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Fragility of local martingale diffusion models of arbitrage and bubbles*

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Abstract For any positive diffusion with minimal regularity, there exists a semimartingale, with uniformly close paths, which is a martingale under an equivalent probability. As a result, in models of asset prices based on such diffusions, arbitrage and bubbles alike disappear under proportional transaction costs, or under small model misspecifications. Thus, local martingale models of arbitrage and bubbles are not robust to small trading and monitoring frictions.

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Introduction

Several recent papers use strict local martingales (local martingales that are not martingales) to model both arbitrage and bubbles in asset prices. Bubbles, this literature suggests, arise if prices are strict local martingales under risk neutral measures. Arbitrage appears if stochastic discount factors are strict local martingales themselves. The idea is almost irresistible: a single delicate concept in Mathematics explains two distinct concepts in Finance, and the link is in the classical duality between payoffs and pricing measures.

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But do strict local martingales offer reliable models for arbitrage and bubbles?

This paper partially answers this question, by interpreting "reliable" as robust to small model mis-specifications or, equivalently, to proportional transaction costs. We show that, once these frictions are acknowledged, bubbles and arbitrage opportunities disappear from a large class of diffusion models. In this sense, models based on strict local martingales generate fragile types of bubbles and arbitrage.

This result offers a word of caution for researchers who seek to employ strict local martingales to design models of financial markets with bubbles and arbitrage opportunities. We show that these phenomena do not arise from the strict local martingale property of asset prices alone, but also from the *joint* assumptions that the market is free of any trading frictions *and* that asset prices are continuously observed with infinite precision. Although arbitrage and bubbles are normally seen as deviations from the perfect market paradigm, in these models they survive only if the market is perfect enough, in that monitoring and trading frictions are absent. Thus, future research in this area will yield more robust results with models based on stochastic processes that lie outside the scope of our result.

A mathematical interpretation of the main result hinges on the subtle but crucial difference in which martingales and local martingales have zero drift. Martingales have zero *global* drift – its expected conditional increments are null. By contrast, a diffusion process is a local martingale when its *local* drift (informally, the coefficient of the dt term) vanishes. While a zero global drift implies a zero local drift, the reverse fails precisely with strict local martingales, in which a nonzero global drift is hidden in the unbounded diffusion term, even with null local drift. We show that a small perturbation in the process and the probability measure can remove this hidden global drift as well.

Importantly, the main result is relevant for both complete and incomplete markets. In a complete market the bubble property boils down to a superreplication price that is strictly lower than the current asset price, and we show that arbitrary small perturbations in the price process make the superreplication price equal to the current asset price. In an incomplete market, we show that *for any* risk neutral measure a small perturbation will restore the true martingale property, and therefore the same issue remains.

Overall, our results bring mixed news. The bad news is that designing models of arbitrage and bubbles that are robust to small frictions is difficult. Perhaps such models will involve strict local martingales, but will either depart from common diffusions, or will include trading constraints or other features that place them beyond the setting of this paper. The good news is that common diffusions with the strict local martingale property, such as quadratic volatility models (for example, see Andersen (2011)) are always free of arbitrage and bubbles, when minimal frictions are accounted for. Thus, if a diffusion fits the data well, is tractable, or both, it need not be excluded from applications for being a strict local martingale. Likewise, if arbitrage or relative arbitrage arises in a model from a stochastic discount factor that is a strict local martingale, then the model is still arbitrage-free with small frictions.

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1 Literature Review

Although the literature on asset bubbles is vast and decades old, the idea of identifying bubbles with local martingales is relatively recent, and has gained momentum only in the last few years. Loewenstein and Willard (2000) are the first to link bubbles to local martingales. In a frictionless, complete market, based on a continuous-time diffusion, they show that bubbles may occur as a result of wealth constraints. Heston, Loewenstein and Willard (2007) further investigate the consequences of bubbles on derivatives prices, and link it to the non-uniqueness of solutions to the standard valuation PDE, which may arise even in common stochastic volatility models. Cox and Hobson (2005) provide diffusion examples of bubbles, and note how standard price relations such as put-call parity may fail if the underlying assets are local martingales. Pal and Protter (2010) construct continuous strict local martingales with Doob's h-transforms, and study connections with bubbles. Madan and Yor (2006) propose to remedy the ostensible option mispricing using an alternative definition of price. Jarrow, Protter and Shimbo (2007) classify bubbles in complete markets into three types. Under a *no dominance* assumption, which entails that no asset can be replicated by trading in the other ones, Jarrow, Protter and Shimbo (2010) argue that bubbles by local martingales arise only in incomplete markets. Bubbles are also implicit in the benchmark pricing approach of Platen (2006), if the underlying market has no equivalent martingale measure. Protter (2013) surveys comprehensively the literature on bubbles by local martingales.

The literature on arbitrage with local martingales starts with Delbaen and Schachermayer (1995), who show that arbitrage arises in the three-dimensional Bessel process, even though its reciprocal is a local martingale, highlighting the importance of the numeraire for arbitrage considerations. Fernholz, Karatzas and Kardaras (2005) derive the existence of arbitrage and relative arbitrage in frictionless markets satisfying a *diversity* property, which means that no asset can overtake in size the remaining ones. Fernholz and Karatzas (2010) study relative arbitrage strategies that are optimal in the sense of maximal multiplication of wealth, and Ruf (2011) characterizes optimal hedging strategies in the presence of arbitrage.

2 Arbitrage and Bubbles

Consider a market with one safe and d risky assets. The safe asset is the numeraire, hence its price is simply one. The prices of the d risky assets are represented by an $\mathbb{R}^d$-valued semimartingale $(S_t)_{t\in[0,T]}$ with positive components, defined on and adapted to a filtered probability space $(\Omega,\mathscr{F},(\mathscr{F}_t)_{t\in[0,T]},P)$ satisfying the usual assumptions of right-continuity and completeness.

Both arbitrage and bubbles depend on the concept of admissible strategy, a strategy that requires a finite credit line.

Definition 2.1 An *admissible strategy* is a predictable, S-integrable process H, such that, for some $c>0$, $\int_0^t H(u)dS(u) \geq -c$ for all $t\in[0,T]$. An asset price S:

i) has *arbitrage* if some admissible strategy H with payoff $X=\int_0^T H(u)dS(u)$ satisfies $P(X\geq 0)=1$ and $P(X>0)>0$.

ii) has *bubbles* if some admissible strategy H satisfies $x+\int_0^T H(u)dS(u) \geq S^i(T)$ for some $i \in \{1,\dots,d\}$ and $x < S^i(0)$.

In plain English, an *arbitrage* is a strategy that delivers a positive nonzero payoff from zero initial capital. A *bubble* arises if some strategy H dominates the payoff $S^i(T)$ starting from less than $S^i(0)$. When this is the case, the i-th asset is "overpriced" at time 0, in that holding it from 0 to T is an inferior plan compared to trading with the strategy H.

The definition of arbitrage is standard. The above definition of bubble first appears in Heston et al. (2007), for both complete and incomplete markets. In a complete market this definition is equivalent to the one used by Loewenstein and Willard (2000) and Cox and Hobson (2005), who define an asset price as a bubble if it is a strict local martingale under the risk-neutral measure[1]. Again in complete markets, Jarrow, Protter and Shimbo (2007) classify bubbles into three types. Definition 2.1 corresponds to bubbles of type three in their setting, while types one and two may arise with an infinite trading horizon. In all these papers, a bubble simply means that the market price is higher than its "fundamental" price, defined as superreplication price.

By contrast, Jarrow et al. (2010) define the fundamental price as the expected value of dividends (including eventual liquidation) under a specific equivalent local martingale measure Q_i, which changes at exogenously specified times τ_i. This definition boils down to the previous one in a complete market, since the martingale measure is unique. In an incomplete market Jarrow et al. (2010) depart from the definition of Heston et al. (2007) by requiring that the market price is greater than some specific risk-neutral expectation (the one corresponding to Q_i), not necessarily all of them. Still, section 5 below shows that the main result is relevant also for this weaker definition. The statistical applications of Jarrow, Kchia and Protter (2011*b*,*c*,*a*) are focused on one-dimensional diffusion models with a unique equivalent local martingale measure, for which the above definition is relevant.

We begin the discussion by recalling that the existence of a martingale measure excludes both arbitrage and bubbles.

Fact 2.2 If S admits an equivalent (true) martingale measure Q, neither arbitrage nor bubbles exist.

Proof The absence of arbitrage follows from the "easy" implication of the Fundamental Theorem of Asset Pricing (Delbaen and Schachermayer, 1994). The absence of bubbles follows from a similar argument: let H be an admissible strategy that superreplicates $S^i(T)$. Then, the wealth process $X(t) = \int_0^t H(u)dS(u)$ is a Q-local martingale (Ansel and Stricker, 1994, Corollary 3.5), hence a supermartingale. Thus,

$$x \geq E_Q\left[x+\int_0^t H(u)dS(u)\right] \geq E_Q[S^i(T)] = S^i(0), \tag{2.1}$$

which contradicts the existence of bubbles.

[1] By the dual characterization of superhedging prices (Ansel and Stricker, 1994; Föllmer and Kabanov, 1998), this definition is equivalent to *ii*) in Definition 2.1.

Thus, bubbles arise if S has an equivalent local martingale measure but not a martingale measure, while arbitrage is possible only if no equivalent local martingale measure exists. The canonical examples of models with arbitrage and bubbles are respectively the three-dimensional Bessel process, and its inverse.

Example 2.3 (Arbitrage) The three-dimensional Bessel process is defined as the (unique strong) solution S of the SDE:

$$dS(t) = \frac{dt}{S(t)} + dW(t), \qquad S(0) = 1, \tag{2.2}$$

where W is a Brownian motion and the filtration is (the augmentation of the one) generated by W, see Chung and Williams (1990, p. 252). The existence of arbitrage in this model was first observed by Delbaen and Schachermayer (1995). Intuitively, the Sharpe ratio $1/S(t)$ explodes as $S(t)$ approaches zero, and arbitrage is obtained increasing positions as $S(t)$ decreases, in the spirit of contrarian strategies. Karatzas and Kardaras (2007) and Ruf (2011) show explicit examples of such strategies.

Example 2.4 (Bubble) The archetypical example of a bubble is the inverse three-dimensional Bessel process, defined as $Z(t) = 1/S(t)$, where $S(t)$ is as in Example 2.3. This process is a supermartingale, but not a martingale, and satisfies the equation:

$$dZ(t) = Z^2(t)dW(t), \qquad Z(0) = 1. \tag{2.3}$$

Ruf (2011, Example 6.3) constructs explicitly a strategy H such that $x + \int_0^T H(t)dZ(t) > Z(T)$ and $x < Z(0)$.

3 The Robustness Question

However precise they might be, models are approximations, and a crucial problem is understanding the robustness of their implications with respect to small perturbations.

Are arbitrage and bubbles robust with respect to small frictions or model perturbations?

Since the answer depends on the class of perturbations, it is important to make a choice with a sound economic interpretation. We consider (multiplicative) pathwise uniform perturbations, which have the dual meaning of small model misspecifications or proportional transaction costs.

Definition 3.1 For $\varepsilon > 0$, two strictly positive processes $S, \tilde{S}$ are *ε-close* if:

$$\frac{1}{1+\varepsilon} \le \frac{\tilde{S}(t)}{S(t)} \le 1+\varepsilon \qquad \text{a.s. for all } t \in [0,T] \tag{3.1}$$

Two $\mathbb{R}^d_{++}$-valued processes are *ε-close* if S^i and $\tilde{S}^i$ are *ε-close* for each $1 \le i \le d$.

This definition embodies model uncertainty, if prices are measured up to a proportional error ε. It also admits an interpretation based on proportional transaction costs. Let us assume in the rest of this section that $S, \tilde{S}$ are $\mathbb{R}$-valued, continuous processes.[2] Indeed, for any predictable, finite-variation strategy θ with $\theta(0) = \theta(T) = 0$, integration by parts yields:

$$\int_0^T \theta(t)d\tilde{S}(t) \geq \int_0^T \theta(t)dS(t) - \frac{\varepsilon}{1+\varepsilon}\int_0^T S(t)d\|\theta\|(t). \tag{3.2}$$

Since the right-hand side is the payoff of the strategy θ when the price is S, but transaction costs are present, it follows that the frictionless payoff for $\tilde{S}$ in the left-hand side is an optimistic estimate of the impact of transaction costs.

In the light of this definition, we now show that the robustness issue for bubbles based on strict local martingales boils down to the following mathematical question:

Question 3.2 Given a strictly positive, continuous process S, is there an ε-close process $\tilde{S}$, which is a martingale under an equivalent probability, for any $\varepsilon > 0$?

The main result of this paper is that the answer to this question is yes for a large class of diffusion processes, in one or more dimensions, which includes most commonly used models in the literature. (In section 6 we will also show an example in which the answer is no.) Before stating the result in the next section, we now explain why a positive answer has important implications both for pricing, hence for bubbles, and for investment, hence for arbitrage.

3.1 Pricing and Bubbles

If the answer to Question 3.2 is yes, any bubble in S is fragile, in two ways. First, since prices are always observed with some small measurement error, due either to asynchronous trading, price granularity, or frictions, the model is observationally indistinguishable from $\tilde{S}$, which has an equivalent martingale measure, hence has no bubbles. Second, any strategy that dominates $S(T)$ starting from $S(0)$ will not survive ε transaction costs, in the following sense:

Proposition 3.3 *If S and $\tilde{S}$ are continuous and ε-close as in Definition 3.1, and $\tilde{S}$ is a martingale under some probability Q equivalent to P, then the superreplication price of $S(T)$ with ε-transaction costs is greater or equal than $S(0)$.*[3] *Thus, the market does not have bubbles with transaction costs, in the sense of Definition 2.1 with $\int_0^T H(u)dS(u)$ replaced by $\int_0^T H(t)dS(t) - \frac{\varepsilon}{1+\varepsilon}\int_0^T S(t)d\|H\|(t)$*

[2] In fact, quasi-left continuity is sufficient, since in this case θ almost surely has no common jumps with either S or $\tilde{S}$, hence the Stieltjes integrals in (3.2) exist.

[3] To be precise, with a bid price less than or equal to $S/(1+\varepsilon)$, and an ask price greater than or equal to $S(1+\varepsilon)$.

Proof Suppose that some predictable, finite variation strategy θ, such that the value process $V(t) := \int_0^t \theta(u)dS(u) - \frac{\varepsilon}{1+\varepsilon}\int_0^t S(u)d\|\theta\|(u)$, $t \in [0,T]$ is bounded below by a constant, satisfies[4]

$$\int_0^T \theta(t)dS(t) - \frac{\varepsilon}{1+\varepsilon}\int_0^T S(t)d\|\theta\|(t) \geq S(T)(1+\varepsilon) - S(0)/(1+\varepsilon) \qquad (3.3)$$

and that the strict inequality holds on a set of positive probability. Then, using (3.2), and passing to the expectation under Q (the martingale measure for $\tilde{S}$), it would follow that

$$0 \geq E_Q\left[\int_0^T \theta(t)d\tilde{S}(t)\right] > E_Q[S(T)(1+\varepsilon) - S(0)/(1+\varepsilon)] \geq E_Q[\tilde{S}(T) - \tilde{S}(0)] = 0\ , \qquad (3.4)$$

which is absurd.

The above proposition shows that, if an ε-close process with an equivalent martingale measure exists for any ε, the bubble property (a superreplication price for $S(T)$ strictly less than $S(0)$) disappears with arbitrarily small transaction costs. Thus, no frictionless strategy starting with less than $S(0)$ can superreplicate with transaction costs, no matter how small. The implication is that the superreplication price is discountinuous at $\varepsilon = 0$, because it is greater than $S(0)$ for any $\varepsilon > 0$, and strictly less than $S(0)$ for $\varepsilon = 0$.

Note the analogy between this phenomenon and the well-known result of Soner, Shreve and Cvitanić (1995), whereby in the Black-Scholes model the minimal superreplication strategy of a call option with arbitrarily small transaction costs is to buy the underlying asset, and hence the superreplication price jumps from the frictionless Black-Scholes price to the current price of the underlying. In both cases, the frictionless superreplication policies generate too much trading costs to be feasible. An interpretation of this result is that, as superreplication does not offer a reliable hedging objective in practice, it also does not offer a reliable definition of bubble.

3.2 Investment and Arbitrage

A positive answer to Question 3.2 implies that any arbitrage by trading in S is fragile, in the twofold sense of model uncertainty or transaction costs.

Proposition 3.4 *If S and $\tilde{S}$ are continuous and ε-close as in Definition 3.1, and $\tilde{S}$ is a martingale under some probability Q equivalent to P, then S is arbitrage-free with ε transaction costs for all strategies θ such that the value process $V(t) := \int_0^t \theta(u)dS(u) - \frac{\varepsilon}{1+\varepsilon}\int_0^t S(u)d\|\theta\|(u)$, $t \in [0,T]$ is bounded below by a constant.*

[4] Equation (3.3) considers the buying price $S(T)(1+\varepsilon)$ at time T and the selling price $S(0)/(1+\varepsilon)$ at time zero, to ensure that the superreplication is robust to the initial cash/stock allocation and to settlement either in cash or in stock. To wit, in the worst case the strategy ends in cash (i.e. $\theta(T) = 0$), and delivering one share requires $S(T)(1+\varepsilon)$. Likewise, in the worst case the strategy begins in cash ($\theta(0) = 0$) but the initial capital is all in stock, and its cash value is $S(0)/(1+\varepsilon)$.

Proof Take any θ as in the statement of this proposition, and initial capital $x = 0$. The process $\int_0^t \theta(u)d\tilde{S}(u)$, $t \in [0,T]$ is clearly a Q-local martingale and also a Q-supermartingale by (3.2). It also follows from (3.2) that $E_Q[V(T)] \leq 0$ which shows that $V(T) = 0$ Q-a.s. and hence also P-a.s.

This proposition leads to a discontinuity of arbitrage profits similar to the one above for superreplication prices. To wit, suppose that S allows a strong form of arbitrage, as in Example 1. In this case, the arbitrage profit has a discontinuity at $\varepsilon = 0$, because it is null for $\varepsilon > 0$, and strictly positive with positive probability for $\varepsilon = 0$.

4 The Fragility of Diffusions

Theorem 4.2 below shows that fragility is the norm for a large class of Markov diffusions. This result hinges on one assumption: the process must be uniquely identified in law by its drift and diffusion coefficients.

Assumption 4.1 Let $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ be measurable and locally bounded, and $\sigma : [0,T] \times \mathbb{R}^d \to \mathbb{R}^{d\times d}$ continuous, and such that $\sigma(t,x)$ is nonsingular for all $(t,x) \in [0,T] \times \mathbb{R}^d$. Assume that, for all $(s,y) \in [0,T) \times \mathbb{R}^d$, the stochastic differential equation

$$d\bar{X}(t) = b(t,\bar{X}(t))dt + \sigma(t,\bar{X}(t))dW(t), \quad \bar{X}(s) = y, \tag{4.1}$$

has a solution, unique in law, in some probability space, on which $W(t)$ is a standard d-dimensional Brownian motion.

Theorem 4.2 *Let $W(t)$ be a d-dimensional Brownian motion on some filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,T]}, P)$ satisfying the usual hypotheses, under Assumption 4.1, let $X(t)$ be a solution to* (4.1) *with $X(0) = x_0$ for some fixed $x_0 \in \mathbb{R}$. Let $\mathscr{F}_t^X$ denote its augmented natural filtration (completed and made right-continuous).*

Define $S^i(t) := \exp(X^i(t))$ for $t \in [0,T]$ and $1 \leq i \leq d$. Then for all $\varepsilon > 0$ there exists a probability $Q \sim P$ and an $\mathbb{R}^d$-valued $(Q, \mathscr{F}_t^X)$ martingale $(\tilde{S}(t))_{t\in[0,T]}$ such that S and $\tilde{S}$ are ε-close. Furthermore, if $\mathscr{F}_t^X$ coincides with the (completed) natural filtration of a Brownian motion, then $\tilde{S}(t)$ has a.s. continuous paths.

Proof See the Appendix.

In plain English, this theorem states that for any positive Markov diffusion $S(t)$ that has some basic regularity, there is another process $\tilde{S}(t)$, arbitrarily close with respect to small multiplicative perturbations, which becomes a *true* martingale under an equivalent probability. In the jargon of the transaction costs literature, $(\tilde{S}, Q)$ is an ε-consistent price system, see e.g. Guasoni, Rásonyi and Schachermayer (2008).

Example 4.3 Take $S(t)$ as in Example 2.3. Itô's formula implies that $X(t) = \ln(S(t))$ is the unique strong solution of

$$dX(t) = \frac{1}{2}e^{-2X(t)}dt + e^{-X(t)}dW(t),\ X(0) = 0.$$

Theorem 4.2 applies and, for any $\varepsilon, T > 0$, there is an arbitrage-free price process $\tilde{S}$ that is ε-close to $S(t)$.

Example 4.4 Let $Z(t)$ be as in Example 2.4. Theorem 4.2 applies again and provides a uniformly ε-close model admitting no bubbles.

Example 4.5 Consider the stochastic volatility model (see for example, Cox and Hobson (2005)):

$$dP(t) = P(t)V(t)dB(t), \quad dV(t) = V(t)\rho dB(t) + V(t)\sqrt{1-\rho^2}dW(t),$$

where $(B(t), W(t))$ is a 2-dimensional standard Brownian motion, and $0 < \rho < 1$. $P(t)$ represents the price of an asset while $V(t)$ its volatility (the dynamics is given under the risk-neutral measure). Fix $P(0), V(0) > 0$. Taking $v(t) := \ln V(t),\ s(t) := \ln P(t)$, they satisfy

$$ds(t) = e^{v(t)}dB(t) - (1/2)e^{2v(t)}dt, \quad dv(t) = \rho dB(t) + \sqrt{1-\rho^2}dW(t) - \frac{1}{2}dt,$$

which, in fact, give explicit formulas for $P(t), V(t)$. Clearly, $S(t) = (P(t), V(t)),\ t \in [0,T]$ satisfies the conditions of Theorem 4.2, in particular, there is $Q \sim P$ and a Q-martingale $\hat{S}(t) := \tilde{S}^1(t),\ t \in [0,T]$ such that

$$\frac{1}{1+\varepsilon} \le \frac{\hat{S}(t)}{P(t)} \le 1+\varepsilon$$

holds a.s. for $t \in [0,T]$. Thus, $P(t)$ is a strict local martingale (see Cox and Hobson (2005)) which admits an arbitrarily close $\hat{S}(t)$ with bubble-free dynamics. This example shows that Theorem 4.2 can also be applied to certain incomplete markets, see the next section for a general result.

5 Incomplete markets

As the dimension of the driving Wiener process equals the number of risky assets, the scope of Theorem 4.2 may appear to be mainly complete markets. In this section we show that in an incomplete setting one may still draw conclusions about strict local martingales (see Example 4.5 above) that are similar in spirit to the complete case.

For an $\mathbb{R}^d$-valued stochastic process $S(t),\ t \in [0,T]$ let $\mathscr{M}(S)$ denote the set of $Q \sim P$ such that S is a Q-local martingale with respect to its natural filtration (made right-continuous). Let $\mathscr{M}(S) \neq \emptyset$. Recall (e.g. Föllmer and Kabanov (1998)), that for a nonnegative random variable G there exists a predictable integrand H such that $x + \int_0^T H(u)dS(u) \ge G$ a.s. iff

$$x \ge \sup_{Q \in \mathscr{M}(S)} E_Q[G].$$

It is then immediate that a bubble in the sense of Definition 2.1 exists iff

$$\sup_{Q \in \mathscr{M}(S)} E_Q[S^i(T)] < S^i(0)$$

for some i. Thus, if in an incomplete market $E_Q[S^i(T)] < S^i(0)$ holds only for one, but not all, martingale measure $Q \in \mathscr{M}(S)$, it is not clear if there is a bubble in the sense of Definition 2.1. Indeed, in an incomplete market Jarrow, Protter and Shimbo (2010) do not adopt the definition of bubble Definition 2.1, which corresponds to the one in Heston et al. (2007). Instead, they select a risk-neutral probability $(Q_i)_{i\geq 1}$, which changes at a sequence of stopping times $(\tau_i)_{i\geq 1}$, and denote by a bubble the difference between the market price S_t and the risk-neutral expectation $S^*_t = E_{Q_i}[S_T]$ on the event $\{\tau_i \leq t < \tau_{i+1}\}$ (in the absence of dividends, which otherwise need to be included in this expectation). Effectively, the sequences $(Q_i)_{i\geq 1}$ and $(\tau_i)_{i\geq 1}$ define a particular market completion, and the question is whether a bubble can arise in such a completion.

We now show that the fragility result extends to this definition, in that, for *any* martingale measure Q, there is another one Q', arbitrarily close to Q in the total variation norm, under which the ε-close process $\tilde{S}$ is a true martingale, and hence there are no bubbles.

Assumption 5.1 Let $N > d$. Let $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ be measurable and locally bounded, $\sigma : [0,T] \times \mathbb{R}^d \to \mathbb{R}^{d\times N}$ continuous such that for each $(t,x) \in [0,T] \times \mathbb{R}^d$, $\sigma(t,x)$ has maximal rank. Assume that, for all $(s,y) \in [0,T) \times \mathbb{R}^d$, the stochastic differential equation

$$dX(t) = b(t,X(t))dt + \sigma(t,X(t))dW(t), \quad X(s) = y, \tag{5.1}$$

has a solution, unique in law, in some probability space, on which $W(t)$ is a standard N-dimensional Brownian motion.

Theorem 5.2 *Let $W(t)$ be N-dimensional Brownian motion on some filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,T]}, P)$ satisfying the usual hypotheses and, under Assumption 5.1, let $X(t)$ be a solution to* (5.1) *for $X(0) = x_0$ for some fixed $x_0 \in \mathbb{R}^d$.*

Define $S^i(t) := \exp(X^i(t))$ for $t \in [0,T]$ and $1 \leq i \leq d$. Then for all $\varepsilon > 0$ there exists an $(0,\infty)^d$-valued $(\tilde{S}(t))_{t\in[0,T]}$ such that S and $\tilde{S}$ are ε-close and $\tilde{S}$ satisfies the following property:

For each $Q \in \mathscr{M}(\tilde{S}) \neq \emptyset$ there is $Q' \sim Q$ such that $\tilde{S}(t)$, $t \in [0,T]$ is a true Q'-martingale and $||Q - Q'||_{tv}$ (distance in the total variation norm) is arbitrarily small.

Proof See the Appendix.

In summary, this result shows that the original incomplete market S has no arbitrage or bubbles (in the sense of Definition 2.1) that are robust to transaction costs and model uncertainty. It also shows that any arbitrary completion of this market satisfies this property, up to a small variation in the pricing measure Q.

6 A Robust Bubble

We now show a local martingale, for which the answer to Question 3.2 is no, and hence Theorem 4.2 does not apply. A variation of this local martingale appears as a counterexample to several plausible statements in Mathematical Finance (Delbaen and Schachermayer, 1998). In the context of equivalent measures changes, it was communicated to Prokaj and Rásonyi (2010) by Christophe Stricker.

The next example is in fact a caricature, in that it shows how distant from a real martingale a local martingale can be. Over the fixed interval $[0,\pi/2]$, this local martingale drops from 1 to $1/2$. Almost surely.

Example 6.1 (Local Martingale Bridge)
Let

$$X(t) := \exp(W(t) - t/2),\ t \geq 0,$$

where $W(t)$ is Brownian motion and $(\mathscr{F}_t)_{t\geq 0}$ is its (completed) natural filtration. Define the (a.s. finite) stopping time

$$\tau := \inf\{t : X(t) = 1/2\},$$

and set

$$S(t) := X(\tau \wedge \tan t),\ 0 \leq t < \pi/2, \quad S(t) = 1/2,\ t \geq \pi/2.$$

Define also $\mathscr{G}_t := \mathscr{F}_{\tan t}$, $0 \leq t < \pi/2$ and $\mathscr{G}_{\pi/2} := \mathscr{F}_\infty$. The process $S(t)$ is a $\mathscr{G}_t$-local martingale (see Prokaj and Rásonyi (2010)). If $\tilde{S}(t)$ were ε-close to $S(t)$ for some $\varepsilon > 0$ then we would necessarily have

$$\frac{1}{2}\frac{1}{1+\varepsilon} \leq \tilde{S}(t) = E_Q[\tilde{S}(T)|\mathscr{F}_t] \leq \frac{1+\varepsilon}{2},$$

a.s. for $0 \leq t \leq \pi/2$, and hence also

$$\frac{1}{2}\frac{1}{(1+\varepsilon)^2} \leq S(t) \leq \frac{(1+\varepsilon)^2}{2} \text{ a.s.}$$

which is absurd, because $S(t)$ is not bounded from above for $0 < t < \pi/2$. Note that $S(t)$ does not satisfy Assumption 4.1, whence Theorem 4.2 does not apply, because the volatility vanishes (and has a discontinuity) at the value of 1/2.

Technically, this is an example of a “robust bubble”: if $0 < \varepsilon < \sqrt{2}-1$, then any $\tilde{S}(t)$ which is ε-close to $S(t)$ satisfies $\tilde{S}(0) > \operatorname{ess\,sup} \tilde{S}(T)$. Of course, $S(T) = 1/2$ is trivially replicated by a position of $1/2$ in the safe asset, with no need to trade $S(t)$. Needless to say, the example is artificial at best, and we do not endorse it as a model of financial bubbles.

7 Conclusion

Frictionless market models based on strict local martingales may lead to arbitrage or bubbles, but these features disappear in a large class of diffusions, under the presence of minimal transaction costs or model misspecifications.

A Proofs

In the sequel we denote by $C_0^\infty([s,T]\times\mathbb{R}^d)$ the space of smooth functions with compact support on $[s,T]\times\mathbb{R}^d$ equipped with the topology of uniform convergence. The following Lemma is essentially (part of) Theorem 10.1.1 of Stroock and Varadhan (2006). That result is stated on the canonical space and we could not find a convenient reference for the present setting.

Lemma A.1 *Let R^s_ω be a regular version of the conditional law of $X(t), t\in[s,T]$ (on $C[s,T]$) with respect to $\mathscr{F}^X_s$. Then, for almost every $\omega\in\Omega$, R^s_ω solves the martingale problem on $[s,T]$ related to* (4.1) *with initial condition $X(s)(\omega)$.*

Proof Recalling the definition of martingale problems from Chapters 6 and 10 of Stroock and Varadhan (2006), we need to prove that, for almost every $\omega\in\Omega$ and for each $f\in C_0^\infty([s,T]\times\mathbb{R}^d)$,

$$f(t,\pi(t,\cdot))-f(s,X(s))-\int_s^t (L_u f)(u,\pi(u,\cdot))du,\quad t\in[s,T]$$

is an $(R^s_\omega,(\mathscr{G}_u)_{u\in[s,T]})$-martingale, where $\pi(u,\cdot), u\in[s,T]$ denote the coordinate mappings on the canonical space $C[s,T]$, $\mathscr{G}_u:=\sigma(\pi(r,\cdot), s\le r\le u)$, and the operator L acts as

$$(L_u f)(u,p):=\frac{\partial}{\partial u}f(u,p)+\frac{1}{2}\sum_{i,j=1}^d(\sigma\sigma^T)_{ij}(u,p)\frac{\partial^2}{\partial p_i\partial p_j}f(u,p)+\sum_{i=1}^d b_i(u,p)\frac{\partial}{\partial p_i}f(u,p).$$

Denote by w a generic element of $C[s,T]$. Then it is clearly sufficient to prove that, for almost all ω,

$$\int_{C[s,T]}\left[f(t,\pi(t,w))-f(r,\pi(r,w))-\int_r^t(L_u f)(u,\pi(u,w))du\right]g(u_1,\dots,u_n)R^s_\omega(dw)=0 \tag{A.1}$$

for suitable, countable collections of $f\in C_0^\infty([s,T]\times\mathbb{R}^d)$ and bounded measurable g and for all rationals $s<r<t\le T$ and $u_i\in(s,r)$, $i=1,\dots,n$. Equality (A.1) follows from

$$E\left[g(X(u_1),\dots,X(u_n))\left[f(t,X(t))-f(r,X(r))-\int_r^t(L_u f)(u,X(u))du\right]|\mathscr{F}^X_s\right]=0$$

a.s. The above formula holds true by Itô's formula, the tower law and

$$E\left[\int_r^t\frac{\partial}{\partial p}f(u,X(u))\sigma(u,X(u))dW(u)|\mathscr{F}_r\right]=0,$$

which is an elementary property of stochastic integrals.

Proof (Proof of Theorem 4.2) To ease notation, consider $d=1$. The same argument carries over to the general case. Recall the concept of conditional full support. A strictly positive adapted process S defined on $(\Omega,\mathscr{F},(\mathscr{F}_t)_{t\in[0,T]},P)$ with continuous paths has conditional full support (CFS) if:

$$\operatorname{supp}P(S|_{[t,T]}\in\cdot|\mathscr{F}_t)=C^+_{S_t}[t,T] \text{ a.s. for all t}\in[0,T],$$

where $\operatorname{supp}\mu$ denotes the support of a measure μ and $C_v[c,d]$ (resp. $C_v^+[c,d]$) is the set of continuous functions f on $[c,d]$ with $f(c)=v$ (resp. strictly positive continuous functions).

Theorem 1.2 of Guasoni, Rásonyi and Schachermayer (2008) states that, for a process satisfying (CFS) there exists $Q\sim P$ and a Q-martingale $\tilde{S}$ (w.r.t. $\mathscr{F}_t^X$) which is ε-close to S. Thus, $M(t):=\tilde{S}(t)E[\frac{dQ}{dP}|\mathscr{F}_t^X]$ is a P-martingale (w.r.t. $\mathscr{F}_t^X$), and if $\mathscr{F}_t^X$ is the natural filtration of some Brownian motion, then $M(t)$ as well as the (positive) P-martingale $E[\frac{dQ}{dP}|\mathscr{F}_t^X]$ must have continuous paths, and so must $\tilde{S}(t)$.

It remains to prove the (CFS) property for $S(t)$ with respect to $\mathscr{F}_t^X$. Let $P^{s,y}$ denote the law of a solution of (4.1) starting from $\bar{X}(s)=y$ on $C[s,T]$. Replacing $\mathscr{F}_t^X$, $t\geq s$ by $\mathscr{F}_t^{\bar{X}}$, $t\geq s$ in Lemma A.1 above the same argument gives that $P^{s,y}$ solves the martingale problem related to (4.1) with initial condition $\bar{X}(s)=y$ (we do not even need to work with conditional expectations in this case, $\mathscr{F}_s^{\bar{X}}$ being trivial). We can now take a regular version of $P^{s,y}$ as provided by Theorem 10.1.1 of Stroock and Varadhan (2006). Let us now notice that uniqueness in law (see Assumption 4.1) and Lemma A.1 necessarily entail $P^{s,X(s)(\omega)}=R^s_\omega$ a.s.

Thus it suffices to show that $\operatorname{supp}R^s_\cdot=\operatorname{supp}P^{s,X(s)}=C_{X(s)}[s,T]$ a.s. for each $0\leq s<T$. To achieve this, we will prove that for all $y\in\mathbb{R}$ and $\eta\in(0,1)$ and $g\in C_y[s,T]$,

$$P^{s,y}(\{f\in C_y[s,T]:\|f-g\|_\infty<\eta\})>0.$$

Set $K:=\|g\|_\infty+1$, $\tilde{b}(t,x):=b(t,x)1_{\{|x|\leq K\}}$ and $\tilde{\sigma}(t,x):=\sigma(t,x)$ for $(t,x)\in[0,T]\times[-K,K]$. Note that, by continuity, there exists $h>0$ such that

$$|\tilde{\sigma}(t,x)|>h \qquad \text{for all } (t,x)\in[0,T]\times[-K,K]. \tag{A.2}$$

Extend $\tilde{\sigma}$ to $[0,T]\times\mathbb{R}$ so that $|\tilde{\sigma}(t,x)|\geq h$ holds for all (t,x) and $\tilde{\sigma}$ remains continuous.

Since $\tilde{b},\tilde{\sigma}$ are bounded, $\tilde{\sigma}$ is bounded away from 0 and continuous, the martingale problem

$$d\tilde{X}(t)=\tilde{b}(t,\tilde{X}(t))dt+\tilde{\sigma}(t,\tilde{X}(t))dW(t), \quad \tilde{X}(s)=y$$

admits unique solution measures $\tilde{P}^{s,y}$ on the space $C_y[s,T]$ for all $0\leq s<T$, see Stroock and Varadhan (1969) and Chapter 7 of Stroock and Varadhan (2006).

Take $\tau(w):=\inf\{t>s:|w(t)|\geq K\}\wedge T$ where w is the generic element of the canonical space $C_y[s,T]$. Let us denote by $\pi(t,\cdot)$ the coordinate mapping on $C_y[s,T]$ corresponding to $s\leq t\leq T$.

Obviously, τ is a $(\mathscr{G}_t)_{t\in[s,T]}$-stopping time (recall that $\mathscr{G}$ is the filtration generated by coordinate mappings). The last part of Theorem 6.1.2 of Stroock and Varadhan (2006) implies that both measure-concatenations $\tilde{P}^{s,y}\otimes_{\tau(\cdot)}P^{\tau(\cdot),\pi(\tau(\cdot),\cdot)}$ and $P^{s,y}\otimes_{\tau(\cdot)}P^{\tau(\cdot),\pi(\tau(\cdot),\cdot)}$ solve the martingale problem (4.1) on $[s,T]$ (see Theorem 6.1.2 of Stroock and Varadhan (2006) for unexplained notation) and hence these measures are equal on $\mathscr{G}_T$. A fortiori, $P^{s,y}|\mathscr{G}_\tau=\tilde{P}^{s,y}|\mathscr{G}_\tau$. The event $\{f\in C_y[s,T]:\|f-g\|_\infty<\eta\}$ is in $\mathscr{G}_\tau$, consequently,

$$P^{s,y}(\{f\in C_y[s,T]:\|f-g\|_\infty<\eta\})=\tilde{P}^{s,y}(\{f\in C_y[s,T]:\|f-g\|_\infty<\eta\}).$$

The functions $\tilde{\sigma}$ and $\tilde{b}$ satisfy the conditions of section 3 in Stroock and Varadhan (1972) hence Lemma 3.1 of Stroock and Varadhan (1972) (or Exercise 6.7.5 of Stroock and Varadhan (2006)) implies that $\tilde{P}^{s,y}(\{f \in C_y[s,T] : \|f-g\|_\infty < \eta\}) > 0$, concluding the proof.

Proof (Proof of Theorem 5.2) Let us define the multifunction

$$D(t,x) := \{u \in \mathbb{R}^{(N-d)\times N} : \det\left(\begin{bmatrix}\sigma(t,x)\\ u\end{bmatrix}\right) > 0\},$$

for $(t,x) \in [0,T]\times\mathbb{R}^d$. Clearly, this multifunction takes convex, nonempty values in a finite-dimensional space and its graph is $\mathscr{B}([0,T]\times\mathbb{R}^d)\otimes\mathscr{B}(\mathbb{R}^{(N-d)\times N})$-measurable. It is also lower semicontinuous. Indeed, by Proposition 2.1 of Michael (1956) it suffices to prove that if $u \in D(t,x)$ and $\varepsilon > 0$ then there is $\delta > 0$ such that for each t',x' with $|t-t'|+|x-x'| < \delta$ there is $u' \in D(t',x')$ with $|u-u'| < \varepsilon$. But this is obvious by the continuity of σ and the determinant function.

Now we may apply Theorem 3.2''' of Michael (1956) which gives us a continuous $\nu : [0,T]\times\mathbb{R}^d \to \mathbb{R}^{(N-d)\times N}$ such that, for all (t,x), $\nu(t,x) \in D(t,x)$ holds.[5] Let us define $\hat{\sigma} : [0,T]\times\mathbb{R}^d \to \mathbb{R}^{N\times N}$ such that its first d rows equal σ and the remaining rows equal ν. Let us also consider $\hat{b} : (t,x) \to (b(x),\mathbf{0})^T$ (where $\mathbf{0}$ denotes an $N-d$ dimensional row vector of zeros). By abuse of notation from now on we write $\hat{b}(t,y) := \hat{b}(t,\bar{y}^d)$ for all $y \in \mathbb{R}^N$ where $\bar{y}^d$ denotes the vector formed by the first d coordinates of y. We define $\hat{\sigma}$ similarly and hence extend $\hat{b},\hat{\sigma}$ over $[0,T]\times\mathbb{R}^N$ (note that the last $N-d$ coordinates are dummy variables only).

Define $\bar{X}^i(t) := X^i(t)$ for $i = 1,\dots,d$ and set

$$\bar{X}^i(t) := \int_0^t \hat{\sigma}_i(s,X(s))dW(s)$$

for $i = d+1,\dots,N$, where $\hat{\sigma}_i$ denotes the ith row of $\hat{\sigma}$. By construction, $\bar{X}(t)$ satisfies

$$d\bar{X}(t) = \hat{b}(t,\bar{X}(t))dt + \hat{\sigma}(t,\bar{X}(t))dW(t), \quad \bar{X}(0) = \hat{x}_0, \tag{A.3}$$

where the first d components of $\hat{x}_0$ coincide with those of x_0 and the remaining components are zero. By Assumption 5.1, $\hat{b},\hat{\sigma}$ clearly satisfy the conditions of Theorem 4.2, hence for $\bar{S}^i(t) = \exp(\bar{X}^i(t))$, $i = 1,\dots,N$ we get an N-dimensional process $\check{S}(t)$ that is ε-close to $\bar{S}(t)$. Notice that the first d coordinates of $\bar{S}(t)$ are precisely $S(t)$. Denoting by $\tilde{S}(t)$ the d-dimensional process consisting of the first d coordinates of $\check{S}(t)$ we clearly have that $\tilde{S}(t)$ is ε-close to $S(t)$.

From Theorem 4.2 we have the existence of a probability $R \sim P$ such that $\check{S}(t)$ is a (true) R-martingale (w.r.t. $\mathscr{F}_t^{\bar{X}}$), in particular, $R \in \mathscr{M}(\tilde{S})$. Note that the construction in Guasoni, Rásonyi and Schachermayer (2008) assures that $\check{S}(T) \in L^2(R)$. A fortiori, $\tilde{S}(t)$ is an $L^2(R)$-martingale, hence $\sup_{t\in[0,T]}|\tilde{S}_t| \in L^2(R)$. Now recall that the set of $Q' \in \mathscr{M}(\tilde{S})$ such that dQ'/dR is bounded is dense in $\mathscr{M}(\tilde{S})$ with respect

[5] Regrettably, in (c) of the statement of Theorem 3.2''' there is a misprint: there should be $\mathscr{D}(Y)$ instead of $\mathscr{K}(Y)$, this is clear from the ensuing remarks. See p. 372 of Michael (1956) for the definition of $\mathscr{D}(Y)$. This family contains finite dimensional convex sets hence we may indeed apply Theorem 3.2''' in the current setting.

to the total variation norm (see Kabanov and Stricker (2001)). Clearly, $\tilde{S}(t)$ satisfies $\sup_{t\in[0,T]} |\tilde{S}_t| \in L^2(Q')$ as well so it is a true Q'-martingale, for all such Q'. This finishes the proof.

Remark A.2 If, in the statements of Theorems 4.2 and 5.2, the solutions X are strong and $\mathscr{F} = \mathscr{F}^W$ then the proof of Lemma A.1 works with the filtration $\mathscr{F}^W$ in lieu of $\mathscr{F}^X$ hence the proofs of Theorems 4.2 and 5.2 yield $\tilde{S}$ which are $\mathscr{F}^W$-martingales, in particular, continuous.

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