# Bilateral Credit Valuation Adjustment for Large Credit Derivatives Portfolios 

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#### Abstract

We obtain an explicit formula for the bilateral counterparty valuation adjustment of a credit default swaps portfolio referencing an asymptotically large number of entities. We perform the analysis under a doubly stochastic intensity framework, allowing for default correlation through a common jump process. The key insight behind our approach is an explicit characterization of the portfolio exposure as the weak limit of measure-valued processes associated to survival indicators of portfolio names. We validate our theoretical predictions by means of a numerical analysis, showing that counterparty adjustments are highly sensitive to portfolio credit risk volatility as well as to default correlation.


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## 1 Introduction

The recent financial crisis has highlighted the importance of counterparty risk valuation in over-the-counter derivatives markets. Indeed, as noted by the Basel Committee on Banking supervision, see Basel III (2010), under Basel II the risk of counterparty default and credit migration were addressed, but mark-to-market losses due to credit valuation adjustments (CVA) were not. Nevertheless, during the financial crisis, roughly two-thirds of losses attributed to counterparty credit risk were due to CVA losses and only about one-third to actual defaults. Credit default swaps (CDSs) have been at the heart of debates between regulator and supervising authorities. They have been claimed to be responsible for increasing significantly the systemic risk in the economy, due to large amounts of traded notional (about $\$ 41$ trillion), and consequently high mark-to-market exposures, see ECB (2009).

The market price of counterparty risk is usually referred to as bilateral credit valuation adjustment, abbreviated with BCVA throughout the paper. This is obtained as the difference between the price of a portfolio transaction, traded between two counterparties assumed default free, and the price of the same portfolio where the default risk of both counterparties is accounted for. Precise estimates of such adjustments are notoriously difficult to obtain, given that they have to be computed at an aggregate portfolio level and are model dependent; as such they depend on the volatility of the underlying portfolio, credit spreads of the counterparties, as well as default correlation. On the other hand, accurate assessments of counterparty risk are essential, given that financial institutions need to mitigate and hedge their counterparty credit exposure. This has originated a significant amount of research, some of which surveyed next.

Brigo et al. (2012) develop an arbitrage free valuation framework for bilateral counterparty risk, inclusive of collateralization. Building on Brigo et al. (2012), Assefa et al. (2011) provide a representation formula for BCVA adjustments for a fully netted and collateralized portfolio of contracts. In a series of two papers, Crepey (2012a) and Crepey (2012b) propose a reduced-form backward stochastic differential equation approach to the problem of pricing and hedging CVA, taking into account funding constraints. Bielecki et al. (2012) develop an analytical framework for dynamic hedging unilateral counterparty risk, without excluding simultaneous defaults. Bielecki et al. (2010) propose a reduced form credit model for dynamically hedging credit default swaptions using credit default swaps. We refer the reader to Capponi (2012) for a survey on counterparty risk valuation and mitigation.

None of the above mentioned studies provides explicit pricing formulas for counterparty valuation adjustments. We provide a rigorous analysis, which culminates into an analytical representation of the BCVA adjustment when the traded portfolio consists of an asymptotically large number of CDS contracts. This extends significantly previous literature, which typically resorts to Monte-Carlo simulation methods to evaluate counterparty risk of credit derivatives portfolios, see Canabarro and Duffie (2004) and, more recently Cesari et al. (2012). Even in the case of single name CDSs portfolios, most of the attempts rely on numerical methods. Bielecki et al. (2012) employ a Markov chain copula model for pricing counterparty risk embedded in a CDS contract. Lipton and Sepp (2009) introduce a structural model with jumps and recover the BCVA adjustment on a CDS contract as the numerical solution of a partial differential equation.

We follow a bottom up approach to default and employ doubly stochastic processes to model the default times of the individual names in the portfolio, as well as of the counterparties, see Bielecki and Rutkowski (2002). The intensity process of each name in the portfolio, as well as of the counterparties, consists of both diffusive and jump components. The diffusive components follow independent mean-reverting CEV type processes. In order to introduce correlation across all portfolio names and counterparties, we assume that the jump process of each name consists of two types of jumps, idiosyncratic and systematic. Idiosyncratic jumps govern the default risk specific to each reference entity, while common jumps model the occurrence of economic events affecting all parties. In summary, our default correlation model falls within the class of the so called conditionally independent default models.

The key innovation in our approach is a fully explicit characterization of the asymptotic portfolio exposure. Besides representing a significant departure from currently available techniques to approximate portfolio exposure, this allows us obtaining explicit representations of the bilateral CVA adjustment. Our "law of large numbers" approximate exposure is recovered as the weak limit of a sequence of weighted empirical measure-valued process associated with the survival indicators of the portfolio entities.

We employ the heavy weak convergence machinery to martingale problems related to measure-valued processes driven by jump-diffusion type processes, as described in Ethier and Kurtz (1986). Although such a machinery is well established in the literature and goes back to the work of Dawson (1983), its application to counterparty risk is novel and requires a detailed mathematical analysis. As already discussed, our default behavior is captured through default intensity processes subject to both systematic and idiosyncratic jumps.

This requires to develop a rigorous analysis to understand the roles that both types of jumps play in the limiting martingale problem.

Following the weak convergence analysis procedures in Ethier and Kurtz (1986) (see Chapter 4 therein), Giesecke et al. (2013a) analyze the default behavior in a large portfolio of interacting firms, while Giesecke et al. (2013b) further develop a numerical method for solving the SPDE describing the law of large numbers limiting behavior. Despite our paper exhibits an overlap with Giesecke et al. (2013a), resulting from both employing the same weak-convergence scheme, there are important differences in the way the various steps are carried out, which are worth noticing. First, in our model correlation among defaults is introduced through a common jump process, whereas in Giesecke et al. (2013a) default correlation is modeled by means of a common diffusion process influencing the intensities of all names, and self-excitation to capture feedback from default. Secondly, we are able to recover a fully explicit expression for the limit measure-valued process (see our Theorem4.6), whereas Giesecke et al. (2013a) only provide an implicit characterization of the limit. Thirdly, while Giesecke et al. (2013a) only have a unique source of jumps coming from self excitation, we need to capture the behavior of both systematic and idiosyncratic jumps. By means of a delicate study (see the analysis leading to Lemma 4.2), we can identify how both jump components are incorporated into the generator of the limiting martingale problem. Interestingly, we find that the limiting killing rate process is purely diffusive with a drift correction given by the sum of the rates of systematic and idiosyncratic jump components.

Cvitanić et al. (2011) consider a model similar to Giesecke et al. (2013a), assuming that intensities are driven by factors following a diffusion process. Using a fixed point argument, they prove a law of large numbers showing that the limiting process tracking the average number of defaults solves an ordinary differential equation. Using a mean-field interaction model, Dai Pra et al. (2009) analyze financial contagion in large networks, and provide characterizations of the entire portfolio loss distribution.

Our default correlation mechanism differs from the ones in the above studies, because we allow for simultaneous jumps of all intensity processes. Our goal is to provide a model which can be calibrated to credit data and at the same time able to match empirical observations. Since the default intensities are estimated firm-by-firm, our model is consistent with an interpretation based on conditional independence where default correlation is built into the common variation of the individual default intensities. Moreover, it has been empirical shown by $\mathrm{Yu}(2005)$ that if the common factor is properly calibrated, the model is able to reproduce the levels of default correlations historically observed.

The rest of the paper is organized as follows. Section 2 introduces the default model. Section 3 gives the general expression for bilateral counterparty valuation adjustments in CDS portfolios. Section 4 develops a weak convergence analysis of the empirical measure associated with the large CDS portfolio. Section 5 provides a law of large numbers approximation formula for the bilateral CVA, under the assumption that all default intensity processes follow CEV dynamics. A fully explicit formula is derived in Section 6 under the empirically relevant specialization of CEV to CIR. Section 7 presents a numerical and economical analysis of our formulas. Section 8 concludes the paper. Technical proofs are delegated to the Appendix.

## 2 The Model

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, where $\mathbb{P}$ denotes the risk-neutral probability measure. The space is endowed with a $K+2$-dimensional Brownian motion ( $\left.W^{(1)}, \ldots, W^{(K)}, W^{(A)}, W^{(B)}\right)$ and $K+3$ independent Poisson processes $\left(\widehat{N}^{(1)}, \ldots, \widehat{N}^{(K)}, \widehat{N}^{(A)}, \widehat{N}^{(B)}, \widehat{N}^{(c)}\right)$, with $K \in \mathbb{N}$ being the number of reference names in the CDS portfolio. The Poisson process $\widehat{N}^{(j)}$ has a constant intensity $\widehat{\lambda}_{j}>0$ for each $j \in$ $\{1, \ldots, K, A, B, c\}$.

We assume that the $K+2$-dimensional Brownian motion is independent of the $K+3$ Poisson processes above. We use a standard construction of the default times, see Lando (2004), based on doubly stochastic point processes, using given strictly positive $\mathbb{F}$-adapted intensity processes $\xi^{(k)}=\left(\xi_{t}^{(k)} ; t \geq 0\right)$ with $k \in$ $\{1, \ldots, K, A, B\}$. The precise details are provided in the following sections.

### 2.1 Intensity Models for Portfolio Names and Counterparties

The default intensity processes of all names in the portfolio, as well as of the two counterparties $A$ and $B$ are given by mean-reverting constant elasticity of variance (CEV) processes with jumps: for $k \in\{1, \ldots, K\}$,

$$
\begin{equation*}
\xi_{t}^{(k)}=\xi_{0}^{(k)}+\int_{0}^{t}\left(\alpha_{k}-\kappa_{k} \xi_{s}^{(k)}\right) \mathrm{d} s+\int_{0}^{t} \sigma_{k}\left(\xi_{s}^{(k)}\right)^{\rho} \mathrm{d} W_{s}^{(k)}+c_{k} \sum_{i=1}^{\widehat{N}_{t}^{(c)}} Y_{i}^{(k)}+d_{k} \sum_{\ell=1}^{\widehat{N}_{t}^{(k)}} \widetilde{Y}_{\ell}^{(k)} \tag{1}
\end{equation*}
$$

and for $l \in\{A, B\}$,

$$
\begin{equation*}
\xi_{t}^{(l)}=\xi_{0}^{(l)}+\int_{0}^{t}\left(\alpha_{l}-\kappa_{l} \xi_{s}^{(l)}\right) \mathrm{d} s+\int_{0}^{t} \sigma_{l}\left(\xi_{s}^{(l)}\right)^{\widehat{\rho}} \mathrm{d} W_{s}^{(l)}+c_{l} \sum_{i=1}^{\widehat{N}_{t}^{(c)}} Y_{i}^{(l)}+d_{l} \sum_{\ell=1}^{\widehat{N}_{t}^{(l)}} \widetilde{Y}_{\ell}^{(l)} \tag{2}
\end{equation*}
$$

For each $j \in\{1, \ldots, K, A, B\}, \xi_{0}^{(j)}>0$. The parameter set $\left(\alpha_{j}, \kappa_{j}, \sigma_{j}, c_{j}, d_{j}\right) \in \mathbb{R}_{+}^{5}$, while $\left(Y_{1}^{(j)}, \ldots, Y_{i}^{(j)}, \ldots\right)$ and $\left(\widetilde{Y}_{1}^{(j)}, \ldots, \widetilde{Y}_{\ell}^{(j)}, \ldots,\right)$ are two independent sequences, each consisting of i.i.d. random variables, and with possibly different distribution functions. The power parameters $\frac{1}{2} \leq \rho, \widehat{\rho}<1$ are, possibly different, elasticity factors. Obviously the SDEs (11) and (2) will reduce to a CIR type process (with jumps) if the elasticity factor $\rho=\widehat{\rho}=\frac{1}{2}$.

It is well known that, when $\rho=\widehat{\rho}=\frac{1}{2}$, the default intensity processes $\xi_{t}^{(j)}, j \in\{1, \ldots, K, A, B\}$, are strictly positive $\mathbb{P}$-a.e. if $2 \alpha_{j} \geq \sigma_{j}^{2}$, see Feller (1951). Next, we deal with the positivity of the default intensity processes when the elasticity factor belongs to $\left(\frac{1}{2}, 1\right)$. Appendix A presents the proof for the CEV process of the $k$-th name given by (11). Obviously, the same proof holds for the CEV process in (2).

Lemma 2.1. For each $k \in\{1, \ldots, K\}$, there exists a unique nonnegative (non-explosive) strong solution $\xi^{(k)}=\left(\xi_{t}^{(k)} ; t \geq 0\right)$ to the stochastic differential equation (SDE) (1). Moreover, we have that $\xi_{t}^{(k)}>0$, $\mathbb{P}$-a.e. for all $t \geq 0$.

### 2.2 Default Times and Market Information

We assume the existence of a sequence of mutually independent unit mean exponential random variables $\left(\Theta_{j} ; j \in\{1, \ldots, K, A, B\}\right)$, defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, independent of the Brownian and Poisson processes. The default times of the $K$-reference names, as well as of the counterparties, are defined by

$$
\begin{equation*}
\tau_{j}=\inf \left\{t \geq 0 ; \int_{0}^{t} \xi_{s}^{(j)} \mathrm{d} s \geq \Theta_{j}\right\}, \quad j \in\{1, \ldots, K, A, B\} \tag{3}
\end{equation*}
$$

The corresponding default indicator processes are given by

$$
\begin{equation*}
H_{t}^{(j)}=\mathbf{1}_{\left\{\tau_{j} \leq t\right\}}, \quad t \geq 0 \tag{4}
\end{equation*}
$$

and the survival indicator processes are denoted by $\bar{H}_{t}^{(j)}=1-H_{t}^{(j)}=1_{\left\{\tau_{j}>t\right\}}$ for each $j \in\{1, \ldots, K, A, B\}$.
Given $t \geq 0$, let $\mathcal{H}_{t}^{(j)}=\sigma\left(H_{s}^{(j)} ; s \leq t\right)$, after completion and regularization on the right, see Belanger et al. (2004). Let $\mathcal{G}_{t}^{(j)}=\mathcal{F}_{t}^{(j)} \vee \mathcal{H}_{t}^{(j)}$ with $j \in\{1, \ldots, K, A, B\}$, where $\mathcal{F}_{t}^{(j)}=\sigma\left(\xi_{s}^{(j)} ; s \leq t\right)$. For $t \geq 0$, the market filtration is given by

$$
\begin{equation*}
\mathcal{G}_{t}^{(K, A, B)}=\mathcal{F}_{t}^{(K, A, B)} \vee \mathcal{H}_{t}^{(K, A, B)}, \quad t \geq 0 \tag{5}
\end{equation*}
$$

where $\mathcal{F}_{t}^{(K, A, B)}=\bigvee_{j \in\{1, \ldots, K, A, B\}} \mathcal{F}_{t}^{(j)}$ and $\mathcal{H}_{t}^{(K, A, B)}=\bigvee_{j \in\{1, \ldots, K, A, B\}} \mathcal{H}_{t}^{(j)}$ with $t \geq 0$. The filtration $\mathcal{F}_{t}^{(K, A, B)}$ is also referred to as the reference filtration, and models all observable market quantities except default events.

A straightforward application of results in Bielecki and Rutkowski (2002) shows that the default times of the reference names and of the counterparties, are conditionally independent given the reference filtration, see Appendix Cor details.

Remark 2.2. The modeling choices adopted above lead to a different stochastic analysis than in Giesecke et al. (2013a) for verifying the various steps of the weak-convergence procedure. As we demonstrate in Section 4. we develop a different method to prove the validity of a stronger compact containment condition of the underlying sequence of measure-valued processes. Our technique consists in transforming such a sequence into a sequence of real-valued stochastic processes defined on the Skorokhod space through test functions (see Li (2010)). Giesecke et al. (2013a) instead prove the compact containment condition for the sequence of their empirical measure-valued process by establishing a compact set of the corresponding functional space.

## 3 Pricing of Bilateral Counterparty Risk in CDS Portfolio

We provide the general arbitrage-free valuation formula for bilateral CVA in portfolios of credit default swaps. Such a formula generalizes the one provided in Brigo et al. (2012), who focus on a portfolio consisting of one credit default swap. We denote by $A$ and $B$ the two counterparties of the trade, and by $T$ the maturity of all contracts. For $0 \leq t \leq T$, we denote by $D(t, T)=e^{-r(T-t)}$ the discount factor from $t$ to $T$, where $r>0$ is the constant interest rate. Define the conditional survival function of the $k$-th name as: for $0 \leq t \leq T$,

$$
\begin{equation*}
G^{(k)}(t, T)=\mathbb{P}\left(\tau_{k}>T \mid \mathcal{G}_{t}^{(K, A, B)}\right), \quad k \in\{1, \ldots, K\} \tag{6}
\end{equation*}
$$

Then, on the event $\left\{\widetilde{\tau}_{K}>t\right\}$, where $\widetilde{\tau}_{K}=\min _{k \in\{1, \ldots, K\}} \tau_{k}$, the CDS price process of the $k$-th reference name is given by

$$
\begin{equation*}
C D S^{(k)}(t, T)=\int_{t}^{T} S_{k} D(t, s) G^{(k)}(t, s) \mathrm{d} s+\int_{t}^{T} L_{k} D(t, s) \mathrm{d} G^{(k)}(t, s) \tag{7}
\end{equation*}
$$

where $S_{k}$ and $L_{k}$ are constants for each $k \in\{1, \ldots, K\}$. The exposure of $A$ to $B$ is given by

$$
\begin{equation*}
\varepsilon^{(K)}(t, T)=\sum_{k=1}^{K} z_{k} C D S^{(k)}(t, T), \quad 0 \leq t \leq T \tag{8}
\end{equation*}
$$

where $z_{k}=1$ if the counterparty $A$ is long on the $k$-th CDS, i.e. $A$ sold the $k$-th CDS to $B$ ( $A$ is receiving the spread payments $S_{k}$ from $B$ ), and $z_{k}=-1$ if the counterparty $A$ is short on the $k$-th CDS, i.e. $A$ bought the $k$-th CDS from $B$ (i.e. $A$ is paying the spread premium $S_{k}$ to $B$ ).

On the event $\left\{\widetilde{\tau}_{K}>t\right\}$, such exposure may be rewritten as

$$
\begin{align*}
\frac{\varepsilon^{(K)}(t, T)}{K}= & \int_{t}^{T} D(t, s) \mathbb{E}\left[\left.\frac{1}{K} \sum_{k=1}^{K} z_{k} S_{k} \bar{H}_{s}^{(k)} \right\rvert\, \mathcal{G}_{t}^{(K, A, B)}\right] \mathrm{d} s \\
& +\int_{t}^{T} D(t, s) \mathrm{d}\left(\mathbb{E}\left[\left.\frac{1}{K} \sum_{k=1}^{K} z_{k} L_{k} \bar{H}_{s}^{(k)} \right\rvert\, \mathcal{G}_{t}^{(K, A, B)}\right]\right), \quad K \in \mathbb{N} . \tag{9}
\end{align*}
$$

where $\mathrm{d} f(t, s)$ denotes the differential of the function $f$ w.r.t. the time variable $s$. On $\{\widetilde{\tau}>t\}$, where $\widetilde{\tau}=\min _{j \in\{1, \ldots, K, A, B\}} \tau_{j}$, the bilateral counterparty valuation adjustment, denoted by BCVA, on the portfolio of $K$ CDSs is given by

$$
\begin{align*}
B C V A^{(K)}(t, T)= & L_{A} \mathbb{E}\left[\mathbf{1}_{\left\{t<\tau_{A} \leq \min \left(\tau_{B}, T\right)\right\}} D\left(t, \tau_{A}\right) \varepsilon_{-}^{(K)}\left(\tau_{A}, T\right) \mid \mathcal{G}_{t}^{(K, A, B)}\right]  \tag{10}\\
& -L_{B} \mathbb{E}\left[\mathbf{1}_{\left\{t<\tau_{B} \leq \min \left(\tau_{A}, T\right)\right\}} D\left(t, \tau_{B}\right) \varepsilon_{+}^{(K)}\left(\tau_{B}, T\right) \mid \mathcal{G}_{t}^{(K, A, B)}\right], \quad 0 \leq t \leq T
\end{align*}
$$

where $x_{+}=x \vee 0$ and $x_{-}=(-x) \vee 0$ for any real number $x \in \mathbb{R}$. Here $L_{A}$ and $L_{B}$ denote the percentage losses incurred when counterparty $A$, respectively $B$, defaults on its obligations.

Notice that the above formula is fully consistent with industry practise, where netting is applied before computing the exposure. Our goal is to provide a rigorous law of large numbers approximation formula for the bilateral CVA above defined.

## 4 Weak Convergence of Portfolio Empirical Measures

We analyze the weak convergence of a sequence of empirical measure-valued processes. The latter are associated with the survival indicators of the large number of reference entities in the CDS portfolio (i.e., $K \longrightarrow \infty)$. We follow a classical martingale approach.

From Section 2 the intensity processes of the $K$-reference entities follow CEV processes extended with jumps given by (11). As in Giesecke et al. (2013a), we first define the 'type' parameter set related to the $K$-intensity processes:

$$
\begin{equation*}
p_{k}=\left(\alpha_{k}, \kappa_{k}, \sigma_{k}, c_{k}, d_{k}, \widehat{\lambda}_{k}\right), \quad k \in\{1, \ldots, K\} \tag{11}
\end{equation*}
$$

taking values in the space $\mathcal{O}_{p}:=\mathbb{R}_{+}^{6}$. Here $\widehat{\lambda}_{k}$ is the intensity of the idiosyncratic component $\widehat{N}^{(k)}$. Throughout the paper, we make the following assumption
(A1) Let $q^{K}=\frac{1}{K} \sum_{k=1}^{K} \delta_{p_{k}}, \eta^{K}=\frac{1}{K} \sum_{k=1}^{K} \delta_{\left(Y_{1}^{(k)}, \widetilde{Y}_{1}^{(k)}\right)}$ and $\phi_{0}^{K}=\frac{1}{K} \sum_{k=1}^{K} \delta_{\xi_{0}^{(k)}}$. Then $q=\lim _{K \rightarrow \infty} q^{K}$, $\eta=\lim _{K \rightarrow \infty} \eta^{K}$ and $\phi_{0}=\lim _{K \rightarrow \infty} \phi_{0}^{K}$ exist in $\mathcal{P}\left(\mathcal{O}_{p}\right), \mathcal{P}\left(\mathbb{R}_{+}^{2}\right)$ and $\mathcal{P}\left(\mathbb{R}_{+}\right)$respectively, where $\mathcal{P}(A)$ denotes all Borel measures $\nu(\cdot)$ defined on $\mathcal{B}(A)$ such that $\nu(A) \leq 1$ for a given space $A$. We use $\delta(\cdot)$ to denote the Dirac-delta measure.

Remark 4.1. If there exists $q^{*}=\left(\alpha^{*}, \kappa^{*}, \sigma^{*}, c^{*}, d^{*}, \widehat{\lambda}^{*}\right) \in \mathcal{O}_{p}$ such that $\lim _{k \rightarrow \infty} p_{k}=q^{*}$, then $q=\delta_{q^{*}}$.
Let the space $\mathcal{O}=\mathcal{O}_{p} \times \mathbb{R}_{+}^{2}$. Define a sequence of measure-valued processes by

$$
\begin{equation*}
\nu_{t}^{(K)}=\frac{1}{K} \sum_{k=1}^{K} \delta_{\left(p_{k},\left(Y_{1}^{(k)}, \widetilde{Y}_{1}^{(k)}\right), \xi_{t}^{(k)}\right)} \bar{H}_{t}^{(k)}, \quad t \geq 0 \tag{12}
\end{equation*}
$$

on $\mathcal{B}(\mathcal{O})$. Let $S=\mathcal{P}(\mathcal{O})$ (i.e., the set of all Borel measures $\nu$ on $\mathcal{B}(\mathcal{O})$ such that $\nu(\mathcal{O}) \leq 1$ ). For any smooth function $f \in C^{\infty}(\mathcal{O})$ and $\nu \in S$, define

$$
\nu(f):=\int_{\mathcal{O}} f(p, y, x) \nu(\mathrm{d} p \times \mathrm{d} y \times \mathrm{d} x)
$$

Obviously, it holds that

$$
\begin{equation*}
\nu_{t}^{(K)}(f)=\int_{\mathcal{O}} f(p, y, x) \nu_{t}^{(K)}(\mathrm{d} p \times \mathrm{d} y \times \mathrm{d} x)=\frac{1}{K} \sum_{k=1}^{K} f\left(p_{k},\left(Y_{1}^{(k)}, \tilde{Y}_{1}^{(k)}\right), \xi_{t}^{(k)}\right) \bar{H}_{t}^{(k)}, \quad t \geq 0 \tag{13}
\end{equation*}
$$

Application of Itô's formula yields that, for all $t \geq 0$,

$$
\begin{align*}
\nu_{t}^{(K)}(f)= & \nu_{0}^{(K)}(f)+\int_{0}^{t} \nu_{s}^{(K)}\left(\mathcal{L}_{11} f\right) \mathrm{d} s+\frac{1}{K} \int_{0}^{t} \sum_{k=1}^{K} f\left(p_{k},\left(Y_{1}^{(k)}, \widetilde{Y}_{1}^{(k)}\right), \xi_{s-}^{(k)}\right) \mathrm{d} \bar{H}_{s}^{(k)} \\
& +\frac{1}{K} \int_{0}^{t} \sum_{k=1}^{K}\left[\sigma_{k} \frac{\partial f}{\partial x}\left(p_{k},\left(Y_{1}^{(k)}, \widetilde{Y}_{1}^{(k)}\right), \xi_{s}^{(k)}\right) \bar{H}_{s}^{(k)}\left(\xi_{s}^{(k)}\right)^{\rho} \mathrm{d} W_{s}^{(k)}\right]  \tag{14}\\
& +\frac{1}{K} \int_{0}^{t} \sum_{k=1}^{K}\left(\left[f\left(p_{k},\left(Y_{1}^{(k)}, \widetilde{Y}_{1}^{(k)}\right), \xi_{s-}^{(k)}+d_{k} \widetilde{Y}_{1}^{(k)}\right)-f\left(p_{k},\left(Y_{1}^{(k)}, \widetilde{Y}_{1}^{(k)}\right), \xi_{s-}^{(k)}\right)\right] \bar{H}_{s}^{(k)} \mathrm{d} \widehat{N}_{s}^{(k)}\right) \\
& +\frac{1}{K} \int_{0}^{t}\left(\sum_{k=1}^{K}\left[f\left(p_{k},\left(Y_{1}^{(k)}, \widetilde{Y}_{1}^{(k)}\right), \xi_{s-}^{(k)}+c_{k} Y_{1}^{(k)}\right)-f\left(p_{k},\left(Y_{1}^{(k)}, \widetilde{Y}_{1}^{(k)}\right), \xi_{s-}^{(k)}\right)\right] \bar{H}_{s}^{(k)}\right) \mathrm{d} \widehat{N}_{s}^{(c)}
\end{align*}
$$

where, for $(p, y, x)=(\alpha, \kappa, \sigma, c, d, \widehat{\lambda}, y, x) \in \mathcal{O}$,

$$
\begin{equation*}
\mathcal{L}_{11} f(p, y, x)=\frac{1}{2} \sigma^{2} x^{2 \rho} \frac{\partial^{2} f}{\partial x^{2}}(p, y, x)+(\alpha-\kappa x) \frac{\partial f}{\partial x}(p, y, x) \tag{15}
\end{equation*}
$$

Next, we analyze the jump behavior of the process $\nu^{(K)}(f)=\left(\nu_{t}^{(K)}(f) ; t \geq 0\right)$ using (14). Consider $M$ smooth functions $f_{m} \in C^{\infty}(\mathcal{O})$, where $m=1, \ldots, M$. For each $K \in \mathbb{N}$, we can define the $M$-dimensional stochastic process

$$
\nu_{t}^{(K)}(\boldsymbol{f}):=\left(\nu_{t}^{(K)}\left(f_{1}\right), \ldots, \nu_{t}^{(K)}\left(f_{M}\right)\right), \quad t \geq 0
$$

Since the Poisson processes $\left(\widehat{N}^{(1)}, \ldots, \widehat{N}^{(K)}, \widehat{N}^{(c)}\right)$ are mutually independent, they cannot experience simultaneous jumps. Hence, the above jump decomposition (14) shows that all components of the $M$-dimensional process $\nu^{(K)}(\boldsymbol{f})$ must jump together with positive probability. More precisely, if the $k$-th $(k \in\{1, \ldots, K\})$

Poisson process $\widehat{N}^{(k)}$ jumps at time $t>0$, the corresponding jump amplitude of the $M$ components of the process $\nu^{(K)}(\boldsymbol{f})$ would be

$$
\begin{aligned}
\boldsymbol{J}_{t}^{(K, k)}\left(Y_{1}^{(k)}, \widetilde{Y}_{1}^{(k)}\right):= & \frac{1}{K}\left(\left[f_{1}\left(p_{k},\left(Y_{1}^{(k)}, \widetilde{Y}_{1}^{(k)}\right), \xi_{t-}^{(k)}+d_{k} \widetilde{Y}_{1}^{(k)}\right)-f_{1}\left(p_{k},\left(Y_{1}^{(k)}, \widetilde{Y}_{1}^{(k)}\right), \xi_{t-}^{(k)}\right)\right], \ldots\right. \\
& \left.\ldots,\left[f_{M}\left(p_{k},\left(Y_{1}^{(k)}, \widetilde{Y}_{1}^{(k)}\right), \xi_{t-}^{(k)}+d_{k} \widetilde{Y}_{1}^{(k)}\right)-f_{M}\left(p_{k},\left(Y_{1}^{(k)}, \widetilde{Y}_{1}^{(k)}\right), \xi_{t-}^{(k)}\right)\right]\right) \bar{H}_{t}^{(k)}
\end{aligned}
$$

The probability that the above event occurs is given by

$$
\frac{\widehat{\lambda}_{k}}{\widehat{\lambda}_{c}+\sum_{k=1}^{K} \widehat{\lambda}_{k}}
$$

The other possibility is that the Poisson process $\widehat{N}^{(c)}$, common to the $K$ intensity processes, jumps at time $t>0$. Then the corresponding common jump amplitudes are given by

$$
\begin{array}{r}
\boldsymbol{J}_{t}^{(K, c)}\left(\boldsymbol{Y}_{1}, \tilde{Y}_{1}\right):=\frac{1}{K}\left(\sum_{k=1}^{K}\left[f_{1}\left(p_{k},\left(Y_{1}^{(k)}, \widetilde{Y}_{1}^{(k)}\right), \xi_{t-}^{(k)}+c_{k} Y_{1}^{(k)}\right)-f_{1}\left(p_{k},\left(Y_{1}^{(k)}, \widetilde{Y}_{1}^{(k)}\right), \xi_{t-}^{(k)}\right)\right] \bar{H}_{t}^{(k)}, \ldots,\right. \\
\left.\ldots, \sum_{k=1}^{K}\left[f_{M}\left(p_{k},\left(Y_{1}^{(k)}, \widetilde{Y}_{1}^{(k)}\right), \xi_{t-}^{(k)}+c_{k} Y_{1}^{(k)}\right)-f_{M}\left(p_{k},\left(Y_{1}^{(k)}, \widetilde{Y}_{1}^{(k)}\right), \xi_{t-}^{(k)}\right)\right] \bar{H}_{t}^{(k)}\right),
\end{array}
$$

where $\boldsymbol{Y}_{1}:=\left(Y_{1}^{(1)}, \ldots, Y_{1}^{(K)}\right)$, and $\tilde{\boldsymbol{Y}}_{1}:=\left(\tilde{Y}_{1}^{(1)}, \ldots, \widetilde{Y}_{1}^{(K)}\right)$. The probability of a common jump is given by

$$
\frac{\widehat{\lambda}_{c}}{\widehat{\lambda}_{c}+\sum_{k=1}^{K} \widehat{\lambda}_{k}}
$$

Based on the above analysis of the behavior of systematic and idiosyncratic jumps in the sequence of empirical measure-valued processes, we define, for any smooth function $\varphi \in C^{\infty}\left(\mathbb{R}^{M}\right)$ and $\nu \in S$,

$$
\begin{equation*}
\Phi(\nu)=\varphi(\nu(\boldsymbol{f})) \tag{16}
\end{equation*}
$$

where $\nu(\boldsymbol{f})=\left(\nu\left(f_{1}\right), \ldots, \nu\left(f_{M}\right)\right) \in \mathbb{R}^{M}$. Moreover, for all functions $\Phi$ of the form (16), define the operator

$$
\begin{equation*}
\mathcal{A} \Phi(\nu):=\sum_{m=1}^{M} \frac{\partial \varphi}{\partial x_{m}}(\nu(\boldsymbol{f}))\left(\nu\left(\mathcal{L}_{1} f_{m}\right)+\nu\left(\mathcal{L}_{21} f_{m}\right)+\widehat{\lambda}_{c} \nu\left(\mathcal{L}_{22} f_{m}\right)\right), \quad \nu \in S \tag{17}
\end{equation*}
$$

where for $p=(\alpha, \kappa, \sigma, c, d, \widehat{\lambda}) \in \mathcal{O}_{p}, x \in \mathbb{R}_{+}$and $y=\left(y_{1}, y_{2}\right) \in \mathbb{R}_{+}^{2}$,

$$
\begin{align*}
\mathcal{L}_{1} f(p, y, x) & =\mathcal{L}_{11} f(p, y, x)-x f(p, y, x) \\
\mathcal{L}_{21} f(p, y, x) & =\widehat{\lambda} d y_{2} \frac{\partial f}{\partial x}(p, y, x)  \tag{18}\\
\mathcal{L}_{22} f(p, y, x) & =c y_{1} \frac{\partial f}{\partial x}(p, y, x)
\end{align*}
$$

We recall that the operator $\mathcal{L}_{11}$ has been defined in (15). Let us analyze the operator in (17). The component $\nu\left(\mathcal{L}_{1} f_{m}\right)$ corresponds to the diffusive part of the limit process with a killing rate $x$, while the component $\nu\left(\mathcal{L}_{21} f_{m}\right)$ is related to the individual jumps of the $K$ default intensity processes. Furthermore, the component $\widehat{\lambda}_{c} \nu\left(\mathcal{L}_{22} f_{m}\right)$ corresponds to the common jump of the $K$ default intensity processes with arrival intensity $\widehat{\lambda}_{c}$. This observation is consistent with our original model setup, inclusive of systematic and idiosyncratic jumps.

Then we have the following
Lemma 4.2. The operator $\mathcal{A}$ given by (17) is the generator of our limit martingale problem in the sense of

$$
\begin{equation*}
\lim _{K \rightarrow \infty} \mathbb{E}\left[\left(\Phi\left(\nu_{t_{n+1}}^{(K)}\right)-\Phi\left(\nu_{t_{n}}^{(K)}\right)-\int_{t_{n}}^{t_{n+1}} \mathcal{A} \Phi\left(\nu_{s}^{(K)}\right) \mathrm{d} s\right) \prod_{j=1}^{n} \Psi_{j}\left(\nu_{t_{j}}^{(K)}\right)\right]=0 \tag{19}
\end{equation*}
$$

whenever $0 \leq t_{1}<\cdots<t_{n+1}<+\infty$ and $\Psi_{1}, \ldots, \Psi_{n} \in B(S)$ (all bounded functions on the space $S$ ).
The proof of the above lemma is reported in Appendix B.2
To prove the weak convergence of the measure-valued process $\nu^{(K)}=\left(\nu_{t}^{(K)} ; t \geq 0\right)$ defined by (12) in $D_{S}([0, \infty))$ as $K \longrightarrow \infty$, the following condition is assumed to hold throughout the paper.
(A2) Let $m_{k}^{Y}:=\mathbb{E}\left[\left|Y_{1}^{(k)}\right|^{4}\right]<+\infty$, and $m_{k}^{\tilde{Y}}:=\mathbb{E}\left[\left|\tilde{Y}_{1}^{(k)}\right|^{4}\right]<+\infty$. Assume that the parameters set $\left(\alpha_{k}, \sigma_{k}, c_{k}, d_{k}, \widehat{\lambda}_{k}, m_{k}^{Y}, m_{k}^{\tilde{Y}}, \xi_{0}^{(k)}\right)$ are bounded by a common constant $C_{p}>0$ for all $k \in\{1,2, \ldots, K\}$.

### 4.1 Relative Compactness of $\left(\nu^{(K)} ; K \in \mathbb{N}\right)$

In order to prove the weak convergence of the family of measure-valued processes $\left(\nu^{(K)} ; K \in \mathbb{N}\right)$ defined by (12), we need to check the relative compactness of $\left(\nu^{(K)} ; K \in \mathbb{N}\right)$. Following the standard procedure specified in Chapter 3 of Ethier and Kurtz (1986), it is enough to verify (a) the compact containment condition and (b) the condition (ii) of Theorem 8.6 of Chapter 3 in Ethier and Kurtz (1986).

We first check the compact containment condition of the family of stochastic processes $\left(\nu^{(K)} ; K \in \mathbb{N}\right)$ whose sample paths are in $D_{S}[0, \infty)$. Notice that we consider the convergence of the martingale problem for functions of the form:

$$
\Phi(\nu)=\varphi\left(\nu\left(f_{1}\right), \ldots, \nu\left(f_{M}\right)\right),
$$

where $\varphi \in C^{\infty}\left(\mathbb{R}^{M}\right)$ and $f_{1}, \ldots, f_{M} \in C^{\infty}(\mathcal{O})$. It, thus, suffices to prove the relatively compactness for $\left(\nu^{(K)}(f) ; K \in \mathbb{N}\right)$ as a stochastic process with sample path in $D_{\mathbb{R}}([0, \infty))$, where $f \in C^{\infty}(\mathcal{O})$ (see, e.g. Li $(2010)$ ). We have the following
Lemma 4.3. Let the assumption (A2) hold. For every $T>0$, it holds that for any $f \in C^{\infty}(\mathcal{O})$,

$$
\begin{equation*}
\sup _{K \in \mathbb{N}} \mathbb{P}\left(\sup _{0 \leq t \leq T}\left|\nu_{t}^{(K)}(f)\right| \geq m\right) \longrightarrow 0 \tag{20}
\end{equation*}
$$

as $m \longrightarrow+\infty$.
The proof of Lemma 4.3 is postponed to Appendix B. Eq. (20) implies the following compact containment condition: for every $\eta>0$ and $T>0$, there exists a compact set $\Gamma_{\eta, T} \subset E$ such that

$$
\begin{equation*}
\inf _{K \in \mathbb{N}} \mathbb{P}\left(\nu_{t}^{(K)} \in \Gamma_{\eta, T}, \forall 0 \leq t \leq T\right)>1-\eta \tag{21}
\end{equation*}
$$

holds for the stochastic process $\left(\nu^{(K)}(f) ; K \in \mathbb{N}\right.$ ) for any $f \in C^{\infty}(\mathcal{O})$ (see Remark 7.3 on Page 129 in Ethier and Kurtz (1986)).

Next we prove that (b) the condition (ii) of Theorem 8.6 of Chapter 3 in Ethier and Kurtz (1986) holds. We have the following lemma, whose proof is reported in Appendix B,

Lemma 4.4. Let $h(u, v)=|u-v| \wedge 1$ for any $u, v \in \mathbb{R}$. Then there exists a positive r.v. $H_{K}(\delta)$ with $\lim _{\delta \rightarrow 0} \sup _{K \in \mathbb{N}} \mathbb{E}\left[H_{K}(\delta)\right]=0$ such that for all $0 \leq t \leq T, 0 \leq u \leq \delta$ and $0 \leq v \leq \delta \wedge t$, it holds that

$$
\begin{equation*}
\mathbb{E}\left[h^{2}\left(\nu_{t+u}^{(K)}(f), \nu_{t}^{(K)}(f)\right) \cdot h^{2}\left(\nu_{t}^{(K)}(f), \nu_{t-v}^{(K)}(f)\right) \mid \bigvee_{k=1}^{K} \mathcal{G}_{t}^{(k)}\right] \leq \mathbb{E}\left[H_{K}(\delta) \mid \bigvee_{k=1}^{K} \mathcal{G}_{t}^{(k)}\right] \tag{22}
\end{equation*}
$$

for each $K \in \mathbb{N}$.
Finally, we need the uniqueness of the martingale problem for the generator ( $\mathcal{A}, \delta_{q \times \eta \times \phi_{0}}$ ) given by (17). This result will be used in the next subsection to identify the limit measure-valued process.

Lemma 4.5. The uniqueness of the martingale problem of the generator ( $\mathcal{A}, \delta_{q \times \eta \times \phi_{0}}$ ) given by (17) holds.
The proof of the above lemma is based on the standard dual argument (see Theorem 4.4.2, pag. 184 and Proposition 4.4.7, pag. 189 in Ethier and Kurtz (1986)). Hence, the details are omitted here.

### 4.2 Limit Measure-Valued Process

Theorem 4.6. Let the measure-valued process $\nu^{(K)}=\left(\nu_{t}^{(K)} ; t \geq 0\right)$ be defined as in (12) for each $K \in \mathbb{N}$. Then $\nu^{(K)} \Longrightarrow \nu$ as $K \longrightarrow \infty$ (i.e. in a large portfolio), where the limit measure-valued process $\nu=\left(\nu_{t} ; t \geq\right.$ 0) is given by

$$
\begin{equation*}
\nu_{t}(A \times B \times C)=\int_{\mathcal{O}} \mathbf{1}_{A \times B}(p, y) \mathbb{E}\left[\exp \left(-\int_{0}^{t} X_{s}(\boldsymbol{p}) \mathrm{d} s\right) \mathbf{1}_{\left\{X_{t}(\boldsymbol{p}) \in C\right\}}\right] q(\mathrm{~d} p) \eta(\mathrm{d} y) \phi_{0}(\mathrm{~d} x) \tag{23}
\end{equation*}
$$

with the sets $A \in \mathcal{B}\left(\mathcal{O}_{p}\right), B \in \mathcal{B}\left(\mathbb{R}_{+}\right)$and $C \in \mathcal{B}\left(\mathbb{R}_{+}\right)$. The (random) measures $q(\mathrm{~d} p), \eta(\mathrm{d} y)$, $\phi_{0}(\mathrm{~d} x)$ are given in Assumption (A1). Here, the process $X(\boldsymbol{p})=\left(X_{t}(\boldsymbol{p}) ; t \geq 0\right)$ satisfies the following SDE:

$$
\begin{equation*}
X_{t}(\boldsymbol{p})=x+\int_{0}^{t}\left(D(\boldsymbol{p})+\alpha-\kappa X_{s}(\boldsymbol{p})\right) \mathrm{d} s+\sigma \int_{0}^{t}\left(X_{s}(\boldsymbol{p})\right)^{\rho} \mathrm{d} W_{s} \tag{24}
\end{equation*}
$$

where $\boldsymbol{p}=(p, y, x) \in \mathcal{O}$ is a parameter set, with $y=\left(y_{1}, y_{2}\right) \in \mathbb{R}_{+}^{2}$ and $p=(\alpha, \kappa, \sigma, c, d, \widehat{\lambda}) \in \mathcal{O}_{p}$. Here the elasticity factor $\frac{1}{2} \leq \rho<1$, and $W=\left(W_{t} ; t \geq 0\right)$ is a Brownian motion. Moreover, the drift rate is given by

$$
\begin{equation*}
D(\boldsymbol{p})=d \widehat{\lambda} y_{2}+c \widehat{\lambda}_{c} y_{1} \tag{25}
\end{equation*}
$$

Remark 4.7. We stress that (23) provides an explicit representation of the limit measure valued process. This contrasts with the limit intensity process provided in Giesecke et al. (2013a), where the recursive dependence on a term coming from self-excitation (see their equations (4.3) and (4.4)) prevents an explicit computation of the limiting measure. Considering that our objective is to obtain closed-form expressions for bilateral CVA adjustments, the explicit form of the limit measure-valued process is of crucial importance. As we demonstrate in Section 5, this allows us obtaining semi-closed form representations for the portfolio exposure under general CEV specifications, and closed-form expressions in case of square root diffusion intensities.

Proof of Theorem4.6. As in standard weak convergence analysis procedures (see Chapter 3 of Ethier and Kurtz (1986)), the weak convergence of $\nu^{(K)} \Longrightarrow \nu$ for some measure-valued process $\nu=\left(\nu_{t} ; t \geq 0\right)$ as $K \longrightarrow \infty$ can be obtained by using Lemma 4.2. Lemma 4.3 and Lemma 4.4 Next, we prove that the limit measure-valued process $\nu(\cdot)$ is given by (23).

First, we note that $D(\boldsymbol{p})>0$ for all $\boldsymbol{p} \in \mathcal{O}$. Using Lemma 2.1, we can immediately show that $X_{t}(\boldsymbol{p}) \geq 0$ for all $t \geq 0$. For any smooth function $f \in C^{\infty}(\mathcal{O})$, it holds that

$$
\nu_{t}(f)=\int_{\mathcal{O}} \mathbb{E}\left[e^{-\int_{0}^{t} X_{s}(\boldsymbol{p}) \mathrm{d} s} f\left(p, y, X_{t}(\mathbf{p})\right)\right] q(\mathrm{~d} p) \eta(\mathrm{d} y) \phi_{0}(\mathrm{~d} x), \quad \forall t \geq 0
$$

Using Itô's formula, we have

$$
\begin{aligned}
& \frac{\partial}{\partial t} \mathbb{E}\left[e^{-\int_{0}^{t} X_{s}(\boldsymbol{p}) \mathrm{d} s} f\left(p, y, X_{t}(\mathbf{p})\right)\right] \\
& \quad=\frac{\partial}{\partial t} \mathbb{E}\left[\int_{0}^{t} e^{-\int_{0}^{s} X_{u}(\boldsymbol{p}) \mathrm{d} u}\left(\mathcal{L}_{1} f\left(p, y, X_{s}(\boldsymbol{p})\right)+D(\mathbf{p}) \mathcal{L}_{2} f\left(p, y, X_{s}(\boldsymbol{p})\right)\right) \mathrm{d} s\right] \\
& \quad=\mathbb{E}\left[e^{-\int_{0}^{t} X_{s}(\boldsymbol{p}) \mathrm{d} s}\left(\mathcal{L}_{1} f\left(p, y, X_{t}(\boldsymbol{p})\right)+D(\boldsymbol{p}) \mathcal{L}_{2} f\left(p, y, X_{t}(\boldsymbol{p})\right)\right)\right]
\end{aligned}
$$

Then, we have the equality

$$
\frac{\mathrm{d} \nu_{t}(f)}{\mathrm{d} t}=\nu_{t}\left(\mathcal{L}_{1} f\right)+\nu_{t}\left(\mathcal{L}_{21} f\right)+\widehat{\lambda}_{c} \nu_{t}\left(\mathcal{L}_{22} f\right)
$$

Using (16) and (17), we have that

$$
\begin{aligned}
\frac{\mathrm{d} \Phi\left(\nu_{t}\right)}{\mathrm{d} t} & =\sum_{m=1}^{M} \frac{\partial \varphi}{\partial x_{m}}\left(\nu_{t}(\boldsymbol{f})\right) \frac{\mathrm{d} \nu_{t}\left(f_{k}\right)}{\mathrm{d} t} \\
& =\sum_{m=1}^{M} \frac{\partial \varphi}{\partial x_{m}}\left(\nu_{t}(\boldsymbol{f})\right)\left(\nu_{t}\left(\mathcal{L}_{1} f_{m}\right)+\nu_{t}\left(\mathcal{L}_{21} f_{m}\right)+\widehat{\lambda}_{c} \nu_{t}\left(\mathcal{L}_{22} f_{m}\right)\right) \\
& =\mathcal{A} \Phi\left(\nu_{t}\right)
\end{aligned}
$$

The above equality implies that, for all functions $\Phi$ of the form (16),

$$
\Phi\left(\nu_{t}\right)=\Phi\left(\nu_{s}\right)+\int_{s}^{t} \Phi\left(\nu_{u}\right) \mathrm{d} u, \quad \forall 0 \leq s<t<+\infty
$$

Hence, the measure $\delta_{\nu}(\cdot)$ satisfies the martingale problem for $\left(\mathcal{A}, \delta_{q \times \eta \times \phi_{0}}\right)$, which is given by (17) due to the uniqueness of the martingale problem for the operator $\left(\mathcal{A}, \delta_{q \times \eta \times \phi_{0}}\right)$ (see Lemma 4.5).

### 4.3 Approximating Formula of Exposure in Large Portfolio

The computation of the exposure in Eq. (19) requires evaluating sums of the form:

$$
\frac{1}{K} \sum_{k=1}^{K} a_{k} \bar{H}_{t}^{(k)}
$$

According to Theorem 4.6, we have the following weak convergence as $K \longrightarrow \infty$,

$$
\begin{equation*}
\frac{1}{K} \sum_{k=1}^{K} a_{k} \bar{H}_{t}^{(k)} \Longrightarrow a^{*} \nu_{t}(\mathcal{O}) \quad 0 \leq t \leq T \tag{26}
\end{equation*}
$$

where

$$
\begin{equation*}
\nu_{t}(\mathcal{O})=\int_{\mathcal{O}} \mathbb{E}\left[\exp \left(-\int_{0}^{t} X_{s}(\boldsymbol{p}) \mathrm{d} s\right)\right] q(\mathrm{~d} p) \eta(\mathrm{d} y) \phi_{0}(\mathrm{~d} x), \quad 0 \leq t \leq T \tag{27}
\end{equation*}
$$

with the process $X(\boldsymbol{p})=\left(X_{t}(\boldsymbol{p}) ; t \geq 0\right)$ satisfying the SDE (24), and ( $\left.a_{k} ; k=1, \ldots, K\right)$ is a sequence of real numbers satisfying $a^{*}=\lim _{K \rightarrow \infty} \frac{1}{K} \sum_{k=1}^{K} a_{k}$, with $a^{*}$ being finite. The killing rate process $X(\boldsymbol{p})$ acts as the limit intensity process of the portfolio of $K$-names, as $K \rightarrow \infty$. Accordingly, we use $\tau_{X}^{*}$ to denote the limit default time of the large portfolio associated to the killing rate $X(\boldsymbol{p})$. Hence in a large pool, $K \longrightarrow \infty$, the default indicator $\mathbf{1}_{\left\{\tau_{X}^{*}>t\right\}}$ plays the role of the average default indicator $\frac{1}{K} \sum_{k=1}^{K} \bar{H}_{t}^{(k)}$.

Using the weak convergence result (26) and noticing that weak convergence implies convergence of expectations, we obtain that as $K \longrightarrow \infty$, for each $t \geq 0$, the conditional expectation on the event $\left\{\tau_{X}^{*}>t\right\}$,

$$
\mathbb{E}\left[\left.\frac{1}{K} \sum_{k=1}^{K} \bar{H}_{s}^{(k)} \right\rvert\, \mathcal{G}_{t}^{(K, A, B)}\right], \quad s>t
$$

approaches the function $\widehat{F}(t, s)$ given by, for $0 \leq t<s \leq T$,

$$
\begin{equation*}
\widehat{F}(t, s):=\mathbb{E}\left[\int_{\mathcal{O}} \mathbb{E}\left[\exp \left(-\int_{t}^{s} X_{u}(\boldsymbol{p}) \mathrm{d} u\right)\right] q(\mathrm{~d} p) \eta(\mathrm{d} y) \phi_{0}(\mathrm{~d} x)\right] \tag{28}
\end{equation*}
$$

Hence, on $\left\{\tau_{X}^{*}>t\right\}$, we can characterize the default time of the limiting portfolio in terms of

$$
\begin{equation*}
\mathbb{P}\left(\tau_{X}^{*}>s \mid \mathcal{G}_{t}^{(K, A, B)}\right)=\widehat{F}(t, s), \quad 0 \leq s \leq t \leq T \tag{29}
\end{equation*}
$$

which represents the conditional survival probability of the portfolio when $K \longrightarrow \infty$.
Moreover, recall the formula for the actual exposure given in Eq. (9). Using Eq. (29), on the event $\left\{\tau_{X}^{*}>t\right\}$ as $K \longrightarrow \infty$, we obtain

$$
\begin{equation*}
\frac{\varepsilon^{(K)}(t, T)}{K} \longrightarrow \bar{\varepsilon}^{(*)}(t, T), \quad 0 \leq t \leq T \tag{30}
\end{equation*}
$$

where

$$
\begin{align*}
\bar{\varepsilon}^{(*)}(t, T) & =S_{z}^{*} \int_{t}^{T} D(t, s) \widehat{F}(t, s) \mathrm{d} s+L_{z}^{*} \int_{t}^{T} D(t, s) \mathrm{d} \widehat{F}(t, s) \\
& =L_{z}^{*}[D(t, T) \widehat{F}(t, T)-1]+\left(S_{z}^{*}+r L_{z}^{*}\right) \int_{t}^{T} D(t, s) \widehat{F}(t, s) \mathrm{d} s \tag{31}
\end{align*}
$$

Here $L_{z}^{*}:=\lim _{K \rightarrow \infty} \frac{1}{K} \sum_{k=1}^{K} z_{k} L_{k}$ and $S_{z}^{*}:=\lim _{K \rightarrow \infty} \frac{1}{K} \sum_{k=1}^{K} z_{k} S_{k}$, both assumed to be finite. We recall the reader that $r$ denotes the constant risk-free rate as defined in Section 3 Eq. (31) is obtained using integration by parts, along with the trivial equality $\widehat{F}(t, t)=1$. For future purposes, we introduce the following quantity

$$
\begin{equation*}
B(\kappa, \sigma ; u)=-\frac{2\left(e^{\varpi u}-1\right)}{2 \varpi+(\kappa+\varpi)\left(e^{\varpi u}-1\right)}, \quad 0 \leq u \leq T \tag{32}
\end{equation*}
$$

where $\kappa, \sigma>0$ and $\varpi=\sqrt{\kappa^{2}+2 \sigma^{2}}$.
Remark 4.8. Consider the special case where $z_{k}=1, L_{k}=L, S_{k}=S$ for all $k \in\{1,2, \ldots, K\}$, and

- $\rho=\frac{1}{2}$, i.e. the intensities of the portfolio names are CIR processes.
- $c_{k}=d_{k}=0$ for all $k \in\{1,2, \ldots, K\}$, i.e., the intensities of the CIR processes do not experience jumps.
- there exists $p^{*}=\left(\alpha^{*}, \kappa^{*}, \sigma^{*}, \widehat{\lambda}^{*}\right) \in \mathcal{O}_{p}$ and $x^{*} \in \mathbb{R}_{+}$such that $\lim _{k \rightarrow \infty} p_{k}=p^{*}$ and $\lim _{k \rightarrow \infty} \xi_{0}^{(k)}=x^{*}$.

Under the above assumption, we write the intensity of the $k$-th name as $\xi^{(k)} \sim \operatorname{CIR}\left(\alpha_{k}, \kappa_{k}, \sigma_{k}\right)$ with $k \in$ $\{1,2, \ldots, K\}$. From (7), we have, on the event $\left\{\tau_{X}^{*}>t\right\}$,

$$
\begin{equation*}
\frac{\varepsilon^{(K)}(t, T)}{K}=L\left[D(t, T) F^{(K)}(t, T)-1\right]+(S+r L) \int_{t}^{T} D(t, s) F^{(K)}(t, s) \mathrm{d} s \tag{33}
\end{equation*}
$$

where for $s>t$,

Moreover, using that $\xi^{(k)} \sim \operatorname{CIR}\left(\alpha_{k}, \kappa_{k}, \sigma_{k}\right)$, we obtain

$$
\begin{equation*}
\mathbb{E}\left[F^{(K)}(t, s)\right]=\frac{1}{K} \sum_{k=1}^{K} e^{A^{(k)}(s-t)+B^{(k)}(s-t) \xi_{0}^{(k)}} \tag{35}
\end{equation*}
$$

where $B^{(k)}(u)=B\left(\kappa_{k}, \sigma_{k} ; u\right)$ has been given in (32) and $A^{(k)}(u)=\alpha_{k} \int_{0}^{u} B\left(\kappa_{k}, \sigma_{k} ; v\right) \mathrm{d} v$. Thus $B^{(k)}(u) \longrightarrow$ $B\left(\kappa^{*}, \sigma^{*} ; u\right)$ and $A^{(k)}(u) \longrightarrow \alpha^{*} \int_{0}^{u} B\left(\kappa^{*}, \sigma^{*} ; v\right) \mathrm{d} v$ as $k \longrightarrow \infty$. Hence

$$
\begin{align*}
\lim _{K \rightarrow \infty} \mathbb{E}\left[F^{(K)}(t, s)\right] & =\exp \left(\alpha^{*} \int_{0}^{s-t} B\left(\kappa^{*}, \sigma^{*} ; v\right) \mathrm{d} v+B\left(\kappa^{*}, \sigma^{*} ; s-t\right) x^{*}\right) \\
& =\mathbb{E}\left[\exp \left(-\int_{t}^{s} X_{v}\left(p^{*}, x^{*}\right) \mathrm{d} v\right)\right] \\
& =\mathbb{E}\left[\int_{\mathcal{O}} \exp \left(-\int_{t}^{s} X_{v}(p, x) \mathrm{d} v\right) \delta\left(p-p^{*}\right) \delta\left(x-x^{*}\right) \mathrm{d} p \mathrm{~d} x\right] \\
& =\widehat{F}(t, s) \tag{36}
\end{align*}
$$

where we have used that $D(\boldsymbol{p})=0$, following by the assumption $c^{*}=\lim _{k \rightarrow \infty} c_{k}=0$ and $d^{*}=\lim _{k \rightarrow \infty} d_{k}=$ 0. It follows from (33) that

$$
\begin{equation*}
\lim _{K \rightarrow \infty} \mathbb{E}\left[\frac{\varepsilon^{(K)}(t, T)}{K}\right]=\bar{\varepsilon}^{(*)}(t, T), \quad 0 \leq t \leq T \tag{37}
\end{equation*}
$$

where $\bar{\varepsilon}^{(*)}(t, T)$ is given in Eq. (31).
Eq. (37) shows that, in case when the default intensities do not jump, the actual $K$-names CDS exposure given by (33) is an asymptotically unbiased estimator of the approximate limiting exposure.

## 5 The Bilateral Credit Valuation Adjustment Formula

This section develops a semi-closed form expression for the "law of large numbers" BCVA on a portfolio of credit default swaps contracts. This is obtained by using formula (30) to approximate the actual exposure.

### 5.1 Joint Survival Probability of Counterparties

Recall that the default intensity processes $\xi^{(A)}=\left(\xi_{t}^{(A)} ; t \geq 0\right)$ and $\xi^{(B)}=\left(\xi_{t}^{(B)} ; t \geq 0\right)$ of counterparties $A$ and $B$ are given by (2). The corresponding default times $\tau_{A}$ and $\tau_{B}$ are defined by (3).

Denote by $G\left(t, t_{A}, t_{B}\right)$ the conditional joint survival probability that the counterparties $A$ and $B$ do not default before time $t_{A} \geq t$ and $t_{B} \geq t$ respectively, given that the $K$ names and the counterparties $A$ and $B$ in the portfolio survive up to time $t \geq 0$.

$$
G\left(t, t_{A}, t_{B}\right)=\mathbb{P}\left(\tau_{A}>t_{A}, \tau_{B}>t_{B} \mid \mathcal{G}_{t}^{(K, A, B)}\right)
$$

Lemma 5.1. Let $t \geq 0$. Then for any $t_{A}, t_{B} \geq t$, we have

$$
\begin{equation*}
G\left(t, t_{A}, t_{B}\right)=\left(\prod_{j \in\{1, \ldots, K, A, B\}} H_{t}^{(j)}\right) \mathbb{E}\left[\exp \left(-\int_{t}^{t_{A}} \xi_{s}^{(A)} \mathrm{d} s-\int_{t}^{t_{B}} \xi_{s}^{(B)} \mathrm{d} s\right) \mid \mathcal{F}_{t}^{(K, A, B)}\right] \tag{38}
\end{equation*}
$$

Moreover, the corresponding conditional joint density of $\left(\tau_{A}, \tau_{B}\right)$ is given by

$$
\begin{equation*}
\frac{\partial^{2} G}{\partial t_{A} \partial t_{B}}\left(t, t_{A}, t_{B}\right)=\left(\prod_{j \in\{1, \ldots, K, A, B\}} H_{t}^{(j)}\right) \mathbb{E}\left[\exp \left(-\int_{t}^{t_{A}} \xi_{s}^{(A)} \mathrm{d} s-\int_{t}^{t_{B}} \xi_{s}^{(B)} \mathrm{d} s\right) \xi_{t_{A}}^{(A)} \xi_{t_{B}}^{(B)} \mid \mathcal{F}_{t}^{(K, A, B)}\right] \tag{39}
\end{equation*}
$$

Proof. On $0 \leq t \leq \widetilde{\tau}:=\min _{j \in\{1, \ldots, K, A, B\}} \tau_{j}$, define the survival probability function for all $K$ names and the counterparties $A, B$ as

$$
\widehat{G}\left(t, t_{1}, \ldots, t_{K}, t_{A}, t_{B}\right)=\mathbb{P}\left(\cap_{j=1}^{K}\left\{\tau_{j}>t_{j}\right\} \cap\left\{\tau_{A}>t_{A}, \tau_{B}>t_{B}\right\} \mid \mathcal{G}_{t}^{(K, A, B)}\right)
$$

By Lemma 9.1.2 in Bielecki and Rutkowski (2002) and LemmaC.1 above, we have that, on the event $\{\tilde{\tau}>t\}$,

$$
\begin{equation*}
\widehat{G}\left(t, t, \ldots, t, t_{A}, t_{B}\right)=\left(\prod_{j \in\{1, \ldots, K, A, B\}} H_{t}^{(j)}\right) \mathbb{E}\left[\exp \left(-\int_{t}^{t_{A}} \xi_{s}^{(A)} \mathrm{d} s-\int_{t}^{t_{B}} \xi_{s}^{(B)} \mathrm{d} s\right) \mid \mathcal{F}_{t}^{(K, A, B)}\right] \tag{40}
\end{equation*}
$$

On the other hand, we also have

$$
\widehat{G}\left(t, t, \ldots, t, t_{A}, t_{B}\right)=\left(\prod_{j=1}^{K} H_{t}^{(j)}\right) G\left(t, t_{A}, t_{B}\right)
$$

Hence it holds that for all $t_{A}, t_{B} \geq t$,

$$
\left(\prod_{j=1}^{K} H_{t}^{(j)}\right) G\left(t, t_{A}, t_{B}\right)=\left(\prod_{j \in\{1, \ldots, K, A, B\}} H_{t}^{(j)}\right) \mathbb{E}\left[\exp \left(-\int_{t}^{t_{A}} \xi_{s}^{(A)} \mathrm{d} s-\int_{t}^{t_{B}} \xi_{s}^{(B)} \mathrm{d} s\right) \mid \mathcal{F}_{t}^{(K, A, B)}\right]
$$

which yields (38).

### 5.2 The BCVA Formula

On the event $\{\widetilde{\tau}>t\}$, for sufficiently large $K$, using the BCVA formula given in (10) we obtain

$$
\begin{align*}
B C V A^{(K, *)}(t, T)= & L_{A} \mathbb{E}\left[\mathbf{1}_{\left\{t<\tau_{A} \leq \min \left(\tau_{B}, T\right)\right\}} \mathbf{1}_{\left\{\tau_{A}<\tau_{X}^{*}\right\}} D\left(t, \tau_{A}\right) \varepsilon_{-}^{(K, *)}\left(\tau_{A}, T\right) \mid \mathcal{G}_{t}^{(K, A, B)}\right]  \tag{41}\\
& -L_{B} \mathbb{E}\left[\mathbf{1}_{\left\{t<\tau_{B} \leq \min \left(\tau_{A}, T\right)\right\}} \mathbf{1}_{\left\{\tau_{B}<\tau_{X}^{*}\right\}} D\left(t, \tau_{B}\right) \varepsilon_{+}^{(K, *)}\left(\tau_{B}, T\right) \mid \mathcal{G}_{t}^{(K, A, B)}\right], 0 \leq t \leq T,
\end{align*}
$$

where $\varepsilon^{(K, *)}(t, T):=K \bar{\varepsilon}^{(*)}(t, T)$ is the "law of large numbers" approximation to the exposure in the large CDS portfolio, given by (31).

Remark 5.2. In market language, the first term of Eq. (41) is often referred to as debit valuation adjustment, and denoted by DVA. The second term in the above formula is often referred to as credit valuation adjustment, and denoted by CVA.

For the above bilateral CVA formula (41), we then have the following semi-closed form representation (where the default intensities of the counterparties satisfy the CEV processes given by (2)):

Theorem 5.3. On the event $\left\{\tau_{X}^{*} \wedge \tau_{A} \wedge \tau_{B}>t\right\}$, the BCVA formula in the large portfolio admits the following semi-closed representation:

$$
\begin{equation*}
B C V A^{(K, *)}(t, T)=L_{A} A^{(K, *)}(t, T)-L_{B} B^{(K, *)}(t, T), \quad 0 \leq t \leq T \tag{42}
\end{equation*}
$$

where, on the event $\left\{\tau_{X}^{*} \wedge \tau_{A} \wedge \tau_{B}>t\right\}:$

$$
\begin{align*}
B^{(K, *)}(t, T) & :=\mathbb{E}\left[\mathbf{1}_{\left\{t<\tau_{B} \leq \min \left(\tau_{A}, T\right)\right\}} \mathbf{1}_{\left\{\tau_{B}<\tau_{X}^{*}\right\}} D\left(t, \tau_{B}\right) \varepsilon_{+}^{(K, *)}\left(\tau_{B}, T\right) \mid \mathcal{G}_{t}^{(K, A, B)}\right] \\
& =\int_{t}^{T} D\left(t, t_{B}\right) \varepsilon_{+}^{(K, *)}\left(t_{B}, T\right) \widehat{F}\left(t, t_{B}\right) H_{1}\left(t_{B}-t, \xi_{t}^{(A)}, \xi_{t}^{(B)}\right) \mathrm{d} t_{B} \tag{43}
\end{align*}
$$

and

$$
A^{(K, *)}(t, T):=\mathbb{E}\left[\mathbf{1}_{\left\{t<\tau_{A} \leq \min \left(\tau_{B}, T\right)\right\}} \mathbf{1}_{\left\{\tau_{A}<\tau_{X}^{*}\right\}} D\left(t, \tau_{A}\right) \varepsilon_{-}^{(K, *)}\left(\tau_{A}, T\right) \mid \mathcal{G}_{t}^{(K, A, B)}\right]
$$

$$
\begin{equation*}
=\int_{t}^{T} D\left(t, t_{A}\right) \varepsilon_{-}^{(K, *)}\left(t_{A}, T\right) \widehat{F}\left(t, t_{A}\right) H_{2}\left(t_{A}-t, \xi_{t}^{(A)}, \xi_{t}^{(B)}\right) \mathrm{d} t_{A} \tag{44}
\end{equation*}
$$

Here the conditional survival functions $\widehat{F}\left(t, t_{B}\right)$ and $\widehat{F}\left(t, t_{A}\right)$ are given by (28) and the functions $H_{1}, H_{2}$ are defined as follows: for $x_{A}, x_{B}>0$,

$$
\begin{align*}
H_{1}\left(t_{B}-t, x_{A}, x_{B}\right) & :=\mathbb{E}\left[\exp \left(-\int_{t}^{t_{B}}\left(\xi_{s}^{(A)}+\xi_{s}^{(B)}\right) \mathrm{d} s\right) \xi_{t_{B}}^{(B)} \mid \xi_{t}^{(A)}=x_{A}, \xi_{t}^{(B)}=x_{B}\right] \\
H_{2}\left(t_{A}-t, x_{A}, x_{B}\right) & :=\mathbb{E}\left[\exp \left(-\int_{t}^{t_{A}}\left(\xi_{s}^{(A)}+\xi_{s}^{(B)}\right) \mathrm{d} s\right) \xi_{t_{B}}^{(A)} \mid \xi_{t}^{(A)}=x_{A}, \xi_{t}^{(B)}=x_{B}\right] \tag{45}
\end{align*}
$$

The proof of Theorem 5.3 is reported in Appendix C Both $B^{(K, *)}$ and $A^{(K, *)}$ have an intuitive economic meaning. Indeed, $B^{(K, *)}$ is the integral over time $s \in[t, T]$ of the positive exposure of the investor to the counterparty at time $s$ weighted by the probability that the limiting portfolio as well as both counterparties survive up to time $s$, and the counterparty defaults at time $s$. When multiplied by $L_{B}$, this precisely identifies the CVA contribution to the BCVA adjustment. A symmetric argument holds for the DVA contribution.

From Eq. (42), we deduce that a fully explicit formula for (41) requires a closed-form representation of (i) the expectations $H_{1}, H_{2}$ defined by (45), and (ii) the survival function $\widehat{F}(t, s)$ associated to the limit default time given in Eq. (28). From the definitions of $H_{1}$ and $H_{2}$, we see immediately that explicit representations can only be obtained when the elasticity factor $\widehat{\rho}=\frac{1}{2}$, i.e. the default intensity processes of both counterparties belong to the affine class. Similarly, to evaluate the expectation in (28), we need the limit process to be affine.

Nevertheless, in the general case when $\rho, \widehat{\rho} \in\left[\frac{1}{2}, 1\right)$ it is possible to accurately evaluate such expectations numerically upon writing the corresponding Feynman-Kac representations, and then solving the resulting PDE. We remark that such a PDE would be two dimensional in the case of (45) and one dimensional for (28). Consequently, the PDE solutions can be accurately and efficiently computed via standard finite difference methods. Alternatively, such expectations may be efficiently estimated using Monte-Carlo methods, see for instance the discretized Euler scheme introduced in Andersen and Andreasen (2000). Notice also that the two dimensional CEV process is an important ingredient of the popular SABR model, hence a plethora of methods are available for computing such expectations, given that they naturally arise when pricing options under SABR.

## 6 Explicit Expression of BCVA

We provide closed-form expressions for $H_{1}, H_{2}$, and $\widehat{F}$, which yield an explicit expression for the law of large numbers bilateral CVA formula. To this purpose, we further specialize the model and specify the empirical measures as well as the distribution of the jump sizes. Throughout the section, we set the elasticity factor $\rho=\widehat{\rho}=\frac{1}{2}$, i.e. choose the default intensities of each name in the portfolio, as well as of investor and counterparty to be CIR processes. Besides mathematical tractability such a choice is empirically relevant, considering that square root diffusion models allow for an automatic calibration of the term structure of credit default swaps. Moreover, they can also be used to calibrate option data, such as caps for the interest rate market and options on CDSs. We refer the reader to Brigo and Alfonsi (2004) for further discussions on this aspect.

### 6.1 BCVA Model Specifications

Let $p^{*}$ be such that $\lim _{k \rightarrow \infty} p_{k}=p^{*}$, where $p^{*}=\left(\alpha^{*}, \kappa^{*}, \sigma^{*}, c^{*}, d^{*}, \widehat{\lambda}^{*}\right) \in \mathcal{O}_{p}$ for all $k \in\{1, \ldots, K\}$. Let $x^{*}>0$ be the limit of the initial intensities of all names, i.e. $\lim _{k \rightarrow \infty} \xi_{0}^{(k)}=x^{*}$. Under the above setting, we have that the limiting measure $q(\mathrm{~d} p)$ is a delta function, i.e. $q(\mathrm{~d} p)=\delta\left(p-p^{*}\right) \mathrm{d} p$ and the limit distribution of the time zero intensity is $\phi_{0}(\mathrm{~d} x)=\delta\left(x-x^{*}\right) \mathrm{d} x$. Let $\mu_{Y}$ and $\mu_{\widetilde{Y}}$ be the probability measures on $\mathbb{R}_{+}$associated with $Y$ and $\widetilde{Y}$, respectively. Next, we characterize the limit jump measure $\eta(\mathrm{d} y)$. To this purpose, we firstly state the following lemma:

Lemma 6.1. Define the empirical measure $\mu_{Y}^{K}(x):=\frac{1}{K} \sum_{k=1}^{K} \delta\left(x-Y^{(k)}\right)$ with $x \in \mathbb{R}_{+}$. Then for each $\varrho \in \mathbb{R}, F_{\mu}^{K}(\varrho) \rightarrow F_{Y}(\varrho)$, as $K \rightarrow \infty$, where $F_{\mu}^{K}(\varrho)$ and $F_{Y}(\varrho)$ denote, respectively, the characteristic function of $\mu_{K}$ and $Y^{(1)}$.

Proof. We have that $F_{\mu}^{K}(\varrho)=\int_{0}^{\infty} e^{\mathrm{i} \varrho x} \mu_{K}(\mathrm{~d} x)=\frac{1}{K} \sum_{k=1}^{K} e^{\mathrm{i} \varrho Y^{(k)}}$, where $\mathrm{i}=\sqrt{-1}$. By the strong law of large numbers, we obtain that $\lim _{K \rightarrow \infty} F_{\mu}^{K}(\varrho)=\mathbb{E}\left[e^{\mathrm{i} \varrho Y^{(1)}}\right]$, a.s. hence proving the statement of the lemma.

By the previous lemma, using the Lévy continuity theorem, we obtain that the measure $\mu_{Y}^{K}$ converges weakly to $\mu_{Y}$, the distribution of the r.v. $Y^{(1)}$, i.e. for every $A \in \mathcal{B}((0, x])$ for which $\mu_{Y}(x)=0$, we have that $\lim _{K \rightarrow \infty} \mu_{Y}^{K}(A)=\mu_{Y}(A)$. Similarly, defining the empirical measure $\mu_{\widetilde{Y}}^{K}(x)=\frac{1}{K} \sum_{k=1}^{K} \delta\left(x-\widetilde{Y}^{(k)}\right)$ with $x \in \mathbb{R}_{+}$, we obtain that $\mu_{\widetilde{Y}}^{K}$ converges weakly to $\mu_{\widetilde{Y}}$. Here, we choose the jump measures to be

$$
\begin{equation*}
\mu_{Y}\left(\mathrm{~d} y_{1}\right)=\gamma_{1} e^{-\gamma_{1} y_{1}}, \quad y_{1} \in \mathbb{R}_{+}, \quad \text { and } \quad \mu_{\widetilde{Y}}\left(\mathrm{~d} y_{2}\right)=\gamma_{2} e^{-\gamma_{2} y_{2}}, \quad y_{2} \in \mathbb{R}_{+} \tag{46}
\end{equation*}
$$

where $Y$ and $\widetilde{Y}$ are two independent exponential random variables with parameters $\gamma_{1}, \gamma_{2}>0$. Hence the corresponding limit measure is $\eta(\mathrm{d} y)=\delta\left(y_{1}-Y\right) \delta\left(y_{2}-\widetilde{Y}\right) \mathrm{d} y_{1} \mathrm{~d} y_{2}$.

We take the elasticity factor to be $\rho=\frac{1}{2}$. This yields intensity processes, which are given by square root diffusion processes with jumps. We remark that the latter are heavily used in CVA computations, see for instance Brigo and Pallavicini (2007) and Bielecki et al. (2012).

Next, we specify the distribution of the jump sizes $\left(Y_{1}^{(A)}, Y_{1}^{(B)}\right)$ of the counterparty intensities due to the common Poisson process. Further, we specify the distribution of the jump sizes $\left(\widetilde{Y}_{1}^{(A)}, \widetilde{Y}_{1}^{(B)}\right)$ of the intensities due to the idiosyncratic Poisson processes of the two counterparties. We assume that $\left(Y_{1}^{(A)}, Y_{1}^{(B)}\right)$ is given by a bivariate exponential distribution with parameters $\gamma_{A}, \gamma_{B}, \gamma_{A B}>0$. Hereafter, we write $\left(Y_{1}^{(A)}, Y_{1}^{(B)}\right) \sim \operatorname{BVE}\left(\gamma_{A}, \gamma_{B}, \gamma_{A B}\right)$. As in Marshall and Olkin 1967), we have $Y^{(i)} \sim \exp \left(\gamma_{i}+\gamma_{A B}\right)$ for $i \in\{A, B\}$ and the correlation of $\left(Y_{1}^{(A)}, Y_{1}^{(B)}\right)$ is given by $\rho_{A B}=\frac{\gamma_{A B}}{\gamma_{0}}$, where $\gamma_{0}=\gamma_{A}+\gamma_{B}+\gamma_{A B}$. Moreover, the moment generating function of the bivariate exponential jump $\left(Y_{1}^{(A)}, Y_{1}^{(B)}\right)$ is given by (see Lemma 3.2 in Marshall and Olkin (1967)):

$$
\begin{equation*}
\Phi\left(\theta_{A}, \theta_{B}\right)=\frac{\left(\gamma_{0}-\theta_{A}-\theta_{B}\right)\left(\gamma_{A}+\gamma_{A B}\right)\left(\gamma_{B}+\gamma_{A B}\right)+\theta_{A} \theta_{B} \gamma_{A B}}{\left(\gamma_{0}-\theta_{A}-\theta_{B}\right)\left(\gamma_{A}+\gamma_{A B}-\theta_{A}\right)\left(\gamma_{B}+\gamma_{A B}-\theta_{B}\right)}, \quad \theta_{A}, \theta_{B} \leq 0 \tag{47}
\end{equation*}
$$

It can be further checked that

$$
\begin{align*}
& \frac{\partial \Phi\left(\theta_{A}, \theta_{B}\right)}{\partial \theta_{A}}=\frac{\left(\gamma_{0}-\theta_{B}\right) \theta_{B} \gamma_{A B}\left(\gamma_{A}+\gamma_{A B}-\theta_{A}\right)+\left(\gamma_{0}-\theta_{A}-\theta_{B}\right)^{2} \gamma_{A B}^{*}+\left(\gamma_{0}-\theta_{A}-\theta_{B}\right) \theta_{A} \theta_{B} \gamma_{A B}}{\left(\gamma_{0}-\theta_{A}-\theta_{B}\right)^{2}\left(\gamma_{A}+\gamma_{A B}-\theta_{A}\right)^{2}\left(\gamma_{B}+\gamma_{A B}-\theta_{B}\right)} \\
& \frac{\partial \Phi\left(\theta_{A}, \theta_{B}\right)}{\partial \theta_{B}}=\frac{\left(\gamma_{0}-\theta_{A}\right) \theta_{A} \gamma_{A B}\left(\gamma_{B}+\gamma_{A B}-\theta_{B}\right)+\left(\gamma_{0}-\theta_{A}-\theta_{B}\right)^{2} \gamma_{A B}^{*}+\left(\gamma_{0}-\theta_{A}-\theta_{B}\right) \theta_{A} \theta_{B} \gamma_{A B}}{\left(\gamma_{0}-\theta_{A}-\theta_{B}\right)^{2}\left(\gamma_{B}+\gamma_{A B}-\theta_{B}\right)^{2}\left(\gamma_{A}+\gamma_{A B}-\theta_{A}\right)} \tag{48}
\end{align*}
$$

where $\gamma_{A B}^{*}=\left(\gamma_{A}+\gamma_{A B}\right)\left(\gamma_{B}+\gamma_{A B}\right)$. Similarly, we assume that $\left(\widetilde{Y}_{1}^{(A)}, \widetilde{Y}_{1}^{(B)}\right) \sim \operatorname{BVE}\left(\widetilde{\gamma}_{A}, \widetilde{\gamma}_{B}, \widetilde{\gamma}_{A B}\right)$, where $\widetilde{\gamma}_{A}, \widetilde{\gamma}_{B}, \widetilde{\gamma}_{A B}>0$.

### 6.2 Closed-Form Representation of $\widehat{F}$

We provide a closed-form expression for the time $t$ conditional survival probability $\widehat{F}(t, s)$, defined in Eq. (28). We state the result in the form of the following proposition.
Proposition 6.2. Under the model setting specified in Section 6.1, the function $\widehat{F}$ in the large portfolio admits the following closed-form representation:

$$
\begin{aligned}
\widehat{F}(t, s)= & \exp \left(x^{*} B_{p^{*}}(s-t)+\alpha^{*} \int_{0}^{s-t} B_{p^{*}}(u) \mathrm{d} u\right) \frac{\gamma_{1}}{\gamma_{1}-c^{*} \widehat{\lambda}_{c} \int_{0}^{s-t} B_{p^{*}}(u) \mathrm{d} u} \\
& \times \frac{\gamma_{2}}{\gamma_{2}-d^{*} \widehat{\lambda}^{*} \int_{0}^{s-t} B_{p^{*}}(u) \mathrm{d} u}
\end{aligned}
$$

where $0 \leq t<s \leq T$, and $B_{p^{*}}(u):=B\left(\kappa^{*}, \sigma^{*} ; u\right)$ with $B$ given by Eq. (32).
The proof of the proposition is reported in Appendix D. In the special case where $Y=\widetilde{Y}$, we obtain

$$
\widehat{F}(t, s)=e^{x^{*} B_{p^{*}}(s-t)} \mathbb{E}\left[\exp \left(\left[\alpha^{*}+\left(d^{*} \widehat{\lambda}^{*}+c^{*} \widehat{\lambda}_{c}\right) Y\right] \int_{0}^{s-t} B_{p^{*}}(u) \mathrm{d} u\right)\right]
$$

$$
\begin{equation*}
=\frac{\gamma}{\gamma-\left(d^{*} \hat{\lambda}^{*}+c^{*} \hat{\lambda}_{c}\right) \int_{0}^{s-t} B_{p^{*}}(u) \mathrm{d} u} \exp \left(x^{*} B_{p^{*}}(s-t)+\alpha^{*} \int_{0}^{s-t} B_{p^{*}}(u) \mathrm{d} u\right), \tag{49}
\end{equation*}
$$

where $\gamma=\gamma_{1}=\gamma_{2}$.

### 6.3 Closed-Form Expressions for $H_{1}$ and $H_{2}$

We provide the closed-form representation of the functions $H_{1}$ and $H_{2}$ defined by (45). As stated earlier, we fix $\widehat{\rho}=\frac{1}{2}$ in (2).
Proposition 6.3. Let $\lambda=\widehat{\lambda}_{A}+\widehat{\lambda}_{B}+\widehat{\lambda}_{c}$. Then, the functions $H_{1}$ and $H_{2}$ defined in (45) admit the following explicit representations:

- For $t \leq t_{B} \leq T$ and $\left(x_{A}, x_{B}\right) \in \mathbb{R}_{+}^{2}$, the function $H_{1}$ admits the closed-form:

$$
\begin{align*}
H_{1}\left(t_{B}-t, x_{A}, x_{B}\right)= & {\left[h_{1}\left(t_{B}-t\right)+h_{A}\left(t_{B}-t\right) x_{A}+h_{B}\left(t_{B}-t\right) x_{B}\right] } \\
& \times \exp \left(\widehat{h}_{1}\left(t_{B}-t\right)+\widehat{h}_{A}\left(t_{B}-t\right) x_{A}+\widehat{h}_{B}\left(t_{B}-t\right) x_{B}\right), \tag{50}
\end{align*}
$$

where the functions $\left(\widehat{h}_{1}(u), \widehat{h}_{A}(u), \widehat{h}_{B}(u)\right)$ are given by: for $0 \leq u \leq t_{B}$,

$$
\begin{align*}
\widehat{h}_{A}(u)= & B\left(\kappa_{A}, \sigma_{A} ; u\right), \\
\widehat{h}_{B}(u)= & B\left(\kappa_{B}, \sigma_{B} ; u\right), \\
\widehat{h}_{1}(u)= & \int_{0}^{u}\left[\alpha_{A} \widehat{h}_{A}(v)+\alpha_{B} \widehat{h}_{B}(v)+\widehat{\lambda}_{c} \Phi\left(c_{A} \widehat{h}_{A}(v), c_{B} \widehat{h}_{B}(v)\right),\right.  \tag{51}\\
& \left.\quad+\widehat{\lambda}_{A} \widetilde{\Phi}\left(d_{A} \widehat{h}_{A}(v), 0\right)+\widehat{\lambda}_{B} \widetilde{\Phi}\left(0, d_{B} \widehat{h}_{B}(v)\right)\right] \mathrm{d} v-\lambda u,
\end{align*}
$$

with $B(\cdot)$ specified in (32) and the functions $\left(h_{1}(u), h_{A}(u), h_{B}(u)\right)$ given by: for $0 \leq u \leq t_{B}$,

$$
\begin{align*}
h_{A}(u) \equiv & 0, \\
h_{B}(u)= & \exp \left(-\kappa_{B} u+\sigma_{B}^{2} \int_{0}^{u} \widehat{h}_{B}(v) \mathrm{d} v\right),  \tag{52}\\
h_{1}(u)= & \int_{0}^{u}\left[\alpha_{B} h_{B}(v)+\widehat{\lambda}_{c} c_{B} h_{B}(v) \frac{\partial \Phi\left(c_{A} \widehat{h}_{A}(v), c_{B} \widehat{h}_{B}(v)\right)}{\partial \theta_{B}}\right. \\
& \left.\quad+\widehat{\lambda}_{B} d_{B} h_{B}(v) \frac{\partial \widetilde{\Phi}\left(0, d_{B} \widehat{h}_{B}(v)\right)}{\partial \theta_{B}}\right] \mathrm{d} v .
\end{align*}
$$

- For $t \leq t_{A} \leq T$ and $\left(x_{A}, x_{B}\right) \in \mathbb{R}_{+}^{2}$, the function $H_{2}$ admits the closed-form:

$$
\begin{align*}
H_{2}\left(t_{A}-t, x_{A}, x_{B}\right)= & {\left[w_{1}\left(t_{A}-t\right)+w_{A}\left(t_{A}-t\right) x_{A}+w_{B}\left(t_{A}-t\right) x_{B}\right] } \\
& \times \exp \left(\widehat{w}_{1}\left(t_{A}-t\right)+\widehat{w}_{A}\left(t_{A}-t\right) x_{A}+\widehat{w}_{B}\left(t_{A}-t\right) x_{B}\right), \tag{53}
\end{align*}
$$

where the functions $\left(\widehat{w}_{1}(u), \widehat{w}_{A}(u), \widehat{w}_{B}(u)\right)$ are given by: for $0 \leq u \leq t_{A}$,

$$
\begin{align*}
\widehat{w}_{A}(u)= & B\left(\kappa_{A}, \sigma_{A} ; u\right), \\
\widehat{w}_{B}(u)= & B\left(\kappa_{B}, \sigma_{B} ; u\right), \\
\widehat{w}_{1}(u)= & \int_{0}^{u}\left[\alpha_{A} \widehat{w}_{A}(v)+\alpha_{B} \widehat{w}_{B}(v)+\widehat{\lambda}_{c} \Phi\left(c_{A} \widehat{w}_{A}(v), c_{B} \widehat{w}_{B}(v)\right),\right.  \tag{54}\\
& \left.\quad+\widehat{\lambda}_{A} \widetilde{\Phi}\left(d_{A} \widehat{w}_{A}(v), 0\right)+\widehat{\lambda}_{B} \widetilde{\Phi}\left(0, d_{B} \widehat{w}_{B}(v)\right)\right] \mathrm{d} v-\lambda u,
\end{align*}
$$

with the functions $\left(w_{1}(u), w_{A}(u), w_{B}(u)\right)$ given by: for $0 \leq u \leq t_{A}$,

$$
\begin{align*}
w_{A}(u) \equiv & \exp \left(-\kappa_{A} u+\sigma_{A}^{2} \int_{0}^{u} \widehat{w}_{A}(v) \mathrm{d} v\right), \\
w_{B}(u) \equiv & 0,  \tag{55}\\
w_{1}(u)= & \int_{0}^{u}\left[\alpha_{A} w_{A}(v)+\widehat{\lambda}_{c} c_{A} w_{A}(v) \frac{\partial \Phi\left(c_{A} \widehat{w}_{A}(v), c_{B} \widehat{w}_{B}(v)\right)}{\partial \theta_{A}}\right. \\
& \left.\quad+\widehat{\lambda}_{A} d_{A} w_{A}(v) \frac{\partial \widetilde{\Phi}\left(d_{A} \widehat{w}_{A}(v), 0\right)}{\partial \theta_{A}}\right] \mathrm{d} v .
\end{align*}
$$

The proof of the above proposition is postponed to Appendix $\mathbb{F}$
The results derived in propositions 6.2 and 6.3 along with Theorem 5.3 yield a closed-form expression for the BCVA formula.

## 7 Numerical and Economic Analysis

We provide a numerical and economic analysis of the BCVA formula. We first analyze the accuracy of the weak limit approximation to the exposure in Subsection 7.1. We then perform a comparative statics analysis of the BCVA adjustment in Subsection 7.2

### 7.1 Quality of Exposure Approximation

We assess the quality of the exposure approximation given by Eq. (30) and (49). We set $K=300$, i.e. sufficiently large, and analyze how our approximate formula for the exposure compares versus the corresponding Monte-Carlo estimate. The latter is obtained by first simulating the multivariate $K+2$ intensity process, including all names in the portfolio and the two counterparties via the Euler scheme, see Andersen et al. (2010). Let us denote by $\tilde{\xi}_{t}^{(k), m}$ the value of $\xi_{t}^{(k)}$ on the $m$-th simulation path. Then, we compute the Monte-Carlo exposure on $\left\{\widetilde{\tau}_{K}>t\right\}$, as

$$
\begin{aligned}
\frac{\tilde{\varepsilon}^{(K)}(t, T)}{K}= & \frac{1}{M} \sum_{m=1}^{M} \frac{1}{K} \sum_{k=1}^{K} z_{k} C D S^{(k)}(t, T) \\
= & \frac{1}{M} \sum_{m=1}^{M}\left[e^{-r(T-t)} \frac{1}{K} \sum_{k=1}^{K} z_{k} L_{k} H_{G}\left(T ; t, \tilde{\xi}_{t}^{(k), m}\right)-\frac{1}{K} \sum_{k=1}^{K} z_{k} L_{k}\right] \\
& +\frac{1}{M} \sum_{m=1}^{M} \int_{t}^{T} e^{-r(s-t)} \frac{1}{K} \sum_{k=1}^{K} z_{k}\left(S_{k}+r L_{k}\right) H_{G}\left(s ; t, \tilde{\xi}_{t}^{(k), m}\right) \mathrm{d} s
\end{aligned}
$$

where the function

$$
H_{G}\left(s ; t, \tilde{\xi}_{t}^{(k), m}\right)=\exp \left(A_{0}^{(k)}(s-t)+B_{0}^{(k)}(s-t) \tilde{\xi}_{t}^{(k), m}\right), \quad s \geq t \geq 0
$$

Here, $B_{0}^{(k)}(u)=B\left(\kappa_{k}, \sigma_{k} ; u\right)$ is given by (32), and the function

$$
A_{0}^{(k)}(u)=\int_{0}^{u}\left[\alpha_{k} B_{0}^{(k)}(v)+\left(\widehat{\lambda}_{c}+\widehat{\lambda}_{k}\right)\left(\frac{\gamma}{\gamma-c_{k} B_{0}^{(k)}(v)}-1\right)\right] \mathrm{d} v
$$

We fix $\alpha^{*}=x^{*} \kappa^{*}, c^{*}=d^{*}, \lambda^{*}=0.5$. Further, $\gamma_{1}=\gamma_{2}=1.5, \widehat{\lambda}_{c}=2.5, r=0.03$. We set $S_{z}^{*}=0.02, L_{z}^{*}=0.4$. We choose $L_{A}=L_{B}=0.4$. Next, we define the sequence of credit risk and contractual parameters of the $K$ names in the portfolio as

$$
\begin{aligned}
\xi_{0}^{(k)} & =x^{*}\left(1+\frac{1}{k}\right) & & \alpha_{k}=\alpha^{*}\left(1+\frac{1}{k}\right) \\
\kappa_{k} & =\kappa^{*}\left(1+\frac{1}{k}\right) & & \sigma_{k}=\sigma^{*}\left(1+\frac{1}{k}\right) \\
c_{k} & =c^{*}\left(1+\frac{1}{k}\right) & & d_{k}=d^{*}\left(1+\frac{1}{k}\right) \\
S_{k} & =S^{*}\left(1+\frac{1}{k}\right) & & L_{k}=L^{*}\left(1-\frac{1}{k}\right) \\
\hat{\lambda}_{k} & =\widehat{\lambda}^{*}\left(1+\frac{1}{k}\right) & &
\end{aligned}
$$

Notice that all parameters are decreasing to their limit, except for $L_{k}$, which increases to its limit (this is done to maintain the realistic assumption that $L_{k} \leq 1$ ). We assume that the counterparty $A$ is equally long on each contract, i.e. $z_{k}=1$.


Figure 1: Monte-Carlo exposure estimate with $K$ names in the portfolio versus "law of large numbers" exposure. The left top panel is associated with $\left(x^{*}, \kappa^{*}, \sigma^{*}, c^{*}, d^{*}\right)=(0.02,0.5,0.01,0,0)$. The left right panel is associated with $\left(x^{*}, \kappa^{*}, \sigma^{*}, c^{*}, d^{*}\right)=(0.5,1.5,0.2,0,0)$. The bottom top panel is associated with $\left(x^{*}, \kappa^{*}, \sigma^{*}, c^{*}, d^{*}\right)=(0.02,0.5,0.2,0.2,0.2)$. The bottom right panel is associated with $\left(x^{*}, \kappa^{*}, \sigma^{*}, c^{*}, d^{*}\right)=$ (0.5, 1.5, 0.2, 0.2, 0.2).


Figure 2: CVA adjustments with respect to $\sigma^{*}$.

Figure 1 shows how the approximation behaves under different configuration settings, where we vary the level of default risk, and allow or not for the presence of jumps in the intensity processes. Figure $\rrbracket$ clearly shows that as time approaches expiration, the exposure decreases in magnitude since both counterparties are exposed to default risk for a shorter time horizon. The top panels show that both in the case when the exposure is positive or negative for the investor, the approximation formula yields very accurate results when compared to the Monte-Carlo estimate. The bottom panels indicate that if the default risk is low (bottom left), then the presence of jumps introduces a small approximation error if the time to horizon is large. However, the mismatch decreases fast and disappears as the time to the horizon decreases. If the portfolio is instead very risky (bottom right panel), then our approximation formula exbibits a perfect match with the Monte-Carlo estimate.

### 7.2 BCVA Economic Analysis

We analyze the behavior of CVA and DVA adjustments computed using formula (41). We use the following parameters for counterparties $A$ and $B: \gamma_{A}=\gamma_{B}=\tilde{\gamma}_{A}=\tilde{\gamma}_{B}=1.5, \gamma_{A B}=\tilde{\gamma}_{A B}=0, \widehat{\lambda}_{A}=\widehat{\lambda}_{B}=0.4, c_{A}=$ $d_{A}=c_{B}=d_{B}=0.3, L_{A}=L_{B}=0.4, \xi_{0}^{(A)}=\xi_{0}^{(B)}=0.2, \kappa_{A}=\kappa_{B}=0.6, \sigma_{A}=\sigma_{B}=0.3, \alpha_{A}=\alpha_{B}=0.4$, $\gamma=2$. We fix the time horizon $T=3$, and assume the following limit parameters for the intensity process: $c^{*}=d^{*}=0.1, \kappa^{*}=0.5, \alpha^{*}=0.01, \sigma^{*}=0.3, \widehat{\lambda}^{*}=0.2, x^{*}=0.02$. The limiting contractual parameters of the CDS are set to $L^{*}=0.4, S^{*}=0.02$. Further, we assume $z_{k}=1$, i.e $A$ is long and $B$ is short on each CDS contract.

Under the above scenario, the trading counterparties have a symmetric credit risk profile, and are riskier than the names in the underlying portfolio. Hence we expect the on-default exposure to be non-negligible. Figure 2 shows that CVA adjustments are increasing in the volatility $\sigma^{*}$. This is expected because larger volatility increases the credit risk of the names in the portfolio, hence increasing the exposure of $A$ to $B$, and resulting in larger CVA adjustments. As the CDS portfolio is always in the money for $A$, and out of money for $B$, DVA adjustments are zero and hence not reported here. When the intensity of the common Poisson process increases, the CVA adjustments decrease. This happens because, when common jumps are more likely to occur, the default likelihood of both portfolio names and counterparties increase. Hence, a smaller number of names would default after either counterparty, implying reduced market exposures and smaller CVA adjustments.

Figure 3 shows that the CVA slightly decreases when the credit risk volatility of the counterparty $B$ increases. When the default intensities are mainly driven by the common Poisson process, i.e. $\widehat{\lambda}_{c}$ is large, then (1) all CDS contracts become less in the money for $A$ and more in the money for $B$, and (2) a larger number of names anticipates $B$ in defaulting. All this reduces the positive on-default exposure of $A$ to $B$. The conclusion is that the CVA decreases, as also confirmed from the graph of Figure 3.

Figure Th $^{2}$ hows that when the underlying portfolio becomes riskier, bilateral counterparty risk inverts the sign. Indeed it changes from being negative (positive CVA and negligible DVA adjustments) to being positive (positive DVA and negligible CVA adjustments). Indeed, when the intensity of the common Poisson process is low, the portfolio default risk is small, and hence the counterparty A would always measure a positive on-default exposure to $B$. As jumps of the common Poisson process occur more frequently, the situation


Figure 3: CVA adjustments with respect to the volatility $\sigma_{B}$.


Figure 4: CVA and DVA adjustments with respect to the intensity $\widehat{\lambda}_{c}$.
reverses and $B$ would measure a positive on-default exposure to $A$. Consequently, the CVA adjustments will become negligible, while the DVA adjustments increasingly higher.

Despite the long counterparty $A$ benefits from his own default when the credit risk level in the underlying portfolio increases (larger DVA adjustments), we next demonstrate that if the credit risk of the names becomes significantly higher than the one of either counterparty, both of them will measure a small exposure to the other. This is illustrated next, by considering a high risk CDS portfolio, with $x^{*}=10, \alpha^{*}=5$ (the remaining parameters are left unchanged). We analyze the impact caused on the adjustments by increasingly larger jump sizes experienced by the intensities of the portfolio names. Clearly, the high default risk of the portfolio makes the on-default exposure of $A$ to $B$ negative, i.e. the portfolio is out-of-money for $A$, which is receiving too a low CDS premium. This immediately explains why the CVA in the left panel of Figure 5 is zero. Although a larger number of portfolio names default before both counterparties, when $A$ defaults his exposure to $B$ is highly negative due to the high default risk of the CDS portfolio. Hence, $A$ would still benefit from his own default, which result in a sizeable DVA adjustment. As $c^{*}$ increases, the intensities of the portfolio names will experience higher jumps, and thus a significantly larger fraction would default before A. As evidenced from Figure 5 this reduces the size of DVA adjustments, especially if the idiosyncratic jumps occur at a high frequency.

## 8 Conclusions

We have developed a rigorous analytical framework for computing the bilateral CVA adjustment on a portfolio of credit default swaps. In the case when the portfolio consists of an asymptotically large number of credit default swaps, we have explicitly characterized the mark-to-market exposure. We have achieved that by means of a weak convergence analysis, relying on martingale arguments, showing that the aggregated intensity process can be recovered as the weak limit of a sequence of weighted empirical measure-valued processes. Using this result, we have provided a semi-closed-form expression for the BCVA adjustment un-


Figure 5: CVA and DVA adjustments with respect to the limiting jump size $c^{*}$.
der a conditionally independent default correlation model, where default intensities follow CEV processes with jumps. By further specializing the CEV to a CIR process enhanced with jumps, we have provided fully explicit expressions for the BCVA, by suitably combining our law of large numbers approximation formula for the exposure and the theory of affine processes. We have provided a detailed numerical analysis to measure the quality of the weak-limit approximation and a comparative statics analysis to interpret the financial meaning of the derived BCVA formula. We have found that our law of large numbers approximation for the portfolio exposure is accurate, regardless of the credit risk levels of the portfolio. From a financial perspective, the CVA adjustments in the large CDS portfolio are increasing in the credit risk volatility of the portfolio names, and highly sensitive to default correlation.

## A Proof of Lemma 2.1

It can be easily checked that the drift and volatility coefficients satisfy the Yamada \& Watanabe condition of Proposition 2.13, Chapter 5 in Karatzas and Shreve (1991). An application of such proposition yields the existence of a unique non-explosive strong solution to the SDE (11).

Next we prove the positivity of this strong solution. Due to the existence of only positive jumps in (1), it is enough to check that the intensity $\xi^{(k)}$ stays positive when $c_{k}=d_{k}=0$. Let $\widetilde{\xi}^{(k)}$ be the corresponding (strong) solution to SDE (11) with $c_{k}=d_{k}=0$. Let $\ell^{a+}(M)=\left(\ell_{t}^{a+}(M) ; t \geq 0\right)$ be the upper local time process of any continuous semi-martingale $M=\left(M_{t} ; t \geq 0\right)$ concentrated on point $a \in \mathbb{R}$. Then the upper local time process $\ell^{a+}(M)$ can be identified by

$$
\ell_{t}^{a+}(M)=\lim _{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_{0}^{t} \mathbf{1}_{\left\{a \leq M_{s}<a+\varepsilon\right\}} \mathrm{d}\langle M, M\rangle_{s}, \quad \forall t \geq 0
$$

We next verify that the upper local time process $\ell^{0+}\left(\widetilde{\xi}^{(k)}\right)$ of the continuous semi-martingale $\widetilde{\xi}^{(k)}$ concentrated on point 0 is zero. When $\rho>\frac{1}{2}$, for all positive $t>0$ and $\varepsilon>0$, we have

$$
\frac{1}{\varepsilon} \int_{0}^{t} \mathbf{1}_{\left\{0 \leq \widetilde{\xi}^{(k)}<\varepsilon\right\}} \mathrm{d}\left\langle\widetilde{\xi}^{(k)}, \widetilde{\xi}^{(k)}\right\rangle_{s}=\frac{\sigma_{k}^{2}}{\varepsilon} \int_{0}^{t} \mathbf{1}_{\left\{0 \leq \widetilde{\xi}^{(k)}<\varepsilon\right\}}\left(\widetilde{\xi}^{(k)}\right)^{2 \rho} \mathrm{~d} s \leq \sigma_{k}^{2} \varepsilon^{2 \rho-1} t
$$

which approaches zero as $\varepsilon \downarrow 0$. This shows that the upper local time process $\ell^{0+}\left(\widetilde{\xi}^{(k)}\right) \equiv 0$ when $\rho>\frac{1}{2}$. For $\rho=\frac{1}{2}$, using the occupation time formula, we get

$$
\begin{aligned}
\int_{\mathbb{R}} \frac{1}{|a|} \mathbf{1}_{\{a \neq 0\}} \ell_{t}^{a+}\left(\widetilde{\xi}^{(k)}\right) \mathrm{d} a & =\sigma_{k}^{2} \int_{0}^{t} \frac{1}{\left|\widetilde{\xi}^{(k)}\right|} \mathbf{1}_{\left\{\left|\widetilde{\xi}^{(k)}\right|>0\right\}}\left|\widetilde{\xi}^{(k)}\right|^{2 \rho} \mathrm{~d} s \\
& =\sigma_{k}^{2} \int_{0}^{t} \mathbf{1}_{\left\{\left|\widetilde{\xi}^{(k)}\right|>0\right\}} \mathrm{d} s \leq \sigma_{k}^{2} t, \quad \forall t \geq 0
\end{aligned}
$$

Note that $|a|^{-1}$ is not integrable in any neighborhood of $a=0$. Then it must hold that $\ell_{t}^{0+}\left(\widetilde{\xi}^{(k)}\right)=0$ for all $t \geq 0$. Using Tanaka's formula, it follows that

$$
\mathbb{E}\left[\left(\widetilde{\xi}_{t \wedge \varsigma_{m}}^{(k)}\right)_{-}\right]=\mathbb{E}\left[\left(\xi_{0}^{(k)}\right)_{-}\right]-\mathbb{E}\left[\int_{0}^{t \wedge s_{m}} \mathbf{1}_{\left\{\widetilde{\xi}_{s}^{(k)} \leq 0\right\}} \mathrm{d} \widetilde{\xi}_{s}^{(k)}\right]+\frac{1}{2} \mathbb{E}\left[\ell_{t \wedge s_{m}}^{0+}\left(\widetilde{\xi}^{(k)}\right)\right]
$$

$$
\begin{aligned}
& =-\alpha_{k} \mathbb{E}\left[\int_{0}^{t \wedge \varsigma_{m}} \mathbf{1}_{\left\{\widetilde{\xi}_{s}^{(k)} \leq 0\right\}} \mathrm{d} s\right]+\kappa_{k} \mathbb{E}\left[\int_{0}^{t \wedge \varsigma_{m}} \mathbf{1}_{\left\{\widetilde{\xi}_{s}^{(k)} \leq 0\right\}} \widetilde{\xi}_{s}^{(k)} \mathrm{d} s\right] \\
& \leq 0, \quad \forall t \geq 0,
\end{aligned}
$$

where $\varsigma_{m}=\inf \left\{t>0 ;\left|\widetilde{\xi}^{(k)}\right| \geq m\right\}$ with $m \in \mathbb{N}$. This implies that $\widetilde{\xi}_{t \wedge \varsigma_{m}}^{(k)} \geq 0, \mathbb{P}$-a.s. for each $m \in \mathbb{N}$. Letting $m \rightarrow+\infty$, we conclude that $\widetilde{\xi}^{(k)} \geq 0$ for all $t \geq 0$.

By virtue of the Feller boundary classification criteria, we have that the boundary 0 is unattainable for $\widetilde{\xi}^{(k)}$ when $\rho>\frac{1}{2}$. Thus the proof of the lemma is complete.

## B Proofs Related to Weak Convergence Analysis

## B. 1 Moment Estimate of Intensities of $K$-Names

Recall that the intensity process $\xi^{(k)}=\left(\xi_{t}^{(k)} ; t \geq 0\right)$ of the $k$-th name follows the CEV process with jumps in (1).
Lemma B.1. Let the assumption (A2) hold. Then for any $T>0$,

$$
\begin{equation*}
\sup _{0 \leq t \leq T, K \in \mathbb{N}} \frac{1}{K} \sum_{k=1}^{K} \mathbb{E}\left[\left|\xi_{t}^{(k)}\right|^{\beta}\right]<+\infty, \tag{B.1}
\end{equation*}
$$

where $1 \leq \beta \leq 4$.
The proof procedure for the moment estimates (B.1) follows straightforward arguments. First, we apply Itô's formula, and then use Hölder inequality, BDG inequality and Gronwall Lemma. The full details are omitted here.

## B. 2 Proof of Lemma 4.2

It follows from the definition (3) of default times, that for each $k \in\{1,2, \ldots, K\}$

$$
\begin{equation*}
\mathcal{M}_{t}^{(k)}:=H_{t}^{(k)}-\int_{0}^{t} \bar{H}_{s}^{(k)} \xi_{s}^{(k)} \mathrm{d} s, \quad t \geq 0 \tag{B.2}
\end{equation*}
$$

is a $\left(\mathbb{P}, \mathcal{G}_{t}^{(k)}\right)$-martingale. Hence the third term on the r.h.s. of Eq. (14) may be rewritten as

$$
\begin{aligned}
& \frac{1}{K} \int_{0}^{t} \sum_{k=1}^{K} f\left(p_{k},\left(Y_{1}^{(k)}, \widetilde{Y}_{1}^{(k)}\right), \xi_{s-}^{(k)}\right) \mathrm{d} \bar{H}_{s}^{(k)}=-\frac{1}{K} \int_{0}^{t} \sum_{k=1}^{K} f\left(p_{k},\left(Y_{1}^{(k)}, \widetilde{Y}_{1}^{(k)}\right), \xi_{s-}^{(k)}\right) \mathrm{d} H_{s}^{(k)} \\
& \quad=-\frac{1}{K} \int_{0}^{t} \sum_{k=1}^{K} f\left(p_{k},\left(Y_{1}^{(k)}, \widetilde{Y}_{1}^{(k)}\right), \xi_{s-}^{(k)}\right) \mathrm{d} \mathcal{M}_{s}^{(k)}-\frac{1}{K} \int_{0}^{t} \sum_{k=1}^{K} \xi_{s}^{(k)} f\left(p_{k},\left(Y_{1}^{(k)}, \widetilde{Y}_{1}^{(k)}\right), \xi_{s-}^{(k)}\right) \bar{H}_{s}^{(k)} \mathrm{d} s .
\end{aligned}
$$

Thus, we can conclude that there exists a (local) martingale $\widehat{\mathcal{M}}^{(K)}=\left(\widehat{\mathcal{M}}_{t}^{(K)} ; t \geq 0\right)$ such that

$$
\begin{aligned}
& \Phi\left(\nu_{t}^{(K)}\right)= \Phi\left(\nu_{0}^{(K)}\right)+\sum_{m=1}^{M} \int_{0}^{t} \frac{\partial \varphi}{\partial x_{m}}\left(\nu_{s}^{(K)}(\boldsymbol{f})\right) \nu_{s}^{(K)}\left(\mathcal{L}_{11} f_{m}\right) \mathrm{d} s+\widehat{\mathcal{M}}_{t}^{(K)} \\
&+ \frac{1}{2 K^{2}} \sum_{m, n=1}^{M} \int_{0}^{t} \frac{\partial^{2} \varphi}{\partial x_{m} \partial x_{n}}\left(\nu_{s}^{(K)}(\boldsymbol{f})\right) \\
& \times\left(\sum_{k=1}^{K} \sigma_{k}^{2} \frac{\partial f_{m}}{\partial x}\left(p_{k},\left(Y_{1}^{(k)}, \widetilde{Y}_{1}^{(k)}\right), \xi_{s}^{(k)}\right) \frac{\partial f_{n}}{\partial x}\left(p_{k},\left(Y_{1}^{(k)}, \widetilde{Y}_{1}^{(k)}\right), \xi_{s}^{(k)}\right)\left(\xi_{s}^{(k)}\right)^{2 \rho} \bar{H}_{s}^{(k)}\right) \mathrm{d} s \\
& \quad-\frac{1}{K} \sum_{m=1}^{M} \int_{0}^{t} \frac{\partial \varphi}{\partial x_{m}}\left(\nu_{s}^{(K)}(\boldsymbol{f})\right)\left(\sum_{k=1}^{K} \xi_{s}^{(k)} f_{m}\left(p_{k},\left(Y_{1}^{(k)}, \widetilde{Y}_{1}^{(k)}\right), \xi_{s}^{(k)}\right) \bar{H}_{s}^{(k)}\right) \mathrm{d} s \\
&+ \sum_{k=1}^{K} \widehat{\lambda}_{k} \int_{0}^{t}\left[\varphi\left(\nu_{s}^{(K)}(\boldsymbol{f})+\boldsymbol{J}_{s}^{(K, k)}\left(Y_{1}^{(k)}, \widetilde{Y}_{1}^{(k)}\right)\right)-\varphi\left(\nu_{s}^{(K)}(\boldsymbol{f})\right)\right] \mathrm{d} s
\end{aligned}
$$

$$
\begin{equation*}
+\widehat{\lambda}_{c} \int_{0}^{t}\left[\varphi\left(\nu_{s}^{(K)}(\boldsymbol{f})+\boldsymbol{J}_{s}^{(K, c)}\left(\boldsymbol{Y}_{1}, \widetilde{\boldsymbol{Y}}_{1}\right)\right)-\varphi\left(\nu_{s}^{(K)} \boldsymbol{f}\right)\right] \mathrm{d} s, \tag{B.3}
\end{equation*}
$$

where $t \geq 0$. Notice that the third line of the above equation may be rewritten as

$$
-\sum_{m=1}^{M} \int_{0}^{t} \frac{\partial \varphi}{\partial x_{m}}\left(\nu_{s}^{(K)}(\boldsymbol{f})\right) \nu_{s}^{(K)}\left(\chi_{0} f_{m}\right) \mathrm{d} s
$$

where $\chi_{0} f(p, y, x)=x f(p, y, x)$ with $(p, y, x) \in \mathcal{O}$. Let $\boldsymbol{J}_{m,{ }^{(K)}}\left(\right.$ resp. $\boldsymbol{J}_{m, r}^{(K, c)}$ ) be the $m$-th component of $\boldsymbol{J}^{(K, k)}\left(Y_{1}^{(k)}, \widetilde{Y}_{1}^{(k)}\right)\left(\right.$ resp. $\left.\boldsymbol{J}^{(K, c)}\left(\boldsymbol{Y}_{1}, \widetilde{\boldsymbol{Y}}_{1}\right)\right)$ with $m=1,2, \ldots, M$. Observe that

$$
\begin{align*}
& \varphi\left(\nu_{s}^{(K)}(\boldsymbol{f})+\boldsymbol{J}_{s}^{(K, k)}\left(Y_{1}^{(k)}, \widetilde{Y}_{1}^{(k)}\right)\right)-\varphi\left(\nu_{s}^{(K)}(\boldsymbol{f})\right) \simeq \sum_{m=1}^{M} \frac{\partial \varphi}{\partial x_{m}}\left(\nu_{s}^{(K)}(\boldsymbol{f})\right) \boldsymbol{J}_{m, s}^{(K, k)} \\
& \simeq \sum_{m=1}^{M} \frac{\partial \varphi}{\partial x_{m}}\left(\nu_{s}^{(K)}(\boldsymbol{f})\right)\left(\frac{\partial f_{m}}{\partial x}\left(p_{k},\left(Y_{1}^{(k)}, \widetilde{Y}_{1}^{(k)}\right), \xi_{s-}^{(k)}\right) \frac{d_{k} \widetilde{Y}_{1}^{(k)}}{K} \bar{H}_{s}^{(k)}\right), \tag{B.4}
\end{align*}
$$

and

$$
\begin{align*}
& \varphi\left(\nu_{s}^{(K)}(\boldsymbol{f})+\boldsymbol{J}_{s}^{(K, c)}\left(\boldsymbol{Y}_{1}\right)\right)-\varphi\left(\nu_{s}^{(K)}(\boldsymbol{f})\right) \simeq \sum_{m=1}^{M} \frac{\partial \varphi}{\partial x_{m}}\left(\nu_{s}^{(K)}(\boldsymbol{f})\right) \boldsymbol{J}_{m, s}^{(K, c)} \\
& \quad \simeq \sum_{m=1}^{M} \frac{\partial \varphi}{\partial x_{m}}\left(\nu_{s}^{(K)}(\boldsymbol{f})\right)\left(\sum_{k=1}^{K} \frac{\partial f_{m}}{\partial x}\left(p_{k},\left(Y_{1}^{(k)}, \widetilde{Y}_{1}^{(k)}\right), \xi_{s-}^{(k)}\right) \frac{c_{k} Y_{1}^{(k)}}{K} \bar{H}_{s}^{(k)}\right) . \tag{B.5}
\end{align*}
$$

Accordingly, the fourth line of (C.21) can be rewritten as

$$
\begin{aligned}
\sum_{m=1}^{M} & \int_{0}^{t} \frac{\partial \varphi}{\partial x_{m}}\left(\nu_{s}^{(K)}(\boldsymbol{f})\right)\left(\frac{1}{K} \sum_{k=1}^{K} \widehat{\lambda}_{k} \frac{\partial f_{m}}{\partial x}\left(p_{k},\left(Y_{1}^{(k)}, \widetilde{Y}_{1}^{(k)}\right), \xi_{s-}^{(k)}\right) d_{k} \widetilde{Y}_{1}^{(k)} \bar{H}_{s}^{(k)}\right) \mathrm{d} s \\
& \simeq \sum_{m=1}^{M} \int_{0}^{t} \frac{\partial \varphi}{\partial x_{m}}\left(\nu_{s}^{(K)}(\boldsymbol{f})\right) \nu_{s}^{(K)}\left(\mathcal{L}_{21} f_{m}\right) \mathrm{d} s,
\end{aligned}
$$

and the fifth line of (C.21) can be rewritten as

$$
\begin{aligned}
& \widehat{\lambda}_{c} \sum_{m=1}^{M} \int_{0}^{t} \frac{\partial \varphi}{\partial x_{m}}\left(\nu_{s}^{(K)}(\boldsymbol{f})\right)\left(\frac{1}{K} \sum_{k=1}^{K} \frac{\partial f_{m}}{\partial x}\left(p_{k},\left(Y_{1}^{(k)}, \widetilde{Y}_{1}^{(k)}\right), \xi_{s-}^{(k)}\right) c_{k} Y_{1}^{(k)} \bar{H}_{s}^{(k)}\right) \mathrm{d} s \\
& \quad \simeq \widehat{\lambda}_{c} \sum_{m=1}^{M} \int_{0}^{t} \frac{\partial \varphi}{\partial x_{m}}\left(\nu_{s}^{(K)}(\boldsymbol{f})\right) \nu_{s}^{(K)}\left(\mathcal{L}_{22} f_{m}\right) \mathrm{d} s,
\end{aligned}
$$

where $a_{K} \simeq b_{K}$ means that $\lim _{K \rightarrow \infty}\left|a_{K}-b_{K}\right|=0$.
Finally, the second line of (C.21) can be rewritten as

$$
\begin{equation*}
\frac{1}{2 K} \sum_{m, n=1}^{M} \int_{0}^{t} \frac{\partial^{2} \varphi}{\partial x_{m} \partial x_{n}}\left(\nu_{s}^{(K)}(\boldsymbol{f})\right) \nu_{s}^{(K)}\left(\chi_{1}\left(\mathcal{L}_{2} f_{m}\right) \cdot \chi_{1}\left(\mathcal{L}_{2} f_{n}\right)\right) \mathrm{d} s, \tag{B.6}
\end{equation*}
$$

where the operator $\chi_{1} f(p, y, x)=\sigma x^{\rho} f(p, y, x)$ and $\mathcal{L}_{2} f(p, y, x)=\frac{\partial f}{\partial x}(p, y, x)$. We now prove that (B.6) approaches zero when $K \longrightarrow \infty$. Indeed, let $\varsigma_{a}^{(K)}=\inf \left\{t \geq 0 ; \bigvee_{k=1}^{K}\left|\xi_{t}^{(k)}\right| \geq a\right\}$ for $a>0$. Then, for each fixed $a>0$,

$$
\left|\frac{1}{2 K} \sum_{m, n=1}^{M} \int_{0}^{t \wedge \varsigma_{a}^{(K)}} \frac{\partial^{2} \varphi}{\partial x_{m} \partial x_{n}}\left(\nu_{s}^{(K)}(\boldsymbol{f})\right) \nu_{s}^{(K)}\left(\chi_{1}\left(\mathcal{L}_{2} f_{m}\right) \cdot \chi_{1}\left(\mathcal{L}_{2} f_{n}\right)\right) \mathrm{d} s\right| \leq \frac{C_{a}}{K} \longrightarrow 0, \quad \text { as } K \longrightarrow \infty .
$$

Letting $a \rightarrow \infty$, we conclude that the quantity in (B.6) approaches zero as $K \longrightarrow+\infty$ since $\varsigma_{a}^{(K)} \longrightarrow+\infty$. Thus we complete the proof of the lemma.

## B. 3 Proof of Lemma 4.3

Let $0 \leq t \leq T$. Recall that the decomposition of $\nu_{t}^{(K)}(f)$ for any $f \in C^{\infty}(\mathcal{O})$ admits the form:

$$
\begin{equation*}
\nu_{t}^{(K)}(f)=\nu_{0}^{(K)}(f)+A_{t}^{(K)}+\widehat{A}_{t}^{(K)}+B_{t}^{(K)}+\widehat{B}_{t}^{(K)}, \tag{B.7}
\end{equation*}
$$

where we have defined

$$
\begin{align*}
A_{t}^{(K)}= & \int_{0}^{t} \nu_{s}^{(K)}\left(\mathcal{L}_{11} f\right) \mathrm{d} s, \\
\widehat{A}_{t}^{(K)}= & \frac{1}{K} \int_{0}^{t} \sum_{k=1}^{K} f\left(p_{k},\left(Y_{1}^{(k)}, \widetilde{Y}_{1}^{(k)}\right), \xi_{s-}^{(k)}\right) \mathrm{d} \bar{H}_{s}^{(k)}, \\
B_{t}^{(K)}= & \frac{1}{K} \int_{0}^{t} \sum_{k=1}^{K}\left[\sigma_{k} \frac{\partial f}{\partial x}\left(p_{k},\left(Y_{1}^{(k)}, \widetilde{Y}_{1}^{(k)}\right), \xi_{s}^{(k)}\right) \bar{H}_{s}^{(k)}\left(\xi_{s}^{(k)}\right)^{\rho} \mathrm{d} W_{s}^{(k)}\right]  \tag{B.8}\\
\widehat{B}_{t}^{(K)}= & \frac{1}{K} \int_{0}^{t} \sum_{k=1}^{K}\left(\left[f\left(p_{k},\left(Y_{1}^{(k)}, \widetilde{Y}_{1}^{(k)}\right), \xi_{s-}^{(k)}+d_{k} \widetilde{Y}_{1}^{(k)}\right)-f\left(p_{k},\left(Y_{1}^{(k)}, \widetilde{Y}_{1}^{(k)}\right), \xi_{s-}^{(k)}\right)\right] \bar{H}_{s}^{(k)} \mathrm{d} \widehat{N}_{s}^{(k)}\right) \\
& +\frac{1}{K} \int_{0}^{t}\left(\sum_{k=1}^{K}\left[f\left(p_{k},\left(Y_{1}^{(k)}, \widetilde{Y}_{1}^{(k)}\right), \xi_{s-}^{(k)}+c_{k} Y_{1}^{(k)}\right)-f\left(p_{k},\left(Y_{1}^{(k)}, \widetilde{Y}_{1}^{(k)}\right), \xi_{s-}^{(k)}\right)\right] \bar{H}_{s}^{(k)}\right) \mathrm{d} \widehat{N}_{s}^{(c)} .
\end{align*}
$$

Then, for any $T>0$, we have

$$
\begin{equation*}
\sup _{0 \leq t \leq T}\left|\nu_{t}^{(K)}(f)\right| \leq \sup _{0 \leq t \leq T}\left|A_{t}^{(K)}\right|+\sup _{0 \leq t \leq T}\left|\widehat{A}_{t}^{(K)}\right|+\sup _{0 \leq t \leq T}\left|B_{t}^{(K)}\right|+\sup _{0 \leq t \leq T}\left|\widehat{B}_{t}^{(K)}\right| . \tag{B.9}
\end{equation*}
$$

Next we estimate the expectation of each term on the r.h.s. of the above equation. First, by the assumption (A2), we have

$$
\begin{aligned}
& \mathbb{E}\left[\sup _{0 \leq t \leq T}\left|A_{t}^{(K)}\right|\right] \\
& \leq \frac{1}{K} \sum_{k=1}^{K} \mathbb{E}\left[\int_{0}^{T}\left|\frac{1}{2} \sigma_{k}^{2}\left(\xi_{s}^{(k)}\right)^{2 \rho} \frac{\partial^{2} f}{\partial x^{2}}\left(p_{k},\left(Y_{1}^{(k)}, \widetilde{Y}_{1}^{(k)}\right), \xi_{s}^{(k)}\right)+\left(\alpha_{k}-\kappa_{k} \xi_{s}^{(k)}\right) \frac{\partial f}{\partial x}\left(p_{k},\left(Y_{1}^{(k)}, \widetilde{Y}_{1}^{(k)}\right), \xi_{s}^{(k)}\right)\right| \mathrm{d} s\right] \\
& \leq \frac{C_{p}^{2}}{2}\left\|\frac{\partial^{2} f}{\partial x^{2}}\right\| \int_{0}^{T}\left(\frac{1}{K} \sum_{k=1}^{K} \mathbb{E}\left[\left(\xi_{s}^{(k)}\right)^{2 \rho}\right]\right) \mathrm{d} s+C_{p}\left\|\frac{\partial f}{\partial x}\right\| \int_{0}^{T}\left(\frac{1}{K} \sum_{k=1}^{K} \mathbb{E}\left[\xi_{s}^{(k)}\right]\right) \mathrm{d} s+C_{p} T\left\|\frac{\partial f}{\partial x}\right\|,
\end{aligned}
$$

where, for a given function $f \in C^{\infty}(\mathcal{O}),\|f\|$ denotes the supremum norm, i.e. $\|f\|=\sup _{(p, y, x) \in \mathcal{O}}|f(p, y, x)|$. The same definition of supremum norm applies to $\left\|\frac{\partial f}{\partial x}\right\|$ and $\left\|\frac{\partial^{2} f}{\partial x^{2}}\right\|$. The constant $C_{p}>0$ is chosen to be $C_{p}=\max _{k \in\{1, \ldots, K\}}\left\{\alpha_{k}, \kappa_{k}, \sigma_{k}, c_{k}, d_{k}, \widehat{\lambda}_{k}, m_{k}^{Y}, m_{k}^{\tilde{Y}}\right\}$, and is finite by assumption (A2).

We can bound the second term on the r.h.s. of Eq. (B.9) as

$$
\mathbb{E}\left[\sup _{0 \leq t \leq T}\left|\widehat{A}_{t}^{(K)}\right|\right] \leq \frac{1}{K} \mathbb{E}\left[\int_{0}^{T} \sum_{k=1}^{K}\left|f\left(p_{k},\left(Y_{1}^{(k)}, \widetilde{Y}_{1}^{(k)}\right), \xi_{s-}^{(k)}\right)\right| \mathrm{d} H_{s}^{(k)}\right] \leq\|f\| \mathbb{E}\left[\frac{1}{K} \sum_{k=1}^{K} H_{T}^{(k)}\right] \leq\|f\|,
$$

where we have used the fact that $\frac{1}{K} \sum_{k=1}^{K} H_{T}^{(k)} \leq 1$ for all $K \in \mathbb{N}$. Using the Burkholder-Davis-Gundy's inequality, we can bound the third term on the r.h.s. of (B.9) as

$$
\begin{aligned}
\mathbb{E}\left[\sup _{0 \leq t \leq T}\left|B_{t}^{(K)}\right|\right] & \leq \frac{1}{K} \mathbb{E}\left[\int_{0}^{T} \sum_{k=1}^{K} \sigma_{k}^{2}\left|\frac{\partial f}{\partial x}\left(p_{k},\left(Y_{1}^{(k)}, \widetilde{Y}_{1}^{(k)}\right), \xi_{s}^{(k)}\right)\right|^{2} \bar{H}_{s}^{(k)}\left(\xi_{s}^{(k)}\right)^{2 \rho} \mathrm{~d} s\right]^{\frac{1}{2}} \\
& \leq C_{p}\left\|\frac{\partial f}{\partial x}\right\| \mathbb{E}\left[\int_{0}^{T} \frac{1}{K} \sum_{k=1}^{K}\left(\xi_{s}^{(k)}\right)^{2 \rho} \mathrm{~d} s\right]^{1 / 2} \\
& \leq \frac{C_{p}}{2}\left\|\frac{\partial f}{\partial x}\right\|\left[\int_{0}^{T} \frac{1}{K} \sum_{k=1}^{K} \mathbb{E}\left[\left(\xi_{s}^{(k)}\right)^{2 \rho}\right] \mathrm{d} s+\frac{1}{K}\right] \\
& \leq \frac{C_{p}}{2}\left\|\frac{\partial f}{\partial x}\right\|\left[\int_{0}^{T} \frac{1}{K} \sum_{k=1}^{K} \mathbb{E}\left[\left(\xi_{s}^{(k)}\right)^{2 \rho}\right] \mathrm{d} s+1\right]
\end{aligned}
$$

Finally, we have

$$
\begin{aligned}
\mathbb{E}\left[\sup _{0 \leq t \leq T}\left|\widehat{B}_{t}^{(K)}\right|\right] \leq & \frac{1}{K} \mathbb{E}\left[\int_{0}^{T} \sum_{k=1}^{K}\left|f\left(p_{k},\left(Y_{1}^{(k)}, \widetilde{Y}_{1}^{(k)}\right), \xi_{s-}^{(k)}+d_{k} \widetilde{Y}_{1}^{(k)}\right)-f\left(p_{k},\left(Y_{1}^{(k)}, \widetilde{Y}_{1}^{(k)}\right), \xi_{s-}^{(k)}\right)\right| \bar{H}_{s}^{(k)} \mathrm{d} \widehat{N}_{s}^{(k)}\right] \\
& +\frac{1}{K} \mathbb{E}\left[\int_{0}^{T} \sum_{k=1}^{K}\left|f\left(p_{k},\left(Y_{1}^{(k)}, \widetilde{Y}_{1}^{(k)}\right), \xi_{s-}^{(k)}+c_{k} Y_{1}^{(k)}\right)-f\left(p_{k},\left(Y_{1}^{(k)}, \widetilde{Y}_{1}^{(k)}\right), \xi_{s-}^{(k)}\right)\right| \bar{H}_{s}^{(k)} \mathrm{d} \widehat{N}_{s}^{(c)}\right] \\
\leq & \left\|\frac{\partial f}{\partial x}\right\| \frac{1}{K} \sum_{k=1}^{K}\left(\lambda_{k}\left(c_{k} \vee d_{k}\right) m_{k}^{Y}\right) \leq C_{p}^{2}\left(C_{p}+\widehat{\lambda}_{c}\right)\left\|\frac{\partial f}{\partial x}\right\|
\end{aligned}
$$

where we have used the mean-value theorem in the last inequality.
Note that $\mathbb{E}\left[\nu_{0}^{(K)}(f)\right] \leq\|f\|$. Using (B.1) in LemmaB. 1 we can find a constant $C=C\left(T,\|f\|,\left\|\frac{\partial f}{\partial x}\right\|,\left\|\frac{\partial^{2} f}{\partial x^{2}}\right\|\right)>$ 0 such that

$$
\sup _{K \in \mathbb{N}} \mathbb{E}\left[\sup _{0 \leq t \leq T}\left|\nu_{t}^{(K)}(f)\right|\right]<C<+\infty
$$

From Chebyshev's inequality, it follows that (20) holds.

## B. 4 Proof of Lemma 4.4

From the decomposition (B.7), it follows that

$$
\begin{align*}
\left(\nu_{t+u}^{(K)}-\nu_{t}^{(K)}\right)(f)= & A_{t+u}^{(K)}-A_{t}^{(K)}+\widehat{A}_{t+u}^{(K)}-\widehat{A}_{t}^{(K)}+B_{t+u}^{(K)}-B_{t}^{(K)} \\
& +\widehat{\mathcal{M}}_{t+u}^{(K)}-\widehat{\mathcal{M}}_{t}^{(K)}+P_{t+u}^{(K)}-P_{t}^{(K)} \tag{B.10}
\end{align*}
$$

where $A_{t}^{(K)}, \widehat{A}_{t}^{(K)}, B_{t+u}^{(K)}$ are given by (B.8) and we have defined

$$
\begin{aligned}
\widehat{\mathcal{M}}_{t}^{(K)}= & \frac{1}{K} \sum_{k=1}^{K} \int_{0}^{t} \int_{\mathbb{R}_{+}}\left[f\left(p_{k}, y, \xi_{s-}^{(k)}+d_{k} y_{2}\right)-f\left(p_{k}, y, \xi_{s-}^{(k)}\right)\right] \bar{H}_{s}^{(k)} \tilde{N}^{(k)}\left(\mathrm{d} s, \mathrm{~d} y_{2}\right) \\
& +\frac{1}{K} \sum_{k=1}^{K} \int_{0}^{t} \int_{\mathbb{R}_{+}}\left[f\left(p_{k}, y, \xi_{s-}^{(k)}+c_{k} y_{1}\right)-f\left(p_{k}, y, \xi_{s-}^{(k)}\right)\right] \bar{H}_{s}^{(k)} \widetilde{N}^{(c)}\left(\mathrm{d} s, \mathrm{~d} y_{1}\right) \\
P_{t}^{(K)}= & \frac{1}{K} \sum_{k=1}^{K} \widehat{\lambda}_{k} \int_{0}^{t} \int_{\mathbb{R}_{+}}\left[f\left(p_{k}, y, \xi_{s-}^{(k)}+d_{k} y_{2}\right)-f\left(p_{k}, y, \xi_{s-}^{(k)}\right)\right] \bar{H}_{s}^{(k)} F_{\widetilde{Y}}^{(k)}\left(\mathrm{d} y_{2}\right) \mathrm{d} s \\
& +\frac{1}{K} \sum_{k=1}^{K} \widehat{\lambda}_{c} \int_{0}^{t} \int_{\mathbb{R}_{+}}\left[f\left(p_{k}, y, \xi_{s-}^{(k)}+c_{k} y_{1}\right)-f\left(p_{k}, y, \xi_{s-}^{(k)}\right)\right] \bar{H}_{s}^{(k)} F_{Y}^{(k)}\left(\mathrm{d} y_{1}\right) \mathrm{d} s
\end{aligned}
$$

Here, for $(t, y)=\left(t, y_{1}, y_{2}\right) \in \mathbb{R}_{+}^{3}, \widetilde{N}^{(k)}\left(\mathrm{d} t, \mathrm{~d} y_{2}\right)$ and $\widetilde{N}^{(c)}\left(\mathrm{d} t, \mathrm{~d} y_{1}\right)$ denote the compensated Poisson random measures associated, respectively, to the systematic compound Poisson process $\sum_{i=1}^{\widehat{N}^{(c)}} Y_{i}^{(k)}$ and to the idiosyncratic one given by $\sum_{\ell=1}^{\widehat{N}^{(k)}} \widetilde{Y}_{\ell}^{(k)}$. Moreover, the measures $F_{Y}^{(k)}\left(\mathrm{d} y_{1}\right)$ and $F_{\widetilde{Y}}^{(k)}\left(\mathrm{d} y_{2}\right)$ are the distributions of the jump amplitude $Y_{1}^{(k)}$ and $\widetilde{Y}_{1}^{(k)}$, respectively. Then

$$
\begin{aligned}
h^{2}\left(\nu_{t+u}^{(K)}(f), \nu_{t}^{(K)}(f)\right) \leq & 8\left[\left|A_{t+u}^{(K)}-A_{t}^{(K)}\right|+\left|\widehat{A}_{t+u}^{(K)}-\widehat{A}_{t}^{(K)}\right|+\left|P_{t+u}^{(K)}-P_{t}^{(K)}\right|\right. \\
& \left.+\left|B_{t+u}^{(K)}-B_{t}^{(K)}\right|^{2}+\left|\widehat{\mathcal{M}}_{t+u}^{(K)}-\widehat{\mathcal{M}}_{t}^{(K)}\right|^{2}\right]
\end{aligned}
$$

First, we have, for $0 \leq u \leq \delta$,

$$
\begin{aligned}
\left|A_{t+u}^{(K)}-A_{t}^{(K)}\right| & \leq \frac{C_{p}^{2}}{2}\left\|\frac{\partial^{2} f}{\partial x^{2}}\right\| \int_{t}^{t+u}\left(\frac{1}{K} \sum_{k=1}^{K}\left(\xi_{s}^{(k)}\right)^{2 \rho}\right) \mathrm{d} s+C_{p}\left\|\frac{\partial f}{\partial x}\right\| \int_{t}^{t+u}\left(\frac{1}{K} \sum_{k=1}^{K} \xi_{s}^{(k)}\right) \mathrm{d} s+C_{p}\left\|\frac{\partial f}{\partial x}\right\| u \\
& \leq \frac{C_{p}^{2}}{4}\left\|\frac{\partial^{2} f}{\partial x^{2}}\right\| \delta^{\frac{1}{4}}\left[1+\int_{0}^{T}\left(\frac{1}{K} \sum_{k=1}^{K}\left(\xi_{s}^{(k)}\right)^{4 \rho}\right) \mathrm{d} s\right]+C_{p}\left\|\frac{\partial f}{\partial x}\right\| \delta
\end{aligned}
$$

$$
+\frac{C_{p}}{2}\left\|\frac{\partial f}{\partial x}\right\| \delta^{\frac{1}{4}}\left[1+\int_{0}^{T}\left(\frac{1}{K} \sum_{k=1}^{K}\left(\xi_{s}^{(k)}\right)^{2}\right) \mathrm{d} s\right]=: H_{K}^{1}(\delta) .
$$

Next, we have

$$
\left|\widehat{A}_{t+u}^{(K)}-\widehat{A}_{t}^{(K)}\right| \leq \frac{1}{K} \sum_{k=1}^{K} \int_{t}^{t+u}\|f\| \mathrm{d} H_{s}^{(k)}=\|f\| \frac{1}{K} \sum_{k=1}^{K}\left(H_{t+u}^{(k)}-H_{t}^{(k)}\right)
$$

Note that the difference $H_{t+u}^{(k)}-H_{t}^{(k)}$ admits the decomposition:

$$
H_{t+u}^{(k)}-H_{t}^{(k)}=\mathcal{M}_{t+u}^{(k)}-\mathcal{M}_{t}^{(k)}+\int_{t}^{t+u} \bar{H}_{s}^{(k)} \xi_{s}^{(n)} \mathrm{d} s
$$

where the martingale $\mathcal{M}^{(k)}=\left(\mathcal{M}_{t}^{(k)} ; t \geq 0\right)$ is defined by (B.2). Then it holds that

$$
\begin{aligned}
\mathbb{E}\left[\left|\widehat{A}_{t+u}^{(K)}-\widehat{A}_{t}^{(K)}\right| \mid \bigvee_{k=1}^{K} \mathcal{G}_{t}^{(k)}\right] & \leq\|f\| \frac{1}{K} \sum_{k=1}^{K} \mathbb{E}\left[\mathcal{M}_{t+u}^{(k)}-\mathcal{M}_{t}^{(k)}+\int_{t}^{t+u} \bar{H}_{s}^{(k)} \xi_{s}^{(k)} \mathrm{d} s \mid \bigvee_{k=1}^{K} \mathcal{G}_{t}^{(k)}\right] \\
& \leq\|f\| \mathbb{E}\left[\left.\int_{t}^{t+u} \frac{1}{K} \sum_{k=1}^{K} \xi_{s}^{(k)} \mathrm{d} s \right\rvert\, \bigvee_{k=1}^{K} \mathcal{G}_{t}^{(k)}\right] \\
& \leq \mathbb{E}\left[\left.\frac{\|f\|}{2} \delta^{\frac{1}{4}}\left(1+\int_{0}^{T} \frac{1}{K} \sum_{k=1}^{K}\left(\xi_{s}^{(k)}\right)^{2} \mathrm{~d} s\right) \right\rvert\, \bigvee_{k=1}^{K} \mathcal{G}_{t}^{(k)}\right] \\
& =\mathbb{E}\left[H_{K}^{2}(\delta) \mid \bigvee_{k=1}^{K} \mathcal{G}_{t}^{(k)}\right]
\end{aligned}
$$

For the third term on the r.h.s. of (B.10), we have

$$
\begin{aligned}
\mathbb{E}\left[\left|B_{t+u}^{(K)}-B_{t}^{(K)}\right|^{2} \mid \bigvee_{k=1}^{K} \mathcal{G}_{t}^{(k)}\right] & =\mathbb{E}\left[\left|B_{t+u}^{(K)}\right|^{2}-\left|B_{t}^{(K)}\right|^{2} \mid \bigvee_{k=1}^{K} \mathcal{G}_{t}^{(k)}\right] \\
& \leq C_{p}^{2}\left\|\frac{\partial f}{\partial x}\right\|^{2} \mathbb{E}\left[\left.\int_{t}^{t+u} \frac{1}{K} \sum_{k=1}^{K}\left(\xi_{s}^{k}\right)^{2 \rho} \mathrm{~d} s \right\rvert\, \bigvee_{k=1}^{K} \mathcal{G}_{t}^{(k)}\right] \\
& \leq \mathbb{E}\left[\left.\frac{C_{p}^{2}}{2}\left\|\frac{\partial f}{\partial x}\right\|^{2} \delta^{\frac{1}{4}}\left(1+\int_{0}^{T} \frac{1}{K} \sum_{k=1}^{K}\left(\xi_{s}^{k}\right)^{4 \rho} \mathrm{~d} s\right) \right\rvert\, \bigvee_{k=1}^{K} \mathcal{G}_{t}^{(k)}\right] \\
& =: \mathbb{E}\left[H_{K}^{3}(\delta) \mid \bigvee_{k=1}^{K} \mathcal{G}_{t}^{(k)}\right] .
\end{aligned}
$$

Next, we consider the fourth term on the r.h.s. of (B.10). Using the equality $(x+y)^{2} \leq 2\left(x^{2}+y^{2}\right)$, the martingale property of $\left(\widehat{\mathcal{M}}_{t}^{(K)} ; t \geq 0\right)$, the mean-value theorem and the assumption (A2), we have

$$
\begin{aligned}
& \mathbb{E}\left[\left|\widehat{\mathcal{M}}_{t+u}^{(K)}-\widehat{\mathcal{M}}_{t}^{(K)}\right|^{2} \mid \bigvee_{k=1}^{K} \mathcal{G}_{t}^{(k)}\right] \\
\leq & 2 \mathbb{E}\left[\left.\frac{1}{K^{2}} \sum_{k=1}^{K} \int_{t}^{t+u} \int_{\mathbb{R}_{+}}\left|f\left(p_{k}, y, \xi_{s-}^{(k)}+d_{k} y_{2}\right)-f\left(p_{k}, y, \xi_{s-}^{(k)}\right)\right|^{2} \bar{H}_{s}^{(k)} \widehat{N}^{(k)}\left(\mathrm{d} s, \mathrm{~d} y_{2}\right)\right|_{k=1} ^{K} \mathcal{G}_{t}^{(k)}\right] \\
& +2 \mathbb{E}\left[\left.\frac{1}{K^{2}} \int_{t}^{t+u} \int_{\mathbb{R}_{+}}\left(\sum_{k=1}^{K}\left|f\left(p_{k}, y, \xi_{s-}^{(k)}+c_{k} y_{1}\right)-f\left(p_{k}, y, \xi_{s-}^{(k)}\right)\right| \bar{H}_{s}^{(k)}\right) \widehat{N}^{(c)}\left(\mathrm{d} s, \mathrm{~d} y_{1}\right) \right\rvert\, \bigvee_{k=1}^{K} \mathcal{G}_{t}^{(k)}\right] \\
\leq & 2 \mathbb{E}\left[\left.\frac{1}{K^{2}}\left\|\frac{\partial f}{\partial x}\right\|^{2} \sum_{k=1}^{K} \widehat{\lambda}_{k} d_{k}^{2} \int_{t}^{t+u} \int_{\mathbb{R}_{+}} y_{2}^{2} F_{\widehat{Y}}^{(k)}\left(\mathrm{d} y_{2}\right) \mathrm{d} s \right\rvert\, \bigvee_{k=1}^{K} \mathcal{G}_{t}^{(k)}\right] \\
& +2 \mathbb{E}\left[\left.\frac{1}{K^{2}}\left\|\frac{\partial f}{\partial x}\right\|^{2} \widehat{\lambda}_{c} \int_{t}^{t+u} \int_{\mathbb{R}_{+}}\left(\sum_{k=1}^{K} c_{k} \bar{H}_{s}^{(k)}\right)^{2} y_{1}^{2} F_{Y}^{(k)}\left(\mathrm{d} y_{1}\right) \mathrm{d} s \right\rvert\, \bigvee_{k=1}^{K} \mathcal{G}_{t}^{(k)}\right]
\end{aligned}
$$

$$
\leq 2 C_{p}^{2}\left(C_{p}+\widehat{\lambda}_{c}\right)\left\|\frac{\partial f}{\partial x}\right\|^{2} \delta=: H_{K}^{4}(\delta)
$$

where $\widehat{N}^{(k)}\left(\mathrm{d} t, \mathrm{~d} y_{2}\right)$ and $\widehat{N}^{(c)}\left(\mathrm{d} t, \mathrm{~d} y_{2}\right)$ denote the Poisson random measures associated, respectively, to the systematic compound Poisson process $\sum_{i=1}^{\widehat{N}^{(c)}} Y_{i}^{(k)}$ and to the idiosyncratic one given by $\sum_{\ell=1}^{\widehat{N}^{(k)}} \widetilde{Y}_{\ell}^{(k)}$. Finally, we have

$$
\begin{aligned}
\left|P_{t+u}^{(K)}-P_{t}^{(K)}\right| \leq & C_{p}\left(C_{p}+\widehat{\lambda}_{c}\right)\left\|\frac{\partial f}{\partial x}\right\| \frac{1}{K} \sum_{k=1}^{K} \int_{t}^{t+u} \int_{\mathbb{R}_{+}} y_{1} F_{Y}^{(k)}\left(\mathrm{d} y_{1}\right) \mathrm{d} s \\
& +2 C_{p}^{2}\left\|\frac{\partial f}{\partial x}\right\| \frac{1}{K} \sum_{k=1}^{K} \int_{t}^{t+u} \int_{\mathbb{R}_{+}} y_{2} F_{\widetilde{Y}}^{(k)}\left(\mathrm{d} y_{2}\right) \mathrm{d} s \\
= & C_{p}\left(3 C_{p}+\widehat{\lambda}_{c}\right)\left\|\frac{\partial f}{\partial x}\right\| \frac{1}{K} \sum_{k=1}^{K} \int_{t}^{t+u}\left(\mathbb{E}\left[Y_{1}^{(k)}\right]+\mathbb{E}\left[\widetilde{Y}_{1}^{(k)}\right]\right) \mathrm{d} s \\
\leq & 2 C_{p}^{2}\left(3 C_{p}+\widehat{\lambda}_{c}\right)\left\|\frac{\partial f}{\partial x}\right\| \delta=: H_{K}^{5}(\delta) .
\end{aligned}
$$

Note that $h^{2}\left(\nu_{t}^{(K)}(f), \nu_{t-v}^{(K)}(f)\right) \leq 1$. Let $H_{K}(\delta)=\sum_{n=1}^{5} H_{K}^{n}(\delta)$. It satisfies $\lim _{\delta \rightarrow 0} \sup _{K \in \mathbb{N}} \mathbb{E}\left[H_{K}(\delta)\right]=0$ and (22) holds, due to the above estimates and (B.1) from Lemma B.1

## C Proofs related to Section 5

Lemma C.1. The default times $\tau_{1}, \ldots, \tau_{K}, \tau_{A}$ and $\tau_{B}$ are conditionally independent. Namely, for any $t_{1}, \ldots, t_{K}, t_{A}, t_{B} \geq 0$ and any $T \geq \max \left\{t_{1}, \ldots, t_{K}, t_{A}, t_{B}\right\}$, it holds that

$$
\begin{equation*}
\mathbb{P}\left(\tau_{1}>t_{1}, \ldots, \tau_{K}>t_{K}, \tau_{A}>t_{A}, \tau_{B}>t_{B} \mid \mathcal{F}_{T}^{(K, A, B)}\right)=\prod_{j \in\{1, \ldots, A, B\}} \exp \left(-\int_{0}^{t_{j}} \xi_{s}^{(j)} \mathrm{d} s\right) \tag{C.1}
\end{equation*}
$$

Proof. The proof is straightforward and follows immediately from the discussion in Section 9.1.1 of Bielecki and Rutkowski (2002).

Proof of Theorem 5.3. First, we define the conditional cumulative distribution function associated to the default times $\left(\tau_{X}^{*}, \tau_{A}, \tau_{B}\right)$. For $\min \left\{t^{*}, t_{A}, t_{B}\right\}>t$, define

$$
\begin{equation*}
P\left(t ; t^{*}, t_{A}, t_{B}\right):=\mathbb{P}\left(\tau_{X}^{*} \leq t^{*}, \tau_{A} \leq t_{A}, \tau_{B} \leq t_{B} \mid \mathcal{G}_{t}^{(K, A, B)}\right) \tag{C.2}
\end{equation*}
$$

Then

$$
\begin{align*}
P\left(t ; t^{*}, t_{A}, t_{B}\right)= & 1-\mathbb{P}\left(\tau_{A}>t_{A} \mid \mathcal{G}_{t}^{(K, A, B)}\right)-\mathbb{P}\left(\tau_{B}>t_{B} \mid \mathcal{G}_{t}^{(K, A, B)}\right) \\
& +\mathbb{P}\left(\tau_{A}>t_{A}, \tau_{B}>t_{B} \mid \mathcal{G}_{t}^{(K, A, B)}\right) \\
& -\left[\mathbb{P}\left(\tau_{X}^{*}>t^{*} \mid \mathcal{G}_{t}^{(K, A, B)}\right)-\mathbb{P}\left(\tau_{A}>t_{A}, \tau_{X}^{*}>t^{*} \mid \mathcal{G}_{t}^{(K, A, B)}\right)\right] \\
& +\left[\mathbb{P}\left(\tau_{B}>t_{B}, \tau_{X}^{*}>t^{*} \mid \mathcal{G}_{t}^{(K, A, B)}\right)-\mathbb{P}\left(\tau_{A}>t_{A}, \tau_{B}>t_{B}, \tau_{X}^{*}>t^{*} \mid \mathcal{G}_{t}^{(K, A, B)}\right)\right] \tag{C.3}
\end{align*}
$$

Using Lemma 5.1. and the definition of the limit default time $\tau_{X}^{*}$ given in Section 4.3, on the event $\left\{\tau_{X}^{*}>\right.$ $\left.t, \tau_{A}>t_{A}, \tau_{B}>t_{B}\right\}$, we have

$$
\begin{equation*}
\mathbb{P}\left(\tau_{X}^{*}>t^{*}, \tau_{A}>t_{A}, \tau_{B}>t_{B} \mid \mathcal{G}_{t}^{(K, A, B)}\right)=\widehat{F}\left(t, t^{*}\right) \mathbb{E}\left[\exp \left(-\int_{t}^{t_{A}} \xi_{s}^{(A)} \mathrm{d} s-\int_{t}^{t_{B}} \xi_{s}^{(B)} \mathrm{d} s\right) \mid \mathcal{F}_{t}^{(K, A, B)}\right] \tag{C.4}
\end{equation*}
$$

By virtue of (C.3), it follows that

$$
\begin{equation*}
\frac{\partial P\left(t ; t^{*}, t_{A}, t_{B}\right)}{\partial t_{B}}=\left(1-\widehat{F}\left(t, t^{*}\right)\right) \frac{\partial P\left(t ; \infty, t_{A}, t_{B}\right)}{\partial t_{B}} \tag{C.5}
\end{equation*}
$$

where

$$
\begin{align*}
\frac{\partial P\left(t ; \infty, t_{A}, t_{B}\right)}{\partial t_{B}}:= & \left.\frac{\partial P\left(t ; t^{*}, t_{A}, t_{B}\right)}{\partial t_{B}}\right|_{t^{*}=\infty}=\mathbb{E}\left[\exp \left(-\int_{t}^{t_{B}} \xi_{s}^{(B)} \mathrm{d} s\right) \xi_{t_{B}}^{(B)} \mid \mathcal{F}_{t}^{(K, A, B)}\right] \\
& -\mathbb{E}\left[\exp \left(-\int_{t}^{t_{A}} \xi_{s}^{(A)} \mathrm{d} s-\int_{t}^{t_{B}} \xi_{s}^{(B)} \mathrm{d} s\right) \xi_{t_{B}}^{(B)} \mid \mathcal{F}_{t}^{(K, A, B)}\right] \tag{C.6}
\end{align*}
$$

Hence we have that

$$
\begin{aligned}
B^{(K, *)}(t, T)= & \int_{t}^{T} \int_{t}^{\infty} \int_{t}^{\infty} \mathbb{E}\left[\left.\mathbf{1}_{\left\{t_{B} \leq t_{A}\right\}} \mathbf{1}_{\left\{t_{B}<t^{*}\right\}} D\left(t, t_{B}\right) \varepsilon_{+}^{(K, *)}\left(t_{B}, T\right) \frac{\partial^{3} P\left(t ; t^{*}, t_{A}, t_{B}\right)}{\partial t^{*} \partial t_{A} \partial t_{B}} \right\rvert\, \mathcal{F}_{t}^{(K, A, B)}\right] \mathrm{d} t^{*} \mathrm{~d} t_{A} \mathrm{~d} t_{B} \\
= & \int_{t}^{T} \int_{t}^{\infty} \mathbb{E}\left[\left.\left.\mathbf{1}_{\left\{t_{B}<t^{*}\right\}} D\left(t, t_{B}\right) \varepsilon_{+}^{(K, *)}\left(t_{B}, T\right) \frac{\partial^{2} P\left(t ; t^{*}, t_{A}, t_{B}\right)}{\partial t^{*} \partial t_{B}}\right|_{t_{A}=t_{B}} ^{\infty} \right\rvert\, \mathcal{F}_{t}^{(K, A, B)}\right] \mathrm{d} t^{*} \mathrm{~d} t_{B} \\
= & \int_{t}^{T} \int_{t}^{\infty} \mathbb{E}\left[\left.\mathbf{1}_{\left\{t_{B}<t^{*}\right\}} D\left(t, t_{B}\right) \varepsilon_{+}^{(K, *)}\left(t_{B}, T\right) \frac{\partial^{2} P\left(t ; t^{*}, \infty, t_{B}\right)}{\partial t^{*} \partial t_{B}}\right|_{t} ^{(K, A, B)}\right] \mathrm{d} t^{*} \mathrm{~d} t_{B} \\
& -\int_{t}^{T} \int_{t}^{\infty} \mathbb{E}\left[\left.\mathbf{1}_{\left\{t_{B}<t^{*}\right\}} D\left(t, t_{B}\right) \varepsilon_{+}^{(K, *)}\left(t_{B}, T\right) \frac{\partial^{2} P\left(t ; t^{*}, t_{B}, t_{B}\right)}{\partial t^{*} \partial t_{B}} \right\rvert\, \mathcal{F}_{t}^{(K, A, B)}\right] \mathrm{d} t^{*} \mathrm{~d} t_{B} \\
= & \int_{t}^{T} D\left(t, t_{B}\right) \varepsilon_{+}^{(K, *)}\left(t_{B}, T\right) \mathbb{E}\left[\left.\frac{\partial P\left(t ; \infty, \infty, t_{B}\right)}{\partial t_{B}}-\frac{\partial P\left(t ; t_{B}, \infty, t_{B}\right)}{\partial t_{B}} \right\rvert\, \mathcal{F}_{t}^{(K, A, B)}\right] \mathrm{d} t_{B} \\
& -\int_{t}^{T} D\left(t, t_{B}\right) \varepsilon_{+}^{(K, *)}\left(t_{B}, T\right) \mathbb{E}\left[\left.\frac{\partial P\left(t ; \infty, t_{B}, t_{B}\right)}{\partial t_{B}}-\frac{\partial P\left(t ; t_{B}, t_{B}, t_{B}\right)}{\partial t_{B}} \right\rvert\, \mathcal{F}_{t}^{(K, A, B)}\right] \mathrm{d} t_{B} \\
= & \int_{t}^{T} D\left(t, t_{B}\right) \varepsilon_{+}^{(K, *)}\left(t_{B}, T\right) \widehat{F}\left(t, t_{B}\right) \mathbb{E}\left[\left.\frac{\partial P\left(t ; \infty, \infty, t_{B}\right)}{\partial t_{B}}-\frac{\partial P\left(t ; \infty, t_{B}, t_{B}\right)}{\partial t_{B}} \right\rvert\, \mathcal{F}_{t}^{(K, A, B)}\right] \mathrm{d} t_{B},(\mathrm{C} .7)
\end{aligned}
$$

where we have used (C.5) to obtain the last equality in (C.7). Using (C.6), we have

$$
\begin{align*}
B^{(K, *)}(t, T)= & \int_{t}^{T} \mathbb{E}\left[D\left(t, t_{B}\right) \varepsilon_{+}^{(K, *)}\left(t_{B}, T\right) \widehat{F}\left(t, t_{B}\right) \exp \left(-\int_{t}^{t_{B}} \xi_{s}^{(A)}+\xi_{s}^{(B)} \mathrm{d} s\right) \xi_{t_{B}}^{(B)} \mid \mathcal{F}_{t}^{(K, A, B)}\right] \mathrm{d} t_{B} \\
& -\int_{t}^{T} \mathbb{E}\left[D\left(t, t_{B}\right) \varepsilon_{+}^{(K, *)}\left(t_{B}, T\right) \widehat{F}\left(t, t_{B}\right) \exp \left(-\int_{t}^{\infty} \xi_{s}^{(A)} \mathrm{d} s-\int_{t}^{t_{B}} \xi_{s}^{(B)} \mathrm{d} s\right) \xi_{t_{B}}^{(B)} \mid \mathcal{F}_{t}^{(K, A, B)}\right] \mathrm{d} t_{B} \tag{C.8}
\end{align*}
$$

For $t \leq t_{B} \leq T$ and $\left(x_{A}, x_{B}\right) \in \mathbb{R}_{+}^{2}$, define the function

$$
\begin{equation*}
\widehat{H}_{1}\left(t_{B}-t, x_{A}, x_{B}\right):=\mathbb{E}\left[\exp \left(-\int_{t}^{\infty} \xi_{s}^{(A)} \mathrm{d} s-\int_{t}^{t_{B}} \xi_{s}^{(B)} \mathrm{d} s\right) \xi_{t_{B}}^{(B)} \mid \xi_{t}^{(A)}=x_{A}, \xi_{t}^{(B)}=x_{B}\right] \tag{C.9}
\end{equation*}
$$

Using the definition of the function $H_{1}$ given in (45) along with Eq. (C.9), we obtain that $B^{(K, *)}(t, T)$ is given by

$$
\begin{align*}
B^{(K, *)}(t, T):= & \mathbb{E}\left[\mathbf{1}_{\left\{t<\tau_{B} \leq \min \left(\tau_{A}, T\right)\right\}} \mathbf{1}_{\left\{\tau_{B}<\tau_{x}^{*}\right\}} D\left(t, \tau_{B}\right) \varepsilon_{+}^{(K, *)}\left(\tau_{B}, T\right) \mid \mathcal{G}_{t}^{(K, A, B)}\right] \\
= & \int_{t}^{T} D\left(t, t_{B}\right) \varepsilon_{+}^{(K, *)}\left(t_{B}, T\right) \widehat{F}\left(t, t_{B}\right) \\
& \quad \times\left(H_{1}\left(t_{B}-t, \xi_{t}^{(A)}, \xi_{t}^{(B)}\right)-\widehat{H}_{1}\left(t_{B}-t, \xi_{t}^{(A)}, \xi_{t}^{(B)}\right)\right) \mathrm{d} t_{B} \tag{C.10}
\end{align*}
$$

Next, we prove that the function $\widehat{H}_{1}$ defined in (C.9) vanishes. To this purpose, let $\xi^{(A, n o j)}=\left(\xi_{t}^{(A, n o j)} ; t \geq\right.$ 0 ) be the CEV process satisfying the SDE:

$$
\begin{equation*}
\mathrm{d} \xi_{t}^{(A, n o j)}=-\kappa_{A} \xi_{t}^{(A, n o j)} \mathrm{d} t+\sigma_{A}\left(\xi_{t}^{(A, n o j)}\right)^{\widehat{\rho}} \mathrm{d} W_{t}^{(A)}, \quad \xi_{t}^{(A, n o j)}=x_{A}>0 \tag{C.11}
\end{equation*}
$$

where $\kappa_{A}, \sigma_{A}>0$ and the elasticity factor $\hat{\rho} \in\left[\frac{1}{2}, 1\right)$ are the parameters specified in (2). Then we have that

$$
0 \leq \mathbb{E}_{t_{B}, x_{A}}\left[\exp \left(-\int_{t_{B}}^{\infty} \xi_{s}^{(A)} \mathrm{d} s\right)\right] \leq \mathbb{E}_{t_{B}, x_{A}}\left[\exp \left(-\int_{t_{B}}^{\infty} \xi_{s}^{(A, n o j)} \mathrm{d} s\right)\right]=: M\left(t_{B}, x_{A}\right)
$$

where $\mathbb{E}_{t_{B}, x_{A}}[\cdot]$ represents the expectation conditional on the underlying state process being equal to $x_{A}$ at time $t_{B}$. Hereafter, we use $\mathbb{E}_{x_{A}}[\cdot]:=\mathbb{E}_{0, x_{A}}[\cdot]$.

We want to verify that $M\left(t_{B}, x_{A}\right)=0$ for fixed $t_{B}, x_{A}>0$. Using the Markov property of $\xi^{(A, n o j)}$,

$$
\begin{equation*}
M\left(t_{B}, x_{A}\right)=M\left(x_{A}\right):=\mathbb{E}_{x_{A}}\left[\exp \left(-\int_{0}^{\infty} \xi_{s}^{(A, n o j)} \mathrm{d} s\right)\right], \quad x_{A}>0 \tag{C.12}
\end{equation*}
$$

Let $B E S Q_{\left(\delta, x_{A}\right)}=\left(B E S Q_{\left(\delta, x_{A}\right)}(t) ; t \geq 0\right)$ denote a squared Bessel process with dimension $\delta>0$. This is a particular CIR process satisfying the SDE:

$$
\begin{equation*}
\mathrm{d} B E S Q_{\left(\delta, x_{A}\right)}(t)=\delta \mathrm{d} t+2 \sqrt{B E S Q_{\left(\delta, x_{A}\right)}(t)} \mathrm{d} W_{t}^{(A)}, \quad B E S Q_{\left(\delta, x_{A}\right)}(0)=x_{A} \tag{C.13}
\end{equation*}
$$

From Proposition 2.3 in Atlan and Lebland (2006), it follows that

$$
\begin{equation*}
\xi_{t}^{(A, n o j)}=e^{-\kappa_{A} t}\left(B E S Q_{\delta, x_{A}^{1 / p}}(a(t))\right)^{p}, \quad t \geq 0 \tag{C.14}
\end{equation*}
$$

where $p=\frac{1}{2(1-\widehat{\rho})}>1, \delta=\frac{2 \widehat{\rho}-1}{\widehat{\rho}-1}$, and the time-changed function $a(t)$ is defined as

$$
\begin{equation*}
a(t)=\frac{(1-\widehat{\rho}) \sigma_{A}^{2}}{2 \kappa_{A}}\left(e^{2(1-\widehat{\rho}) \kappa_{A} t}-1\right)=\frac{1}{l_{A}}\left(e^{\left(\kappa_{A} / p\right) t}-1\right) . \tag{C.15}
\end{equation*}
$$

Here $l_{A}=\frac{2 \kappa_{A}}{(1-\hat{\rho}) \sigma_{A}^{2}}$. Then

$$
\begin{equation*}
M\left(x_{A}\right)=\mathbb{E}_{x_{A}}\left\{\exp \left[-\int_{0}^{\infty} e^{-\kappa_{A} s}\left(B E S Q_{\delta, x_{A}^{1 / p}}(a(s))\right)^{p} \mathrm{~d} s\right]\right\} \tag{C.16}
\end{equation*}
$$

Set the time variable $v=a(s)$. Then $s=a^{-1}(v)=\frac{p}{\kappa_{A}} \log \left\{l_{A} v+1\right\}$. Observing that $a(0)=0$, we obtain that

$$
\begin{equation*}
M\left(x_{A}\right)=\mathbb{E}_{x_{A}}\left\{\exp \left[-\frac{p l_{A}}{\kappa_{A}} \int_{0}^{\infty} \frac{1}{\left(l_{A} v+1\right)^{p+1}}\left(B E S Q_{\delta, x_{A}^{1 / p}}(v)\right)^{p} \mathrm{~d} v\right]\right\} \tag{C.17}
\end{equation*}
$$

For any $T>0$, define

$$
\begin{align*}
M_{T}\left(x_{A}\right) & :=\mathbb{E}_{x_{A}}\left\{\exp \left[-\frac{p l_{A}}{\kappa_{A}} \int_{0}^{T} \frac{1}{\left(l_{A} v+1\right)^{p+1}}\left(B E S Q_{\delta, x_{A}^{1 / p}}(v)\right)^{p} \mathrm{~d} v\right]\right\} \\
& \leq \mathbb{E}_{x_{A}}\left\{\exp \left[-\frac{p l_{A}}{\kappa_{A}} \frac{1}{\left(l_{A} T+1\right)^{p+1}} \int_{0}^{T}\left(B E S Q_{\delta, x_{A}^{1 / p}}(v)\right)^{p} \mathrm{~d} v\right]\right\} \\
& =g(T) \mathbb{E}_{x_{A}}\left\{\exp \left[-\int_{0}^{T}\left(B E S Q_{\delta, x_{A}^{1 / p}}(v)\right)^{p} \mathrm{~d} v\right]\right\} \tag{C.18}
\end{align*}
$$

where the function

$$
g(T)=\exp \left(\frac{p l_{A}}{\kappa_{A}} \frac{1}{\left(l_{A} T+1\right)^{p+1}}\right), \quad \text { and hence } \lim _{T \rightarrow \infty} g(T)=1
$$

Next, we prove the following limit:

$$
\begin{equation*}
\lim _{T \rightarrow \infty} \mathbb{E}_{x_{A}}\left\{\exp \left[-\int_{0}^{T}\left(B E S Q_{\delta, x_{A}^{1 / p}}(v)\right)^{p} \mathrm{~d} v\right]\right\}=0 \tag{C.19}
\end{equation*}
$$

Let $\nu=\frac{\delta-2}{2}$ and hence $\nu=-p$. In terms of (C.13), we have

$$
\begin{equation*}
\mathrm{d} B E S Q_{\left(\delta, x_{A}\right)}(t)=2(\nu+1) \mathrm{d} t+2 \sqrt{B E S Q_{\left(\delta, x_{A}\right)}(t)} \mathrm{d} W_{t}^{(A)}, \quad B E S Q_{\left(\delta, x_{A}\right)}(0)=x_{A} \tag{C.20}
\end{equation*}
$$

Notice that $\nu<0$ and $p \geq 1$. Using Lemma 2.1 in Cetin (2012), we have that, for any $\alpha>0$, the process

$$
\begin{equation*}
M_{t}^{(u)}:=u\left(\sqrt{X_{t}}\right)\left(X_{t}\right)^{p} \exp \left(-\frac{\alpha}{2} \int_{0}^{t}\left(X_{s}\right)^{p} \mathrm{~d} s\right), \quad t \geq 0 \tag{C.21}
\end{equation*}
$$

is a (local) martingale, where $X=\left(X_{t} ; t \geq 0\right)$ denotes any squared Bessel process starting at $x>0$ with the above dimension $\delta>0$ and the function $u(\cdot)$ satisfies the ODE:

$$
\begin{equation*}
x^{2} u^{\prime \prime}(x)+x u^{\prime}(x)-u(x)\left(p^{2}+\alpha x^{2(p+1)}\right)=0, \quad x>0 \tag{C.22}
\end{equation*}
$$

Then for the stopping time $\tau_{R}:=\inf \left\{t \geq 0 ; X_{t} \geq R\right\}$ with $R \geq x, x=x_{A}^{1 / p}$ and $X_{t}=B E S Q_{\left(\delta, x_{A}^{1 / p}\right)}(t)$, it holds that

$$
\mathbb{E}_{x_{A}}\left[M_{t \wedge \tau_{R}}\right]=u\left(x_{A}^{1 /(2 p)}\right) x_{A}^{1 / 2}, \quad \text { for all } t>0
$$

Letting $t \longrightarrow \infty$, it follows that

$$
\begin{equation*}
\mathbb{E}_{x_{A}}\left[\exp \left(-\frac{\alpha}{2} \int_{0}^{\tau_{R}}\left(X_{s}\right)^{p} \mathrm{~d} s\right)\right]=\frac{u\left(x_{A}^{1 /(2 p)}\right) x_{A}^{1 / 2}}{u(\sqrt{R}) R^{p / 2}} \tag{C.23}
\end{equation*}
$$

Since we are considering the case $R \geq x_{A}^{1 / p}$, where $x_{A}^{1 / p}$ is the starting value of the squared Bessel process $B E S Q_{\left(\delta, x_{A}^{1 / p}\right)}$, we must have that $R \rightarrow u(\sqrt{R}) R^{p / 2}$ is an increasing function. This is the case because when $R$ increases, $\tau_{R}$ would be increasing hence $\mathbb{E}_{x_{A}}\left[\exp \left(-\frac{\alpha}{2} \int_{0}^{\tau_{R}}\left(X_{s}\right)^{p} \mathrm{~d} s\right)\right]$ is decreasing w.r.t. $R$. Using Eq. (2.8) in Cetin (2012), we deduce that

$$
\begin{equation*}
\mathbb{E}_{x_{A}}\left[\exp \left(-\frac{\alpha}{2} \int_{0}^{\tau_{R}}\left(X_{s}\right)^{p} \mathrm{~d} s\right)\right]=\frac{u_{0}\left(x_{A}^{1 /(2 p)}\right) x_{A}^{1 / 2}}{u_{0}(\sqrt{R}) R^{p / 2}} \tag{C.24}
\end{equation*}
$$

where the function $u_{0}(\cdot)$ is defined as

$$
u_{0}(x)=I_{\frac{p}{p+1}}\left(\frac{1}{p+1} \sqrt{\alpha} x^{p+1}\right), \quad x>0
$$

Here $I_{b}(\cdot)$ represents the modified Bessel function of the first kind with $b>-\frac{1}{2}$, defined by

$$
\begin{equation*}
I_{b}(x)=\frac{(x / 2)^{b}}{\Gamma\left(b+\frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)} \int_{-1}^{1} e^{-x t}\left(1-t^{2}\right)^{b-\frac{1}{2}} \mathrm{~d} t, \quad x>0 \tag{C.25}
\end{equation*}
$$

Using the fact that $\lim _{x \rightarrow \infty} I_{b}(x)=+\infty$ and taking $R \longrightarrow \infty$ in (C.24), it follows that for any $\alpha>0$,

$$
\begin{equation*}
\mathbb{E}_{x_{A}}\left[\exp \left(-\frac{\alpha}{2} \int_{0}^{\infty}\left(X_{s}\right)^{p} \mathrm{~d} s\right)\right]=0 \tag{C.26}
\end{equation*}
$$

Accordingly, the limit equality (C.19) is proven by taking the parameter $\alpha=2$ in (C.26) and hence $M\left(t_{B}, x_{A}\right)=0$. This results in $\mathbb{E}_{t_{B}, x_{A}}\left[\exp \left(-\int_{t_{B}}^{\infty} \xi_{s}^{(A)} \mathrm{d} s\right)\right]=0$. For $t \leq t_{B} \leq T$, using the tower property, it follows that

$$
\begin{align*}
& \widehat{H}_{1}\left(t_{B}-t, \xi_{t}^{(A)}, \xi_{t}^{(B)}\right)=\mathbb{E}\left[\exp \left(-\int_{t}^{\infty} \xi_{s}^{(A)} \mathrm{d} s-\int_{t}^{t_{B}} \xi_{s}^{(B)} \mathrm{d} s\right) \xi_{t_{B}}^{(B)} \mid \mathcal{F}_{t}^{(K, A, B)}\right] \\
& \quad=\mathbb{E}\left\{\mathbb{E}\left[\exp \left(-\int_{t}^{\infty} \xi_{s}^{(A)} \mathrm{d} s-\int_{t}^{t_{B}} \xi_{s}^{(B)} \mathrm{d} s\right) \xi_{t_{B}}^{(B)} \mid \mathcal{F}_{t_{B}}^{(K, A, B)}\right] \mid \mathcal{F}_{t}^{(K, A, B)}\right\} \\
& \quad=\mathbb{E}\left\{\exp \left(-\int_{t}^{t_{B}}\left(\xi_{s}^{(A)}+\xi_{s}^{(B)}\right) \mathrm{d} s\right) \xi_{t_{B}}^{(B)} \mathbb{E}\left[\exp \left(-\int_{t_{B}}^{\infty} \xi_{s}^{(A)} \mathrm{d} s\right) \mid \mathcal{F}_{t_{B}}^{(K, A, B)}\right] \mid \mathcal{F}_{t}^{(K, A, B)}\right\} \\
& \quad=0 \tag{C.27}
\end{align*}
$$

This yields Eq. (43). Similarly for the second term on the r.h.s. of (41), we have, on the event $\left\{\tau_{X}^{*} \wedge \tau_{A} \wedge \tau_{B}>\right.$ $t\}$,

$$
\begin{aligned}
A^{(K, *)}(t, T)= & \int_{t}^{T} \\
& D\left(t, t_{A}\right) \varepsilon_{-}^{(K, *)}\left(t_{A}, T\right) \widehat{F}\left(t, t_{A}\right) \\
& \times\left(H_{2}\left(t_{A}-t, \xi_{t}^{(A)}, \xi_{t}^{(B)}\right)-\widehat{H}_{2}\left(t_{A}-t, \xi_{t}^{(A)}, \xi_{t}^{(B)}\right)\right) \mathrm{d} t_{A}
\end{aligned}
$$

where the function $H_{2}$ is defined as in (45) and the function

$$
\widehat{H}_{2}\left(t_{A}-t, x_{A}, x_{B}\right):=\mathbb{E}\left[\exp \left(-\int_{t}^{t_{A}} \xi_{s}^{(A)} \mathrm{d} s-\int_{t}^{\infty} \xi_{s}^{(B)} \mathrm{d} s\right) \xi_{t_{B}}^{(A)} \mid \xi_{t}^{(A)}=x_{A}, \xi_{t}^{(B)}=x_{B}\right]
$$

Using a symmetric argument to the one used to show that $\widehat{H}_{1}\left(t_{B}-t, x_{A}, x_{B}\right)=0$, it follows that $\widehat{H}_{2}\left(t_{A}-\right.$ $\left.t, x_{A}, x_{B}\right)=0$. Hence, we obtain that $A^{(K, *)}(t, T)$ is given by (44). This completes the proof of the theorem yielding the law of large number approximation formula for the BCVA (41).

## D Solutions to Riccati Equations

Lemma D.1. The explicit solution to the following Riccati equation:

$$
\begin{align*}
B^{\prime}(u) & =-\kappa B(u)+\frac{1}{2} \sigma^{2} B^{2}(u)-1 \\
B(0) & =0 \tag{D.1}
\end{align*}
$$

is given by

$$
\begin{equation*}
B(\kappa, \sigma ; u)=-\frac{2\left(e^{\varpi u}-1\right)}{2 \varpi+(\kappa+\varpi)\left(e^{\varpi u}-1\right)}, \quad 0 \leq u \leq T \tag{D.2}
\end{equation*}
$$

where $\kappa>0, \sigma>0$ and $\varpi=\sqrt{\kappa^{2}+2 \sigma^{2}}$.
Lemma D.2. Let the real number $b \neq 0$. Then the explicit solution of the following Riccati equation:

$$
\begin{align*}
\beta^{\prime}(u) & =-\kappa \beta(u)+\frac{1}{2} \sigma^{2} \beta^{2}(u)-1 \\
\beta(0) & =b \tag{D.3}
\end{align*}
$$

is given by

$$
\begin{equation*}
\beta(\kappa, \sigma, b ; u)=B(\kappa, \sigma ; u)+e^{\phi(u)} \frac{1}{\frac{1}{b}-\frac{\sigma^{2}}{2} \int_{0}^{u} e^{\phi(v)} \mathrm{d} v}, \tag{D.4}
\end{equation*}
$$

where the function $B(\kappa, \sigma ; u)$ is given by (D.2), and

$$
\phi(u)=\sigma^{2} \int_{0}^{u} B(\kappa, \sigma ; v) \mathrm{d} v-\kappa u, \quad 0 \leq u \leq T
$$

Moreover, we have

$$
\begin{align*}
\int_{0}^{u} e^{\phi(v)} \mathrm{d} v & =\int_{0}^{u} e^{\varpi v}\left(\frac{2 \varpi}{(\varpi-\kappa)+e^{\varpi v}(\kappa+\varpi)}\right)^{2} \mathrm{~d} v \\
& =\frac{2}{\kappa+\varpi \operatorname{Coth}\left(\frac{\varpi u}{2}\right)} \tag{D.5}
\end{align*}
$$

where $\operatorname{Coth}(u)=\frac{\cosh (u)}{\sinh (u)}$ gives the hyperbolic cotangent of $u$.
Proof. For $b \neq 0$, we define the following function

$$
\begin{equation*}
f(u)=B(\kappa, \sigma ; u)+e^{\phi(u)} \frac{1}{C-\frac{\sigma^{2}}{2} \int_{0}^{u} e^{\phi(v)} \mathrm{d} v} \tag{D.6}
\end{equation*}
$$

where $C$ is an unspecified real constant. Then

$$
\begin{aligned}
f^{\prime}(u) & =B^{\prime}(\kappa, \sigma ; u)+\phi^{\prime}(u)(f(u)-B(\kappa, \sigma ; u))+\frac{\sigma^{2}}{2}(f(u)-B(\kappa, \sigma ; u))^{2} \\
& =B^{\prime}(\kappa, \sigma ; u)+\kappa B(\kappa, \sigma ; u)-\frac{\sigma^{2}}{2} B^{2}(\kappa, \sigma ; u)-\kappa f(u)+\frac{\sigma^{2}}{2} f^{2}(u) \\
& =-1-\kappa f(u)+\frac{\sigma^{2}}{2} f^{2}(u)
\end{aligned}
$$

This yields that the function given by (D.6) is the general solution to the above Riccati equation. Taking the initial condition $\beta(0)=b$ into account, we have the constant $C=\frac{1}{b}$ in (D.6), since $B(\kappa, \sigma ; 0)=0$.
Lemma D.3. Let $b \in \mathbb{R}$ and $a_{\ell}>0$. Then the explicit solution to the following Riccati equation:

$$
\begin{align*}
\widehat{\beta}^{\prime}(u) & =-\kappa \widehat{\beta}(u)+\frac{1}{2} \sigma^{2} \widehat{\beta}^{2}(u)-a_{\ell} \\
\widehat{\beta}(0) & =b \tag{D.7}
\end{align*}
$$

is given by

$$
\begin{equation*}
\widehat{\beta}\left(\kappa, \sigma, a_{\ell}, b ; u\right)=a_{\ell} \cdot \beta\left(\kappa, \sigma \sqrt{a_{\ell}}, \frac{b}{a_{\ell}} ; u\right), \tag{D.8}
\end{equation*}
$$

where the function $\beta(\kappa, \sigma, b ; u)$ is given by (D.4).

Proof. Let $g(u)=\frac{\widehat{\beta}(u)}{a_{\ell}}$. Then the function $g(u)$ satisfies the Riccati equation (D.7) with coefficient $\sigma$ and initial value $b$ replaced by $\sigma \sqrt{a_{\ell}}$ and $\frac{b}{a_{\ell}}$ respectively. Thus $g(u)=\beta\left(\kappa, \sigma \sqrt{a_{\ell}}, \frac{b}{a_{\ell}} ; u\right)$ and hence the solution (D.8) follows.

Lemma D.4. Assume the default intensities $\xi^{(A)}$ and $\xi^{(B)}$ of the two counterparties to be CIR processes (i.e., the elasticity factor $\widehat{\rho}=\frac{1}{2}$ in (2)). Define the conditional expectations:

$$
\begin{equation*}
Q_{t, T} g\left(x_{A}, x_{B}\right):=\mathbb{E}\left[\exp \left(-\int_{t}^{T} \ell\left(\xi_{s}^{(A)}, \xi_{s}^{(B)}\right) \mathrm{d} s\right) g\left(\xi_{T}^{(A)}, \xi_{T}^{(B)}\right) \mid \xi_{t}^{(A)}=x_{A}, \xi_{t}^{(B)}=x_{B}\right] \tag{D.9}
\end{equation*}
$$

where $\ell\left(x_{A}, x_{B}\right)$ and $g\left(x_{A}, x_{B}\right)$ are two measurable functions satisfying the form specified as in the following lemma. Assume that the functions $\ell\left(x_{A}, x_{B}\right)$ and $g\left(x_{A}, x_{B}\right)$ are of the following forms on $\left(x_{A}, x_{B}\right) \in \mathbb{R}_{+}^{2}$,

$$
\begin{align*}
\ell\left(x_{A}, x_{B}\right) & =a_{\ell} x_{A}+b_{\ell} x_{B}+c_{\ell} \\
g\left(x_{A}, x_{B}\right) & =\left(a_{g}+b_{g} x_{A}+c_{g} x_{B}\right) e^{d_{g}+e_{g} x_{A}+f_{g} x_{B}} \tag{D.10}
\end{align*}
$$

where $d_{g}, e_{g}, f_{g}$ and $a_{i}, b_{i}, c_{i}$ for $i \in\{\ell, g\}$ are real constants. Then we have

$$
\begin{align*}
Q_{t, T} g\left(x_{A}, x_{B}\right)= & {\left[\theta_{A B}(T-t)+\theta_{A}(T-t) x_{A}+\theta_{B}(T-t) x_{B}\right] } \\
& \times e^{\beta_{A B}(T-t)+\beta_{A}(T-t) x_{A}+\beta_{B}(T-t) x_{B}}, \quad 0 \leq t \leq T \tag{D.11}
\end{align*}
$$

where the unspecified functions in (D.11) satisfy the following generalized Riccati equations:

$$
R\left(a_{\ell}, b_{\ell}, c_{\ell}\right):\left\{\begin{array}{l}
-\beta_{A}^{\prime}(u)-\kappa_{A} \beta_{A}(u)+\frac{1}{2} \sigma_{A}^{2} \beta_{A}^{2}(u)-a_{\ell}=0  \tag{D.12}\\
-\beta_{B}^{\prime}(u)-\kappa_{B} \beta_{B}(u)+\frac{1}{2} \sigma_{B}^{2} \beta_{B}^{2}(u)-b_{\ell}=0 \\
\alpha_{A} \beta_{A}(u)+\alpha_{B} \beta_{B}(u)-\lambda-c_{\ell}+\widehat{\lambda}_{c} \Phi\left(c_{A} \beta_{A}(u), c_{B} \beta_{B}(u)\right) \\
\quad+\widehat{\lambda}_{A} \widetilde{\Phi}\left(d_{A} \beta_{A}(u), 0\right)+\widehat{\lambda}_{B} \widetilde{\Phi}\left(0, d_{B} \beta_{B}(u)\right)=\beta_{A B}^{\prime}(u)
\end{array}\right.
$$

and

$$
\left\{\begin{array}{l}
-\theta_{A}^{\prime}(u)-\kappa_{A} \theta_{A}(u)+\sigma_{A}^{2} \theta_{A}(u) \beta_{A}(u)=0  \tag{D.13}\\
-\theta_{B}^{\prime}(u)-\kappa_{B} \theta_{B}(u)+\sigma_{B}^{2} \theta_{B}(u) \beta_{B}(u)=0 \\
\alpha_{A} \theta_{A}(u)+\alpha_{B} \theta_{B}(u)+\widehat{\lambda}_{c} c_{A} \theta_{A}(u) \frac{\partial \Phi\left(c_{A} \beta_{A}(u), c_{B} \beta_{B}(u)\right)}{\partial \theta_{A}} \\
\quad+\widehat{\lambda}_{c} c_{B} \theta_{B}(u) \frac{\partial \Phi\left(c_{A} \beta_{A}(u), c_{B} \beta_{B}(u)\right)}{\partial \theta_{B}}+\widehat{\lambda}_{A} d_{A} \theta_{A}(u) \frac{\partial \widetilde{\Phi}\left(d_{A} \beta_{A}(u), 0\right)}{\partial \theta_{A}} \\
\quad+\widehat{\lambda}_{B} d_{B} \theta_{B}(u) \frac{\partial \widetilde{\Phi}\left(0, d_{B} \beta_{B}(u)\right)}{\partial \theta_{B}}=\theta_{A B}^{\prime}(u) .
\end{array}\right.
$$

Here the function $\Phi\left(\theta_{A}, \theta_{B}\right)$ denotes the moment generating function of the bivariate random variable $\left(Y_{1}^{(A)}, Y_{1}^{(B)}\right)$ defined by

$$
\begin{equation*}
\Phi\left(\theta_{A}, \theta_{B}\right)=\int_{\mathbb{R}_{+}^{2}} e^{\theta_{A} y_{A}+\theta_{B} y_{B}} F_{A B}\left(\mathrm{~d} y_{A}, \mathrm{~d} y_{B}\right), \quad \theta_{A}, \theta_{B} \leq 0 \tag{D.14}
\end{equation*}
$$

Similarly, $\widetilde{\Phi}\left(\theta_{A}, \theta_{B}\right)$ denotes the moment generating function of the bivariate random variable $\left(\widetilde{Y}_{1}^{(A)}, \widetilde{Y}_{1}^{(B)}\right)$ defined by

$$
\begin{equation*}
\widetilde{\Phi}\left(\theta_{A}, \theta_{B}\right)=\int_{\mathbb{R}_{+}^{2}} e^{\theta_{A} \widetilde{y}_{A}+\theta_{B} \widetilde{y}_{B}} \widetilde{F}_{A B}\left(\mathrm{~d} \widetilde{y}_{A}, \mathrm{~d} \widetilde{y}_{B}\right), \quad \theta_{A}, \theta_{B} \leq 0 \tag{D.15}
\end{equation*}
$$

Here, $F_{A B}\left(\mathrm{~d} y_{A}, \mathrm{~d} y_{B}\right)$ and $\widetilde{F}_{A B}\left(\mathrm{~d} \widetilde{y}_{A}, \mathrm{~d} \widetilde{y}_{B}\right)$ are the joint distribution functions of $\left(Y_{1}^{(A)}, Y_{1}^{(B)}\right)$, and $\left(\widetilde{Y}_{1}^{(A)}, \widetilde{Y}_{1}^{(B)}\right)$ respectively.

In addition, $\lambda=\widehat{\lambda}_{A}+\widehat{\lambda}_{B}+\widehat{\lambda}_{c}$, with $\widehat{\lambda}_{A}$ and $\widehat{\lambda}_{B}$ being the individual intensities associated to counterparties $A$ and $B$, while $\widehat{\lambda}_{c}$ is the intensity of the common Poisson process. The initial conditions of the unspecified functions in (D.11) are given by

$$
\begin{equation*}
\theta_{A B}(0)=a_{g}, \theta_{A}(0)=b_{g}, \theta_{B}(0)=c_{g}, \beta_{A B}(0)=d_{g}, \beta_{A}(0)=e_{g}, \beta_{B}(0)=f_{g} \tag{D.16}
\end{equation*}
$$

Proof. Applying the Feynman-Kac formula to (D.9), it follows that the function $Q_{t, T} g\left(x_{A}, x_{B}\right)$ satisfies on $\left(x_{A}, x_{B}\right) \in \mathbb{R}_{+}^{2}$,

$$
\left(\frac{\partial}{\partial t}+\mathcal{L}\right) f\left(t, x_{A}, x_{B}\right)=\ell\left(x_{A}, x_{B}\right) f\left(t, x_{A}, x_{B}\right)
$$

$$
\begin{equation*}
f\left(T, x_{A}, x_{B}\right)=g\left(x_{A}, x_{B}\right) \tag{D.17}
\end{equation*}
$$

where the integro-differential operator $\mathcal{L}$ acting on $h \in C^{2}\left(\mathbb{R}_{+}^{2}\right)$ is given by

$$
\begin{align*}
\mathcal{L} h\left(x_{A}, x_{B}\right)= & \frac{1}{2} \sigma_{A}^{2} x_{A} \frac{\partial^{2} h}{\partial x_{A}^{2}}\left(x_{A}, x_{B}\right)+\frac{1}{2} \sigma_{B}^{2} x_{B} \frac{\partial^{2} h}{\partial x_{B}^{2}}\left(x_{A}, x_{B}\right) \\
& +\left(\alpha_{A}-\kappa_{A} x_{A}\right) \frac{\partial h}{\partial x_{A}}\left(x_{A}, x_{B}\right)+\left(\alpha_{B}-\kappa_{B} x_{B}\right) \frac{\partial h}{\partial x_{B}}\left(x_{A}, x_{B}\right)-\lambda h\left(x_{A}, x_{B}\right) \\
& +\widehat{\lambda}_{A} \int_{\mathbb{R}_{+}} h\left(x_{A}+d_{A} \widetilde{y}_{A}, x_{B}\right) \widetilde{F}_{A}\left(\mathrm{~d} \widetilde{y}_{A}\right)+\widehat{\lambda}_{B} \int_{\mathbb{R}_{+}} h\left(x_{A}, x_{B}+d_{B} \widetilde{y}_{B}\right) \widetilde{F}_{B}\left(\mathrm{~d} \widetilde{y}_{B}\right) \\
& +\widehat{\lambda}_{c} \int_{\mathbb{R}_{+}^{2}} h\left(x_{A}+c_{A} y_{A}, x_{B}+c_{B} y_{B}\right) F_{A B}\left(\mathrm{~d} y_{A}, \mathrm{~d} y_{B}\right) . \tag{D.18}
\end{align*}
$$

Plugging the solution form (D.11) into the PIDE (D.17), we obtain

$$
\begin{aligned}
\frac{\partial f}{\partial t}\left(t, x_{A}, x_{B}\right)= & -f\left(t, x_{A}, x_{B}\right)\left[\beta_{A B}^{\prime}(T-t)+\beta_{A}^{\prime}(T-t) x_{A}+\beta_{B}^{\prime}(T-t) x_{B}\right] \\
& -\left(\theta_{A B}^{\prime}(T-t)+\theta_{A}^{\prime}(T-t) x_{A}+\theta_{B}^{\prime}(T-t) x_{B}\right) e^{\beta_{A B}(T-t)+\beta_{A}(T-t) x_{A}+\beta_{B}(T-t) x_{B}} \\
\frac{\partial f}{\partial x_{A}}\left(t, x_{A}, x_{B}\right)= & f\left(t, x_{A}, x_{B}\right) \beta_{A}(T-t)+\theta_{A}(T-t) e^{\beta_{A B}(T-t)+\beta_{A}(T-t) x_{A}+\beta_{B}(T-t) x_{B}} \\
\frac{\partial f}{\partial x_{B}}\left(t, x_{A}, x_{B}\right)= & f\left(t, x_{A}, x_{B}\right) \beta_{B}(T-t)+\theta_{B}(T-t) e^{\beta_{A B}(T-t)+\beta_{A}(T-t) x_{A}+\beta_{B}(T-t) x_{B}} \\
\frac{\partial^{2} f}{\partial x_{A}^{2}}\left(t, x_{A}, x_{B}\right)= & f\left(t, x_{A}, x_{B}\right) \beta_{A}^{2}(T-t)+2 \theta_{A}(T-t) \beta_{A}(T-t) e^{\beta_{A B}(T-t)+\beta_{A}(T-t) x_{A}+\beta_{B}(T-t) x_{B}} \\
\frac{\partial^{2} f}{\partial x_{B}^{2}}\left(t, x_{A}, x_{B}\right)= & f\left(t, x_{A}, x_{B}\right) \beta_{B}^{2}(T-t)+2 \theta_{B}(T-t) \beta_{B}(T-t) e^{\beta_{A B}(T-t)+\beta_{A}(T-t) x_{A}+\beta_{B}(T-t) x_{B}}
\end{aligned}
$$

and

$$
\begin{aligned}
& \int_{\mathbb{R}_{+}} f\left(t, x_{A}+d_{A} \widetilde{y}_{A}, x_{B}\right) \widetilde{F}_{A}\left(\mathrm{~d} \widetilde{y}_{A}\right)=f\left(t, x_{A}, x_{B}\right) \int_{\mathbb{R}_{+}} e^{\beta_{A}(T-t) d_{A} \widetilde{y}_{A}} \widetilde{F}_{A}\left(\mathrm{~d} \widetilde{y}_{A}\right) \\
& +e^{\beta_{A B}(T-t)+\beta_{A}(T-t) x_{A}+\beta_{B}(T-t) x_{B}} \int_{\mathbb{R}_{+}} \theta_{A}(T-t) d_{A} \widetilde{y}_{A} e^{\beta_{A}(T-t) d_{A} \widetilde{y}_{A}} \widetilde{F}_{A}\left(\mathrm{~d} \widetilde{y}_{A}\right) \\
& \int_{\mathbb{R}_{+}} f\left(t, x_{A}, x_{B}+d_{B} \widetilde{y}_{B}\right) \widetilde{F}_{B}\left(\mathrm{~d} \widetilde{y}_{B}\right)=f\left(t, x_{A}, x_{B}\right) \int_{\mathbb{R}_{+}} e^{\beta_{B}(T-t) d_{B} \widetilde{y}_{B}} \widetilde{F}_{B}\left(\mathrm{~d} \widetilde{y}_{B}\right) \\
& +e^{\beta_{A B}(T-t)+\beta_{A}(T-t) x_{A}+\beta_{B}(T-t) x_{B}} \int_{\mathbb{R}_{+}} \theta_{B}(T-t) d_{B} \widetilde{y}_{B} e^{\beta_{B}(T-t) d_{B} \widetilde{y}_{B}} \widetilde{F}_{B}\left(\mathrm{~d} \widetilde{y}_{B}\right) \\
& \int_{\mathbb{R}_{+}^{2}} f\left(t, x_{A}+c_{A} y_{A}, x_{B}+c_{B} y_{B}\right) F_{A B}\left(\mathrm{~d} y_{A}, \mathrm{~d} y_{B}\right)=f\left(t, x_{A}, x_{B}\right) \\
& \quad \times \int_{\mathbb{R}_{+}^{2}} e^{\beta_{A}(T-t) c_{A} y_{A}+\beta_{B}(T-t) c_{B} y_{B}} F_{A B}\left(\mathrm{~d} y_{A}, \mathrm{~d} y_{B}\right) \\
& \quad+e^{\beta_{A B}(T-t)+\beta_{A}(T-t) x_{A}+\beta_{B}(T-t) x_{B}} \\
& \quad \times \int_{\mathbb{R}_{+}^{2}}\left(\theta_{A}(T-t) c_{A} y_{A}+\theta_{B}(T-t) c_{B} y_{B}\right) e^{\beta_{A}(T-t) c_{A} y_{A}+\beta_{B}(T-t) c_{B} y_{B}} F_{A B}\left(\mathrm{~d} y_{A}, \mathrm{~d} y_{B}\right)
\end{aligned}
$$

In order for (D.17) to hold, we need that for all $u \in[0, T]$ and $\left(x_{A}, x_{B}\right) \in \mathbb{R}_{+}^{2}$, the following two equalities are satisfied

$$
\begin{align*}
x_{A}[ & \left.-\beta_{A}^{\prime}(u)-\kappa_{A} \beta_{A}(u)+\frac{1}{2} \sigma_{A}^{2} \beta_{A}^{2}(u)-a_{\ell}\right]+x_{B}\left[-\beta_{B}^{\prime}(u)-\kappa_{B} \beta_{B}(u)+\frac{1}{2} \sigma_{B}^{2} \beta_{B}^{2}(u)-b_{\ell}\right] \\
& +\alpha_{A} \beta_{A}(u)+\alpha_{B} \beta_{B}(u)-\beta_{A B}^{\prime}(u)-\lambda-c_{\ell}+\widehat{\lambda}_{c} \Phi\left(c_{A} \beta_{A}(u), c_{B} \beta_{B}(u)\right) \\
& +\widehat{\lambda}_{A} \widetilde{\Phi}\left(d_{A} \beta_{A}(u), 0\right)+\widehat{\lambda}_{B} \widetilde{\Phi}\left(0, d_{B} \beta_{B}(u)\right)=0 \tag{D.19}
\end{align*}
$$

and

$$
\begin{aligned}
& +x_{A}\left[-\theta_{A}^{\prime}(u)-\kappa_{A} \theta_{A}(u)+\sigma_{A}^{2} \theta_{A}(u) \beta_{A}(u)\right]+x_{B}\left[-\theta_{B}^{\prime}(u)-\kappa_{B} \theta_{B}(u)+\sigma_{B}^{2} \theta_{B}(u) \beta_{B}(u)\right] \\
& -\theta_{A B}^{\prime}(u)+\alpha_{A} \theta_{A}(u)+\alpha_{B} \theta_{B}(u)
\end{aligned}
$$

$$
\begin{align*}
& +\widehat{\lambda}_{c} c_{A} \theta_{A}(u) \frac{\partial \Phi\left(c_{A} \beta_{A}(u), c_{B} \beta_{B}(u)\right)}{\partial \theta_{A}}+\widehat{\lambda}_{c} c_{B} \theta_{B}(u) \frac{\partial \Phi\left(c_{A} \beta_{A}(u), c_{B} \beta_{B}(u)\right)}{\partial \theta_{B}} \\
& +\widehat{\lambda}_{A} d_{A} \theta_{A}(u) \frac{\partial \widetilde{\Phi}\left(d_{A} \beta_{A}(u), 0\right)}{\partial \theta_{A}}+\widehat{\lambda}_{B} d_{B} \theta_{B}(u) \frac{\partial \widetilde{\Phi}\left(0, d_{B} \beta_{B}(u)\right)}{\partial \theta_{B}}=0 \tag{D.20}
\end{align*}
$$

Hence the unspecified functions in (D.11) satisfy the Riccati equations (D.12) and (D.13). From the terminal condition in (D.17), we further have the initial conditions given by (D.16). This completes the proof of the lemma.

The following results give the explicit solutions to the generalized Riccati equations (D.12) and (D.13).
Lemma D.5. Assume that the initial values $\beta_{A}(0)=e_{g} \leq 0$ and $\beta_{B}(0)=f_{g} \leq 0$. If the constants $a_{\ell}, b_{\ell}>0$, then the generalized Riccati equation (D.12) admits the explicit solution given by

$$
\begin{align*}
& \beta_{A}(u)=a_{\ell} \cdot \beta\left(\kappa_{A}, \sigma_{A} \sqrt{a_{\ell}}, \frac{e_{g}}{a_{\ell}} ; u\right) \\
& \beta_{B}(u)=b_{\ell} \cdot \beta\left(\kappa_{B}, \sigma_{B} \sqrt{b_{\ell}}, \frac{f_{g}}{b_{\ell}} ; u\right) \tag{D.21}
\end{align*}
$$

and

$$
\begin{align*}
& \beta_{A B}(u)=\int_{0}^{u}\left[\alpha_{A} \beta_{A}(v)+\alpha_{B} \beta_{B}(v)+\widehat{\lambda}_{c} \Phi\left(c_{A} \beta_{A}(v), c_{B} \beta_{B}(v)\right)\right. \\
&\left.+\widehat{\lambda}_{A} \widetilde{\Phi}\left(d_{A} \beta_{A}(v), 0\right)+\widehat{\lambda}_{B} \widetilde{\Phi}\left(0, d_{B} \beta_{B}(v)\right)\right] \mathrm{d} v+d_{g}-\left(\lambda+c_{\ell}\right) u \tag{D.22}
\end{align*}
$$

where the function $\beta(\kappa, \sigma, b ; u)$ is given by (D.4) and $\Phi\left(\theta_{A}, \theta_{B}\right), \widetilde{\Phi}\left(\theta_{A}, \theta_{B}\right)$ are the moment generating functions defined in (D.14) and (D.15) with $\theta_{A}, \theta_{B} \leq 0$.

Proof. The solutions $\left(\beta_{A}(u), \beta_{B}(u)\right)$ given in (D.21) can be obtained by an immediate application of Lemma D.3 Note that the bivariate random variables $\left(Y_{1}^{(A)}, Y_{1}^{(B)}\right)$ and $\left(\widetilde{Y}_{1}^{(A)}, \widetilde{Y}_{1}^{(B)}\right)$ associated to the jumps of the counterparties, are assumed to take values on $\mathbb{R}_{+}^{2}$. Then the corresponding moment generating function $\Phi\left(\theta_{A}, \theta_{B}\right)$ exists if $\theta_{A}, \theta_{B} \leq 0$. From Lemma D.2, it follows that the solution (D.8) given by

$$
\beta(\kappa, \sigma, b ; u)=B(\kappa, \sigma ; u)+e^{\phi(u)} \frac{1}{\frac{1}{b}-\frac{\sigma^{2}}{2} \int_{0}^{u} e^{\phi(v)} \mathrm{d} v} \leq 0, \quad \forall 0 \leq u \leq T
$$

provided the initial value $\beta(\kappa, \sigma, b ; 0)=b \leq 0$. This is because $B(\kappa, \sigma ; u) \leq 0$ for all $0 \leq u \leq T$, by (D.2). Hence if the initial values $\beta_{A}(0)=e_{g} \leq 0$ and $\beta_{B}(0)=f_{g} \leq 0$ in (D.21), then $\Phi\left(c_{A} \beta_{A}(u), c_{B} \beta_{B}(u)\right)$, $\widetilde{\Phi}\left(d_{A} \beta_{A}(u), 0\right)$ and $\widetilde{\Phi}\left(0, d_{B} \beta_{B}(u)\right)$ exist since $c_{A}, c_{B}, d_{A}, d_{B}>0$, and can be computed using (D.14) and (D.15). Hence, we can derive the solution $\beta_{A B}(u)$ to the third equation in (D.12), which is given by (D.22).

Based on the above explicit solution to the generalized Riccati equation (D.12), we immediately have
Lemma D.6. The generalized Riccati equation (D.13) admits the explicit solution given by

$$
\begin{align*}
& \theta_{A}(u)=b_{g} \exp \left(-\kappa_{A} u+\sigma_{A}^{2} \int_{0}^{u} \beta_{A}(v) \mathrm{d} v\right) \\
& \theta_{B}(u)=c_{g} \exp \left(-\kappa_{B} u+\sigma_{B}^{2} \int_{0}^{u} \beta_{B}(v) \mathrm{d} v\right) \tag{D.23}
\end{align*}
$$

and

$$
\begin{align*}
\theta_{A B}(u)= & a_{g}+\int_{0}^{u}\left[\alpha_{A} \theta_{A}(v)+\alpha_{B} \theta_{B}(v)+\widehat{\lambda}_{c} c_{A} \theta_{A}(v) \frac{\partial \Phi\left(c_{A} \beta_{A}(v), c_{B} \beta_{B}(v)\right)}{\partial \theta_{A}}\right. \\
& +\widehat{\lambda}_{c} c_{B} \theta_{B}(v) \frac{\partial \Phi\left(c_{A} \beta_{A}(v), c_{B} \beta_{B}(v)\right)}{\partial \theta_{B}}+\widehat{\lambda}_{A} d_{A} \theta_{A}(v) \frac{\partial \widetilde{\Phi}\left(d_{A} \beta_{A}(v), 0\right)}{\partial \theta_{A}} \\
& \left.+\widehat{\lambda}_{B} d_{B} \theta_{B}(v) \frac{\partial \widetilde{\Phi}\left(0, d_{B} \beta_{B}(v)\right)}{\partial \theta_{B}}\right] \mathrm{d} v, \tag{D.24}
\end{align*}
$$

where $0 \leq u \leq T$.

## E Proof of Proposition 6.2

Recall from Eq. (28) that, for $0 \leq t \leq s \leq T$,

$$
\widehat{F}(t, s)=\mathbb{E}\left[\int_{\mathcal{O}} \mathbb{E}\left[\exp \left(-\int_{t}^{s} X_{u}(\boldsymbol{p}) \mathrm{d} u\right)\right] q(\mathrm{~d} p) \eta(\mathrm{d} y) \phi_{0}(\mathrm{~d} x)\right],
$$

where the 'type' parameter $\boldsymbol{p}=(p, y, x) \in \mathcal{O}$ with $p=(\alpha, \kappa, \sigma, c, d, \widehat{\lambda}) \in \mathcal{O}_{p}$. The limit process $X(\boldsymbol{p})=$ $\left(X_{t}(\boldsymbol{p}) ; t \geq 0\right)$ is a shifted square root diffusion process given by

$$
\begin{equation*}
X_{t}(\boldsymbol{p})=x+\int_{0}^{t}\left[D(\boldsymbol{p})+\alpha-\kappa X_{u}(\boldsymbol{p})\right] \mathrm{d} u+\sigma \int_{0}^{t}\left(X_{u}(\boldsymbol{p})\right)^{1 / 2} \mathrm{~d} W_{u} \tag{E.1}
\end{equation*}
$$

where the drift $D(\boldsymbol{p})$ is given by (25), i.e., $D(\boldsymbol{p})=d y_{2} \widehat{\lambda}+c y_{1} \widehat{\lambda}_{c}$.
Note that the limit process $X(\boldsymbol{p})$ is an affine process. Using Lemma D.4, we have

$$
\mathbb{E}\left[\exp \left(-\int_{t}^{s} X_{u}(\boldsymbol{p}) \mathrm{d} u\right) \mid X_{t}(\boldsymbol{p})=x\right]=\exp \left(A_{\boldsymbol{p}}(s-t)+B_{\boldsymbol{p}}(s-t) x\right), \quad 0 \leq t \leq s,
$$

where the functions $A_{p}(u)$ and $B_{p}(u)$ satisfy the following system of Riccati equations

$$
\left\{\begin{array}{l}
-A_{p}^{\prime}(u)+(D(\boldsymbol{p})+\alpha) B_{\boldsymbol{p}}(u)=0,  \tag{E.2}\\
-B_{\boldsymbol{p}}^{\prime}(u)-\kappa B_{\boldsymbol{p}}(u)+\frac{1}{2} \sigma^{2} B_{\boldsymbol{p}}^{2}(u)-1=0,
\end{array}\right.
$$

with the following initial conditions

$$
\begin{equation*}
A_{p}(0)=B_{p}(0)=0 \tag{E.3}
\end{equation*}
$$

From Lemma D. 1 it follows that the solution to the second equation of the Riccati system (E.2) is given by

$$
B_{\boldsymbol{p}}(u)=-\frac{2\left(e^{\varpi u}-1\right)}{2 \varpi+(\kappa+\varpi)\left(e^{\varpi u}-1\right)}, \quad 0 \leq u \leq s,
$$

where $\varpi=\sqrt{\kappa^{2}+2 \sigma^{2}}$. Using the first equation of the Riccati system (E.2) and the initial conditions (E.3), it follows that

$$
e^{A_{\boldsymbol{p}}(s)}=\exp \left[(\alpha+D(\boldsymbol{p})) \int_{0}^{s} B_{\boldsymbol{p}}(u) \mathrm{d} u\right],
$$

where

$$
\int_{0}^{s} B_{p}(u) \mathrm{d} u=\frac{2 T}{\varpi-\kappa}+\frac{4}{\varpi^{2}-\kappa^{2}} \log \left[\frac{2 \varpi}{\left(1+e^{\varpi T}\right) \varpi+\left(e^{\varpi T}-1\right) \kappa}\right]
$$

Hence, we obtain

$$
\begin{array}{rl}
\int_{\mathcal{O}} & \mathbb{E}\left[\exp \left(-\int_{t}^{s} X_{u}(\boldsymbol{p}) \mathrm{d} u\right)\right] q(\mathrm{~d} p) \eta(\mathrm{d} y) \phi_{0}(\mathrm{~d} x) \\
& =\int_{\mathcal{O}} \exp \left(A_{\boldsymbol{p}}(s-t)+B_{\boldsymbol{p}}(s-t) x\right) q(\mathrm{~d} p) \eta(\mathrm{d} y) \phi_{0}(\mathrm{~d} x) \\
& =e^{x^{*} B_{p^{*}(s-t)}} \int_{\mathbb{R}_{+}^{2}} e^{A_{\left(p^{*}, y_{1}, y_{2}\right)}(s-t)} \delta\left(y_{1}-Y\right) \mathrm{d} y_{1} \delta\left(y_{2}-\widetilde{Y}\right) \mathrm{d} y_{2} \\
& =e^{x^{*} B_{p^{*}(s-t)+A_{\left(p^{*}, Y, \tilde{Y}\right)}(s-t)}} \\
=e^{x^{*} B_{p^{*}(s-t)}} \exp \left(\left[\alpha^{*}+d^{*} \hat{\lambda}^{*} \widetilde{Y}+c^{*} \widehat{\lambda}_{c} Y\right] \int_{0}^{s-t} B_{p^{*}}(u) \mathrm{d} u\right) .
\end{array}
$$

Using the independence of the exponential random variables $Y$ and $\widetilde{Y}$, we have

$$
\widehat{F}(t, s)=e^{x^{*} B_{p^{*}}(s-t)} \mathbb{E}\left[\exp \left(\left[\alpha^{*}+d^{*} \widehat{\lambda}^{*} \widetilde{Y}+c^{*} \widehat{\lambda}_{c} Y\right] \int_{0}^{s-t} B_{p^{*}}(u) \mathrm{d} u\right)\right]
$$

$$
\begin{align*}
= & \exp \left(x^{*} B_{p^{*}}(s-t)+\alpha^{*} \int_{0}^{s-t} B_{p^{*}}(u) \mathrm{d} u\right) \\
& \times \mathbb{E}\left[\exp \left(\widetilde{Y} d^{*} \widehat{\lambda}^{*} \int_{0}^{s-t} B_{p^{*}}(u) \mathrm{d} u\right)\right] \mathbb{E}\left[\exp \left(Y c^{*} \widehat{\lambda}_{c} \int_{0}^{s-t} B_{p^{*}}(u) \mathrm{d} u\right)\right] \\
= & \exp \left(x^{*} B_{p^{*}}(s-t)+\alpha^{*} \int_{0}^{s-t} B_{p^{*}}(u) \mathrm{d} u\right) \frac{\gamma_{1}}{\gamma_{1}-c^{*} \widehat{\lambda}_{c} \int_{0}^{s-t} B_{p^{*}}(u) \mathrm{d} u} \\
& \times \frac{\gamma_{2}}{\gamma_{2}-d^{*} \widehat{\lambda}^{*} \int_{0}^{s-t} B_{p^{*}}(u) \mathrm{d} u}, \tag{E.4}
\end{align*}
$$

since $\int_{0}^{u} B_{p^{*}}(z) \mathrm{d} z<0$ for all $u>0$. Hence, the proof of the Lemma is complete.

## F Proof of Proposition 6.3

For $t \leq t_{B} \leq T$ and $\left(x_{A}, x_{B}\right) \in \mathbb{R}_{+}^{2}$, using ( (D.9), we obtain

$$
\begin{align*}
H_{1}\left(t_{B}-t, x_{A}, x_{B}\right):= & \mathbb{E}\left[\exp \left(-\int_{t}^{t_{B}}\left(\xi_{s}^{(A)}+\xi_{s}^{(B)}\right) \mathrm{d} s\right) \xi_{t_{B}}^{(B)} \mid \xi_{t}^{(A)}=x_{A}, \xi_{t}^{(B)}=x_{B}\right] \\
= & {\left[h_{1}\left(t_{B}-t\right)+h_{A}\left(t_{B}-t\right) \xi_{t}^{(A)}+h_{B}\left(t_{B}-t\right) \xi_{t}^{(B)}\right] } \\
& \times \exp \left(\widehat{h}_{1}\left(t_{B}-t\right)+\widehat{h}_{A}\left(t_{B}-t\right) \xi_{t}^{(A)}+\widehat{h}_{B}\left(t_{B}-t\right) \xi_{t}^{(B)}\right) \tag{F.1}
\end{align*}
$$

where the functions $\left(\widehat{h}_{1}(u), \widehat{h}_{A}(u), \widehat{h}_{B}(u)\right)$ satisfy the generalized Riccati equation $R(1,1,0)$ given by (D.12), while the functions $\left(h_{1}(u), h_{A}(u), h_{B}(u)\right)$ satisfy the generalized Riccati equation given by (D.13). The initial conditions are given by

$$
\begin{equation*}
h_{1}(0)=h_{A}(0)=\widehat{h}_{1}(0)=\widehat{h}_{A}(0)=\widehat{h}_{B}(0)=0, \text { and } h_{B}(0)=1 \tag{F.2}
\end{equation*}
$$

Solving the corresponding Riccati equations via Lemma D. 5 and Lemma D. 6 respectively, we have the solutions $\left(\widehat{h}_{1}(u), \widehat{h}_{A}(u), \widehat{h}_{B}(u)\right)$ and $\left(h_{1}(u), h_{A}(u), h_{B}(u)\right)$ are (51) and (52) respectively.

By virtue of Lemma D. 4 we have, for $t \leq t_{A} \leq T$,

$$
\begin{align*}
H_{2}\left(t_{A}-t, x_{A}, x_{B}\right)= & {\left[w_{1}\left(t_{A}-t\right)+w_{A}\left(t_{A}-t\right) x_{A}+w_{B}\left(t_{A}-t\right) x_{B}\right] } \\
& \times \exp \left(\widehat{w}_{1}\left(t_{A}-t\right)+\widehat{w}_{A}\left(t_{A}-t\right) x_{A}+\widehat{w}_{B}\left(t_{A}-t\right) x_{B}\right), \tag{F.3}
\end{align*}
$$

where the functions $\left(\widehat{w}_{1}(u), \widehat{w}_{A}(u), \widehat{w}_{B}(u)\right)$ satisfy the generalized Riccati equation $R(1,1,0)$ given by (D.12), while the functions $\left(w_{1}(u), w_{A}(u), w_{B}(u)\right)$ satisfy the generalized Riccati equation given by (D.13). The initial conditions are given by

$$
\begin{equation*}
w_{1}(0)=w_{B}(0)=\widehat{w}_{1}(0)=\widehat{w}_{A}(0)=\widehat{w}_{B}(0)=0, \text { and } w_{A}(0)=1 \tag{F.4}
\end{equation*}
$$

Solving the corresponding Riccati equations by using Lemma D. 5 and Lemma D. 6 respectively, we have the solutions $\left(\widehat{w}_{1}(u), \widehat{w}_{A}(u), \widehat{w}_{B}(u)\right)$ and $\left(w_{1}(u), w_{A}(u), w_{B}(u)\right)$ are given by (54) and (55) respectively. Hence, the proof of the proposition is complete.

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