Drift dependence of optimal trade execution strategies under transient price impact

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Abstract

We give a complete solution to the problem of minimizing the expected liquidity costs in presence of a general drift when the underlying market impact model has linear transient price impact with exponential resilience. It turns out that this problem is well-posed only if the drift is absolutely continuous. Optimal strategies often do not exist, and when they do, they depend strongly on the derivative of the drift. Our approach uses elements from singular stochastic control, even though the problem is essentially non-Markovian due to the transience of price impact and the lack in Markovian structure of the underlying price process. As a corollary, we give a complete solution to the minimization of a certain cost-risk criterion in our setting.

1 Introduction

Standard asset pricing models like the Black–Scholes model assume that asset prices are given exogenously and are unaffected by the trading behavior of economic agents. In reality, however, many trades are large enough to feed back on asset prices so that price impact and the resulting liquidity costs cannot be ignored. In such a situation, one aims at minimizing the liquidity costs from trade execution by constructing suitable trading strategies. The problem of computing such trading strategies is called the *optimal trade execution problem*.

To deal with price impact quantitatively, several stochastic market impact models have been proposed in recent years. In the first model class, which goes back to Bertsimas & Lo (1998) and Almgren & Chriss (1999, 2000), price impact is modeled by combining convex transaction costs with a linear permanent price impact term. While these models make computations feasible and lead to relatively nice and robust trading strategies, they do not adequately model the empirically observed transience of price impact. Transience means that price impact is strongest immediately after being triggered and that it subsequently decays in time. This effect

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is well-established empirically, it can be measured, and it is widely believed that the decay of price impact follows some general laws; see, e.g., Gatheral (2010), Lehalle & Dang (2010), Moro et al. (2009), and the references therein. Therefore, several models for transient price impact have been proposed in recent years. To our knowledge, the first models were proposed by Bouchaud et al. (2004) and Obizhaeva & Wang (2013). The latter is a linear price impact model with exponential decay of price impact and seems to be the first transient-price impact model used for computing optimal trade execution strategies. Two different extensions were given to the case of nonlinear transient price impact. The first was proposed by Alfonsi et al. (2010) and further developed by Alfonsi & Schied (2010) and Predoiu et al. (2011). The second extension is due to Gatheral (2010) and, besides nonlinearity, also allows for more general decay patterns than exponential decay. Let us also mention related research by Bayraktar & Ludkovski (2011), Bouchard et al. (2011), Kharroubi & Pham (2010), and Guéant et al. (2012).

Since transience of price impact is more realistic than the combination of transaction costs with linear permanent impact, one might guess that market impact models with transient price impact perform better in practice than those of Bertsimas & Lo (1998) and Almgren & Chriss (1999, 2000). But what can be said about their mathematical stability and robustness in comparison to these older models? This is an important question because of the high degree of uncertainty in the estimation of market microstructure parameters. Gatheral (2010) addressed this question by analyzing the possible non-existence of optimal trade execution strategies for certain parameters. As shown by Alfonsi & Schied (2010) and further discussed in Gatheral et al. (2011), these results depend strongly on the way in which nonlinearity of price impact is modeled. Therefore stability investigations with respect to other model features have been carried out in the case of linear price impact. Moreover, for liquid stocks linear price impact can also be a very good approximation to reality as shown empirically by Blais & Protter (2010). Alfonsi et al. (2012) investigate the dependence of optimal trade execution strategies on the decay kernel that models the temporal decay of price impact. They find that discretetime strategies react in a very sensitive manner to the choice of this decay kernel and that price impact must decay as a convex nonincreasing function of time so as to exclude certain irregularities of optimal strategies. This observation implies in particular that in practice the decay of price impact cannot be estimated in a nonparametric way.

An extension of the results in Alfonsi et al. (2012) to continuous time was given by Gatheral et al. (2012). Finally, assuming exponential decay of price impact, Fruth et al. (2011) analyze the specific form and regularity of optimal trade execution strategies when liquidity can be timedependent or even stochastic. An analysis pertaining specifically to regularity issues arising in this context has recently been given by Klöck (2012).

When investigating a particular model aspect, it is important to keep the remaining features of the model simple. For instance, to analyze the existence or nonexistence of price manipulation strategies as in Gatheral (2010) or Alfonsi et al. (2012), it is necessary to assume that the underlying price process is a martingale. There are additional reasons why it may be natural to make this martingale assumption; see, e.g., the discussion in Alfonsi et al. (2012). But there are also good reasons to allow for a nonvanishing drift in unaffected asset prices. For instance, an economic agent may be aware of the trading activities of another market participant. These trading activities will create price impact, which from the point of view of our economic agent will be perceived as a drift in asset prices. Moreover, for several reasons, the economic agent may have a rather accurate estimate of this drift. For instance, some trade execution algorithms create characteristic order patterns and therefore allow for an inference of their future trading trajectory. We refer to Schöneborn & Schied (2009) for a study of a multi-agent situation in the Almgren–Chriss framework. In this paper, we aim at continuing the investigation of the stability of models for transient price impact by focusing on the dependence of optimal trade execution strategies on a possible *drift* of the underlying unaffected price process. In doing this, we will allow for rather general dynamics of the drift and in particular allow for jumps and a non-Markovian structure. This is important because the price impact patterns of optimal trade execution strategies with transient price impact have precisely these features and, as mentioned above, the price impact of another market participant is perhaps the most common source for the presence of a drift. On the other hand, we will keep the remaining features of the drift from the effects created by other model features. We therefore use the linear continuous-time model of Obizhaeva & Wang (2013) (in the version of Gatheral et al. (2012)) with exponential decay of price impact and the problem we are looking at is the minimization of the expected costs.

Theorem 1, our main result, shows that this optimal trade execution problem is very sensitive with respect to the drift. The expected costs will be equal to negative infinity as soon as the drift is not absolutely continuous, a fact that will have strong impact when market impact is generated by several market participants. Moreover, even when the drift is absolutely continuous, optimal strategies will typically not exist if strategies are understood in the sense of Gatheral et al. (2012). We therefore extend the class of admissible strategies by allowing strategies to be semimartingales. We show that unique optimal trade execution strategies may exist in this class of strategies, but the number of shares to be held depends directly on the derivative of the drift at each time and thus may fluctuate strongly. This sensitivity of strategies is particularly striking when compared to the relatively robust drift dependence of optimal trade execution strategies in the Almgren–Chriss framework, which was found by Schied (2011).

Our problem of minimizing the expected costs in the presence of a drift turns out to be also of interest from a purely mathematical point of view. Our approach uses elements from singular stochastic control, although the problem is basically non-Markovian due to both the transience of price impact and the lack in Markovian structure of the underlying price process. We deal with the first type of non-Markovianity by using an auxiliary 'impact process' E_t^X that, under the specific assumption of exponential decay of price impact, leads to a Markovian structure for the dynamics of transient price impact. We then guess a formula for the optimal expected costs conditional at time $t \ge 0$ where an arbitrary impact E_t^X is given as initial condition. With this formula at hand, we can then use a verification argument. The control problem is 'singular' since our controls are semimartingale strategies, which enter the value function as integrators of stochastic integrals. A similar technique was recently used in Alfonsi & Schied (2012) to compute optimal strategies for general, completely monotone decay kernels but without drift in the unaffected price process. As an application of our results, we also obtain a complete solution for the minimization of a cost-risk criterion that was recently proposed in Gatheral & Schied (2011).

2 Statement of results

2.1 Model setup

A market impact model is a model for an economic agent who can move asset prices. As long as this agent is not active, asset prices are determined by the actions of the other market participants and are described by the *unaffected price process* S^0 . We assume that S^0 is a squareintegrable càdlàg semimartingale defined on a given filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ satisfying the usual conditions. We also assume that \mathcal{F}_0 is \mathbb{P} -trivial, i.e., every \mathcal{F}_0 -measurable random variable is \mathbb{P} -a.s. constant. We will use the linear market impact model with exponential decay of price impact proposed by Obizhaeva & Wang (2013). More precisely, we will use the zero-spread version of this model that was suggested in Gatheral et al. (2012); we refer to Alfonsi & Schied (2010) for a discussion of the possible re-introduction of a bid-ask spread.

The actual asset price will depend on the strategy chosen by the trader. Such a strategy will be an adapted stochastic process $X = (X_t)_{t \ge 0^-}$ that describes the number of shares held by the trader at each time. Following Gatheral et al. (2012), we call X admissible if the following conditions are satisfied:

- (a) the function $t \to X_t$ is right-continuous¹ and adapted;
- (b) the function $t \to X_t$ has finite and \mathbb{P} -a.s. bounded total variation;
- (c) there exists a *liquidation time* $T \ge 0$ such that $X_t = 0$ \mathbb{P} -a.s. for all $t \ge T$.

Such a strategy has the interpretation that the value X_{0-} stands for an initially given amount of shares that needs to be liquidated by time T. When X is nonincreasing, it is a pure sell strategy. When it is nondecreasing, it is a pure buy strategy. A general admissible strategy is the sum of a sell and a buy strategy and therefore is of bounded variation. This shows that condition (b) is economically meaningful. With $\mathcal{X}_{BV}(x,T)$ we will denote the class of all strategies that are admissible in this sense for a fixed liquidation time $T \geq 0$ and that satisfy $X_{0-} = x$.

When the admissible strategy X is used, the price S_t^X will be

$$S_t^X = S_t^0 + \eta \int_{[0,t)} e^{-\rho(t-s)} \, dX_s,\tag{1}$$

where $\rho > 0$, the function $e^{-\rho t}$ describes the temporal decay of price impact, and the parameter η describes its magnitude. Clearly we can set $\eta := 1$ without loss of generality. Following Gatheral et al. (2012), we define the liquidation costs of $X \in \mathcal{X}_{BV}(x,T)$ as

$$\mathcal{C}(X) := \int_{[0,T]} S_t^0 \, dX_t + \int_{[0,T]} \int_{[0,t]} e^{-\rho(t-s)} \, dX_s \, dX_t + \frac{1}{2} \sum_{t \in [0,T]} (\Delta X_t)^2.$$
(2)

Remark 1 (Economic motivation of the cost functional $\mathcal{C}(\cdot)$). Let us follow Alfonsi et al. (2012) and Gatheral et al. (2012) in motivating the cost functional (2). For a continuous strategy $X \in \mathcal{X}_{BV}(x,T)$, $\mathcal{C}(X)$ equals $\int_0^T S_t^X dX_t$ and can thus be easily understood as the accumulated costs of buying dX_t shares at price S_t^X at each time t. For general X, a nonzero jump ΔX_t can be interpreted as a large market order which shifts the asset price by eating into a block-shaped limit order book. Its execution therefore incurs the following costs:

$$\int_{S_t^X}^{S_t^X + \Delta X_t} y \, dy = S_t^X \Delta X_t + \frac{1}{2} (\Delta X_t)^2 = S_t^0 \Delta X_t + \int_{[0,t)} e^{-\rho(t-s)} \, dX_s \, \Delta X_t + \frac{1}{2} (\Delta X_t)^2.$$

We assume here that the order ΔX_t is executed immediately *after* a jump of S_t^0 in case both jumps nominally occur at the same time, an assumption that is economically natural since it precludes arbitrage-like exploitation of price jumps. Decomposing a general strategy into its continuous part and its jumps thus leads to the definition (2). An alternative derivation of (2), based on a continuous-time limit of discrete-time cost functionals, will be provided by Lemma 1 in the more general framework of semimartingale strategies.

¹Although Gatheral et al. (2012) consider the left-continuous modification of X, our definitions of both price process and costs coincide with the one in Gatheral et al. (2012). See Remark 2 for a detailed discussion of right versus left continuity.

The problem of minimizing the expected costs, $\mathbb{E}[\mathcal{C}(X)]$, over $X \in \mathcal{X}_{BV}(x,T)$ is called the *optimal trade execution problem*. When S^0 is a square-integrable martingale, this problem admits the unique solution

$$X_t = \frac{x(1 + \rho(T - t))}{2 + \rho T}, \qquad 0 \le t < T.$$
(3)

That is, X has an initial jump at t = 0 of size $\Delta X_0 = \frac{-x}{2+\rho T}$, continuous trading at rate $dX_t = \frac{-x\rho}{2+\rho T} dt$ in (0,T), and a terminal jump of size $\Delta X_T = \Delta X_0$. This formula was found by Obizhaeva & Wang (2013) (see also Example 2.12 in Gatheral et al. (2012) for a short proof).

Remark 2. Gatheral et al. (2012) consider the left-continuous modification of admissible strategies. Since the respective formulas (1) and (2) for the price process and the costs of a strategy $X \in \mathcal{X}_{BV}(x,T)$ depend only on the measure dX_t , it is just a matter of notational convention whether to choose the right- or left-continuous modification of X. In particular, our formulas for the price process S^X and the costs $\mathcal{C}(X)$ are the same as those in Gatheral et al. (2012). Later on, however, we will consider a larger class of semimartingale strategies, and since semimartingales are right-continuous by default and for good reason, we must adopt the convention of right continuity so as to be consistent between our two classes of strategies.

As can be seen from the formula (3), optimal strategies will typically have jumps at times t = 0 and t = T. For right-continuous strategies, we need to include the possibility of an initial jump by allowing for an initial value X_{0-} that can be different from X_0 . Similarly, for the left-continuous modification of strategies used in Gatheral et al. (2012), the terminal jump must be accommodated by allowing for a nonzero value of X_T and by requiring the modified liquidation constraint $X_{T+} = 0$. So both conventions require us to impose conditions on the limits of X_t when t approaches a boundary point of the actual trading interval [0, T] from outside this interval.

Here, our goal is to study the minimization of the expected costs $\mathbb{E}[\mathcal{C}(X)]$ when S^0 has an additional drift. This topic is of intrinsic mathematical interest, and we refer to the introduction of this paper for an account of our economic motivation to study this problem. We assume henceforth that S^0 is a càdlàg semimartingale with decomposition

$$S_t^0 = S_0 + M_t + A_t, (4)$$

where S_0 is a constant, M is a square-integrable càdlàg martingale with $M_0 = 0$, and A is an adapted process with $A_0 = 0$ and locally square-integrable total variation, i.e., for every T > 0 we have $\mathbb{E}[|A|_{[0,T]}^2] < \infty$ when $|A|_{[0,T]}$ denotes the total variation of A over the interval [0,T]. There is in fact no loss of generality in assuming that A is predictable (see Proposition I.4.23 in Jacod & Shiryaev (2003)).

It will turn out that the presence of A increases the complexity of the optimal trade execution problem significantly. In particular, optimal execution strategies in $\mathcal{X}_{BV}(x,T)$ will exist only under very restrictive assumptions on A. For instance, they will not exist even in the simple case in which S^0 is a diffusion model,

$$dS_t^0 = \sigma(S_t^0) \, dW_t + b(S_t^0) \, dt,$$

with nonconstant drift coefficient $b(\cdot)$. We therefore need to extend our class of admissible trading strategies.

Definition 1. An admissible semimartingale strategy is a bounded² right-continuous semimartingale X for which there exists a liquidation time $T \ge 0$ such that $X_t = 0$ P-a.s. for all $t \ge T$. By $\mathcal{X}_{\text{sem}}(x,T)$ we denote the class of all admissible semimartingale strategies X with $X_{0-} = x$ and liquidation time T.

Note that $\mathcal{X}_{BV}(x,T)$ is a subset of $\mathcal{X}_{sem}(x,T)$. While semimartingale strategies are standard in frictionless asset pricing models, their application in a high-frequency market impact model is economically less natural than strategies of bounded variation, because they can no longer be written as the superposition of buying and selling strategies.

Given a semimartingale strategy $X \in \mathcal{X}_{sem}(x,t)$, we need to extend the definitions (1) and (2) for the corresponding price process and the resulting liquidation costs. These formulas and our further analysis will involve stochastic integrals in which X appears both as integrand and as integrator. Therefore, we first need to clarify how stochastic integrals must be understood in view of our requirement $X_{0-} = x \neq 0$.

Remark 3 (On the definition of stochastic integrals). It is a common assumption in the literature on stochastic integration that semimartingales X may jump at t = 0, but a typical convention is to assume $X_{0-} = 0$. With this convention, a stochastic integral $X_- \cdot Y$, as defined, e.g., in Protter (2004), will not depend on the initial jump of the integrator Y at time t = 0, and so there is no ambiguity in writing $(X_- \cdot Y)_t = \int_0^t X_{s-} dY_s$. When the value X_{0-} is nonzero, as it is the case for the semimartingale strategies defined above, one must carefully distinguish whether an initial jump of the integrator is or is not part of a stochastic integral. This has been done, e.g., by Meyer (1976), from where we adopt the convention of writing $\int_{[0,t]} X_{s-} dY_s$ or $\int_{(0,t]} X_{s-} dY_s$, respectively, when the initial jump is or is not part of the stochastic integral. We then have

$$\int_{[0,t]} X_{s-} dY_s = X_{0-} \Delta Y_0 + \int_{(0,t]} X_{s-} dY_s \quad \text{and} \quad [X,Y]_0 = \Delta X_0 \Delta Y_0.$$
(5)

The integration by parts formula for stochastic integrals becomes

$$X_t Y_t = X_{0-} Y_{0-} + \int_{[0,t]} X_{s-} \, dY_s + \int_{[0,t]} Y_{s-} \, dX_s + [X,Y]_t \tag{6}$$

see (Meyer 1976, p. 303). When $Z_t := \int_{[0,t]} X_{s-} dY_s$ is a stochastic integral, we set $Z_{0-} := 0$ by default.

Given a semimartingale strategy $X \in \mathcal{X}_{\text{sem}}(x,T)$, the price S_t^X at time t can be defined just as in (1) when $\int_{[0,t)} e^{-\rho(t-s)} dX_s$ denotes the left-hand limit, E_{t-}^X , of the generalized Ornstein-Uhlenbeck process

$$E_t^X := e^{-\rho t} \int_{[0,t]} e^{\rho s} \, dX_s, \qquad t \ge 0.$$
(7)

We now turn to the definition of the liquidation costs of the semimartingale strategy X. We will motivate our definition by an approximation from the discrete-time case. To this end, we

²The requirement that X is bounded is natural from an economic point of view, because the total number of available shares is finite for every stock.

take $N \in \mathbb{N}$, let $t_k^N := kT/N$ for k = 0, ..., N and define the following sequence of discrete trades:

$$\xi_0^N := X_0 - X_{0-}$$
 and, for $k = 1, \dots, N$, $\xi_k^N := X_{t_k^N} - X_{t_{k-1}^N}$

Then, $\boldsymbol{\xi}^N := (\xi_k^N)$ is an admissible trading strategy in the sense of Alfonsi et al. (2012). In Proposition 1 of Alfonsi et al. (2012) and its proof, the costs incurred by the discrete-time strategy $\boldsymbol{\xi}^N$ were derived as

$$\mathcal{C}^{N}(\boldsymbol{\xi}^{N}) = \sum_{k=0}^{N} \left(S_{t_{k}^{N}}^{0} \xi_{k}^{N} + \sum_{i=0}^{k-1} e^{-\rho(t_{k}^{N} - t_{i}^{N})} \xi_{i}^{N} \xi_{k}^{N} + \frac{1}{2} (\xi_{k}^{N})^{2} \right).$$

The economic motivation of this formula is analogous to the one given in Remark 1. In fact $\mathcal{C}^{N}(\boldsymbol{\xi}^{N})$ coincides with $\mathcal{C}(X^{N})$, when $X^{N} \in \mathcal{X}_{BV}(x,T)$ denotes the step function with jumps described by $\boldsymbol{\xi}^{N}$. We have the following asymptotics of these costs when our time grid becomes finer.

Lemma 1 (Liquidation costs of a semimartingale strategy). As $N \uparrow \infty$, we have

$$\mathcal{C}^{N}(\boldsymbol{\xi}^{N}) \longrightarrow \int_{[0,T]} S_{t-}^{0} \, dX_{t} + [S^{0}, X]_{T} + \int_{[0,T]} E_{t-}^{X} \, dX_{t} + \frac{1}{2} [X]_{T} =: \mathcal{C}(X)$$

in probability, where $\mathcal{C}(X)$ is independent of the (arbitrary) choice of the value S_{0-}^0 , and E^X is the generalized Ornstein-Uhlenbeck process from (7).

We therefore define $\mathcal{C}(X)$ as the *liquidation costs* incurred by $X \in \mathcal{X}_{\text{sem}}(x,T)$. Note that $\mathcal{C}(X)$ reduces to the liquidation costs defined in (2) when $X \in \mathcal{X}_{\text{BV}}(x,T)$. Moreover, it follows from (5) that $\mathcal{C}(X)$ is indeed independent of the particular choice of S_{0-}^0 .

2.2 Minimizing the expected costs

The optimization problem we are interested in is the minimization of the expected costs,

$$\mathbb{E}[\mathcal{C}(X)] = \mathbb{E}\left[\int_{[0,T]} S_{t-}^0 dX_t + [S^0, X]_T + \int_{[0,T]} E_{t-}^X dX_t + \frac{1}{2}[X]_T\right],\tag{8}$$

over all strategies X that belong to $\mathcal{X}_{\text{sem}}(x,T)$ or to $\mathcal{X}_{\text{BV}}(x,T)$. To state its solution, let $Z = (Z_t)$ be a càdlàg version of the martingale

$$-\mathbb{E}\Big[A_T + \rho \int_0^T A_s \, ds \, \Big| \, \mathcal{F}_t \,\Big],$$

which exists due to our assumption that $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ satisfies the usual conditions. We also define the semimartingale Y as

$$Y_t := Z_t + \rho \int_0^t A_s \, ds + \big(1 + \rho(T - t)\big) A_t.$$

Theorem 1. When A is \mathbb{P} -a.s. absolutely continuous on [0, T) with square-integrable derivative $A'_t = dA_t/dt$, i.e., when $A_t = \int_0^t A'_s ds$ for $0 \le t < T$ and $\mathbb{E}[\int_0^T (A'_t)^2 dt] < \infty$, then

$$\inf_{X \in \mathcal{X}_{BV}(x,T)} \mathbb{E}\left[\mathcal{C}(X)\right] = \inf_{X \in \mathcal{X}_{sem}(x,T)} \mathbb{E}\left[\mathcal{C}(X)\right] \tag{9}$$

$$= -xS_0 + \frac{x^2}{2+\rho T} + \frac{xY_0}{2+\rho T} - \frac{\rho}{4} \mathbb{E}\left[\int_0^T \left(\frac{Y_s}{2+\rho(T-s)} - \frac{1}{\rho}A'_s\right)^2 ds\right],$$

and

$$\inf_{X \in \mathcal{X}_{BV}(x,T)} \mathbb{E}\left[\mathcal{C}(X)\right] = \inf_{X \in \mathcal{X}_{sem}(x,T)} \mathbb{E}\left[\mathcal{C}(X)\right] = -\infty$$
(10)

otherwise.

When in addition A' is bounded, the second infimum in (9) can be attained only if A' is a (right-continuous) semimartingale, and the unique optimal strategy is then given by

$$X_{t} = \frac{x(1+\rho(T-t)) - \frac{1}{2}(1+\rho t)Y_{0}}{2+\rho T} - \frac{1}{2}\int_{(0,t]}\varphi(s)\,dZ_{s} + \frac{1}{2\rho}A_{t}' - \rho\int_{0}^{t}\left(\frac{1}{2}\int_{(0,s]}\varphi(r)\,dZ_{r} + \frac{1}{2}A_{s}\right)ds,$$
(11)

where $\varphi(t) := (2 + \rho(T - t))^{-1}$. In particular the first infimum in (9) can only be attained when $\frac{1}{\rho}A'_t - \int_{(0,t]} \varphi(s) dZ_s$ is \mathbb{P} -a.s. right-continuous and of finite variation on [0,T].

Remark 4. From an economic point of view, the fact that it is possible to generate arbitrarily negative expected costs for drift processes that are not absolutely continuous might indicate a market inefficiency that arises when trading takes place on a much shorter time scale than the resilience of price impact. The market then becomes inefficient, because its resilient reaction to a price shock is delayed in comparison to the trading activities of the economic agent; see also Remarks 2 and 3 in Alfonsi et al. (2012). This becomes particularly apparent when the drift is generated by the trading behavior of a large fundamental seller, who is subject to predatory trading by a high-frequency trader; see Remark 6 below.

We refer to Lemma 6 in Section 3 for the details of constructing a strategy with arbitrarily negative expected costs when A is not absolutely continuous.

The situation in Theorem 1 simplifies significantly when A' is a martingale:

Corollary 1. Suppose that A is of the form $A_t = \int_0^t A'_s ds$ for a bounded càdlàg martingale A'. Then the optimal strategy (11) becomes

$$X_t = \frac{x(1+\rho(T-t))}{2+\rho T} + \frac{1}{4\rho}(2+\rho(T-t))A'_t + \frac{1}{4}(1+\rho(T-t))A_t.$$
 (12)

Note that the strategy (12) can be computed in a pathwise manner without reference to the particular distribution of A; see Figure 1. This special case highlights the ambiguous and seemingly contradictory nature of the robustness of the optimal strategy: this strategy reacts very sensitively to structural features of the price process, i.e., to the martingale property of A', but once this structural requirement is satisfied, the strategy is completely independent of the law of A'. When A vanishes, this strategy reduces to the Obizhaeva–Wang solution (3).

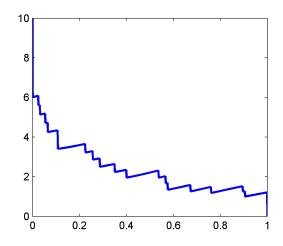


Figure 1: Optimal strategy (12) when $\rho = 2$ and A' is a compensated Poisson process with intensity $\lambda = 20$.

Remark 5 (Comparison with Almgren–Chriss model). It is interesting to compare the optimal strategy (9) with the one for the corresponding Almgren–Chriss model. In the latter model, strategies must be absolutely continuous. Given such a strategy X, the price process takes the form

$$\widetilde{S}_t^X = S_t^0 + \eta \dot{X}_t + \gamma (X_t - X_0),$$

where η and γ are two nonnegative constants. When $X_0 = x$ and $X_T = 0$, the corresponding liquidation costs are

$$\widetilde{\mathcal{C}}(X) = \int_0^T S_t^0 \dot{X}_t \, dt + \eta \int_0^T \dot{X}_t^2 \, dt + \frac{\gamma}{2} x^2.$$

In our setting, there is always a unique strategy that minimizes the expected liquidation costs $\mathbb{E}[\tilde{\mathcal{C}}(X)]$ and it is given by

$$X_t = \frac{T-t}{T} \left(x - \frac{1}{2\eta} \int_0^t \frac{T}{(T-s)^2} \mathbb{E} \left[\int_s^T (T-u) \, dA_u \, \Big| \, \mathcal{F}_s \right] \, ds \right);$$

see Corollary 2 in Schied (2011). Here the drift A enters the optimal strategy basically in integrated form, and so one can expect that possible misspecifications of the drift may average out to some extent. This relatively stable behavior should be compared to the direct dependence of the strategy (9) on the derivative of the drift. \diamond

Remark 6 (A two-player situation). As discussed in the Introduction, an important source for a drift in the asset price process S^0 can be the trading activity of another large market participant ("the seller"). There are various reasons why another economic agent ("the predator") may get good estimates for the resulting drift. For instance, some trade execution algorithms create characteristic order patterns and therefore allow for an inference of their future trading trajectory. But there are also other possibilities as discussed in Schöneborn & Schied (2009).

Suppose that the seller aims at liquidating a position of $x \neq 0$ shares by time T > 0. Suppose moreover, for simplicity, that the unaffected asset price S^0 is a square-integrable martingale so that the seller will use the liquidation strategy X^* from (3). The predator will then perceive the unaffected price process $\tilde{S}^0 = S^0 + E^{X^*}$, which is no longer a martingale but has the drift E^{X^*} . Since X^* has a terminal jump, also the resulting 'drift' E^{X^*} will jump by the same amount at time T. So if the predator faces a more relaxed time constraint than the seller, which is a natural assumption, the predator will perceive a drift that is not absolutely continuous and, by Theorem 1, will have the possibility of making arbitrary large expected profits. Similar results will also hold when S^0 has a nonvanishing drift.

2.3 Minimization of a cost-risk criterion

As a corollary to Theorem 1, we can also find optimal strategies for the linear risk criterion that was proposed in Gatheral & Schied (2011) for the Almgren–Chriss framework with a riskneutral geometric Brownian motion as unaffected price process. When in our model S^0 is a risk-neutral geometric Brownian motion, i.e.,

$$S_t^0 = S_0 e^{\sigma W_t - \frac{1}{2}\sigma^2 t} \qquad \text{for some } \sigma > 0 \text{ and a Brownian motion } W, \tag{13}$$

the same reasoning as in Gatheral & Schied (2011) motivates the minimization of a cost-risk functional of the form -

$$\mathbb{E}\Big[\mathcal{C}(X) + \lambda \int_0^T S_t^X X_t \, dt \Big],\tag{14}$$

where λ has the same sign as $X_{0-} = x$. The parameter λ is typically derived from the Value at Risk of a unit asset position under the assumption of log-normal future returns. As argued in Remark 2.2 of Gatheral & Schied (2011), one could obtain the same cost-risk functional (but perhaps with a different value for λ) if Value at Risk is replaced by a coherent risk measure or by any other positively homogeneous risk measure.

Optimal strategies for the cost-risk functional (14) in the Almgren–Chriss framework have the advantages of being sensitive to changes in the asset price, easily computable in closed form, and possess completely transparent reactions to parameter changes. In addition, they have a striking robustness property: they are independent of the actual law of S^0 as long as S^0 is a martingale. Thus they may be optimal even when the law of S^0 is not of the particular form (13). A disadvantage is that optimal strategies can switch sign, in which case the interpretation of the cost-risk functional (14) breaks down. But, as discussed in Section 4 of Gatheral & Schied (2011), the probability that strategies become negative will be small with reasonable parameter choices.

Corollary 2. The minimization of the cost-risk functional (14) is equivalent to the minimization of the expected costs for the new price process

$$\widetilde{S}_t^0 = \frac{\rho}{\rho + \lambda} \bigg(S_t^0 - \lambda \int_0^t S_s^0 \, ds \bigg).$$

In particular, the statements of Theorem 1 carry over to the minimization of the cost-risk functional (14) when A is replaced by

$$\widetilde{A}_t = \frac{\rho}{\rho + \lambda} \left(A_t - \lambda \int_0^t S_s^0 \, ds \right).$$

When S^0 is a bounded martingale, then $\widetilde{A}'_t = -\frac{\rho\lambda}{\rho+\lambda}S^0_t$ is also a martingale. Thus, by Corollary 1 the optimal strategy that minimizes the cost-risk criterion (13) simply becomes

$$X_t^* = \frac{x(1+\rho(T-t))}{2+\rho T} - \frac{\rho\lambda}{\rho+\lambda} \bigg(\frac{1}{4\rho} (2-\rho(T-t))S_t^0 + \frac{1}{4} (1+\rho(T-t)) \int_0^t S_s^0 \, ds \bigg).$$

This strategy can be computed in a pathwise manner and is completely independent of the particular law of the martingale S^0 . It thus minimizes the cost-risk criterion (13) whenever \mathbb{P} is a martingale measure for S^0 . When S^0 is not bounded but just a square-integrable martingale, then X^* will not be an admissible semimartingale strategy in the sense of Definition 1. Nevertheless, in this special case, one can show that X^* attains the optimum of the cost-risk criterion and thus can still be regarded as an optimal strategy. We leave the details to the reader.

3 Proofs

To simplify the notation, we will drop the superscript X in E^X throughout the proofs when there is no ambiguity about the strategy X used in the definition of $E = E^X$.

Proof of Lemma 1. We first note that

$$\sum_{k=0}^{N} S_{t_{k}^{N}}^{0} \xi_{k}^{N} = S_{0}^{0} \Delta X_{0} + \sum_{k=1}^{N} S_{t_{k-1}^{N}}^{0} (X_{t_{k}^{N}} - X_{t_{k-1}^{N}}) + \sum_{k=1}^{N} (S_{t_{k}^{N}}^{0} - S_{t_{k-1}^{N}}^{0}) (X_{t_{k}^{N}} - X_{t_{k-1}^{N}}).$$

By Theorems II.5.21 and II.5.23 in Protter (2004), this expression converges in probability to

$$S_0^0 \Delta X_0 + \int_{(0,T]} S_{t-}^0 dX_t + [S^0, X]_T - \Delta S_0^0 \Delta X_0 = \int_{[0,T]} S_{t-}^0 dX_t + [S^0, X]_T.$$

Similarly,

$$\sum_{k=0}^{N} (\xi_{k}^{N})^{2} = (\Delta X_{0})^{2} + \sum_{k=1}^{N} (X_{t_{k}^{N}} - X_{t_{k-1}^{N}})^{2} \longrightarrow [X]_{T},$$

in probability.

When defining

$$\widetilde{E}_t^N := \sum_{i=0}^{N-1} e^{\rho t_i^N} (X_{t_{i+1}^N \wedge t} - X_{t_i^N \wedge t})$$

then \widetilde{E}_t^N is the Riemann approximation of a stochastic integral with a deterministic and continuous integrand, and hence $\widetilde{E}^N \to \int_{(0,\cdot]} e^{\rho s} dX_s$ uniformly on compacts in probability (ucp) (Jacod & Shiryaev 2003, Proposition I.4.44). It follows that

$$E_t^N := e^{-\rho t} \left(e^{-\rho T/N} \Delta X_0 + \widetilde{E}_t^N \right) \longrightarrow E_t \qquad \text{ucp as } N \uparrow \infty.$$

Moreover,

$$\begin{split} \sum_{i=0}^{k-1} e^{-\rho(t_k^N - t_i^N)} \xi_i^N &= e^{-\rho t_k^N} \Big(\Delta X_0 + \sum_{i=1}^{k-1} e^{\rho t_i^N} (X_{t_i^N} - X_{t_{i-1}^N}) \Big) \\ &= e^{-\rho t_{k-1}^N} \Big(e^{-\rho T/N} \Delta X_0 + \sum_{i=0}^{k-2} e^{\rho t_i^N} (X_{t_{i+1}^N} - X_{t_i^N}) \Big) = E_{t_{k-1}^N}^N. \end{split}$$

Therefore,

$$\begin{split} \sum_{k=0}^{N} \sum_{i=0}^{k-1} e^{-\rho(t_{k}^{N} - t_{i}^{N})} \xi_{i}^{N} \xi_{k}^{N} &= \sum_{k=1}^{N} \sum_{i=0}^{k-1} e^{-\rho(t_{k}^{N} - t_{i}^{N})} \xi_{i}^{N} \xi_{k}^{N} = \sum_{k=1}^{N} E_{t_{k-1}}^{N} (X_{t_{k}^{N}} - X_{t_{k-1}^{N}}) \\ &= \int_{(0,T]} (E^{N})_{t}^{\sigma_{N}} dX_{t}, \end{split}$$

where, using the notation from Section II.5 of Protter (2004), for a process Y we let

$$Y^{\sigma_N} := Y_0 1\!\!1_{\{0\}} + \sum_{k=0}^{N-1} Y_{t_k^N} 1\!\!1_{(t_k^N, t_{k+1}^N]} \quad \text{and} \quad \int_{(0,T]} Y_t^{\sigma_N} \, dX_t = \sum_{k=0}^{N-1} Y_{t_k^N} (X_{t_{k+1}^N} - X_{t_k^N}).$$

Now

$$\int_{(0,T]} (E^N)_t^{\sigma_N} dX_t = \int_{(0,T]} E_t^{\sigma_N} dX_t + \int_{(0,T]} \left((E^N)_t^{\sigma_N} - E_t^{\sigma_N} \right) dX_t.$$
(15)

The first integral on the right converges to $\int_{(0,T]} E_{t-} dX_t$ in probability by Theorem II.5.21 of Protter (2004). To deal with the second integral on the right, we note that $\sup_{0 \le t \le T} |E_t^N - E_t| \le \varepsilon$ implies that also $\sup_{0 \le t \le T} |(E^N)_t^{\sigma_N} - E_t^{\sigma_N}| \le \varepsilon$. Thus, $(E^N)^{\sigma_N} - E^{\sigma_N} \to 0$ ucp. The continuity of the stochastic integral with respect to ucp convergence (Protter 2004, p. 59) therefore implies that the rightmost integral in (15) tends to zero in probability for $N \uparrow \infty$. We thus obtain that

$$\sum_{k=0}^{N} \sum_{i=0}^{k-1} e^{-\rho(t_k^N - t_i^N)} \xi_i^N \xi_k^N \longrightarrow \int_{(0,T]} E_{t-} \, dX_t = \int_{[0,T]} E_{t-} \, dX_t$$

in probability (here we have used the fact that $E_{0-} = 0$ by our convention on stochastic integrals made at the end of Remark 3). Putting everything together yields the assertion.

Now we start preparing for the proof of Theorem 1, which will rely on a series of lemmas. The basic idea underlying the proof is the verification argument appearing in the next lemma. The nature of the verification argument becomes apparent when taking $\alpha_t := A'_t$ in Lemma 2. The key to the argument is the following formula for the remaining costs of optimally liquidating the asset position X_t over (t, T], taking into account a given volume impact E_t . This volume impact E_t can be thought of as the volume impact generated by using a strategy X throughout [0, t] that leads to the asset position X_t at time t. The formula is

$$-\frac{1}{2}E_{t}^{2} + \varphi(t)(X_{t} - E_{t})^{2} + \varphi(t)(X_{t} - E_{t})Y_{t} - \rho \mathbb{E}\bigg[\int_{t}^{T} \left(\frac{1}{2}\varphi(s)Y_{s} - \frac{1}{2\rho}A_{s}'\right)^{2} ds \,\Big|\,\mathcal{F}_{t}\bigg].$$
(16)

This formula needs to be guessed; we are not aware of a method by which it can be derived analytically. Once this formula has been guessed, we can proceed by the following standard verification argument, which is also used, e.g., in Section 6.6.1 of Pham (2009): We show that the costs (16) plus the costs generated by using X over [0, t] is submartingale for any strategy X and a true martingale if X is an optimal strategy.

Let us recall the definition

$$\varphi(t) = \frac{1}{2 + \rho(T - t)}.$$

Lemma 2. Fix $X \in \mathcal{X}_{sem}(x,T)$, and let α_t be any progressively measurable process with $\mathbb{E}[\int_0^T \alpha_t^2 dt] < \infty$. We furthermore let $Z^{\alpha} = (Z_t^{\alpha})$ be a càdlàg version of the martingale

$$-\mathbb{E}\bigg[\int_0^T \alpha_s \, ds + \rho \int_0^T \int_0^s \alpha_r \, dr \, ds \, \Big| \, \mathcal{F}_t \bigg],$$

and we define

$$Y_t^{\alpha} := Z_t^{\alpha} + \rho \int_0^t \int_0^s \alpha_r \, dr \, ds + \left(1 + \rho(T-t)\right) \int_0^t \alpha_s \, ds.$$

Then

$$\mathbb{E}\left[\mathcal{C}(X)\right] = -xS_0 + \varphi(0)x^2 + \varphi(0)xZ_0^{\alpha} - \rho\mathbb{E}\left[\int_0^T \left(\frac{1}{2}\varphi(s)Y_s^{\alpha} - \frac{1}{2\rho}\alpha_s\right)^2 ds\right] \\ + \mathbb{E}\left[\int_0^T X_t\alpha_t dt - \int_{(0,T]} X_{t-} dA_t\right] \\ + \rho\mathbb{E}\left[\int_0^T \left\{\varphi(t)X_t + (1-\varphi(t))E_t + \frac{1}{2}\varphi(t)Y_t^{\alpha} - \frac{1}{2\rho}\alpha_t\right\}^2 dt\right].$$
(17)

Proof. We note first that Jensen's inequality implies $(\int_0^T \alpha_t dt)^2 \leq T \int_0^T \alpha_t^2 dt$. Hence,

$$\mathbb{E}[(Z_T^{\alpha})^2] = \mathbb{E}\left[\left(\int_0^T \alpha_t \, dt + \rho \int_0^T \int_0^t \alpha_s \, ds \, dt\right)^2\right] \le \mathbb{E}\left[\left((1+\rho T)\sqrt{T\int_0^T \alpha_s^2 \, ds}\right)^2\right] < \infty, \ (18)$$

and in turn $\mathbb{E}[\int_0^T (Y_t^{\alpha})^2 dt] < \infty$. So all expressions in (17) are well-defined. We now define for $X \in \mathcal{X}_{sem}(x,T)$

$$\widetilde{C}_t^X := \int_{[0,t]} S_{t-}^0 \, dX_t + [S^0, X]_t + \int_{[0,t]} E_{s-} \, dX_s + \frac{1}{2} [X]_t$$

Then \widetilde{C}_t^X describes the costs incurred by using the strategy X throughout the time interval [0, t]. Next, we use our guess (16) for the costs of optimally liquidating the amount $x = X_t$ by trading over (t, T] when an initial volume impact of size $\varepsilon = E_t$ is given at time t. It leads to defining the function

$$V^{\alpha}(t,x,\varepsilon) := -\frac{1}{2}\varepsilon^{2} + \varphi(t)(x-\varepsilon)^{2} + \varphi(t)(x-\varepsilon)Y_{t}^{\alpha}$$

which describes these optimal costs less the integral term in (16), which does not depend on $x = X_t$ or $\varepsilon = E_t$. By adding \widetilde{C}_t^X we get the process

$$C_t^X := \widetilde{C}_t^X + V^\alpha(t, X_t, E_t).$$
(19)

We will now compute the Itô differential dC_t^X . Our computation will mainly rely on Itô's product rule in the form (6). For the computation, it will be helpful to collect a few auxiliary formulas in advance. For instance, it follows from the definition of Y^{α} that

$$dY_t^{\alpha} = dZ_t^{\alpha} + (1 + \rho(T - t))\alpha_t dt = dZ_t^{\alpha} + \frac{1 - \varphi(t)}{\varphi(t)}\alpha_t dt.$$
⁽²⁰⁾

Using $E_t = e^{-\rho t} \int_{[0,t]} e^{\rho s} dX_s$ from (7), the fact that $E_{0-} = 0$ (which follows from our corresponding convention for stochastic integrals), and integration by parts yields

$$E_t = X_t - x - \rho \int_0^t E_s \, ds, \qquad 0 \le t \le T.$$
 (21)

It follows in particular that the process $E_t - X_t$ does not jump throughout [0, T] and that on this interval $d(E_t - X_t) = -\rho E_t dt$. We also note that $d\varphi(t) = \rho \varphi(t)^2 dt$.

Recalling the fact that $M_0 = A_0 = 0$, we now choose values S_{0-}^0 , M_{0-} , and A_{0-} in \mathbb{R} that satisfy

$$S_0 - S_{0-}^0 = \Delta S_0^0 = \Delta M_0 + \Delta A_0 = -M_{0-} - A_{0-}$$
(22)

but can otherwise be arbitrary. We also choose an arbitrary value $Z_{0-}^{\alpha} \in \mathbb{R}$. We then have on [0,T]

$$d\widetilde{C}_{t}^{X} = S_{t-}^{0} dX_{t} + d[S^{0}, X]_{t} + E_{t-} dX_{t} + \frac{1}{2} d[X]_{t}$$

= $d(S_{t}^{0}X_{t}) - X_{t-} dM_{t} - X_{t-} dA_{t} + E_{t-} dX_{t} + \frac{1}{2} d[X]_{t}.$ (23)

Hence, a lengthy but straightforward calculation gives

$$dC_{t}^{X} = d(S_{t}^{0}X_{t}) - X_{t-} dM_{t} - X_{t-} dA_{t} + \varphi(t)(X_{t-} - E_{t-}) dZ_{t}^{\alpha} + \rho \left\{ E_{t}^{2} + \varphi(t)^{2}(X_{t} - E_{t})^{2} + 2\varphi(t)(X_{t} - E_{t})E_{t} + \frac{1}{\rho}(1 - \varphi(t))(X_{t} - E_{t})\alpha_{t} + \varphi(t)E_{t}Y_{t}^{\alpha} + \varphi(t)^{2}(X_{t} - E_{t})Y_{t}^{\alpha} \right\} dt = d(S_{t}^{0}X_{t}) - X_{t-} dM_{t} + \varphi(t)(X_{t-} - E_{t-}) dZ_{t}^{\alpha} + \rho \left\{ \varphi(t)X_{t} + (1 - \varphi(t))E_{t} + \frac{1}{2}\varphi(t)Y_{t}^{\alpha} - \frac{1}{2\rho}\alpha_{t} \right\}^{2} dt + X_{t-}(\alpha_{t} dt - dA_{t}) - \rho \left(\frac{1}{2}\varphi(t)Y_{t}^{\alpha} - \frac{1}{2\rho}\alpha_{t}\right)^{2} dt.$$
(24)

Due to the regularity of their sample paths, all stochastic processes involved have \mathbb{P} -a.s. at most countably many discontinuities. The set of jump times therefore has Lebesgue measure zero. Hence we can replace left-hand limits in terms such as $\alpha_{t-} dt$ by their regular values, i.e., we can write $\alpha_t dt$. Moreover, by (22) and the fact that $X_T = 0$,

$$\int_{[0,T]} d(S_t^0 X_t) - \int_{[0,T]} X_{t-} dM_t = S_T^0 X_T - S_{0-}^0 X_{0-} - X_{0-} \Delta M_0 - \int_{(0,T]} X_{t-} dM_t$$

$$= -xS_0 + X_{0-} \Delta A_0 - \int_{(0,T]} X_{t-} dM_t.$$
(25)

Building the integral $\int_{[0,T]} dC_t^X$ thus yields

$$C_{T}^{X} - C_{0-}^{X} = -xS_{0} - \int_{(0,T]} X_{t-} dM_{t} + \int_{[0,T]} \varphi(t)(X_{t-} - E_{t-}) dZ_{t}^{\alpha} + \int_{0}^{T} X_{t} \alpha_{t} dt - \int_{(0,T]} X_{t-} dA_{t} + \rho \int_{0}^{T} \left\{ \varphi(t)X_{t} + (1 - \varphi(t))E_{t} + \frac{1}{2}\varphi(t)Y_{t}^{\alpha} - \frac{1}{2\rho}\alpha_{t} \right\}^{2} dt$$

$$-\rho \int_{0}^{T} \left(\frac{1}{2}\varphi(t)Y_{t}^{\alpha} - \frac{1}{2\rho}\alpha_{t} \right)^{2} dt.$$
(26)

The stochastic integral $L_u := \int_{(0,u]} X_{t-} dM_t$ satisfies $\mathbb{E}[[L]_T] = \mathbb{E}[\int_{(0,T]} X_{t-}^2 d[M]_t] < \infty$ since M is a square-integrable martingale and X is bounded by definition. Hence L is a true martingale and satisfies $\mathbb{E}[L_T] = L_0 = 0$. Now we show that

$$\mathbb{E}\Big[\int_{[0,T]} \varphi(t)(X_{t-} - E_{t-}) \, dZ_t^{\alpha}\Big] = \varphi(0)x(Z_0^{\alpha} - Z_{0-}^{\alpha}).$$
(27)

Taking expectations in (26) will then yield the assertion, because, on the one hand, $\tilde{C}_{0-}^X = 0$, $Y_{0-}^{\alpha} = Z_{0-}^{\alpha}$, and so $C_{0-}^X = V(t-, X_{0-}, E_{0-}) = \varphi(0)x^2 + \varphi(0)xZ_{0-}^{\alpha}$. On the other hand,

$$Y_T^{\alpha} = 0 \qquad \mathbb{P}\text{-a.s.} \tag{28}$$

and so $C_T^X = \widetilde{C}_T^X + V^{\alpha}(T, 0, E_T) = \mathcal{C}(X)$ P-a.s.

To show (27), we use (18) and Doob's quadratic maximal inequality to conclude that Z^{α} is a square-integrable martingale. Moreover, the boundedness of X, the identity (21), and Gronwall's lemma yield that E is bounded as well. Thus, the stochastic integral $N_u := \int_{[0,u]} \varphi(t)(X_{t-} - E_{t-}) dZ_t^{\alpha}$ is a true martingale. Together with $N_0 = \varphi(0) x \Delta Z_0^{\alpha}$ this shows (27) and thus concludes the proof.

Remark 7. In the preceding proof, we have only used the facts that Y^{α} satisfies (20) for some square-integrable martingale Z^{α} and the identity (28). But these two identities already determine Y^{α} and Z^{α} .

In the next lemma, we derive an explicit formula for a strategy for which the last term in (17) vanishes. When we can take $\alpha_t = A'_t$, this strategy will be the optimal strategy.

Lemma 3. Suppose that α is a bounded semimartingale and that Y^{α} and Z^{α} are as in Lemma 2. Then for all $x \in \mathbb{R}$ and T > 0 there exists a unique strategy $X \in \mathcal{X}_{sem}(x,T)$ such that

$$\varphi(t)X_t + (1 - \varphi(t))E_t + \frac{1}{2}\varphi(t)Y_t^{\alpha} - \frac{1}{2\rho}\alpha_t = 0 \quad \text{for a.e. } t \in [0, T).$$
(29)

Moreover, for $0 \le t < T$, X is given by

$$X_{t} = \frac{x(1+\rho(T-t)) - \frac{1}{2}(1+\rho t)Y_{0}^{\alpha}}{2+\rho T} - \frac{1}{2}\int_{(0,t]}\varphi(s) dZ_{s}^{\alpha} + \frac{1}{2\rho}\alpha_{t} + \rho\int_{0}^{t} \left(-\frac{1}{2}\int_{(0,s]}\varphi(r) dZ_{r}^{\alpha} - \frac{1}{2}\int_{0}^{s}\alpha_{r} dr\right) ds.$$
(30)

Furthermore, X has the initial jump

$$\Delta X_0 = \frac{1}{2\rho} \alpha_0 - \frac{x + \frac{1}{2} Y_0^{\alpha}}{2 + \rho T}$$
(31)

and the terminal jump

$$\Delta X_T = -\frac{x + \frac{1}{2}Y_0^{\alpha}}{2 + \rho T} - \frac{1}{2} \int_{(0,T)} \varphi(s) \, dZ_s^{\alpha} - \frac{1}{2} \int_0^T \alpha_s \, ds + \frac{1}{2}Y_{T-}^{\alpha} - \frac{1}{2\rho}\alpha_{T-}.$$
 (32)

In particular, X belongs to $\mathcal{X}_{BV}(x,T)$ when both Z and α are of finite variation.

Proof. When (29) is not already satisfied at t = 0-, then an initial jump is needed so that (29) is satisfied immediately after the jump, because all processes in (29) are right-continuous. Taking the limit $t \downarrow 0$ in (29) and using the identities $X_0 = x + \Delta X_0$ and $E_0 = \Delta X_0$ yields

$$\Delta X_0 = -\frac{1}{2}\varphi(0)Y_0^{\alpha} + \frac{1}{2\rho}\alpha_0 - \varphi(0)x = \frac{1}{2\rho}\alpha_0 - \frac{x + \frac{1}{2}Y_0^{\alpha}}{2 + \rho T}.$$
(33)

We now solve for the dynamics of X on (0,T). Dividing (29) by φ and taking differentials yields

$$0 = dX_t - \rho E_t dt + (1 + \rho(T - t)) dE_t + \frac{1}{2} dY_t^{\alpha} + \frac{1}{2} \alpha_t dt - \frac{1}{2\rho\varphi(t)} d\alpha_t$$

= $\frac{1}{\varphi(t)} dE_t + \frac{1}{2} dY_t^{\alpha} + \frac{1}{2} \alpha_t dt - \frac{1}{2\rho\varphi(t)} d\alpha_t,$ (34)

where we have used the identity $dE_t = dX_t - \rho E_t dt$ in the second step. We can now informally solve (34) for dE_t and then obtain that for $t \in [0, T)$

$$E_t = \Delta X_0 + \int_{(0,t]} dE_s = -\frac{x + \frac{1}{2}Y_0^{\alpha}}{2 + \rho T} - \frac{1}{2} \int_{(0,t]} \varphi(s) \, dZ_s^{\alpha} - \frac{1}{2} \int_0^t \alpha_s \, ds + \frac{1}{2\rho} \alpha_t, \qquad (35)$$

where we have used the fact that $E_0 = \Delta X_0$, (33), and (20). To make this argument rigorous, we define Λ_t so that (34) becomes $0 = \int_{(0,t]} \frac{1}{\varphi(s)} dE_s + \Lambda_t$ after integration. Then we use the associativity of the stochastic integral (Protter 2004, Theorem II.5.19) to get

$$0 = \int_{(0,t]} \varphi(s) \frac{1}{\varphi(s)} \, dE_s + \int_{(0,t]} \varphi(s) \, d\Lambda_s = E_t - E_0 + \int_{(0,t]} \varphi(s) \, d\Lambda_s.$$

When taking differentials again, we arrive at (35).

Now (21) and (35) yield that for $t \in [0, T)$

$$X_{t} = x + E_{t} + \rho \int_{0}^{t} E_{s} ds$$

= $\frac{x(1 + \rho(T - t)) - \frac{1}{2}(1 + \rho t)Y_{0}^{\alpha}}{2 + \rho T} - \frac{1}{2} \int_{(0,t]} \varphi(s) dZ_{s}^{\alpha} + \frac{1}{2\rho} \alpha_{t}$
 $+ \rho \int_{0}^{t} \left(-\frac{1}{2} \int_{(0,s]} \varphi(r) dZ_{r}^{\alpha} - \frac{1}{2} \int_{0}^{s} \alpha_{r} dr \right) ds$

and this proves (30). It is moreover clear from the proof that any strategy in $\mathcal{X}_{sem}(x,T)$ satisfying (29) must be of this form, which gives uniqueness.

Now we turn to proving our formula for the terminal jump. Taking left-hand limits $t \uparrow T$ in (29) and using that $\varphi(T-) = \varphi(T) = 1/2$ yields

$$0 = X_{T-} + E_{T-} + \frac{1}{2}Y_{T-}^{\alpha} - \frac{1}{\rho}\alpha_{T-}$$

= $X_{T-} - \frac{x + \frac{1}{2}Y_{0}^{\alpha}}{2 + \rho T} - \frac{1}{2}\int_{(0,T)} \varphi(s) dZ_{s}^{\alpha} - \frac{1}{2}\int_{0}^{T} \alpha_{s} ds + \frac{1}{2}Y_{T-}^{\alpha} - \frac{1}{2\rho}\alpha_{T-},$

where we have used (35) in the second step. Since $X_T = 0$ we have $\Delta X_T = -X_{T-}$, and our formula follows.

Now we show that X is admissible, i.e., we must show that X is bounded. To this end, integration by parts yields that

$$\int_{(0,u]} \varphi(t) \, dZ_t^{\alpha} = \varphi(t) Z_t^{\alpha} - \varphi(0) Z_0^{\alpha} - \int_0^u Z_t^{\alpha} \varphi'(t) \, dt. \tag{36}$$

Since α is bounded, so is Z^{α} . Moreover, φ and φ' are bounded as well. It hence follows that $\int_{(0,\cdot]} \varphi(t) dZ_t^{\alpha}$ is bounded. But all other terms in (30) are bounded by assumption. Therefore X is an admissible strategy.

Lemma 4. Suppose that M is a given constant, α is a semimartingale satisfying $\mathbb{E}[\int_0^T \alpha_t^2 dt] \leq M$, and Y^{α} , Z^{α} , and X are as in Lemma 2. Then there exists a constant C that depends only on M, x, ρ , and T such that

$$\mathbb{E}\left[\sup_{0 \le t < T} \left(X_t - \frac{1}{2\rho}\alpha_t\right)^2\right] \le C.$$

Moreover, $|Y_0^{\alpha}| \leq (1 + \rho T)\sqrt{MT}$.

Proof. We get from (18) that $\mathbb{E}[(Z_T^{\alpha})^2] \leq MT(1+\rho T)^2$. Doob's quadratic maximal inequality therefore yields that $Z^* := \sup_{0 \leq t \leq T} |Z_t^{\alpha}|$ satisfies $\mathbb{E}[(Z^*)^2] \leq 4MT(1+\rho T)^2$. We furthermore have

$$Y_0^{\alpha} = Z_0^{\alpha} \le \sqrt{\mathbb{E}[(Z_T^{\alpha})^2]} \le (1 + \rho T)\sqrt{MT}$$

and

$$\sup_{0 \le t \le T} |Y_t^{\alpha}| \le Z^* + (1 + 2\rho T) \sqrt{T \int_0^T \alpha_t^2 \, dt},$$

and so

$$\mathbb{E}\Big[\sup_{0 \le t \le T} (Y_t^{\alpha})^2\Big] \le 8MT(1+\rho T)^2 + 2(1+2\rho T)^2 MT.$$

Next, we get from (36) that

$$\sup_{0 \le t \le T} \left| \int_{(0,t]} \varphi(s) \, dZ_s^{\alpha} \right| \le 2Z^*.$$

Whence,

$$\mathbb{E}\Big[\sup_{0\le t\le T}\Big(\int_{(0,t]}\varphi(s)\,dZ_s^{\alpha}\Big)^2\Big]\le 16MT(1+\rho T)^2.$$

Since (30) holds for $0 \le t < T$, we now easily get the assertion.

We will say that α is a *bounded elementary process* if it is of the form

$$\alpha_t = \alpha_0 1\!\!1_{\{0\}}(t) + \sum_{i=1}^N \alpha_i 1\!\!1_{[\tau_i, \tau_{i+1})}(t),$$

where $N \in \mathbb{N}$, the (τ_i) are stopping times with $0 \leq \tau_1 \leq \cdots \leq \tau_{N+1} < \infty$, and the coefficients α_i are bounded \mathcal{F}_{τ_i} -measurable random variables.

Lemma 5. Let α be a bounded elementary process, $x \in \mathbb{R}$, T > 0, and consider the corresponding strategy $X \in \mathcal{X}_{sem}(x,T)$ constructed in Lemma 3. Then for each $\varepsilon > 0$ there exists a strategy $\widetilde{X} \in \mathcal{X}_{BV}(x,T)$ such that $|\mathbb{E}[\mathcal{C}(X)] - \mathbb{E}[\mathcal{C}(\widetilde{X})]| < \varepsilon$.

Proof. First, we recall from (36) that $N_u := \int_{(0,u]} \varphi(t) dZ_t^{\alpha}$ is a bounded càdlàg martingale with $N_0 = 0$. We set $N_t = 0$ for t < 0 and define

$$N_t^n := n \int_{t-\frac{1}{n}}^t N_s \, ds, \qquad n \in \mathbb{N}$$

Then N_t^n is continuous, bounded uniformly in n and t, and of bounded variation in t. Furthermore, $N_t^n \to N_{t-}$ for all $t \ge 0$ as $n \uparrow \infty$. Thus, when defining $X_{0-}^n := x$ and $X_t^n := X_t + \frac{1}{2}(N_t - N_t^n)$ we have $X_{t-}^n \to X_{t-}$ for all t boundedly. Moreover, we get from (30) that, for $0 \le t < T$,

$$X_t^n = \frac{x(1+\rho(T-t)) - \frac{1}{2}(1+\rho t)Y_0^\alpha}{2+\rho T} - \frac{1}{2}N_t^n + \frac{1}{2\rho}\alpha_t - \rho \int_0^t \left(N_s + \frac{1}{2}\int_0^s \alpha_r \, dr\right) ds,$$

and so X^n is of bounded variation.

Now we set $E_t^n := \int_{[0,t]} e^{-\rho(t-s)} dX_s^n$. Integrating by parts as in (21) yields $E_t^n = X_t^n - x - \rho \int_0^t E_s^n ds$. Therefore,

$$|E_{t-}^n - E_{t-}| \le |X_{t-}^n - X_{t-}| + \rho \int_0^t |E_{s-}^n - E_{s-}| \, ds,$$

and so Gronwall's inequality (in the extended form of, e.g., Lemma 2.7 in Teschl (2012)) implies that

$$|E_{t-}^n - E_{t-}| \le |X_{t-}^n - X_{t-}| + \rho \int_0^t e^{\rho s} |X_{s-}^n - X_{s-}| \, ds.$$

Thus, also $E_{t-}^n \to E_{t-}$ boundedly.

With $|A|_{[0,t]}$ denoting the total variation of A over [0,t], we get from (17) that

$$\mathbb{E}[\mathcal{C}(X^{n})] - \mathbb{E}[\mathcal{C}(X)] | \\
\leq \mathbb{E}\left[\int_{0}^{T} |X_{t}^{n} - X_{t}| |\alpha_{t}| dt + \int_{[0,T]} |X_{t-}^{n} - X_{t-}| d|A|_{[0,t]}\right] \\
+ \rho \left| \mathbb{E}\left[\int_{0}^{T} \left\{\varphi(t)X_{t}^{n} + (1 - \varphi(t))E_{t}^{n} + \frac{1}{2}\varphi(t)Y_{t}^{\alpha} - \frac{1}{2\rho}\alpha_{t}\right\}^{2} dt\right] \\
- \mathbb{E}\left[\int_{0}^{T} \left\{\varphi(t)X_{t} + (1 - \varphi(t))E_{t} + \frac{1}{2}\varphi(t)Y_{t}^{\alpha} - \frac{1}{2\rho}\alpha_{t}\right\}^{2} dt\right] \right|.$$
(37)

Dominated convergence implies that the right-hand side converges to zero when $n \uparrow \infty$.

Lemma 6. Fix T > 0 and suppose that A is not \mathbb{P} -a.s. absolutely continuous on [0, T). That is, A is not \mathbb{P} -a.s. of the form $A_t = \int_0^t A'_s ds$ for some progressively measurable process A' and $0 \le t < T$. Then, for any $x \in \mathbb{R}$,

$$\inf_{\mathcal{X}_{\mathrm{BV}}(x,T)} \mathbb{E}[\mathcal{C}(X)] = -\infty.$$

Proof. Let us define two finite measures Q and Q^A on $([0,T) \times \Omega, \mathcal{B}[0,T) \otimes \mathcal{F})$ by

$$\int f \, dQ = \int \int_0^T f(t,\omega) \, dt \, \mathbb{P}(d\omega), \qquad \int f \, dQ^A = \int \int_{[0,T)} f(t,\omega) \, dA_t(\omega) \, \mathbb{P}(d\omega),$$

where f is a bounded measurable function on $([0, T) \times \Omega, \mathcal{B}[0, T) \otimes \mathcal{F})$. Since A is not absolutely continuous, there exists a bounded measurable function $\overline{\psi} \ge 0$ on $[0, T) \times \Omega$ such that $\int \overline{\psi} \, dQ = 0$ and $\int \overline{\psi} \, dQ^A = 1$. By the predictability of A and Theorem 57 in Chapter VI of Dellacherie & Meyer (1982), we may replace $\overline{\psi}$ by its predictable projection, ψ , and still have $\int \psi \, dQ = 0$ and $\int \psi \, dQ^A = 1$.

It follows from Theorem II.4.10 in Protter (2004) and a monotone class argument that the left-hand limits of bounded elementary processes are dense with respect to $(Q + Q^A)$ -a.e. convergence in the class of predictable processes. Moreover, bounded elementary processes are clearly of finite total variation. By approximating $(K + 1)\psi$ for some $K \in \mathbb{N}$, we hence get that there exists a bounded elementary process $\alpha \geq 0$ such that

$$1 \ge \int \alpha_{-}^{2} dQ = \mathbb{E} \Big[\int_{0}^{T} \alpha_{t-}^{2} dt \Big] = \mathbb{E} \Big[\int_{0}^{T} \alpha_{t}^{2} dt \Big]$$
(38)

and

$$K \leq \int \alpha_{-} dQ^{A} = \mathbb{E} \Big[\int_{[0,T]} \alpha_{t-} dA_{t} \Big].$$

Now let $X \in \mathcal{X}_{sem}(x,T)$ be the corresponding strategy constructed in Lemma 3. We denote by Ξ the random variable

$$\Xi := \sup_{0 \le t < T} \left| X_t - \frac{1}{2\rho} \alpha_t \right| = \sup_{0 < t \le T} \left| X_{t-} - \frac{1}{2\rho} \alpha_{t-} \right|.$$
(39)

By Lemma 4, $\mathbb{E}[\Xi^2]$ is bounded by a constant C that depends only on x, ρ , and T. By Lemma 2, (29), and (4), the expected costs of the strategy X can be estimated as follows:

$$\mathbb{E}\left[\mathcal{C}(X)\right] = -xS_0 + \varphi(0)x^2 + \varphi(0)xY_0^{\alpha} - \rho \mathbb{E}\left[\int_0^T \left(\frac{1}{2}\varphi(s)Y_s^{\alpha} - \frac{1}{2\rho}\alpha_s\right)^2 ds\right] \\ + \mathbb{E}\left[\int_0^T X_t\alpha_t \, dt - \int_{(0,T]} X_{t-} \, dA_t\right] \\ \leq -xS_0 + x^2 + x(1+\rho T)\sqrt{MT} + \mathbb{E}\left[\frac{1}{2\rho}\int_0^T \alpha_t^2 \, dt + \Xi\int_0^T |\alpha_t| \, dt\right] \quad (40) \\ - \mathbb{E}\left[\frac{1}{2\rho}\int_{[0,T]} \alpha_{t-} \, dA_t - \Xi|A|_{[0,T]}\right] \\ \leq \widetilde{C} - \frac{K}{2\rho},$$

where $|A|_{[0,T]}$ denotes again the total variation of A on [0,T] and \widetilde{C} is a constant depending only on x, ρ , T, and $\mathbb{E}[|A|_{[0,T]}^2]$. Here we have also used (38). Since K was arbitrary, it follows that $\inf \mathbb{E}[\mathcal{C}(X)] = -\infty$ when the infimum is taken over $X \in \mathcal{X}_{sem}(x,T)$. An application of Lemma 5 shows that $\mathcal{X}_{sem}(x,T)$ can be replaced by $\mathcal{X}_{BV}(x,T)$. **Lemma 7.** Fix T > 0 and suppose that A is \mathbb{P} -a.s. of the form $A_t = \int_0^t A'_s ds$ for some progressively measurable process A' but that $\mathbb{E}[\int_0^T (A'_t)^2 dt] = \infty$. Then, for any $x \in \mathbb{R}$,

$$\inf_{\mathcal{X}_{\mathrm{BV}}(x,T)} \mathbb{E}[\mathcal{C}(X)] = -\infty.$$

Proof. We have

$$\infty = \sqrt{\mathbb{E}\Big[\int_0^T (A'_t)^2 \, dt\Big]} = \sup_{\psi} \mathbb{E}\Big[\int_0^T \psi_t A'_t \, dt\Big]$$

where the supremum is taken over all progressively measurable ψ with $\mathbb{E}[\int_0^T \psi_t^2 dt] \leq 1$. By a monotone class argument, the supremum over these ψ can be replaced by a supremum over all bounded elementary processes α with $\mathbb{E}[\int_0^T \alpha_t^2 dt] \leq 1$. For every K > 0 there hence exists a bounded elementary process α such that

$$\mathbb{E}\Big[\int_0^T \alpha_t^2 dt\Big] \le 1 \qquad \text{and} \qquad \mathbb{E}\Big[\int_0^T \alpha_t A_t' dt\Big] \ge K \tag{41}$$

Let $X \in \mathcal{X}_{sem}(x,T)$ be the corresponding strategy constructed in Lemma 3 and define Ξ as in (39). Then, due to (41),

$$\begin{split} \mathbb{E}\left[\mathcal{C}(X)\right] &= -xS_0 + \varphi(0)x^2 + \varphi(0)xY_0^{\alpha} - \rho \mathbb{E}\left[\int_0^T \left(\frac{1}{2}\varphi(s)Y_s^{\alpha} - \frac{1}{2\rho}\alpha_s\right)^2 ds\right] \\ &+ \mathbb{E}\left[\int_0^T X_t(\alpha_t - A_t') dt\right] \\ &\leq -xS_0 + x^2 + x\sqrt{MT(1+\rho T)^2} + \mathbb{E}\left[\frac{1}{2\rho}\int_0^T \alpha_t^2 dt + \Xi\int_0^T |\alpha_t| dt\right] \\ &- \mathbb{E}\left[\frac{1}{2\rho}\int_{[0,T]} \alpha_t A_t' dt - \Xi |A|_{[0,T]}\right] \leq \widetilde{C} - \frac{K}{2\rho}, \end{split}$$

where \widetilde{C} is a constant depending only on x, ρ , T, and $\mathbb{E}[|A|^2_{[0,T]}]$. Since K was arbitrary, it follows that $\inf \mathbb{E}[\mathcal{C}(X)] = -\infty$ when the infimum is taken over $X \in \mathcal{X}_{\text{sem}}(x,T)$. An application of Lemma 5 shows that $\mathcal{X}_{\text{sem}}(x,T)$ can be replaced by $\mathcal{X}_{\text{BV}}(x,T)$.

Proof of Theorem 1. By Lemmas 6 and 7 we may concentrate on the case in which A is absolutely continuous on [0,T) with square-integrable derivative A'. Taking $\alpha_t := A'_t$ in Lemma 2 yields that for any strategy X,

$$\mathbb{E}\left[\mathcal{C}(X)\right] = -xS_0 + \varphi(0)x^2 + \varphi(0)xY_0 - \rho\mathbb{E}\left[\int_0^T \left(\frac{1}{2}\varphi(s)Y_s - \frac{1}{2\rho}A_s'\right)^2 ds\right] \\ + \rho\mathbb{E}\left[\int_0^T \left\{\varphi(t)X_t + (1-\varphi(t))E_t + \frac{1}{2}\varphi(t)Y_t - \frac{1}{2\rho}A_t'\right\}^2 dt\right] \\ \ge -xS_0 + \varphi(0)x^2 + \varphi(0)xY_0 - \rho\mathbb{E}\left[\int_0^T \left(\frac{1}{2}\varphi(s)Y_s - \frac{1}{2\rho}A_s'\right)^2 ds\right]$$
(42)

with equality if and only if (29) holds with $\alpha = A'$. Since X was arbitrary, we have

$$\inf_{X \in \mathcal{X}_{\text{sem}}(x,T)} \mathbb{E}[\mathcal{C}(X)] \ge \varphi(0)x^2 + \varphi(0)xY_0 - \rho \mathbb{E}\left[\int_0^T \left(\frac{1}{2}\varphi(s)Y_s - \frac{1}{2\rho}A'_s\right)^2 ds\right].$$

To show the converse inequality, take bounded elementary processes α^n such that

$$\delta_n := \mathbb{E}\Big[\int_0^T (\alpha_t^n - A_t')^2 \, dt\,\Big] \longrightarrow 0.$$

In particular there exists a constant M such that for all n we have $\mathbb{E}[\int_0^T (\alpha_t^n)^2 dt] \leq M$. Let X^n be the strategy constructed for α^n in Lemma 3. By Lemma 2 we have

$$\mathbb{E}\left[\mathcal{C}(X^{n})\right] = -xS_{0} + \varphi(0)x^{2} + \varphi(0)xY_{0}^{\alpha^{n}} - \rho\mathbb{E}\left[\int_{0}^{T}\left(\frac{1}{2}\varphi(s)Y_{s}^{\alpha^{n}} - \frac{1}{2\rho}\alpha_{s}^{n}\right)^{2}ds\right] + \mathbb{E}\left[\int_{0}^{T}X_{t}^{n}(\alpha_{t}^{n} - A_{t}')dt\right].$$
(43)

By Jensen's inequality, we have

$$\mathbb{E}\left[\left(\sup_{t\leq T}\left|\int_{0}^{t}\alpha_{s}^{n}\,ds-\int_{0}^{t}A_{s}'\,ds\right|\right)^{2}\right]\leq T\delta_{n},\\\mathbb{E}\left[\left(\sup_{t\leq T}\left|\int_{0}^{t}\int_{0}^{s}\alpha_{r}^{n}\,dr\,ds-\int_{0}^{t}\int_{0}^{s}A_{r}'\,dr\,ds\right|\right)^{2}\right]\leq T^{3}\delta_{n}.$$

In particular, we have $\mathbb{E}[(Z_T^{\alpha^n} - Z_T)^2] \to 0$. It therefore follows from Doob's quadratic maximal inequality that $\sup_{t \leq T} |Z_t^{\alpha^n} - Z_t| \to 0$ in $L^2(\mathbb{P})$ and hence moreover that $\sup_{t \leq T} |Y_t^{\alpha^n} - Y_t| \to 0$ in $L^2(\mathbb{P})$. Consequently, the right-hand side of the first line in (43) converges to

$$-xS_0 + \varphi(0)x^2 + \varphi(0)xY_0 - \rho \mathbb{E}\left[\int_0^T \left(\frac{1}{2}\varphi(s)Y_s - \frac{1}{2\rho}A'_s\right)^2 ds\right].$$

Furthermore, by the Cauchy–Schwarz inequality,

$$\left|\mathbb{E}\Big[\int_0^T X_t^n(\alpha_t^n - A_t')\,dt\,\Big]\right| \le \sqrt{\delta_n \cdot \mathbb{E}\Big[\int_0^T (X_t^n)^2\,dt\,\Big]}.$$

It is a consequence of Lemma 4 that $\mathbb{E}[\int_0^T (X_t^n)^2 dt]$ is bounded uniformly in n, and so it follows that $\mathbb{E}[\int_0^T X_t^n(\alpha_t^n - A_t') dt] \to 0$. We thus have proved that

$$\mathbb{E}\left[\mathcal{C}(X^n)\right] \longrightarrow -xS_0 + \varphi(0)x^2 + \varphi(0)xY_0 - \rho \mathbb{E}\left[\int_0^T \left(\frac{1}{2}\varphi(s)Y_s - \frac{1}{2\rho}A'_s\right)^2 ds\right].$$

This completes the proof of our formula for the optimal expected costs.

As for optimal strategies, we have already remarked above that equality in (42) can hold only when

$$0 = \varphi(t)X_t + (1 - \varphi(t))E_t + \frac{1}{2}\varphi(t)Y_t - \frac{1}{2\rho}A'_t \qquad \mathbb{P}\text{-a.s. for all } t \in [0, T).$$
(44)

By Lemma 3, there exists a unique strategy in $\mathcal{X}_{\text{sem}}(x,T)$ satisfying this condition when A' is a bounded semimartingale. This strategy is given by (11). When A' is not a semimartingale, it is clear that (44) cannot be satisfied by a semimartingale X. Also, X will be of finite variation if and only if $\varphi(t)Y_t - \frac{1}{2\rho}A'_t$ is of finite variation. This concludes the proof of Theorem 1. \Box Proof of Corollary 1: A computation shows that

$$-Z_t = (1 + \rho(T - t))A_t + \frac{1}{2}(2 + \rho(T - t))(T - t)A'_t + \rho \int_0^t A_s \, ds,$$

and so in particular $Y_0 = Z_0 = -\frac{1}{2}(2 + \rho T)TA'_0$. From the integration-by-parts formula, dZ_t must satisfy

$$-dZ_t = \frac{1}{2}(2 + \rho(T - t))(T - t) \, dA'_t + \Lambda_t \, dt,$$

for a suitable process Λ_t . But both Z_t and A'_t are martingales, and so we must have $\Lambda_t = 0$ (alternatively, the fact $\Lambda_t = 0$ can also be verified by a direct computation). Consequently,

$$-\int_{(0,s]}\varphi(u)\,dZ_u = \frac{1}{2}\int_{(0,s]}(T-u)\,dA'_u = \frac{1}{2}\Big((T-s)A'_s - TA'_0 + A_s\Big).$$

This identity yields that

$$-\int_0^t \int_{(0,s]} \varphi(u) \, dZ_u \, ds = \frac{1}{2} \bigg((T-t)A_t - TA_0't + 2\int_0^t A_s \, ds \bigg).$$

Plugging these formulas into (11) yields the assertion after a short computation.

Proof of Corollary 2. We start by simplifying the cost-risk functional (14). First, we can assume $S_{0-}^0 = 0$ without loss of generality and therefore write

$$\int_{[0,T]} S_{t-}^0 dX_t + [S^0, X]_T = -\int_{[0,T]} X_{t-} dS_t^0$$

as in (23). Then we can write $S_t^X = S_t^0 + E_{t-}$. When defining

$$\widehat{S}_t := S_t^0 - \lambda \int_0^t S_s^0 \, ds,$$

we can thus write

$$\mathbb{E}\Big[\mathcal{C}(X) + \lambda \int_0^T S_t^X X_t \, dt\Big]$$

$$= \mathbb{E}\left[-\int_{[0,T]} X_{t-} \, d\widehat{S}_t + \int_{[0,T]} E_{t-} \, dX_t + \frac{1}{2}[X]_T + \lambda \int_0^T E_t X_t \, dt\right].$$
(45)

To simplify (45) further, we let

$$\overline{E}_t := \int_0^t E_s \, ds = \frac{1}{\rho} \Big(X_t - x - E_t \Big),$$

where we have used (21) in the second step, and set $\overline{E}_{0-} = 0$. Then

$$\int_{0}^{T} E_{t} X_{t} dt = \int_{[0,T]} X_{t} d\overline{E}_{t} = X_{T} \overline{E}_{T} - X_{0-} \overline{E}_{0-} - \int_{[0,T]} \overline{E}_{t-} dX_{t}$$
$$= \frac{1}{\rho} \int_{[0,T]} E_{t-} dX_{t} - \frac{1}{\rho} \int_{[0,T]} X_{t-} dX_{t} - \frac{x^{2}}{\rho}$$
$$= \frac{1}{\rho} \left(\int_{[0,T]} E_{t-} dX_{t} + \frac{1}{2} [X]_{T} - \frac{1}{2} x^{2} \right).$$

It follows that

$$\mathbb{E}\Big[\mathcal{C}(X) + \lambda \int_0^T S_t^X X_t \, dt\Big]$$

= $-\frac{\lambda}{2\rho} x^2 + \Big(1 + \frac{\lambda}{\rho}\Big) \mathbb{E}\left[\int_{[0,T]} \widetilde{S}_{t-}^0 \, dX_t + [\widetilde{S}^0, X]_T + \int_{[0,T]} E_{t-} \, dX_t + \frac{1}{2}[X]_T\right],$

where

$$\widetilde{S}_t^0 = \frac{\widehat{S}_t}{1 + \frac{\lambda}{\rho}} = \frac{\rho}{\rho + \lambda} \bigg(S_t^0 - \lambda \int_0^t S_s^0 \, ds \bigg).$$

This concludes the proof.

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