{c_t}{\sqrt{\langle N\rangle_t}}\right) - \int_0^t \frac{\tilde{F}'(t)}{\sqrt{\langle N\rangle_t - \langle N\rangle_s}} \varphi\left(\frac{c_t - c_s}{\sqrt{\langle N\rangle_t - \langle N\rangle_s}}\right) dU_s$$

Proof See Appendix. $\square$

4 Valuation of default-sensitive claims

In this section, we propose a pricing method for financial products when agents face information imperfections and the (H)-hypothesis holds, as in the previously presented model. The main idea is to consider the loss-given-default process as a default-free security. Then, we show that a default-sensitive security may be considered as a default-free security with the same characteristics but paying a flow of "negative dividends." These "dividends" are fractions of the loss-given-default process and paid whenever the $(\mathcal{F}_t)$-conditional default probability is increasing, i.e., on the set

$\{t : dU_t > 0\}$. In the case of intensity-based models, this “dividend” flow is continuous.

4.1 Assumptions on the financial market

We recall that interest rates are supposed to be deterministic.We consider an arbitrage-free financial market comprising three types of securities: default-free, default-sensitive, and defaultable, which will be defined to be compatible with our imperfect information model.

Definition 4.1 For a fixed maturity T, a *default-free $\mathcal{F}_T$-contingent claim* is a non-negative, square-integrable, $\mathcal{F}_T$-measurable random variable ξ_T. A *default-free security price* $(\xi_t)_{t\in[0,T]}$ is an $(\mathcal{F}_t)$-adapted process describing the price of a default-free $\mathcal{F}_T$-contingent claim ξ_T.

The default-free securities prices are supposed to be adapted to the filtration generated by the Brownian motion β, since the information relative to the default time does not impact default-free securities under the (H)-hypothesis.

Denote by $B(t,T)$ the discounting factor corresponding to the period $[t,T]$, i.e., $B(0,t)^{-1}$ is the value of the savings account at time t and $B(t,T) = B(0,T)/B(0,t)$. Note that this relies on the assumption that we have deterministic interest rates. We now define $\mathbb{P}^*$ as an equivalent measure under which the discounted default-free security prices are $(\mathcal{F}_t)$-martingales. When the process X represents the value of the assets of the firm, the following assumption is commonly used in structural models:

(**E**) The discounted process $(X_t\ B(0,t))_{t\geq 0}$ is a $\mathbb{P}^*$-martingale.

In the following, we assume that (**E**) and (**A**$'$) hold for the dynamics of the process X under $\mathbb{P}^*$.

Before default occurs, market participants try to infer the true value of the fundamental process X from their available information, namely the observed process Y and the news that the process X has not yet crossed the default barrier.

Definition 4.2 The *risk-neutral estimate* of the variable X_t is the $\mathcal{F}_t$-measurable random variable $\hat{X}_t$ defined by

$$\hat{X}_t := \frac{1}{1-U_t^*}\mathbb{E}^{\mathbb{P}^*}[\mathbf{1}_{(\tau>t)}X_t|\mathcal{F}_t], \tag{4.1}$$

where we set $U_t^* = \mathbb{P}^*(\tau \leq t|\mathcal{F}_t)$.

It is easy to check (see, for instance, [7], III.41, p. 58) that

$$\mathbf{1}_{(\tau>t)}\mathbb{E}^{\mathbb{P}^*}\left[X_t\middle|\mathcal{F}_t^\tau\right] = \mathbf{1}_{(\tau>t)}\hat{X}_t. \tag{4.2}$$

Hence, before the default time, the process $\hat{X} = (\hat{X}_t)_{t\geq 0}$ represents a risk-neutral estimate of the unobservable process X given the market information, and it should play an important role for pricing, as assumed in the following definition of defaultable claims.

Definition 4.3 For a fixed maturity T, a *default-sensitive contingent claim* is an $\mathcal{F}_T^\tau$-measurable random variable of the form

$$c_T := \mathbf{1}_{(\tau>T)}\xi_T + \mathbf{1}_{(\tau\le T)}\tilde{\xi}_T,$$

where ξ_T and $\tilde{\xi}_T$ are two default-free $\mathcal{F}_T$-contingent claims. A *defaultable contingent claim* is an integrable $\mathcal{F}_T^\tau$-measurable random variable of the form

$$d_T := \mathbf{1}_{(\tau>T)} f(\hat{X}_T) + \mathbf{1}_{(\tau\le T)} g(X_\tau)/B(\tau, T), \tag{4.3}$$

where f and g are two Borel functions.

Note that in our definition of defaultable claims, we assume that in the case of default, the recovery payment $g(X_\tau)$ is immediately invested up to time T in default-free bonds.

We wish to emphasize that we consider defaultable claims as a different class from default-sensitive claims. Thus, corporate bonds, equity, and their derivatives issued by the firm are defaultable claims; in the case of default, the recovery is established as a function of the defaulted firm value and priority rules. But a portfolio containing a corporate bond secured by a credit default swap is a default-sensitive claim: the holder receives a contractual compensation in case of default, hence the portfolio is independent of the value of the defaulted firm. Nevertheless, this portfolio is not default-free, as its composition varies once the default event has occurred. Lastly, a corporate bond secured by a total return swap behaves like a default-free bond: the return on the portfolio does not depend on the occurrence of the default event.

The main difference with complete information models is that in the presence of imperfect information, we suppose the defaultable claims valuated to be using the estimation $\hat{X}_T$ whenever the firm is not in the default state. Let us provide a simple example in order to illustrate the definition of defaultable claims under imperfect information.

Example 4.4 Suppose that the firm has a simple debt structure composed of equity and zero bonds with maturity T and nominal value N. In structural models with complete information, the payoff structure used for equity (assuming zero recovery) is

$$E_T = \mathbf{1}_{(\tau>T)}(X_T - N)^+.$$

Instead, our formulation incorporates the fact that on the set $\{\tau > T\}$, the true value of assets is not revealed to investors; hence we propose the payoff

$$\hat{E}_T = \mathbf{1}_{(\tau>T)}(\hat{X}_T - N)^+.$$

4.2 The valuation methodology

Beside the different representation of the payoffs of defaultable claims, our approach fundamentally differs from the perfect information case in the way these payoffs are priced, since we shall use a different filtration.

Proposition 4.5 (Prices of default-sensitive and defaultable claims) *Suppose that, at time $t < T$, default has not yet occurred. Then the price of a default-sensitive claim is given by*

$$c_t = \xi_t - \mathbb{E}^{\mathbb{P}^*}\left[\int_t^T B(t,s)(\xi_s - \tilde{\xi}_s)\frac{dU_s^*}{1-U_t^*}\middle|\mathcal{F}_t\right],$$

where $(\xi_t - \tilde{\xi}_t)_{0\le t\le T}$ is the loss-given-default process; the price of a defaultable claim is given by

$$d_t = v_f(t) - \mathbb{E}^{\mathbb{P}^*}\left[\int_t^T B(t,s)\big[v_f(s) - g\big(b(s)\big)\big]\frac{dU_s^*}{1-U_t^*}\middle|\mathcal{F}_t\right],$$

where $v_f(t)$ is the price at date t of a default-free security with the same characteristics.

Proof $(\mathcal{F}_t^\tau)$-informed agents will valuate the payoff of default-sensitive claims under the risk-neutral expectation conditional on the filtration $(\mathcal{F}_t^\tau)$. On the set $\{\tau > t\}$, we obtain

$$\begin{aligned} c_t &= B(t,T)\mathbb{E}^{\mathbb{P}^*}\big[\mathbf{1}_{(\tau>T)}\xi_T + \mathbf{1}_{(\tau\le T)}\xi_T'\big|\mathcal{F}_t^\tau\big] \\ &= \xi_t - \frac{\mathbb{E}^{\mathbb{P}^*}[\mathbb{P}^*(t<\tau\le T|\mathcal{F}_T)\pi_T|\mathcal{F}_t]}{(1-U_t^*)B(0,t)} \end{aligned} \tag{4.4}$$

with $\pi_t = (\xi_t - \tilde{\xi}_t)B(0,t)$, $t > 0$, being the discounted loss-given-default process. We integrate by parts the product inside the expectation in (4.4), observing that the process $(\mathbb{P}^*(t<\tau\le u|\mathcal{F}_u))_{u\in(t,\infty)}$ is predictable and increasing, and obtain

$$\mathbb{P}^*(t<\tau\le T|\mathcal{F}_T)\pi_T = \int_t^T \mathbb{P}^*(t<\tau\le u|\mathcal{F}_u)\,d\pi_u + \int_t^T \pi_u\,dU_u^*.$$

The martingale property of the discounted process π implies that the first term in the above expression is an $(\mathcal{F}_t)$-local martingale. It is in fact a true martingale, since the integrand $\mathbb{P}^*(t<\tau\le u|\mathcal{F}_u)$ is bounded. Taking the $\mathcal{F}_t$-conditional expectation in the formula above, we obtain the result.

As in the case of default-sensitive claims, we can obtain the price of a defaultable claim as an expectation via

$$\begin{aligned} d_t &= B(t,T)\mathbb{E}^{\mathbb{P}^*}\left[\mathbf{1}_{(\tau>T)}f(\hat{X}_T) + \mathbf{1}_{(t<\tau\le T)}\frac{g(X_\tau)}{B(\tau,T)}\middle|\mathcal{F}_t^\tau\right] \\ &= B(t,T)\mathbb{E}^{\mathbb{P}^*}\big[f(\hat{X}_T)\big|\mathcal{F}_t\big] \\ &\quad - \frac{B(t,T)}{1-U_t^*}\mathbb{E}^{\mathbb{P}^*}\left[\int_t^T\left(f(\hat{X}_T) - \frac{g(b(u))}{B(u,T)}\right)dU_u^*\middle|\mathcal{F}_t\right]. \end{aligned}$$

The first term corresponds to the price at date t of a default-free security which pays $f(\hat{X}_T)$ at date T. We denote this price by $v_f(t)$ and obtain the second formula. □

Example 4.6 (Equity and zero-coupon bond valuation) The price $\hat{E}_t$ of equity with payoff $\hat{E}_T$ defined in Example 4.4 is given by

$$\hat{E}_t = C_t(\hat{X}, N) - \mathbb{E}^{\mathbb{P}^*}\left[\int_t^T \frac{B(t,s)C_s(\hat{X}, N)}{1-U_t^*}\, dU_s^* \middle| \mathcal{F}_t\right], \quad \tau > t,$$

where $C_t(\hat{X}, N)$ is the time t Black–Scholes price of an European call option with underlying asset $\hat{X}$ and strike price N.

Let $D(t,T)$ be the price of a defaultable zero-coupon bond with face value one dollar and maturity date T. We assume that in case of default, a recovery payment $\delta_\tau \in [0,1]$ is instantaneously made with $\delta_\tau = \frac{\alpha b(\tau)}{N}$, N representing the nominal value of all outstanding debt and $\alpha \in [0,1]$ being a parameter accounting for the seniority of the bond. Then the price of the zero-coupon bond at time t, conditional on no default, is

$$D(t,T) = B(t,T) - \mathbb{E}^{\mathbb{P}^*}\left[\int_t^T B(t,s)(1-\delta_s)\frac{dU_s^*}{1-U_t^*} \middle| \mathcal{F}_t\right], \quad \tau > t.$$

5 Numerical illustrations

We now study some particular credit spreads for the cases where the process X is either a geometric Brownian motion or an Ornstein–Uhlenbeck process. To simplify the analysis, we consider zero-recovery rates for bonds, so that, at time t, the spread for the maturity T is given by

$$-\ln\left(\frac{\mathbb{P}^*(t < \tau \le T | \mathcal{F}_t)}{1-U_t^*}\right)\Big/(T-t).$$

We also illustrate with a generalization of [12] to a stochastic unobservable barrier how new models can easily be created in our framework.

5.1 The value of the assets is a geometric Brownian motion

Suppose that the fundamental process X is the value of the assets of the firm, $X_t = F(B_t, t)$ with $F(x,t) = x_0 e^{(r-\sigma^2/2)t+\sigma x}$ (i.e., X is a geometric Brownian motion under the risk-neutral measure).

We choose a constant default barrier $b \in (0, x_0)$ and suppose for the observation process the form

$$dY_t = rY_t\, dt + \sigma_1 Y_t\, dB_t, \quad Y_0 = x_0,$$

where $\sigma_1 = \sqrt{\sigma^2 + s^2}$ and $\beta_t = \frac{\sigma B_t + s B_t'}{\sigma_1}$. This means that a noise affects the observation of the firm's returns of the assets of the firm. This situation was first analyzed by Duffie and Lando [8] (hereafter DL) with a discrete observation process. Hence, we have here a generalization for a continuous observation process.

We choose our base case parameters like those of DL, namely

$$t = 1; \qquad \sigma = 0.05; \qquad s = 0.1; \qquad r = 0.03; \qquad x_0 = 86.3; \qquad b(t) = 76.$$

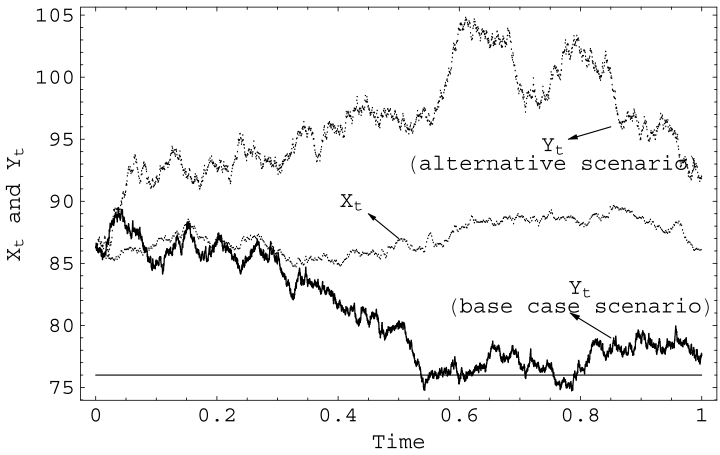

Fig. 1 The base case realization of the observed process Y (*solid path*) up to the current time $t = 1$ and an alternative realization (*dashed path*)

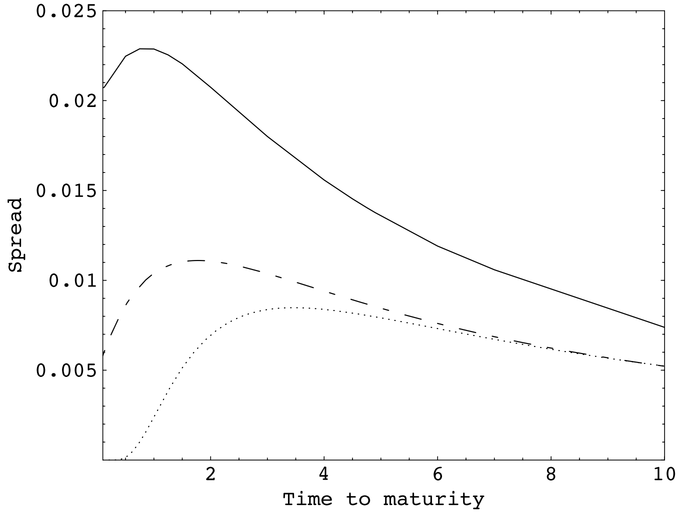

Fig. 2 Credit spreads generated by our **base case scenario** (*solid curve*), compared to spreads obtained with complete information (*dotted curve*) and to spreads generated by DL model (*dot dashed curve*)

We suppose that, at time $t = 1$, agents observe the realization of the process Y shown in Fig. 1 (solid curve); an alternative scenario for the realization of Y (dashed curve) is displayed for subsequent comparison, as is the unobservable realization of the true value of assets X.

Figure 2 displays the credit spreads generated by our model, the DL model, and the corresponding complete information model. In our model, the entire path of the process Y on [0, 1] is the available information used for computing the default probabilities; for the DL model, the available information contains only the points Y_0

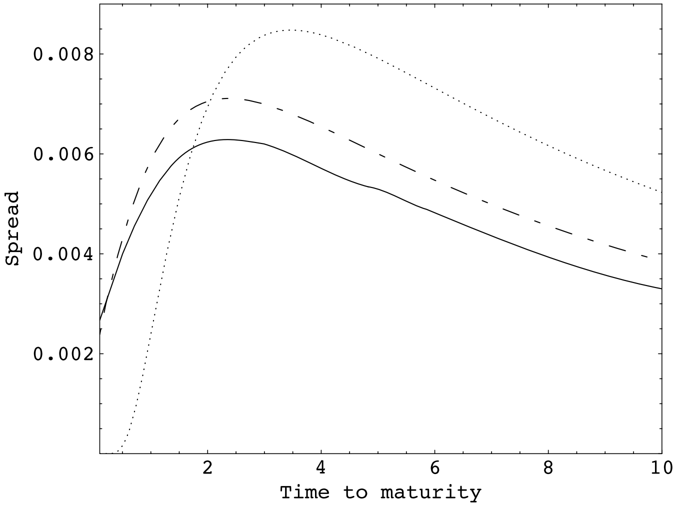

Fig. 3 Credit spreads generated by our **alternative scenario** (*solid curve*), compared to spreads obtained with complete information (*dotted curve*) and to spreads generated by DL model (*dot dashed curve*)

and Y_1; and for the complete information case, we have assumed that the process X is observable.

Because the default time is totally inaccessible, the short term spreads are always higher in our model and in DL compared to the complete information short term spreads. For the medium term, however, other configurations are possible: when the firm value is critical (X close to b), then the sole observation of Y may lead to thinner spreads for medium term than those computed with complete information. In this way, we capture the situations when default comes as a surprise and produces large losses.

Also, comparing our model with DL, we can say that our model keeps the memory of all the observed path of the process Y when computing spreads and default probabilities. Thus, if between Y_0 and Y_1 the process experiences "bad" excursions, the spreads are likely to be larger than DL; the contrary will happen if over the period of observation the process Y is "well-behaved" (see Fig. 3, generated under the alternative scenario for Y).

This path-dependence feature of default probabilities is very important. Note that it is implicit in reduced-form models, since their implementation needs calibration of the hazard process to historical data on bonds. The advantage of imperfect information models is to model explicitly the role of the past information in the current prices combined with the use of a fundamental economic process for predicting the default event. In structural models, default probabilities do not present a path-dependent feature, since the fundamental process is usually Markov, hence only the current state of the diffusion impacts the default probability. Thus, information imperfections may explain the non-Markovian patterns of the default probabilities documented in the literature, such as the "rating momentum," which is the dependence of the new rating on the previous ones instead of the last observed one (see, for instance, [18] or [20]).

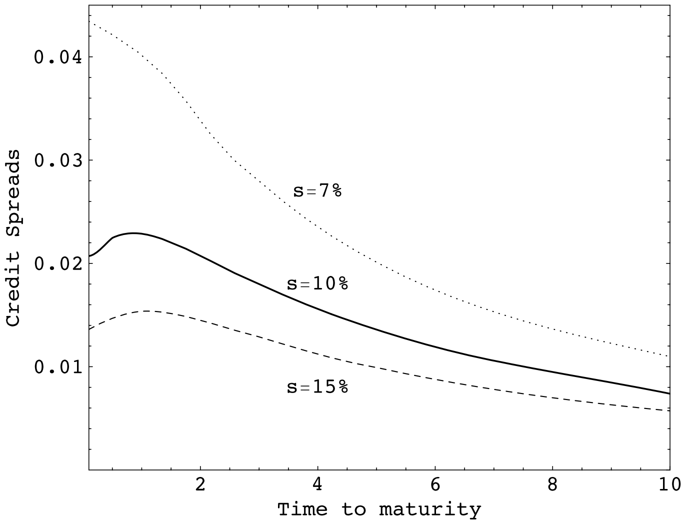

Fig. 4 Impact of the noise volatility s on the credit spreads

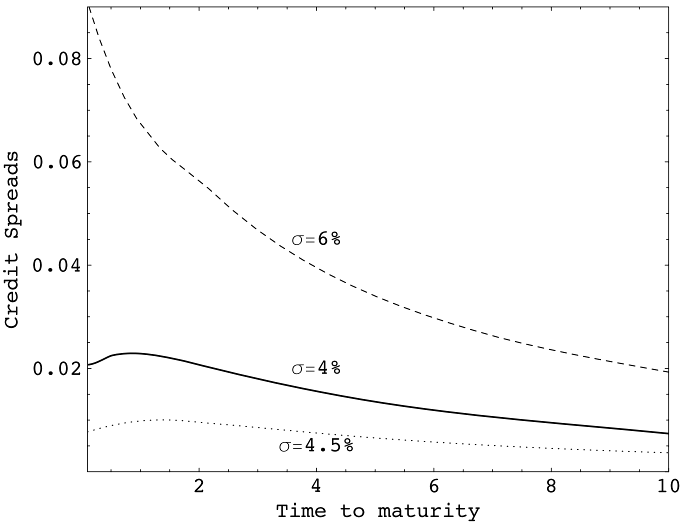

Fig. 5 Impact of the assets' volatility σ on the credit spreads

Figure 4 shows the impact of the volatility of the noise s on the credit spreads, conditional on the observation of the base case realization of Y from Fig. 1. The variable s has a negative impact on the accuracy of the observation process: the lower s, the higher are the chances for X to be not far from Y. Figures 5 and 6 show that the impact of the firm's fundamental parameters (drift r and volatility σ) on the default probability is similar to the situation of structural models: r is negatively related, and σ is positively related to the credit spreads.

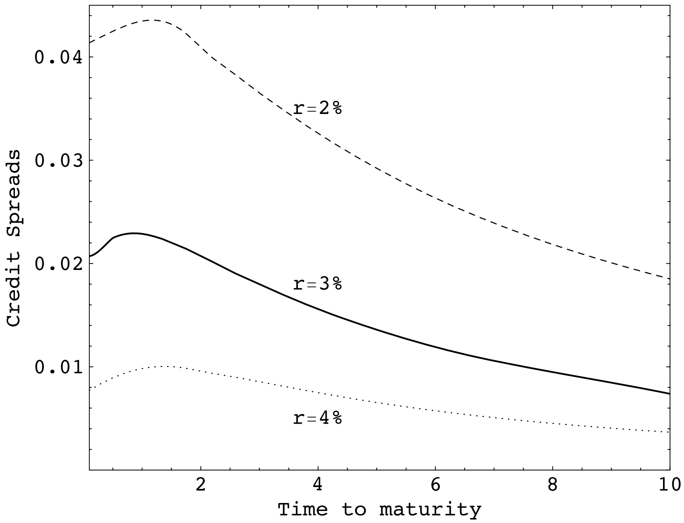

Fig. 6 Impact of the assets' risk-neutral drift r on the credit spreads

5.2 The value of the assets is an Ornstein–Uhlenbeck process

We now suppose that the fundamental process follows an Ornstein–Uhlenbeck process under the risk-neutral measure, i.e.,

$$X_t = F(m_t, t) = \theta + (x_0 - \theta + m_t)e^{-\lambda t},$$

where $m_t = \sigma \int_0^t e^{\lambda u}\, dB_u$ and θ, λ, and σ are constants. Note that condition ($\mathbf{A}'$) is satisfied, since m is a Gaussian $(\mathcal{G}_t)$-martingale. We define $\tau = \inf(t : X_t = 0)$, so that the default barrier is $b(t) \equiv 0$.

A structural model of this type is presented in [6], where the fundamental process is the log-leverage process. The firm is supposed to continuously adjust its leverage ratio towards a target level. We will keep the same framework but rather interpret X as minus the log-leverage process, a positive process up to the default time. It follows that the parameter θ has here an interesting financial interpretation, since $-\theta$ is the expected long term log-leverage ratio of the firm.

We choose to define the observation process as

$$dY_t = \lambda(\theta - Y_t)\, dt + \sigma_1\, dB_t, \quad Y_0 = x_0,$$

with $\sigma_1 = \sqrt{\sigma^2 + s^2}$ and $\beta_t = (\sigma B_t + s B'_t)/\sigma_1$. The processes defined in Remark 3.6 take here the particular forms

$$M'_t = \frac{\sigma\sigma_1}{\sigma + \eta} \int_0^t e^{\lambda u}\, dB_u,$$

$$N'_t = \frac{\sigma\eta}{\sigma + \eta} \int_0^t e^{\lambda u}\, dB_u + \frac{\sigma s}{\sigma + \eta} \int_0^t e^{\lambda u}\, dB'_u$$

with $\eta = s^2/\sigma$.

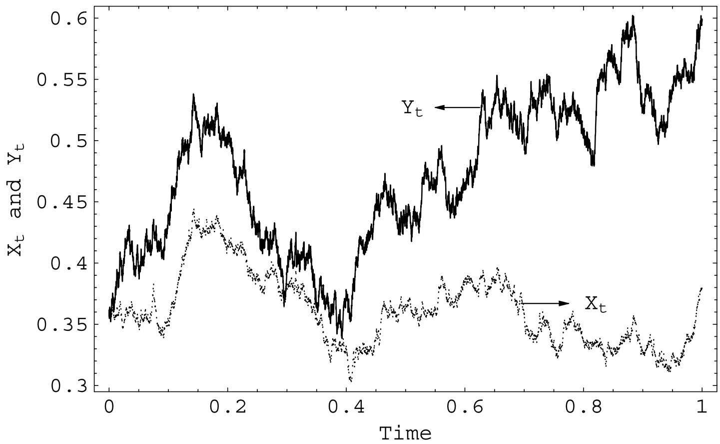

Fig. 7 A realization of the processes Y and X up to the current time $t = 1$

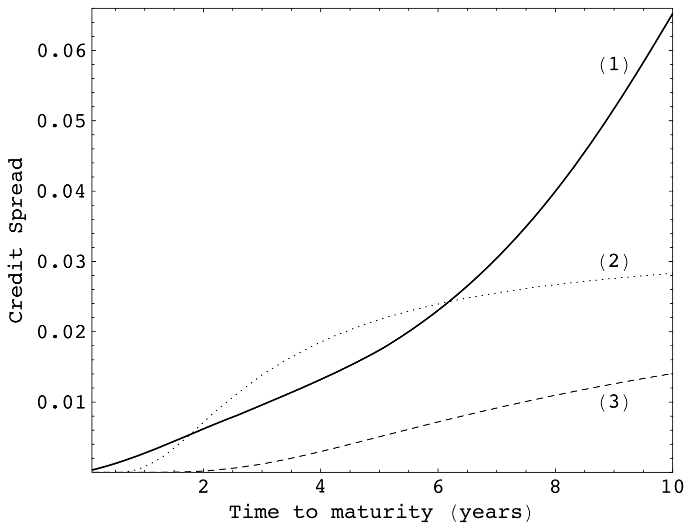

Fig. 8 Credit spreads generated by our model: (*1*) market spreads with imperfect observation; (*2*) spreads computed with complete information versus (*3*) spreads with structural modeling

To illustrate our model we consider the base case parameters

$$t = 1; \qquad x_0 = 0.35; \qquad \lambda = 0.18; \qquad \theta = 0.35; \qquad \sigma = 0.12; \qquad s = 0.16,$$

so that the observable process Y has diffusion parameters similar to those estimated in [6] under the risk-neutral measure.

Figure 7 displays a path of the observable process Y and the unobservable process X generated using the base case parameters. The corresponding credit spreads are depicted in Fig. 8, where we also show the spreads computed using a complete information (i.e., the process X is observable) and spreads generated by the structural model (i.e., only the process Y is observable, but we do not take into account the noise when pricing). For the last two curves, we use an estimation of the first passage time distribution of an OU process over a constant barrier (same methodology as in [6]).

5.3 A model with a stochastic and unobservable barrier

Giesecke and Goldberg [12] (hereafter GG) model a situation where the market observes the true value of the firm's assets but not the default threshold, which is supposed to be a random variable with a known distribution. Our framework permits to generalize the GG model by choosing for the unobservable barrier a stochastic process with known diffusion parameters. We thus avoid facing zero short term spreads whenever the firm's assets are above their historical minimum level, which was the case in GG.

We illustrate with one example how to fit such a model into our framework. Suppose that the firm's value is an observable process driven by a Brownian motion with drift,

$$dY_t = \mu\, dt + \sigma_1\, dB_t,$$

while the default threshold k is unobservable for market participants and driven by a mean-reverting process,

$$dk_t = \lambda(Y_t - k_t - \gamma)\, dt + \xi\, dW_t,$$

reflecting the fact that the firm has a target barrier level of $Y_t - \gamma$ with γ a positive constant. The processes β and W are correlated Brownian motions with $\langle \beta, W\rangle_t = \rho t$, $|\rho| < 1$. Now, we may set the fundamental process to be $X = Y - k$ and the barrier to $b(t) \equiv 0$. Note that the process X is indeed unobservable, since it is not adapted with respect to the filtration generated by Y. Moreover, the process X satisfies assumption ($\mathbf{A}'$), since it can be shown that $X_t = \theta + (x_0 - \theta + m_t)e^{-\lambda t}$ with

$$m_t = \sigma \int_0^t e^{\lambda s}\, dB_s,$$

$$B_t = \frac{\sigma_1 \beta_t + \xi W_t}{\sqrt{\sigma_1^2 + \xi^2 - 2\rho\xi\sigma_1}},$$

$$\theta = \frac{\mu}{\lambda} + \gamma,$$

i.e., X is an Ornstein–Uhlenbeck process. Hence, we recover the model presented before with a different interpretation of the parameters.

6 Conclusion

We have provided a framework for modeling the informational noise affecting the market perception of the firm's fundamentals; the corresponding pricing formulas are derived. Information imperfections have an impact on the investors' perception of the real risks of their positions, hence the model predicts credit spreads which are quite different from the perfect information models. An important feature we exhibit is that investors will always price a short-term default-risk premium, since the default time becomes a totally inaccessible stopping time.

Acknowledgements We are grateful to Marc Yor, Nizar Touzi, and two anonymous referees for carefully reading this paper and for their helpful comments.

Appendix

Proof of Lemma 2.1 It is obvious that $\{\tau \leq t\} \notin \mathcal{F}_t$, hence τ is not an $(\mathcal{F}_t)$-stopping time. Since the filtration $(\mathcal{G}_t)$ is Brownian, τ is a $(\mathcal{G}_t)$-predictable stopping time. Also, by the definition of the filtration $(\mathcal{F}_t^\tau)$, τ is an $(\mathcal{F}_t^\tau)$-stopping time, and we now prove that it is totally inaccessible.

Like all stopping times, τ has a unique decomposition into an accessible stopping time, say τ_A, and a totally inaccessible stopping time, say τ_B, such that $\tau = \tau_A \wedge \tau_B$ (see [7], III.41, p. 58). Let us remark that from the construction of $(\mathcal{F}_t^\tau)$ it follows that τ_A is also an $(\mathcal{F}_t)$-stopping time. To get this property, note first that prior to τ, all predictable $(\mathcal{F}_t^\tau)$-stopping times are also $(\mathcal{F}_t)$-stopping times (see [16]) and second that an accessible time will be equal to some predictable time on a partition of Ω, except negligible sets, implying that it will also be an $(\mathcal{F}_t)$-stopping time.

Using the notation from the proof of the Proposition 3.3, we have, for any function h and $t \geq 0$, that $\pi^{\tilde{\mathbf{P}},h}(t) < 1$, since T_h as the first hitting time of a Brownian motion is not a bounded stopping time. Hence, $U_t < 1$ for any $t \geq 0$ (see equality (3.8)).

However, for $t \geq \tau_A$, we have $U_t = 1$, since $\tau \leq \tau_A$ a.s. and τ_A is an $(\mathcal{F}_t)$-stopping time. This being impossible, $\mathbb{P}(\tau_A < \infty) = 0$ and $\tau = \tau_B$ a.s., which proves the result. □

Proof of Lemma 3.7 By definition, the intensity is given by the limit

$$\lambda_t = \lim_{T \downarrow t} \frac{\mathbb{P}(t < \tau \leq T | \mathcal{F}_t^\tau)}{T - t} = \frac{\mathbf{1}_{(\tau > t)}}{(1 - U_t)} \lim_{T \downarrow t} \frac{\mathbb{P}(t < \tau \leq T | \mathcal{F}_t)}{T - t} = \frac{\mathbf{1}_{(\tau > t)}}{(1 - U_t)} \widetilde{\lambda}_t.$$

The limit $\widetilde{\lambda}_t$ can easily be found from (3.14) when dividing both sides by $T - t$ and taking the limit when $T \downarrow t$. Indeed, the left-hand side of (3.14) becomes

$$\lim_{T \downarrow t} \frac{1}{T - t} \int_t^T \Phi\left(\frac{\tilde{F}(T) - \tilde{F}(s)}{\sqrt{T - s}}\right) d\mathbb{P}(\tau \leq s | \mathcal{F}_t) = \lim_{h \to 0} \frac{\mathbb{P}(t < \tau \leq T | \mathcal{F}_t)}{T - t} = \widetilde{\lambda}_t,$$

since $\lim_{T \to t} \Phi(\frac{\tilde{F}(T) - \tilde{F}(t)}{\sqrt{T - t}}) = 1$. Then, the limit of the right-hand side of (3.14) can be shown to equal

$$\frac{\tilde{F}'(t)}{\sqrt{\langle N \rangle_t}} \varphi\left(\frac{c_t}{\sqrt{\langle N \rangle_t}}\right) - \int_0^t \frac{\tilde{F}'(t)}{\sqrt{\langle N \rangle_t - \langle N \rangle_s}} \varphi\left(\frac{c_t - c_s}{\sqrt{\langle N \rangle_t - \langle N \rangle_s}}\right) dU_s$$

by using a Taylor expansion. □

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