Representation of the penalty term of dynamic concave utilities

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November 4, 2008

Abstract

In the context of a Brownian filtration and with a fixed finite time horizon, we will provide a representation of the penalty term of general dynamic concave utilities (hence of dynamic convex risk measures) by applying the theory of g-expectations.

1 Introduction

Coherent risk measures were introduced by Artzner et al. [2] in finite sample spaces and later by Delbaen [15] and [16] in general probability spaces. The aim of this financial tool is to quantify the intertemporal riskiness which an investor would face at a maturity date T in order to decide if this risk could be acceptable for him or not. The family of coherent risk measures were extended later by Föllmer and Schied [23], [24] and Frittelli and Rosazza Gianin [25], [26] to the class of convex risk measures.

$g-$expectations were introduced by Peng [33] as solutions of a class of nonlinear Backward Stochastic Differential Equations (BSDE, for short), a class which was first studied by Pardoux and Peng [32]. Financial applications and particular cases were discussed in detail by El Karoui et al. [21].

As shown by Rosazza Gianin [38], the families of static risk measures and of $g-$expectations are not disjoint. Indeed, under suitable hypothesis

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†This research was sponsored by a grant of Credit Suisse as well as by a grant NCCR-Finrisk. The text only reflects the opinion of the authors.
Part of the research was done while this author was visiting China in 2005, 2006 and 2007. The hospitality of Shandong University is greatly appreciated.

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¶This work was done while this author was appointed at the Università di Napoli "Federico II", Italy. Part of this research was carried out during her visiting in China in 2006 and at ETH in Zürich in 2004, 2006 and 2007. The warm hospitality of Shandong University and of ETH is gratefully acknowledged.

on the functional g, $g-$expectations provide examples of coherent and/or convex static risk measures. Furthermore, by defining "dynamic risk measure" as a "map" which quantifies at any intermediate time t the riskiness which will be faced at maturity T, a class of dynamic risk measures can be obtained by means of conditional $g-$expectations. In particular, any dynamic risk measure induced by a conditional $g-$expectation satisfies a "time-consistency property" (in line with the notion introduced by Koopmans [30] and Duffie and Epstein [20]) or, in the language of Artzner et al. [3], a "recursivity property". Further discussions on dynamic risk measures can be found in Artzner et al. [3], Barrieu and El Karoui [5], Bion-Nadal [6], Cheridito et al. [10], [11], Cheridito and Kupper [13], Detlefsen and Scandolo [19], Frittelli and Rosazza Gianin [26] and Klöppel and Schweizer [29], among many others.

The main aim of this paper is to represent the penalty term of general dynamic concave utilities (hence of dynamic convex risk measures) in the context of a Brownian filtration, a fixed finite time horizon T and under the assumption of the existence of an equivalent probability measure with zero penalty. By applying the theory of g-expectations, we will finally prove that the penalty term is of the following form:

$$c_{s,t}(Q) = E_Q\left[\int_s^t f(u, q_u)du|\mathcal{F}_s\right]$$

(see the exact statement in Theorem 5).

The paper is organised as follows. Some well-known results on BSDE and on risk measures are recalled in Section 2. Section 3 contains the main result of the paper, that is the representation of the penalty term of suitable dynamic concave utilities. As we will see later, this representation will be obtained by applying the theory of g-expectations.

2 Notation and preliminaries

Let $(B_t)_{t\geq 0}$ be a standard $d-$dimensional Brownian motion defined on the probability space $(\Omega, \mathcal{F}, P)$ and $\{\mathcal{F}_t\}_{t\geq 0}$ be the augmented filtration generated by $(B_t)_{t\geq 0}$.

In the sequel, we will identify a probability measure $Q \ll P$ with its Radon-Nykodim density $\frac{dQ}{dP}$. Furthermore, because of the choice of the Brownian setting, we will also identify a probability measure Q equivalent to P with the predictable process $(q_t)_{t\in[0,T]}$ induced by the stochastic exponential, i.e. such that

$$E_P\left[\frac{dQ}{dP}|\mathcal{F}_t\right] = \mathcal{E}(q.B)_t \triangleq \exp\left(-\frac{1}{2}\int_0^t \|q_s\|^2 ds + \int_0^t q_s dB_s\right) \qquad (1)$$

(see Proposition VIII.1.6 of Revuz and Yor [35]).

Consider now a function

$$\begin{array}{cccccccccc} g: & \mathbb{R}^+ & \times & \Omega & \times & \mathbb{R} & \times & \mathbb{R}^d & \to & \mathbb{R} \\ & (t, & & \omega, & & y, & & z) & \longmapsto & g(t,\omega,y,z) \end{array}$$

satisfying at least the following assumptions (as in Coquet et al. [14], but without imposing a priori an horizon of time T). To simplify the notations, we will often write $g(t, y, z)$ instead of $g(t, \omega, y, z)$.

Basic assumptions on g:

(A) g is Lipschitz in (y, z), i.e. there exists a constant $\mu > 0$ such that, $(dt \times dP) - a.s.$, for any $(y_0, z_0), (y_1, z_1) \in \mathbb{R} \times \mathbb{R}^d$,

$$|g(t, y_0, z_0) - g(t, y_1, z_1)| \leq \mu(|y_0 - y_1| + \|z_0 - z_1\|).$$

(B) For all $(y, z) \in \mathbb{R} \times \mathbb{R}^d$, $g(\cdot, y, z)$ is a predictable process such that for any finite $T > 0$ it holds $E[\int_0^T (g(t, \omega, y, z))^2 dt] < +\infty$ for any $y \in \mathbb{R}$ and $z \in \mathbb{R}^d$.

(C) $(dt \times dP)-$a.s., $\forall y \in \mathbb{R}$, $g(t, y, 0) = 0$.

Once the horizon of time $T > 0$ is fixed, Pardoux and Peng [32] introduced the following Backward Stochastic Differential Equation (BSDE, for short):

$$\begin{cases} -dY_t &= g(t, Y_t, Z_t)dt - Z_t dB_t \\ \quad Y_T &= \xi, \end{cases}$$

where ξ is a random variable in $L^2(\Omega, \mathcal{F}_T, P)$. Moreover, they showed (see also El Karoui et al. [21]) that there exists a unique solution $(Y_t, Z_t)_{t \in [0,T]}$ of predictable stochastic processes (the former $\mathbb{R}$-valued, the latter $\mathbb{R}^d$-valued) such that $E[\int_0^T Y_t^2 dt] < +\infty$ and $E[\int_0^T \|Z_t\|^2 dt] < +\infty$.

Peng [33] defined the $g-$expectation of ξ as:

$$\mathcal{E}_g(\xi) \triangleq Y_0$$

and the conditional $g-$expectation of ξ at time t as:

$$\mathcal{E}_g(\xi|\mathcal{F}_t) \triangleq Y_t.$$

When $g(t, y, z) = \mu\|z\|$ (with $\mu > 0$), $\mathcal{E}_g$ will be denoted by $\mathcal{E}^\mu$.

In the sequel, we will only consider essentially bounded random variables ξ, i.e. $\xi \in L^\infty(\Omega, \mathcal{F}_T, P)$.

Further assumptions on g

(1_g) g does not depend on y

(2_g) g is convex in z:
$\forall \alpha \in [0, 1], \forall z_0, z_1 \in \mathbb{R}^d, \quad (dt \times dP)-$a.s.:
$g(t, \alpha z_0 + (1 - \alpha)z_1) \leq \alpha g(t, z_0) + (1 - \alpha)g(t, z_1).$

In the sequel, we will write *"g with the usual assumptions"* when g satisfies hypothesis (A)-(C) and (1_g)-(2_g).

Some sufficient conditions for a functional to be induced by a g-expectation are provided by Coquet et al. [14]. Before recalling this result, we will introduce what is needed.

Definition 1 ***(Coquet et al. [14])*** *A functional $\mathcal{E} : L^2(\mathcal{F}_T) \to \mathbb{R}$ is called an $\mathcal{F}$-*consistent expectation *if it satisfies the following properties:*

(i) constancy*: $\mathcal{E}(c) = c$, for any $c \in \mathbb{R}$;*

(ii) strict monotonicity*: if $\xi \geq \eta$, then $\mathcal{E}(\xi) \geq \mathcal{E}(\eta)$. Moreover, if $\xi \geq \eta$: $\xi = \eta$ if and only if $\mathcal{E}(\xi) = \mathcal{E}(\eta)$;*

(iii) consistency*: for any $\xi \in L^2(\mathcal{F}_T)$ and $t \in [0, T]$ there exists a random variable $\mathcal{E}(\xi|\mathcal{F}_t) \in L^2(\mathcal{F}_t)$ such that for any $A \in \mathcal{F}_t$ it holds*

$$\mathcal{E}(\xi 1_A) = \mathcal{E}\left(\mathcal{E}(\xi|\mathcal{F}_t) 1_A\right).$$

Again in the terminology of [14], $\mathcal{E}$ is said to satisfy translation invariance (or to be monetary) if for any $t \in [0, T]$:

$$\mathcal{E}(\xi + \eta|\mathcal{F}_t) = \mathcal{E}(\xi|\mathcal{F}_t) + \eta, \quad \forall \xi \in L^2(\mathcal{F}_T), \eta \in L^2(\mathcal{F}_t);$$

while it is said to be $\mathcal{E}^\mu$-dominated (for some $\mu > 0$) if:

$$\mathcal{E}(\xi + \eta) - \mathcal{E}(\xi) \leq \mathcal{E}^\mu(\eta), \quad \forall \xi, \eta \in L^2(\mathcal{F}_T).$$

Theorem 2 ***(Coquet et al.; Theorem 7.1; [14])*** *Let $\mathcal{E}$ be an $\mathcal{F}$-consistent expectation.*

If $\mathcal{E}$ satisfies translation invariance and if it is dominated by some $\mathcal{E}^\mu$ with $\mu > 0$, then it is induced by a conditional g-expectation, that is there exists a function g satisfying (A)-(C), (1_g) such that for any $t \in [0, T]$

$$\mathcal{E}(\xi|\mathcal{F}_t) = \mathcal{E}_g(\xi|\mathcal{F}_t), \quad \forall \xi \in L^2(\mathcal{F}_t).$$

Some relevant extensions of such a result can be found in Peng [34] and in Hu et al. [27], while some applications to risk measures can be found in Rosazza Gianin [38]. The last author, in particular, showed that g-expectations (respectively, conditional g-expectations) provide static (respectively, dynamic) risk measures. More precisely, the following result holds true. For definitions, representations and details on (static) risk measures an interested reader can see Artzner et al. [2], Delbaen [15], [16], Föllmer and Schied [23], [24], Frittelli and Rosazza Gianin [25], among many others.

Proposition 3 ***(Rosazza Gianin; Proposition 11; [38])*** *If g satisfies the usual assumptions (including convexity in z), then the risk measure ρ_g defined as*

$$\rho_g(X) \triangleq \mathcal{E}_g(-X)$$

is a convex risk measure satisfying monotonicity, constancy and translation invariance.

Moreover: if g also satisfies positive homogeneity in z, then ρ_g is coherent.

In view of the result above, some sufficient conditions for a risk measure to be induced by a g-expectation have been found in [38] as an application of Theorem 2.

Note that, at least in the sublinear case and under some suitable assumptions, one can prove a one-to-one correspondence between the functional g and the m-stable set of generalized scenarios $\mathcal{S}$ of the suitable risk measure. Hence, one may find (as an application of the results of Delbaen [17] on m-stable sets) a one-to-one correspondence between time-consistent coherent risk measures and conditional g-expectation. See also Chen and Epstein [8].

In the sequel, we will prefer to work with concave utilities instead of convex risk measures. Note that, given a risk measure ρ, the associated monetary utility functional (or, shortly, utility) is defined as $u \triangleq -\rho$.

3 Representation of the penalty term of dynamic concave utilities

In the sequel, we will still work in a Brownian setting, hence $\mathcal{F}_0$ is trivial. Let T be a fixed finite time horizon. Given two stopping times σ and τ such that $0 \le \sigma \le \tau \le T$, consider a *concave monetary utility* functional $u_{\sigma,\tau} : L^\infty(\mathcal{F}_\tau) \to L^\infty(\mathcal{F}_\sigma)$, i.e. a functional satisfying

(a) monotonicity: if $\xi, \eta \in L^\infty(\mathcal{F}_\tau)$ and $\xi \le \eta$, then $u_{\sigma,\tau}(\xi) \le u_{\sigma,\tau}(\eta)$

(b) translation invariance: $u_{\sigma,\tau}(\xi+\eta) = u_{\sigma,\tau}(\xi)+\eta$ for any $\xi \in L^\infty(\mathcal{F}_\tau)$ and $\eta \in L^\infty(\mathcal{F}_\sigma)$

(c) concavity: $u_{\sigma,\tau}(\alpha\xi + (1-\alpha)\eta) \ge \alpha u_{\sigma,\tau}(\xi) + (1-\alpha)u_{\sigma,\tau}(\eta)$ for any $\xi, \eta \in L^\infty(\mathcal{F}_\tau)$ and $\alpha \in [0, 1]$

(d) $u_{\sigma,\tau}(0) = 0$

$(u_{\sigma,\tau})_{0\le\sigma\le\tau\le T}$ is called a *dynamic concave utility.* In particular, $u_{0,T} : L^\infty(\mathcal{F}_T) \to \mathbb{R}$. The acceptance set $\mathcal{A}_{\sigma,\tau}$ induced by $u_{\sigma,\tau}$ is defined as $\mathcal{A}_{\sigma,\tau} \triangleq \{\xi \in L^\infty(\mathcal{F}_\tau) : u_{\sigma,\tau}(\xi) \ge 0\}$. To simplify notations, we will often write u_t instead of $u_{t,T}$.

On $(u_{\sigma,\tau})_{0\le\sigma\le\tau\le T}$ we will assume the following:

Assumption (e): $(u_{\sigma,\tau})_{0\le\sigma\le\tau\le T}$ is continuous from above (or it satisfies the Fatou property), i.e. for any decreasing sequence $(\xi_n)_{n\in\mathbb{N}}$ in $L^\infty(\mathcal{F}_\tau)$ such that $\lim_n \xi_n = \xi$ it holds true that $\lim_n u_{\sigma,\tau}(\xi_n) = u_{\sigma,\tau}(\xi)$.

Assumption (f): $(u_{\sigma,\tau})_{\sigma,\tau}$ is time-consistent, i.e. for all stopping times σ, τ, υ with $0 \le \sigma \le \tau \le \upsilon \le T$:

$$u_{\sigma,\upsilon}(\xi) = u_{\sigma,\tau}(u_{\tau,\upsilon}(\xi)), \quad \forall \xi \in L^\infty(\mathcal{F}_\upsilon).$$

Assumption (g): $(u_{\sigma,\tau})_{\sigma,\tau}$ satisfies

$$u_{\sigma,\tau}(\xi 1_A + \eta 1_{A^c}) = u_{\sigma,\tau}(\xi)1_A + u_{\sigma,\tau}(\eta)1_{A^c}, \forall \xi, \eta \in L^\infty(\mathcal{F}_\tau), \forall A \in \mathcal{F}_\sigma. \tag{2}$$

Assumption (h): $c_t(P) = 0$ for any $t \in [0, T]$

It is straightforward to check that this last condition is equivalent to: $E_P[\xi|\mathcal{F}_t] \ge 0$ for any $\xi \in \mathcal{A}_t$. Furthermore, $c_0(P) = 0$ can be replaced by the hypothesis that there is a probability measure Q equivalent to P satisfying $c_0(Q) = 0$.

Note that, up to a sign, dynamic concave utilities satisfying the assumptions above correspond to normalized time-consistent dynamic risk measures $(\rho_{\sigma,\tau})_{0\leq\sigma\leq\tau\leq T}$ studied, for instance, in Bion-Nadal [6] in a general setting. More precisely, it holds $u_{\sigma,\tau} = -\rho_{\sigma,\tau}$.

By Bion-Nadal [6] and Detlefsen and Scandolo [19], it is known that, under the assumptions above and in the setting of a general filtration,

$$\begin{aligned} u_{s,t}(\xi) &= \text{ess.inf}_{Q\sim P, Q=P \text{ on } \mathcal{F}_s}\{E_Q[\xi|\mathcal{F}_s] + c_{s,t}(Q)\} \\ &= \text{ess.inf}_{Q\in\mathcal{P}_{s,t}}\{E_Q[\xi|\mathcal{F}_s] + c_{s,t}(Q)\} \end{aligned} \tag{3}$$

for any $0 \leq s \leq t \leq T$, where

$$\begin{aligned} c_{s,t}(Q) &= \text{ess.sup}_{\xi\in L^\infty(\mathcal{F}_t)}\{E_Q[-\xi|\mathcal{F}_s] + u_{s,t}(\xi)\} \\ \mathcal{P}_{s,t} &= \{Q \text{ on } (\Omega, \mathcal{F}_t) : Q \sim P, Q = P \text{ on } \mathcal{F}_s\}. \end{aligned}$$

In particular:

$$\begin{aligned} u_t(\xi) &= \text{ess.inf}_{Q\sim P; Q=P \text{ on } \mathcal{F}_t}\{E_Q[\xi|\mathcal{F}_t] + c_{t,T}(Q)\} \\ u_0(\xi) &= \inf_{Q\sim P}\{E_Q[\xi] + c_{0,T}(Q)\} \end{aligned}$$

where $c_t(Q) \triangleq c_{t,T}(Q) = \text{ess.sup}_{\xi\in\mathcal{A}_t} E_Q[-\xi|\mathcal{F}_t] \geq 0$ and $\mathcal{A}_t$ denotes the acceptance set induced by u_t. Note that $c_{0,T}(Q) = \sup_{\xi\in L^\infty}\{E_Q[-\xi] + u_{0,T}(\xi)\}$, hence $c_{0,T}$ is lower semi-continuous and is the Fenchel-Legendre transform of u.

Furthermore, Bion-Nadal (see Theorem 3 in [6]) proved that $(\rho_{t,T})_{t\in[0,T]}$ (hence $(u_{t,T})_{t\in[0,T]}$) admits a càdlàg modification. We will prove in the Appendix that the same is true for $(c_{t,T})_{t\in[0,T]}$.

Note that in [6] and [19] the representation (3) was shown with $Q \ll P$ instead of $Q \sim P$. Nevertheless, assumption (h) guarantees that the representation (3) also holds true (for a proof see Klöppel and Schweizer [29] and, in discrete-time, Cheridito et al. [12] and Föllmer and Penner [22]).

Remark 4 *It is evident that if $(u_t)_{t\geq 0}$ is time-consistent, if $u_t(0) = 0$ and if it satisfies condition (2), then*

$$u_0(\xi 1_A) = u_0(u_t(\xi 1_A)) = u_0(u_t(\xi)1_A)$$

for any $t \geq 0$, $\xi \in L^\infty(\mathcal{F}_T)$ and $A \in \mathcal{F}_t$.

It is therefore clear that if $(u_{\sigma,\tau})_{\sigma,\tau}$ is time-consistent, then everything is defined by u_0. The relevance of time-consistency of the dynamic concave utility is also underlined by the following results. On one hand, as shown by Delbaen [17] and Cheridito et al. [12], time-consistency is indeed equivalent to the *decomposition property* of acceptable sets, that is

$$\mathcal{A}_{\sigma,\upsilon} = \mathcal{A}_{\sigma,\tau} + \mathcal{A}_{\tau,\upsilon}$$

for all stopping times σ, τ, υ such that $0 \leq \sigma \leq \tau \leq \upsilon \leq T$. On the other hand, both the properties above are equivalent to the *cocycle property* of the penalty term c, that is

$$c_{\sigma,\upsilon}(Q) = c_{\sigma,\tau}(Q) + E_Q[c_{\tau,\upsilon}(Q)|\mathcal{F}_\sigma]$$

for all stopping times σ, τ, υ such that $0 \le \sigma \le \tau \le \upsilon \le T$ (see Bion-Nadal [6] for the definition and the proof).

In the sequel, we use the terminology of Rockafellar [36] on convex functions. Our aim is now to prove the following result.

Theorem 5 *Let $(u_{\sigma,\tau})_{0\le\sigma\le\tau\le T}$ be a dynamic concave utility satisfying the assumptions above.*

(i) For all stopping times σ, τ such that $0 \le \sigma \le \tau \le T$ and for any probability measure Q equivalent to P:

$$c_{\sigma,\tau}(Q) = E_Q\left[\int_\sigma^\tau f(u, q_u)du|\mathcal{F}_\sigma\right] \tag{4}$$

for some suitable function $f : [0,T] \times \Omega \times \mathbb{R}^d \to [0, +\infty]$ such that $f(t, \omega, \cdot)$ is proper, convex and lower semi-continuous.

(ii) For all stopping times σ, τ such that $0 \le \sigma \le \tau \le T$ and $\xi \in L^\infty(\mathcal{F}_T)$, the dynamic concave utility in (3) can be represented as

$$u_{\sigma,\tau}(\xi) = \text{ess.inf}_{Q\in\mathcal{P}_{\sigma,\tau}} E_Q\left[\xi + \int_\sigma^\tau f(u, q_u)du|\mathcal{F}_\sigma\right].$$

Remark 6 *For a dynamic concave utilities satisfying assumptions (e), (g), (h), from Theorem 1 of Bion-Nadal [6] it follows that Theorem 5(i) is equivalent to time-consistency (assumption (f)) of $(u_{\sigma,\tau})_{0\le\sigma\le\tau\le T}$.*

Remark 7 *In an incomplete market, the lower price $\inf_{Q\in\mathcal{M}} E_Q[\xi]$ (where $\mathcal{M}$ denotes the set of all risk-neutral probability measures) defines a utility satisfying all our properties but it is not given by a g-expectation. See Delbaen [17] for details about how to get f.*

The proof of Theorem 5 will be decomposed into several steps as outlined below.

Set

$$u^n_{s,t}(\xi) = \text{ess.inf}_{Q\sim P;\|q\|\le n}\{E_Q[\xi|\mathcal{F}_s] + c_{s,t}(Q)\}. \tag{5}$$

Note that (by definition of u^n and by assumption (h)) for any $\xi \in L^\infty(\mathcal{F}_T)$ it holds $u^0_t(\xi) = E_P[\xi|\mathcal{F}_t]$ and $u^n_t(\xi) \le E_P[\xi|\mathcal{F}_t]$.

Remark 8 *The reason why the truncated utility u^n has been defined as above is due to the fact that the set $\{Q \sim P; \|q\| \le n\}$ is weakly compact. This argument will be useful in the proof of Proposition 9.*

Proposition 9 *Suppose that the dynamic concave utility $(u_{\sigma,\tau})_{0\le\sigma\le\tau\le T}$ satisfies the assumptions above. Then:*

(i) u^n is a dynamic concave utility satisfying assumptions (e)-(g). Moreover, the acceptance sets induced by u^n satisfy the decomposition property and

$$c^n_{s,t}(Q) = \begin{cases} c_{s,t}(Q); & \textit{if} \quad \|q\| \le n \\ +\infty; & \textit{otherwise} \end{cases} \tag{6}$$

satisfies the cocycle property and $c^n_{s,t}(P) = 0$.

(ii) u^n *is induced by a conditional* g_n*-expectation, i.e.*

$$u_t^n(\xi) = -\mathcal{E}_{g_n}(-\xi|\mathcal{F}_t)$$

for some convex function g_n *satisfying the usual conditions and such that* $g_n(\cdot, \cdot, z)$ *is predictable for any* $z \in \mathbb{R}^d$*. In other words,* u^n *satisfies the following BSDE*

$$\begin{cases} du_t^n(\xi) &= g_n(t, Z_t^n)dt - Z_t^n dB_t \\ u_T^n(\xi) &= \xi \end{cases} \tag{7}$$

(iii) For any probability measure $Q \sim P$ *such that* $\|q\| \leq n$ *it holds that for any* $0 \leq s \leq t \leq T$*:*

$$\begin{aligned} c_{0,t}^n(Q) &= E_Q\left[\int_0^t f_n(u, q_u)du\right] \\ c_{s,t}^n(Q) &= E_Q\left[\int_s^t f_n(u, q_u)du|\mathcal{F}_s\right] \end{aligned}$$

where $f_n : [0, T] \times \Omega \times \mathbb{R}^d \to [0, +\infty]$ *is induced (by duality) by* g_n *and* $f_n(t, \omega, \cdot)$ *is proper, convex and lower semi-continuous.*

(iv) The sequence of convex functions g_n *is increasing in* n*.*

(v) The sequence of f_n *is decreasing in* n *and, for any* $n \geq 0$*,* $f_n(t, \omega, q) = +\infty$ *for* $\|q\| > n$*.*
Furthermore, once (t, ω) *is fixed, for any* q *either there exists* $n \geq 0$ *such that*

$$f_n(t, \omega, q) = f_m(t, \omega, q) = f(t, \omega, q) < +\infty, \quad \forall m \geq n$$

or for all $n \geq 0$

$$f_n(t, \omega, q) = +\infty = f(t, \omega, q),$$

for some function $f : [0, T] \times \Omega \times \mathbb{R}^d \to [0, +\infty]$*.*
Hence $f(t, \omega, x) = \inf_n f_n(t, \omega, x)$ *and it is such that* $f(t, \omega, \cdot)$ *is proper, convex and lower semi-continuous.*

Proof.

(i) From the representation (5) it follows that u^n is a dynamic concave utility which is continuous from above (see Detlefsen and Scandolo [19] and Klöppel and Schweizer [29]). Still from (5) one deduces that $u_{\sigma,\tau}^n(\xi 1_A) = u_{\sigma,\tau}^n(\xi)1_A$ for any $\xi \in L^\infty(\mathcal{F}_T)$, $0 \leq \sigma \leq \tau \leq T$ and $A \in \mathcal{F}_\sigma$. Hence, by Proposition 2.9 of Detlefsen and Scandolo [19], also assumption (g) is satisfied.
The cocycle property of c^n and time-consistency of u^n follow from

$$c_{s,t}^n(Q) = \begin{cases} c_{s,t}(Q); & \text{if} \quad \|q\| \leq n \\ +\infty; & \text{otherwise} \end{cases}$$

and from Theorem 1 of Bion-Nadal [6].
Since for the probability measure P it holds $q^P \equiv 0$, $c_{s,t}^n(P) = c_{s,t}(P) = 0$.
The decomposition property of acceptance sets is due to Theorem 4.6 of Cheridito et al. [12] and, later, to Theorem 1 of Bion-Nadal [6].

(ii) Set $\pi_t^n(\xi) \triangleq -u_t^n(-\xi) = \text{ess.sup}_{Q\sim P;\|q\|\leq n}\{E_Q[\xi|\mathcal{F}_t] - c_t(Q)\}$.

From (i), $(\pi^n_{\sigma,\tau})_{0\leq\sigma\leq\tau\leq T}$ is time-consistent. Furthermore, it is easy to check that it satisfies monotonicity, translation invariance and constancy (this last follows from the assumption $c_t(P) = 0$).

Moreover, π_0^n satisfies strict monotonicity. This property follows from weak compactness of the set $\{Q \sim P : \|q\| \leq n\}$ (see Remark 8). In order to verify strict monotonicity, consider $\eta \geq \xi$ such that $P(\eta > \xi) > 0$. Since $\pi_0^n(\xi) = E_Q[\xi] - c_0(Q)$ for some $Q \sim P$ such that $\|q\| \leq n$, $\pi_0^n(\eta) \geq E_Q[\eta] - c_0(Q) > E_Q[\xi] - c_0(Q) = \pi_0^n(\xi)$.

Finally, we will show that π_0^n is dominated by some $\mathcal{E}^\mu$. For any $\xi, \eta \in L^\infty(\mathcal{F}_T)$

$$\begin{aligned}
&\pi_0^n(\xi+\eta) - \pi_0^n(\xi) \\
&= \sup_{Q:\|q\|\leq n}\{E_Q[\xi+\eta] - c_0(Q)\} - \sup_{Q:\|q\|\leq n}\{E_Q[\xi] - c_0(Q)\} \\
&\leq \sup_{Q:\|q\|\leq n} E_Q[\eta] = \mathcal{E}^n(\eta).
\end{aligned}$$

The last equality follows from Lemma 3 of Chen and Peng [9] ($\mathbb{R}$ case) which may be extended to $\mathbb{R}^d$.

By the arguments above and Remark 4, $(\pi_t^n)_{t\geq 0}$ satisfies the hypothesis of Theorem 2. Hence there exists a functional $g_n : [0,T] \times \Omega \times \mathbb{R}^d \to \mathbb{R}$ satisfying assumptions (A)-(C), (1_g) and such that $\pi_t^n(\xi) = \mathcal{E}_{g_n}(\xi|\mathcal{F}_t)$.

It can be checked that $g_n(\cdot,\cdot,z)$ is predictable for any $z \in \mathbb{R}^d$ (see also Theorem 3.1 of Peng [34]). Furthermore, since π_t^n is a convex functional, by Theorem 3.2 of Jiang [28] it follows that $g_n(t,\omega,\cdot)$ has to be convex. Hence

$$\begin{aligned}
u_t^n(\xi) &= -\mathcal{E}_{g_n}(-\xi|\mathcal{F}_t) \\
u_0^n(\xi) &= -\mathcal{E}_{g_n}(-\xi)
\end{aligned}$$

for some function g_n satisfying the usual conditions. It is therefore immediate to check that u^n satisfies the BSDE in (7).

Moreover: for almost all (t,ω) it holds that the set $\{z \in \mathbb{R}^d : g_n(t,\omega,z) \leq \alpha\}$ is closed for any $\alpha \in \mathbb{R}$. The closure of such a set (or, equivalently, the lower semi-continuity of $g_n(t,\omega,\cdot)$) is due to the fact that g_n is Lipschitz with constant n (see the arguments above and Theorem 2). Hence $g_n(t,\omega,\cdot)$ is convex, proper and lower semi-continuous.

(iii) Set now

$$f_n(t,\omega,q) \triangleq \sup_{z\in\mathbb{R}^d}\{q\cdot z - g_n(t,\omega,z)\}. \tag{8}$$

Note that $f_n(t,\omega,q) \geq 0$ (take for instance $z = 0$ in the definition of f_n) and, because of the assumption $c_0(P) = 0$, $f_n(t,0) = 0$.

Since $g_n(t,\omega,z)$ is predictable (by item (ii)),

$$f_n(t,\omega,q) = \sup_{z\in\mathbb{R}^d}\{q\cdot z - g_n(t,\omega,z)\} = \sup_{z\in\mathbb{Q}^d}\{q\cdot z - g_n(t,\omega,z)\}$$

is predictable for any $q \in \mathbb{R}^d$ (as supremum of countably many predictable elements). Note that $\|q\| > n$ implies $f_n(t,\omega,q) = +\infty$ (by (8)).

Since $g_n(t,\omega,\cdot)$ is convex, proper and lower semi-continuous (see above) and $f_n(t,\omega,\cdot)$ is the convex conjugate of $g_n(t,\omega,\cdot)$, i.e. $f_n(q) = g_n^*(q)$, also f_n is convex, proper and lower semi-continuous (see Rockafellar [36]).

As a consequence of the dual representation of a g-expectation in Theorem 7.4 of Barrieu and El Karoui [5] we get

$$c_{0,T}^n(Q) = E_Q\left[\int_0^T f_n(u,q_u)du\right]$$

for any probability measure $Q \sim P$ such that $\|q\| \leq n$.

It remains to show that $c_{s,t}^n(Q) = E_Q\left[\int_s^t f_n(u,q_u)du|\mathcal{F}_s\right]$ for any $0 \leq s \leq t \leq T$ and for any probability measure $Q \sim P$ such that $\|q\| \leq n$. Also this result can be deduced by Theorem 7.4 of Barrieu and El Karoui [5]. Nevertheless, since the proof will be useful later, we postpone it to Lemma 10.

(iv) It is easy to check that the sequences of u_0^n and of c_0^n are decreasing in $n \in \mathbb{N}$. By applying the Converse Comparison Theorem on BSDE (see Briand et al. [7]) and Lemma 2.1 of Jiang [28], we will show that the sequence of convex functions g_n (which induce u^n) is increasing in n.

In order to prove the thesis above we will proceed in a similar way as in Jiang [28]. By definition of u^n, $u_{0,T}^n(\xi) \geq u_{0,T}^{n+1}(\xi)$ as well as $u_{s,T}^n(\xi) \geq u_{s,T}^{n+1}(\xi)$ hold true for any $\xi \in L^\infty(\mathcal{F}_T)$. By item (ii) we deduce therefore that for any $\xi \in L^\infty(\mathcal{F}_T)$

$$\begin{aligned} \mathcal{E}_{g_n}(\xi) &\leq \mathcal{E}_{g_{n+1}}(\xi) \\ \mathcal{E}_{g_n}(\xi|\mathcal{F}_s) &\leq \mathcal{E}_{g_{n+1}}(\xi|\mathcal{F}_s) \end{aligned} \tag{9}$$

Denote now by $\mathcal{E}_g^{s,t}$ the conditional g-expectation at time s with final time t. To apply successfully Lemma 2.1 of Jiang [28] we need to verify that

$$\mathcal{E}_{g_n}^{s,t}(\xi) \leq \mathcal{E}_{g_{n+1}}^{s,t}(\xi), \quad \forall s,t \in [0,T] \text{ with } s \leq t, \quad \forall \xi \in L^\infty(\mathcal{F}_t). \tag{10}$$

Condition (10) has already been established for $(s,t) = (0,T)$ and $(s,t) = (s,T)$. Consider now the case $(s,t) = (0,t)$. Since (see Peng [33] for details) $\mathcal{E}_g^{s,t}(\eta) = \mathcal{E}_g^{s,T}(\eta)$ for any $\eta \in L^\infty(\mathcal{F}_t)$, from (9) we deduce that $\mathcal{E}_{g_n}^{0,t}(\xi) \leq \mathcal{E}_{g_{n+1}}^{0,t}(\xi)$ for any $0 \leq t \leq T$ and $\xi \in L^\infty(\mathcal{F}_t)$. For general (s,t), inequality (9) can be checked as above. Indeed, for any $\xi \in L^\infty(\mathcal{F}_t)$ it holds that $\mathcal{E}_{g_n}^{s,t}(\xi) = \mathcal{E}_{g_n}^{s,T}(\xi) \leq \mathcal{E}_{g_{n+1}}^{s,T}(\xi) = \mathcal{E}_{g_{n+1}}^{s,t}(\xi)$.
Set now

$$\mathcal{S}^z(g) \triangleq \{t \in [0,T) : g(t,z) = L^1 - \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon}\mathcal{E}_g^{t,t+\varepsilon}(z(B_{t+\varepsilon} - B_t))\}.$$

From Lemma 2.1 of Jiang [28] it follows that

$$m([0,T) \setminus \mathcal{S}^z(g_i)) = 0 \quad \forall z \in \mathbb{R}^d$$

for $i = n, n+1$, where m denotes the Lebesgue measure on $[0,T]$.
By the arguments above it follows that for any $z \in \mathbb{R}^d$

$$\text{if } t \in \mathcal{S}^z(g_n) \cap \mathcal{S}^z(g_{n+1}) \neq \emptyset \qquad \Rightarrow g_n(t,z) \le g_{n+1}(t,z) \quad P\text{-a.s.}$$

and

$$\begin{aligned} & m([0,T] \setminus (\mathcal{S}^z(g_n) \cap \mathcal{S}^z(g_{n+1}))) \\ & = m(([0,T] \setminus \mathcal{S}^z(g_n)) \cup ([0,T] \setminus \mathcal{S}^z(g_{n+1}))) = 0 \end{aligned}$$

Hence, by proceeding as in Jiang [28] it can be checked that for any $z \in \mathbb{R}^d$

$$g_n(t,z) \le g_{n+1}(t,z) \quad (dt \times dP)\text{-a.s.}$$

Positivity of any g_n is due to the fact that $u_t^0(\xi) = E_P[\xi|\mathcal{F}_t] = -\mathcal{E}_{g_0}(-\xi|\mathcal{F}_t)$ where $g_0 \equiv 0$. By the same arguments above, therefore, $g_n \ge g_0 \equiv 0$.

(v) From (iii) and (iv) it then follows that the sequence of f_n is decreasing in n.

Consider again the measurable space $([0,T] \times \Omega, \mathcal{P}, m \times P)$, where $\mathcal{P}$ denotes the predictable σ-algebra and m denotes the Lebesgue measure on $[0,T]$. Denote by $\overline{\mathcal{P}}$ the completion of $\mathcal{P}$.

Take $N > 0$ and, for any $\varepsilon > 0$, set

$$\begin{aligned} & E = E_{N,\varepsilon} \\ & \triangleq \left\{ (t,\omega,q) \in [0,T] \times \Omega \times \mathbb{R}^d | \begin{array}{c} \|q\| \le n; \\ f_{n+1}(t,\omega,q) + \varepsilon < f_n(t,\omega,q) \le N \end{array} \right\} \end{aligned}$$

and $\pi(E)$ its projection on $[0,T] \times \Omega$. Note that $E \in \overline{\mathcal{P}} \otimes \mathcal{B}(\mathbb{R}^d)$.

From the Measurable Selection Theorem (see Aumann [4] and Aliprantis and Border [1]), $\pi(E) \in \overline{\mathcal{P}}$ and there exists a $\overline{\mathcal{P}}$- measurable $\overline{q} : \pi(E) \to \mathbb{R}^d$ such that $(t,\omega,\overline{q}(t,\omega)) \in E$ for $(m \times P)$-a.e. $(t,\omega) \in \pi(E)$. Set now $\overline{q} = 0$ on $\pi(E)^c$. To such a $\overline{q}$ it is therefore possible to associate a $q : [0,T] \times \Omega \to \mathbb{R}^d$ which is $\mathcal{P}$-measurable and equal to $\overline{q}$ $(m \times P)$-almost everywhere.

Let Q be the probability measure associated to q as above. By definition, $\|q\| \le n$. Hence, $c_{0,T}^n(Q) = c_{0,T}^{n+1}(Q) = c_{0,T}(Q) < +\infty$. Furthermore, by definition of E it follows that

$$\begin{aligned} c_{0,T}^n(Q) & = E_Q\left[\int_0^T f_n(u,q_u)du\right] \\ & = E_Q\left[\int_0^T f_n(u,q_u)1_{\pi(E)}du\right] + E_Q\left[\int_0^T f_n(u,q_u)1_{\pi(E)^c}du\right] \\ & = E_Q\left[\int_0^T f_n(u,q_u)1_{\pi(E)}du\right] \\ & \ge E_Q\left[\int_0^T [f_{n+1}(u,q_u)+\varepsilon]1_{\pi(E)}du\right] \\ & = E_Q\left[\int_0^T f_{n+1}(u,q_u)1_{\pi(E)}du\right] + \varepsilon(m \times Q)(\pi(E)) \\ & = c_{0,T}^{n+1}(Q) + \varepsilon(m \times Q)(\pi(E)) \end{aligned}$$

If $(m \times Q)(\pi(E)) > 0$, then $c_{0,T}^{n+1}(Q) + \tilde{\varepsilon} < c_{0,T}^n(Q) < +\infty$, that is a contradiction. Hence, $(m \times Q)(\pi(E)) = 0$, i.e. $(m \times Q)(\{(t,\omega) : N \ge f_n(t,\omega,q_t) > f_{n+1}(t,\omega,q_t) + \varepsilon\}) = 0$.

By letting N tend to $+\infty$, from the arguments above and since $Q \sim P$, it follows that if $f_n < +\infty$ on $\{x : \|x\| \leq n\}$

$$f_n = f_{n+1} \quad (m \times dP)\text{-a.s.}$$

and hence $f_n = f_{n+1} = f$ $(m \times dP)$-a.s. for some functional f. I.e.

$$f_n(t, \omega, x) = f_{n+1}(t, \omega, x) = f(t, \omega, x) \quad (m \times dP)\text{-a.s.}, \qquad \text{for } \|x\| \leq n.$$

Furthermore, we may conclude that, once (t, ω) is fixed, for any q either (1) there exists $n \geq 0$ such that $f_n(t, \omega, q) < +\infty$ (hence $f_m(t, \omega, q) = f(t, \omega, q) < +\infty$ for any $m \geq n$ and $m \geq \|q\|$) or (2) for all $n \geq 0$ it holds $f_n(t, \omega, q) = +\infty = f(t, \omega, q)$. Hence

$$f(t, \omega, x) = \inf_n f_n(t, \omega, x).$$

By the properties of the sequence of f_n, it follows that $f(\cdot, \cdot, 0) = 0$. It remains to prove that $f(t, \omega, \cdot)$ is proper, convex and lower semi-continuous. Properness of $f(t, \omega, \cdot)$ is trivial. Since $f(t, \omega, x) = \lim_n f_n(t, \omega, x) = \inf_n f_n(t, \omega, x)$ for almost all (t, ω) and any f_n is predictable and convex in x, it is easy to check that also f is predictable and convex in x.
Furthermore, for almost all (t, ω) the set $\{q \in \mathbb{R}^d : f(t, \omega, q) \leq \alpha\}$ is closed for any $\alpha \in \mathbb{R}$. Take indeed a sequence $\{q^k\}_{k \geq 0}$ such that $q^k \to_k q$ and $f(t, \omega, q^k) \leq \alpha$. There exists $N \in \mathbb{N}$ such that $\|q^k\| \leq N$ for all k. Hence

$$f(t, \omega, q^k) = f_N(t, \omega, q^k) \leq \alpha.$$

Since $f_N(t, \omega, \cdot)$ is lower semi-continuous, $f(t, \omega, q) = f_N(t, \omega, q) \leq \underline{\lim}_k f_N(t, \omega, q^k) \leq \alpha$. Hence also $f(t, \omega, \cdot)$ is lower semi-continuous.

■

Lemma 10 *If $c^n_{0,T}(Q) = E_Q\left[\int_0^T f_n(u, q_u)du\right]$ holds for any probability measure $Q \sim P$ such that $\|q\| \leq n$, then also $c^n_{s,t}(Q) = E_Q\left[\int_s^t f_n(u, q_u)du | \mathcal{F}_s\right]$ holds for any $0 \leq s \leq t \leq T$ and for any probability measure $Q \sim P$ such that $\|q\| \leq n$.*

Proof. Let Q be a probability measure equivalent to P and such that $\|q\| \leq n$. Consider the case where $s = 0$ and take the probability measure $\overline{Q}$ corresponding to the following $\overline{q}$:

$$\overline{q}_u = \begin{cases} q_u; & \text{if } 0 \leq u \leq t \\ 0; & \text{if } t < u \leq T \end{cases}$$

obtained by pasting Q and P. It is clear that $\|\overline{q}\| \leq n$. From the cocycle property of c^n established in Proposition 9(i) it follows that

$$\begin{aligned} c^n_{0,T}(\overline{Q}) &= c^n_{0,t}(\overline{Q}) + E_{\overline{Q}}[c^n_{t,T}(\overline{Q})] \\ &= c^n_{0,t}(\overline{Q}) + E_{\overline{Q}}[c^n_{t,T}(P)] = c^n_{0,t}(\overline{Q}). \end{aligned}$$

From the arguments above it follows that

$$\begin{aligned} c^n_{0,t}(Q) &= c^n_{0,t}(\overline{Q}) = c^n_{0,T}(\overline{Q}) \\ &= E_{\overline{Q}}\left[\int_0^T f_n(u,\overline{q}_u)du\right] \\ &= E_{\overline{Q}}\left[\int_0^t f_n(u,\overline{q}_u)du\right] = E_Q\left[\int_0^t f_n(u,q_u)du\right]. \end{aligned}$$

We will now come back to the general case. Consider the probability measure Q^* obtained by pasting Q and P as follows:

$$q^*_u = \begin{cases} 0; & \text{if } 0 \le u \le s \\ q_u 1_A + 0 \cdot 1_{A^c}; & \text{if } s < u \le T \end{cases},$$

with $A \in \mathcal{F}_s$. On one hand, we deduce that $c^n_{0,s}(Q^*) = c^n_{0,s}(P) = 0$, while for any $s < t \le T$

$$\begin{aligned} c^n_{0,t}(Q^*) &= E_{Q^*}\left[\int_0^t f_n(u,q^*_u)du\right] \\ &= E_{Q^*}\left[1_A \int_s^t f_n(u,q_u)du\right] \\ &= E_P\left[E_Q\left[1_A \int_s^t f_n(u,q_u)du|\mathcal{F}_s\right]\right] \\ &= E_P\left[1_A E_Q\left[\int_s^t f_n(u,q_u)du|\mathcal{F}_s\right]\right]. \end{aligned}$$

On the other hand, from the cocycle property $E_{Q^*}[c^n_{s,t}(Q^*)] = c^n_{0,t}(Q^*) - c^n_{0,s}(Q^*)$, hence

$$\begin{aligned} c^n_{0,t}(Q^*) &= c^n_{0,t}(Q^*) - c^n_{0,s}(Q^*) = E_{Q^*}[c^n_{s,t}(Q^*)] \\ &= E_{Q^*}[E_{Q^*}[c^n_{s,t}(Q^*)|\mathcal{F}_s]] \\ &= E_P[1_A E_Q[c^n_{s,t}(Q)|\mathcal{F}_s]]. \end{aligned}$$

Since the set A is arbitrary, we deduce that for any $A \in \mathcal{F}_s$

$$E_P\left[1_A E_Q\left[\int_s^t f_n(u,q_u)du|\mathcal{F}_s\right]\right] = E_P[1_A E_Q[c^n_{s,t}(Q)|\mathcal{F}_s]],$$

hence

$$c^n_{s,t}(Q) = E_Q[c^n_{s,t}(Q)|\mathcal{F}_s] = E_Q\left[\int_s^t f_n(u,q_u)du|\mathcal{F}_s\right].$$

■

Lemma 11 *For any probability measure Q equivalent to P it holds true that*

$$\begin{aligned} c_{0,T}(Q) &\le E_Q\left[\int_0^T f(u,q_u)du\right] \\ c_{t,T}(Q) &\le E_Q\left[\int_t^T f(u,q_u)du|\mathcal{F}_t\right] \end{aligned}$$

Proof. We will start proving the inequality for $c_{0,T}(Q)$.

Case 1: $\int_0^T f(u,q_u)du$ is bounded.

Consider the probability measure Q^n corresponding to $q^n \triangleq q 1_{\|q\| \le n}$. Since $\int_0^T f(u,q_u)du$ is bounded (by assumption), from Proposition 9(iii) it follows that

$$\begin{aligned} \lim_n c_{0,T}(Q^n) &= \lim_n E_{Q^n}\left[\int_0^T f_n(u,q^n_u)du\right] \\ &= \lim_n E_{Q^n}\left[\int_0^T f(u,q_u)1_{\|q\| \le n}du\right] \\ &= \lim_n E_Q\left[\frac{dQ^n}{dQ}\int_0^T f(u,q_u)1_{\|q\| \le n}du\right] \\ &= E_Q\left[\int_0^T f(u,q_u)du\right] < +\infty \end{aligned}$$

Since $\frac{dQ^n}{dP} \to_n^{L^1} \frac{dQ}{dP}$, by lower semi-continuity of $c_{0,T}(Q)$ it follows that

$$c_{0,T}(Q) \leq \liminf_n c_{0,T}(Q^n) \leq E_Q\left[\int_0^T f(u, q_u)du\right].$$

Case 2: $\int_0^T f(u, q_u)du \in L^1(Q)$.

For any $n \in \mathbb{N}$, set $\sigma_n \triangleq \inf\{t \geq 0 : \int_0^t f(u, q_u)du \geq n\}$. Then σ_n is a stopping time and $\sigma_n \uparrow T$.

Set Q^{σ_n} the probability measure corresponding to $\frac{dQ^{\sigma_n}}{dP} = \mathcal{E}(q \cdot B)^{\sigma_n}$. It is easy to check that $\frac{dQ^{\sigma_n}}{dP} \to_n^{L^1} \frac{dQ}{dP}$. Furthermore,

$$E_{Q^{\sigma_n}}\left[\int_0^{\sigma_n} f(u, q_u^{\sigma_n})du\right] = E_Q\left[\int_0^{\sigma_n} f(u, q_u)du\right] \to_n E_Q\left[\int_0^T f(u, q_u)du\right],$$

where the equality above is due to the fact that q and q^{σ_n} coincide on the stochastic interval $[\![0, \sigma_n]\!]$. By applying the arguments above, we obtain

$$\begin{aligned} c_{0,T}(Q) &\leq \liminf_n c_{0,T}(Q^{\sigma_n}) \leq \liminf_n E_{Q^{\sigma_n}}\left[\int_0^{\sigma_n} f(u, q_u^{\sigma_n})du\right] \\ &\leq E_Q\left[\int_0^T f(u, q_u)du\right]. \end{aligned}$$

Case 3: General case.

In general, if $\int_0^T f(u, q_u)du \notin L^1(Q)$, then $E_Q\left[\int_0^T f(u, q_u)du\right] = +\infty$. Hence $c_{0,T}(Q) \leq E_Q\left[\int_0^T f(u, q_u)du\right]$.

The inequality $c_{t,T}(Q) \leq E_Q\left[\int_t^T f(u, q_u)du | \mathcal{F}_t\right]$ can be checked by proceeding as in the proof of Lemma 10. ■

Lemma 12 *Let Q be a probability measure equivalent to P and such that $c_{0,T}(Q) < +\infty$. If $\{\tau_n\}_{n\geq 0}$ is a sequence of stopping times such that $P(\tau_n < T) \to_n 0$, then $c_0(Q^{\tau_n}) \uparrow c_0(Q)$, where Q^{τ_n} is defined by $\frac{dQ^{\tau_n}}{dP} = E_P\left[\frac{dQ}{dP} | \mathcal{F}_{\tau_n}\right]$.*

Proof. On one hand, by the cocycle property and by the definition of Q^{τ_n} it follows

$$\begin{aligned} c_{0,T}(Q) &= c_{0,\tau_n}(Q) + E_Q[c_{\tau_n,T}(Q)] \\ &\geq c_{0,\tau_n}(Q) = c_{0,T}(Q^{\tau_n}) \end{aligned}$$

On the other hand, by the lower semi-continuity of c_0 and by $\frac{dQ^{\tau_n}}{dP} \to_n^{L^1} \frac{dQ}{dP}$ it holds $c_{0,T}(Q) \leq \liminf_n c_{0,T}(Q^{\tau_n})$. So $\lim_n c_{0,T}(Q^{\tau_n}) = c_{0,T}(Q)$. ■

Lemma 13 *Consider a general setting where the filtration satisfies the usual hypothesis but it is not necessarily a Brownian filtration.*

Let Q be a probability measure equivalent to P such that $c_{0,T}(Q) < +\infty$ and $(c_t(Q))_{t\in[0,T]}$ is right-continuous.

Then there exists a unique increasing, predictable process $(A_t)_{t\in[0,T]}$ (depending on Q) such that $A_0 = 0$ and

$$c_t(Q) = E_Q[A_T - A_t | \mathcal{F}_t], \quad \forall t \in [0, T], \tag{11}$$

i.e. $c_t(Q)$ is a Q-Potential.

Proof. By Theorem VII.8 of Dellacherie and Meyer [18], equality (11) holds true if $(c_t(Q))_{t\in[0,T]}$ is a positive Q-supermartingale of class (D), that is $(c_\sigma(Q))_{\sigma\in S}$ is uniformly integrable where S is the family of all stopping times smaller or equal to T.

The process $(c_t(Q))_{t\in[0,T]}$ is clearly adapted and positive and, by hypothesis and from the cocycle property, $c_t(Q)\in L^1(Q)$ for any $t\in[0,T]$. By the cocycle property we deduce that for any $0\le s\le t\le T$

$$E_Q[c_{t,T}(Q)|\mathcal{F}_s]=c_{s,T}(Q)-c_{s,t}(Q)\le c_{s,T}(Q),$$

i.e. $(c_t(Q))_{t\in[0,T]}$ is a Q-supermartingale. Furthermore, $c_T(Q)=0$. It remains to show that $(c_t(Q))_{t\in[0,T]}$ is of class (D). This proof is postponed to the Appendix. ∎

Remark 14 *Since in our setting $(c_t(Q))_{t\in[0,T]}$ is càdlàg (see the Appendix for the proof), as a particular case of the previous Lemma it follows that equation (11) holds for a càdlàg $(A_t)_{t\in[0,T]}$.*

Note that from (11) it follows that $c_{t,u}(Q)=E_Q\left[A_u-A_t|\mathcal{F}_t\right]$ for any $0\le t\le u\le T$. Furthermore, the assumption $c_t(P)=0$ implies that for $Q=P$ we have $A=A^P=0$.

Lemma 15 *Let σ,τ be two stopping times such that $0\le\sigma\le\tau\le T$ and Q^1,Q^2 be two probability measures equivalent to P. Denote by A^1,A^2 the corresponding increasing processes as in (11).*

Let Q be the probability measure induced by

$$q=\begin{cases} q^1, & \text{on } H^1=]\!]0,\sigma]\!]\cup]\!]\tau,T]\!] \\ q^2, & \text{on } H^2=]\!]\sigma,\tau]\!] \end{cases}$$

and denote by A the corresponding process as in (11).

Then

$$dA=dA^1|_{H^1}+dA^2|_{H^2}=1_{H^1}dA^1+1_{H^2}dA^2. \tag{12}$$

Proof. Consider an arbitrary $t\in[0,T]$.

For $t\ge\tau$: we have that $c_t(Q)=c_t^1(Q^1)=E_{Q^1}\left[A_T^1-A_t^1|\mathcal{F}_t\right]$.

For $\sigma\le t<\tau$: from the cocycle property we deduce that

$$\begin{aligned} c_t(Q) &= c_{t,\tau}(Q)+E_Q\left[c_{\tau,T}(Q)|\mathcal{F}_t\right] \\ &= E_{Q^2}\left[A_\tau^2-A_t^2|\mathcal{F}_t\right]+E_{Q^2}\left[E_{Q^1}[A_T^1-A_\tau^1|\mathcal{F}_\tau]|\mathcal{F}_t\right] \\ &= E_Q\left[A_\tau^2-A_t^2+A_T^1-A_\tau^1|\mathcal{F}_t\right] \\ &= E_Q\left[\int_{(t,T]}\left(1_{H^1}dA^1+1_{H^2}dA^2\right)|\mathcal{F}_t\right]. \end{aligned}$$

For $t\le\sigma$: from the cocycle property and from the case above we deduce that

$$\begin{aligned} c_t(Q) &= c_{t,\sigma}(Q)+E_Q\left[c_{\sigma,T}(Q)|\mathcal{F}_t\right] \\ &= E_{Q^1}\left[A_\sigma^1-A_t^1|\mathcal{F}_t\right] \\ &\quad +E_{Q^1}\left[E_Q\left[\int_{]\!]\sigma,T]\!]}\left(1_{H^1}dA^1+1_{H^2}dA^2\right)|\mathcal{F}_\sigma\right]|\mathcal{F}_t\right] \\ &= E_Q\left[\int_{(t,T]}\left(1_{H^1}dA^1+1_{H^2}dA^2\right)|\mathcal{F}_t\right]. \end{aligned}$$

Since $A_t\triangleq\int_{(0,t]}\left(1_{H^1}dA^1+1_{H^2}dA^2\right)$ is càdlàg, predictable and increasing, $(A_t)_{t\in[0,T]}$ is the process associated to Q in the sense of (11). ∎

Corollary 16 *Let $\sigma_1, \sigma_2, \ldots, \sigma_n, \tau_1, \tau_2, \ldots, \tau_n$ be stopping times such that $0 \le \sigma_1 \le \tau_1 \le \sigma_2 \le \tau_2 \le \ldots \le \sigma_n \le \tau_n \le T$ and let Q be a probability measure equivalent to P and whose corresponding increasing process is denoted by A. Set*

$$H \triangleq]\!]\sigma_1, \tau_1]\!] \cup]\!]\sigma_2, \tau_2]\!] \cup \ldots \cup]\!]\sigma_n, \tau_n]\!]. \tag{13}$$

Let Q^H be the probability measure induced by $q^H = q1_H$ and denote by A^H the corresponding process as in (11). Then

$$dA^H = 1_H dA. \tag{14}$$

Proof. The proof of this result is a repeated application of Lemma 15 (with $Q^1 = P$ and $Q^2 = Q$). ■

Lemma 17 *Let Q be a probability measure equivalent to P and A be the associated increasing process.*

Then there exists a sequence $(\tau^n)_{n\in\mathbb{N}}$ of stopping times such that

(i) $\frac{dQ^{]\!]0,\tau^n]\!]}}{dP} \to_n^{L^1} \frac{dQ}{dP}$, where $Q^{]\!]0,\tau^n]\!]}$ denotes the probability measure induced by $q^{]\!]0,\tau^n]\!]} = q1_{]\!]0,\tau^n]\!]}$;

(ii) $c_{0,T}(Q^{]\!]0,\tau^n]\!]}) \uparrow c_{0,T}(Q)$;

(iii) A_{τ^n} is bounded.

Proof. For any $n \in \mathbb{N}$ set $\sigma^n \triangleq \inf\{t \ge 0 : A_t \ge n\}$. Hence σ^n is a predictable stopping time. For any fixed n, take now a sequence $(\tau^{n,m})_{m\in\mathbb{N}}$ such that $\tau^{n,m}$ is increasing (in m), $\tau^{n,m} < \sigma^n$ on $\{\sigma^n > 0\}$ and $\tau^{n,m} \uparrow \sigma^n$. By definition of σ^n and from $\tau^{n,m} < \sigma^n$ it follows that $A_{\tau^{n,m}} \le n$.

For any $\varepsilon > 0$ small enough, take now n and consequently m big enough to have $\|\frac{dQ^{]\!]0,\tau^{n,m}]\!]}}{dP} - \frac{dQ}{dP}\|_1 \le \varepsilon$. For such indexes set $\tau^{(n)} \triangleq \tau^{n,m}$.

Take now $\tau^n \triangleq \max_{k\le n} \tau^{(k)}$. It can be checked that $(\tau^n)_{n\in\mathbb{N}}$ is an increasing sequence of stopping times and that $A_{\tau^n} \le n$ (since also $\tau^n < \sigma^n$). Furthermore, since $\sigma^n = T$ for sufficiently big n and $\tau^n \uparrow T$, property (i) follows. Property (ii) can be checked as usual (see, for instance, the proof of Lemma 12). ■

Lemma 18 *Let Q be a probability measure equivalent to P and let A be the associated increasing process. Suppose that A is bounded. Let H be a predictable set.*

Suppose that $\mathcal{E}(q1_H \cdot B)$ is a uniformly integrable martingale. Set $\frac{dQ^H}{dP} \triangleq \mathcal{E}(q1_H \cdot B)_T$ and denote by A^H the associated increasing process.

Then

$$dA^H \le dA, \tag{15}$$

hence $A^H_T \le A_T$.

Proof. First of all, we recall that the sets of the same form as in (13) form an algebra $\mathcal{A}$ (Boolean algebra) and that the σ-algebra $\mathcal{P}$ of predictable sets is generated by $\mathcal{A}$.

Consider now any predictable set $H \in \mathcal{P}$ satisfying the hypothesis above. If $H \in \mathcal{A}$, we already know that $dA^H = 1_H dA$ (from Corollary 16). We will consider now the general case.

Consider two stopping times σ, τ such that $0 \leq \sigma \leq \tau \leq T$ and take a sequence $(H^n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$ such that

$$\begin{array}{l} E_Q\left[\int_0^T |1_{H^n} - 1_H| dA\right] \to_n 0 \\ E\left[\int_0^T |1_{H^n} - 1_H| dt\right] \to_n 0. \end{array} \tag{16}$$

Set Q^{H^n} the probability measure induced by $q^n = q1_{H^n}$ and denote by A^{H^n} the associated increasing process. Again from Corollary 16 it follows that $dA^{H^n} = 1_{H^n} dA$ (since $H^n \in \mathcal{A}$). By (16) we have that $\frac{dQ^{H^n}}{dP} \to_n^{L^1} \frac{dQ^H}{dP}$.

By lower semi-continuity of c and by (11), we get

$$\begin{aligned} E_{Q^H}\left[A^H_\tau - A^H_\sigma | \mathcal{F}_\sigma\right] &= c_{\sigma,\tau}(Q^H) \\ &\leq \liminf_n c_{\sigma,\tau}(Q^{H^n}) \\ &= \liminf_n E_{Q^{H^n}}\left[A^{H^n}_\tau - A^{H^n}_\sigma | \mathcal{F}_\sigma\right]. \end{aligned}$$

Since $\int_{\rrbracket\sigma,\tau\rrbracket} 1_{H^n} dA \to_n \int_{\rrbracket\sigma,\tau\rrbracket} 1_H dA$, $\int_{\rrbracket\sigma,\tau\rrbracket} 1_{H^n} dA$ is uniformly bounded and $\mathcal{E}(1_{\rrbracket\sigma,\tau\rrbracket\cap H^n} q \cdot B) \to_n^{L^1} \mathcal{E}(1_{\rrbracket\sigma,\tau\rrbracket\cap H} q \cdot B)$, then

$$E_{Q^{H^n}}\left[\int_{\rrbracket\sigma,\tau\rrbracket} 1_{H^n} dA | \mathcal{F}_\sigma\right] \to_n E_{Q^H}\left[\int_{\rrbracket\sigma,\tau\rrbracket} 1_H dA | \mathcal{F}_\sigma\right]. \tag{17}$$

From (16) and (17) it follows that

$$\begin{aligned} E_{Q^H}\left[A^H_\tau - A^H_\sigma | \mathcal{F}_\sigma\right] &= E_{Q^H}\left[\int_{\rrbracket\sigma,\tau\rrbracket} dA^H | \mathcal{F}_\sigma\right] \\ &\leq \liminf_n E_{Q^{H^n}}\left[A^{H^n}_\tau - A^{H^n}_\sigma | \mathcal{F}_\sigma\right] \\ &= E_{Q^H}\left[\int_{\rrbracket\sigma,\tau\rrbracket} 1_H dA | \mathcal{F}_\sigma\right], \end{aligned}$$

hence $E_{Q^H}\left[\int_{\rrbracket\sigma,\tau\rrbracket} dA^H | \mathcal{F}_\sigma\right] \leq E_{Q^H}\left[\int_{\rrbracket\sigma,\tau\rrbracket} 1_H dA | \mathcal{F}_\sigma\right]$.

The same inequality holds if we replace $\rrbracket\sigma,\tau\rrbracket$ with any element $K \in \mathcal{A}$ (it is sufficient to sum over intervals of the same form as in (13)), that is

$$E_{Q^H}\left[\int_0^T 1_K dA^H | \mathcal{F}_\sigma\right] \leq E_{Q^H}\left[\int_0^T 1_K 1_H dA | \mathcal{F}_\sigma\right]. \tag{18}$$

Moreover, by passing to the limit we obtain that inequality (18) holds true for any $K \in \mathcal{P}$. So we get $dA^H \leq 1_H dA$ as stochastic measures on $(0, T]$, hence $A^H_T \leq A_T$. ■

Lemma 19 *Let Q be a probability measure equivalent to P and suppose that the corresponding increasing process A is bounded.*

If H^n is predictable, $H^n \uparrow (0,T]\times\Omega$ and Q^{H^n} is the probability measure induced by $q^{H^n} = q1_{H^n}$, then

$$c_{0,T}(Q^{H^n}) \to_n c_{0,T}(Q).$$

Proof. We already know (by Lemma 18) that

$$dA^{H^n} \leq 1_{H^n} dA. \tag{19}$$

From $\frac{dQ^{H^n}}{dP} \to_n^{L^1} \frac{dQ}{dP}$, inequality (19) and lower semi-continuity of $c_{0,T}$ we get

$$\begin{aligned} c_{0,T}(Q) &\leq \liminf_n c_{0,T}(Q^{H^n}) \\ &= \liminf_n E_{Q^{H^n}} \left[A_T^{H^n} \right] \\ &\leq \liminf_n E_{Q^{H^n}} \left[\int_{(0,T]} 1_{H^n} dA \right] \end{aligned}$$

Since $\int_{(0,T]} 1_{H^n} dA$ is bounded and $\frac{dQ^{H^n}}{dP} \to_n^{L^1} \frac{dQ}{dP}$, we have that

$$\begin{aligned} c_{0,T}(Q) &\leq \liminf_n c_{0,T}(Q^{H^n}) \leq \liminf_n E_{Q^{H^n}} \left[\int_{(0,T]} 1_{H^n} dA \right] \\ &= E_Q \left[\int_{(0,T]} dA \right] = c_{0,T}(Q), \end{aligned}$$

hence $c_{0,T}(Q^{H^n}) \to_n c_{0,T}(Q)$. ■

Theorem 20 *Let Q be a probability measure equivalent to P and let A be the associated increasing process. Then there exists a sequence $(Q^n)_{n\in\mathbb{N}}$ of probability measures with q^n bounded such that $\frac{dQ^n}{dP} \to_n^{L^1} \frac{dQ}{dP}$ and $c_{0,T}(Q^n) \to_n c_{0,T}(Q)$.*

Proof. From the arguments above (and by stopping arguments) we may suppose A bounded.

For any $n \in \mathbb{N}$ take $H^n \triangleq \{\|q\| \leq n\}$ and set Q^n the probability measure induced by $q^n = q 1_{H^n}$. Hence H^n is predictable and $H^n \uparrow (0,T] \times \Omega$, it satisfies the hypothesis of Lemma 19. It follows that $\frac{dQ^n}{dP} \to_n^{L^1} \frac{dQ}{dP}$ and (by Lemma 19) that $c_{0,T}(Q^n) \to_n c_{0,T}(Q)$. ■

We are now ready to prove the representation of the penalty term c in terms of f (see Theorem 5).

Proof. (of Theorem 5) Since (ii) is a straightforward consequence of (i) and of the representation in (3), it remains to show that

$$c_{0,T}(Q) = E_Q \left[\int_0^T f(t, q_t) dt \right]. \tag{20}$$

By Lemma 11, we already know that $c_{0,T}(Q) \leq E_Q \left[\int_0^T f(u, q_u) du \right]$ for any probability measure $Q \sim P$.

Suppose that $\int_0^T f(t, \omega, q_t) dt \in L^1(Q)$. For any $n \in \mathbb{N}$ set $\sigma_n \triangleq \inf\{t \geq 0 : \int_0^t f(u, q_u) \geq n\}$. $(\sigma_n)_{n\geq 0}$ is a sequence of stopping times such that $\sigma_n \uparrow T$.

Take now a sequence $(Q^m)_{m\in\mathbb{N}}$ of probability measures as in Theorem 20. Then

$$\begin{array}{ll} c_{0,T}(Q) & \leq E_Q\left[\int_0^T f(u,q_u)du\right] \\ & = \lim_n E_Q\left[\int_0^{\sigma_n} f(u,q_u)du\right] \\ & \leq \sup_n \lim_m E_{Q^m}\left[\int_0^{\sigma_n} f(u,q_u)1_{\|q\|\leq m}du\right] \\ & \leq \lim_m \sup_n E_{Q^m}\left[\int_0^{\sigma_n} f(u,q_u)1_{\|q\|\leq m}du\right] \\ & = \lim_m E_{Q^m}\left[\int_0^T f(u,q_u)1_{\|q\|\leq m}du\right] \\ & = \lim_m c^m_{0,T}(Q^m) = \lim_m c_{0,T}(Q^m) = c_{0,T}(Q), \end{array}$$

where the last equality is due to Theorem 20. Equality (20) has therefore been established for $\int_0^T f(t,q_t)dt \in L^1(Q)$.

If $\int_0^T f(t,\omega,q_t)dt \notin L^1(Q)$, by Fatou's Lemma we get

$$\begin{array}{ll} c_{0,T}(Q) & \leq E_Q\left[\int_0^T f(t,q_t)dt\right] \\ & \leq \liminf_m E_{Q^m}\left[\int_0^T f(t,q_t)1_{\|q\|\leq m}dt\right] \\ & = \liminf_m c^m_{0,T}(Q^m) \\ & = \liminf_m c_{0,T}(Q^m) = c_{0,T}(Q), \end{array}$$

hence $c_{0,T}(Q) = E_Q\left[\int_0^T f(t,q_t)dt\right] = +\infty$. The representation of $c_{s,t}(Q)$ (hence of $c_{\sigma,\tau}(Q)$) can be deduced as usual. ∎

Acknowledgements The authors thank two anonymous referees for useful comments that improved this paper.

4 Appendix

Let Q be a probability measure equivalent to P and such that $c_{0,T}(Q) < +\infty$.

In the following, we will prove that $(c_{t,T}(Q))_{t\in[0,T]}$ is of class (D) and that it admits a càdlàg modification.

The following corollary of Lemma 12 will be useful later.

Corollary 21 $\sup\{E_Q[c_{\tau,T}(Q)] | \tau$ *stopping time s.t.* $P(\tau < T) \leq \frac{1}{n}\} \to_n 0$

The following result is a straightforward consequence of the cocycle property of c.

Lemma 22 *Denote by S the family of all stopping times smaller or equal to T.*

The family $(c_{\sigma,T}(Q))_{\sigma\in S}$ satisfies the following property: given any pair of stopping times σ,τ such that $0 \leq \sigma \leq \tau \leq T$ then $c_{\sigma,T}(Q) \geq E_Q[c_{\tau,T}(Q)|\mathcal{F}_\sigma]$.

Lemma 23 *The family $(c_{\sigma,T}(Q))_{\sigma\in S}$ is Q-uniformly integrable.*

Proof. We have to prove that

$$\lim_{n\to+\infty}\sup_{\sigma\in S}\int_{c_{\sigma,T}(Q)>n} c_{\sigma,T}(Q)dQ=0. \tag{21}$$

Consider an arbitrary stopping time $\sigma\in S$ and set

$$\sigma^{(n)}=\begin{cases}\sigma; & \text{if } c_{\sigma,T}(Q)>n\\ T; & \text{if } c_{\sigma,T}(Q)\le n\end{cases}$$

By the cocycle property we get

$$\begin{aligned}c_0(Q) &= c_0(Q^{\sigma^{(n)}})+E_Q[c_{\sigma^{(n)},T}(Q)]\\ &\ge E_Q[c_{\sigma^{(n)},T}(Q)]\\ &= \int_{c_{\sigma,T}(Q)>n} c_{\sigma,T}(Q)dQ \ge nP(c_{\sigma,T}(Q)>n).\end{aligned}$$

Hence $P(c_{\sigma,T}(Q)>n)\le\frac{c_0(Q)}{n}$ uniformly in σ, so we get

$$\begin{aligned}0 &\le \sup_{\sigma\in S}\int_{c_{\sigma,T}(Q)>n} c_{\sigma,T}(Q)dQ\\ &\le \sup\{E_Q[c_{\tau,T}(Q)] \mid \tau \text{ stopping time s.t. } P(\tau<T)\le\tfrac{c_0(Q)}{n}\}\end{aligned}$$

Since the last term tends to 0 as $n\to+\infty$ by Corollary 21, (21) follows. ■

Lemma 24 *Let $\varepsilon>0$ such that $E_Q[-\xi]>c_0(Q)-\varepsilon$ with $\xi\in\mathcal{A}_{0,T}$. Then for any pair of stopping times σ,τ such that $0\le\sigma\le\tau\le T$ it holds that:*

$$E_Q[c_{\sigma,\tau}(Q)]\le E_Q[u_\sigma(\xi)-u_\tau(\xi)]+\varepsilon.$$

Proof. By translation invariance of $(u_{t,T})_{t\in[0,T]}$ it follows that $u_{\tau,T}(\xi-u_{\tau,T}(\xi))=0$, hence $\xi-u_\tau(\xi)\in\mathcal{A}_{\tau,T}$. Furthermore, by time-consistency and translation invariance of u and by $\xi\in\mathcal{A}_{0,T}$ it follows that $u_\tau(\xi)-u_\sigma(\xi)\in\mathcal{A}_{\sigma,T}$ and that $u_\sigma(\xi)\in\mathcal{A}_{0,\sigma}$.

The cocycle property or, equivalently, the decomposition property $\mathcal{A}_{0,T}=\mathcal{A}_{0,\sigma}+\mathcal{A}_{\sigma,\tau}+\mathcal{A}_{\tau,T}$ implies that

$$\begin{aligned}c_0(Q) &= E_Q[c_{0,\sigma}(Q)]+E_Q[c_{\sigma,\tau}(Q)]+E_Q[c_{\tau,T}(Q)]\\ &\ge E_Q[-u_\sigma(\xi)]+E_Q[u_\sigma(\xi)-u_\tau(\xi)]+E_Q[u_\tau(\xi)-\xi]\\ &\ge E_Q[-\xi]\ge c_0(Q)-\varepsilon,\end{aligned}$$

where the first inequality follows from $c_{t,T}(Q)=\operatorname{ess.sup}_{\xi\in\mathcal{A}_{t,T}}E_Q\left[-\xi|\mathcal{F}_t\right]$. By proceeding as above we get

$$\begin{aligned}c_0(Q) &\ge E_Q[-u_\sigma(\xi)]+E_Q[u_\sigma(\xi)-u_\tau(\xi)]+E_Q[u_\tau(\xi)-\xi]\\ &\ge c_0(Q)-\varepsilon\\ &\ge E_Q[-u_\sigma(\xi)]+E_Q[c_{\sigma,\tau}(Q)]+E_Q[u_\tau(\xi)-\xi]-\varepsilon,\end{aligned}$$

hence $E_Q[c_{\sigma,\tau}(Q)]\le E_Q[u_\sigma(\xi)-u_\tau(\xi)]+\varepsilon$. ■

Lemma 25 *Let $\sigma\in S$. If $\{\sigma_n\}_{n\in\mathbb{N}}$ is a sequence of stopping times such that $\sigma_n\downarrow\sigma$, then $E_Q[c_{\sigma,\sigma_n}(Q)]\to_n 0$.*

Proof. Suppose by contradiction that $E_Q[c_{\sigma,\sigma_n}(Q)]$ does not tend to 0 as $n \to +\infty$. Hence there exists $\varepsilon > 0$ such that $E_Q[c_{\sigma,\sigma_n}(Q)] \geq \varepsilon > 0$ for any $n \in \mathbb{N}$. Take now $\xi \in \mathcal{A}_{0,T}$ such that $E_Q[-\xi] \geq c_0(Q) - \frac{\varepsilon}{2}$. Hence, by Lemma 24,

$$E_Q[u_\sigma(\xi) - u_{\sigma_n}(\xi)] \geq E_Q[c_{\sigma,\sigma_n}(Q)] - \frac{\varepsilon}{2} \geq \frac{\varepsilon}{2}$$

for any $n \in \mathbb{N}$. This leads to a contradiction since $(u_{t,T})_{t\in[0,T]}$ admits a càdlàg version with $u_{\sigma_n}(\xi) \to_n u_\sigma(\xi)$ in $L^1(Q)$ (see Lemma 4 of Bion-Nadal [6]). ■

By the cocycle property it is easy to deduce the following result from the one above.

Corollary 26 *Let $\sigma \in S$. If $\{\sigma_n\}_{n\in\mathbb{N}}$ is a sequence of stopping times such that $\sigma_n \downarrow \sigma$, then $E_Q[c_{\sigma_n,T}(Q)] \to_n E_Q[c_{\sigma,T}(Q)]$.*

Lemma 27 *$(c_{t,T}(Q))_{t\in[0,T]}$ admits a càdlàg modification. Furthermore, if $(\overline{c}_t)_{t\in[0,T]}$ denotes this modification, for any stopping time $\sigma \in S$ it holds $c_{\sigma,T}(Q) = \overline{c}_\sigma$ a.s.*

We remark that this ends the proof of the statement in the beginning of the appendix.

Proof. We already know that $(c_{t,T}(Q))_{t\in[0,T]}$ is a positive Q-supermartingale (see the proof of Lemma 13) and that for any sequence $\{t_n\}_{n\in\mathbb{N}}$ in $[0,T]$ and such that $t_n \downarrow t$ it holds $E_Q[c_{t_n,T}(Q)] \to_n E_Q[c_{t,T}(Q)]$ (by Corollary 26). By Theorem 4 at page 76 of Dellacherie and Meyer [18] it follows that $(c_{t,T}(Q))_{t\in[0,T]}$ admits a càdlàg modification. This implies that for any stopping time $\sigma \in S$ taking rational values it holds $\overline{c}_\sigma = c_{\sigma,T}(Q)$ a.s.. For a general stopping time $\sigma \in S$ there exists a sequence $\{\sigma_n\}_{n\in\mathbb{N}}$ of finite stopping times taking rational values and such that $\sigma_n \downarrow \sigma$. Hence

$$\lim_{n\to+\infty} c_{\sigma_n,T}(Q) = \lim_{n\to+\infty} \overline{c}_{\sigma_n} = \overline{c}_\sigma \quad a.s. \tag{22}$$

where the last equality follows from the fact that $(\overline{c}_t)_{t\in[0,T]}$ is càdlàg.

It remains to prove that $c_{\sigma,T}(Q) = \lim_{n\to+\infty} c_{\sigma_n,T}(Q)$. This proof is quite standard and we include it for completeness. By the cocycle property it follows that $(c_{\sigma_n,T}(Q), \mathcal{F}_{\sigma_n})_{n\in\mathbb{N}}$ is a positive reversed Q-supermartingale (see Neveu [31]). By Proposition V-3-11 of Neveu [31], $c_{\sigma_n,T}(Q)$ converges as $n \to +\infty$ to a positive $\mathcal{F}_\sigma$-measurable random variable η and $E_Q\left[c_{\sigma_n,T}(Q)|\mathcal{F}_\sigma\right] \to_n \eta$ a.s.. Since $E_Q\left[c_{\sigma_n,T}(Q)|\mathcal{F}_\sigma\right] \leq c_{\sigma,T}(Q)$, we get $\eta \leq c_{\sigma,T}(Q)$. Furthermore, by Q-uniform integrability of $(c_{\sigma_n,T}(Q))_{n\in\mathbb{N}}$ (see Lemma 23) we get

$$E_Q[c_{\sigma,T}(Q)] = \lim_n E_Q[c_{\sigma_n,T}(Q)] = E_Q[\eta]$$

where the first equality is due to Corollary 26. By the arguments above it follows that $\eta = c_{\sigma_n}(Q)$ a.s., hence the thesis. ■

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