# Forward equations for option prices in semimartingale models

Amel Bentata and Rama Cont

Laboratoire de Probabilités et Modèles Aléatoires, CNRS - Université de Paris VI and Columbia University, New York.

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#### Abstract

We derive a forward partial integro-differential equation for prices of call options in a model where the dynamics of the underlying asset under the pricing measure is described by a -possibly discontinuous- semimartingale. This result generalizes Dupire's forward equation to a large class of non-Markovian models with jumps.

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Since the seminal work of Black, Scholes and Merton [7, 30] partial differential equations (PDE) have been used as a way of characterizing and efficiently computing option prices. In the Black-Scholes-Merton model and various extensions of this model which retain the Markov property of the risk factors, option prices can be characterized in terms of solutions to a backward PDE, whose variables are time (to maturity) and the value of the underlying asset. The use of backward PDEs for option pricing has been extended to cover options with path-dependent and early exercise features, as well as to multifactor models (see e.g. [1]). When the underlying asset exhibit jumps, option prices can be computed by solving an analogous partial integro-differential equation (PIDE) [2, 14].

A second important step was taken by Dupire [15, 16, 18] who showed that when the underlying asset is assumed to follow a diffusion process

$$dS_t = S_t \sigma(t, S_t) dW_t$$

prices of call options (at a given date  $t_0$ ) solve a forward PDE

$$\frac{\partial C_{t_0}}{\partial T}(T,K) = -r(T)K\frac{\partial C_{t_0}}{\partial K}(T,K) + \frac{K^2\sigma(T,K)^2}{2}\frac{\partial^2 C_{t_0}}{\partial K^2}(T,K)$$

on  $[t_0, \infty[\times]0, \infty[$  in the strike and maturity variables, with the initial condition

$$\forall K > 0 \quad C_{t_0}(t_0, K) = (S_{t_0} - K)_+.$$

This forward equation allows to price call options with various strikes and maturities on the same underlying asset, by solving a *single* partial differential equation. Dupire's forward equation also provides useful insights into the *inverse problem* of calibrating diffusion models to observed call and put option prices [6].

Given the theoretical and computational usefulness of the forward equation, there have been various attempts to extend Dupire's forward equation to other types of options and processes, most notably to Markov processes with jumps [2, 10, 12, 26, 9]. Most of these constructions use the Markov property of the underlying process in a crucial way (see however [27]).

As noted by Dupire [17], the forward PDE holds in a more general context than the backward PDE: even if the (risk-neutral) dynamics of the underlying asset is not necessarily Markovian, but described by a continuous Brownian martingale

$$dS_t = S_t \delta_t dW_t$$
,

then call options still verify a forward PDE where the diffusion coefficient is given by the local (or effective) volatility function  $\sigma(t, S)$  given by

$$\sigma(t,S) = \sqrt{E[\delta_t^2 | S_t = S]}.$$

This method is linked to the "Markovian projection" problem: the construction of a Markov process which mimicks the marginal distributions of a martingale

[5, 23, 29]. Such "mimicking processes" provide a method to extend the Dupire equation to non-Markovian settings.

We show in this work that the forward equation for call prices holds in a more general setting, where the dynamics of the underlying asset is described by a – possibly discontinuous – semimartingale. Our parametrization of the price dynamics is general, allows for stochastic volatility and does *not* assume jumps to be independent or driven by a Lévy process, although it includes these cases. Also, our derivation does not require ellipticity or non-degeneracy of the diffusion coefficient. The result is thus applicable to various stochastic volatility models with jumps, pure jump models and point process models used in equity and credit risk modeling.

Our result extends the forward equation from the original diffusion setting of Dupire [16] to various examples of non-Markovian and/or discontinuous processes and implies previous derivations of forward equations [2, 10, 9, 12, 16, 17, 26, 28] as special cases. Section 2 gives examples of forward PIDEs obtained in various settings: time-changed Lévy processes, local Lévy models and point processes used in portfolio default risk modeling. In the case where the underlying risk factor follows, an Itô process or a Markovian jump-diffusion driven by a Lévy process, we retrieve previously known forms of the forward equation. In this case, our approach gives a rigorous derivation of these results under precise assumptions in a unified framework. In some cases, such as index options (Sec. 2.5) or CDO expected tranche notionals (Sec. 2.6), our method leads to a new, more general form of the forward equation valid for a larger class of models than previously studied [3, 12, 35].

The forward equation for call options is a PIDE in one (spatial) dimension, regardless of the number of factors driving the underlying asset. It may thus be used as a method for reducing the dimension of the problem. The case of index options (Section 2.5) in a multivariate jump-diffusion model illustrates how the forward equation projects a high dimensional pricing problem into a one-dimensional state equation.

# 1 Forward PIDEs for call options

## 1.1 General formulation of the forward equation

Consider a (strictly positive) semimartingale S whose dynamics under the pricing measure  $\mathbb P$  is given by

$$S_T = S_0 + \int_0^T r(t)S_{t-}dt + \int_0^T S_{t-}\delta_t dW_t + \int_0^T \int_{-\infty}^{+\infty} S_{t-}(e^y - 1)\tilde{M}(dt \ dy), (1)$$

where r(t) > 0 represents a (deterministic) bounded discount rate,  $\delta_t$  the (random) volatility process and M is an integer-valued random measure with compensator

$$\mu(dt dy; \omega) = m(t, dy, \omega) dt,$$

representing jumps in the log-price, and  $\tilde{M} = M - \mu$  is the compensated random measure associated to M (see [13] for further background). Both the volatility  $\delta_t$  and m(t, dy), which represents the intensity of jumps of size y at time t, are allowed to be stochastic. In particular, we do *not* assume the jumps to be driven by a Lévy process or a process with independent increments. The specification (1) thus includes most stochastic volatility models with jumps.

We assume the following conditions:

**Assumption 1** (Full support).  $\forall t \geq 0$ , supp $(S_t) = [0, \infty[$ .

**Assumption 2** (Integrability condition).

$$\forall T > 0, \quad \mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T \delta_t^2 dt + \int_0^T dt \int_{\mathbb{R}} (e^y - 1)^2 m(t, dy)\right)\right] < \infty. \tag{H}$$

The value  $C_{t_0}(T, K)$  at  $t_0$  of a call option with expiry  $T > t_0$  and strike K > 0 is given by

$$C_{t_0}(T, K) = e^{-\int_{t_0}^T r(t) dt} E^{\mathbb{P}}[\max(S_T - K, 0) | \mathcal{F}_{t_0}].$$
 (2)

As argued in Section 1.2, under Assumption (H), the expectation in (2) is finite. Our main result is the following:

**Theorem 1** (Forward PIDE for call options). Let  $\psi_t$  be the exponential double tail of the compensator m(t, dy)

$$\psi_t(z) = \begin{cases} \int_{-\infty}^z dx \ e^x \int_{-\infty}^x m(t, du), & z < 0 ;\\ \int_z^{+\infty} dx \ e^x \int_x^\infty m(t, du), & z > 0 \end{cases}$$
 (3)

and let  $\sigma: [t_0, T] \times \mathbb{R}^+ - \{0\} \mapsto \mathbb{R}^+, \ \chi: [t_0, T] \times \mathbb{R}^+ - \{0\} \mapsto \mathbb{R}^+$  be measurable functions such that for all  $t \in [t_0, T]$ 

$$\begin{cases}
\sigma(t, S_{t-}) &= \sqrt{\mathbb{E}\left[\delta_t^2 | S_{t-}\right]}; \\
\chi_{t, S_{t-}}(z) &= \mathbb{E}\left[\psi_t(z) | S_{t-}\right] & a.s.
\end{cases}$$
(4)

Under assumption (H), the call option price  $(T, K) \mapsto C_{t_0}(T, K)$ , as a function of maturity and strike, is a solution (in the sense of distributions) of the partial integro-differential equation

$$\frac{\partial C_{t_0}}{\partial T}(T,K) = -r(T)K\frac{\partial C_{t_0}}{\partial K}(T,K) + \frac{K^2\sigma(T,K)^2}{2}\frac{\partial^2 C_{t_0}}{\partial K^2}(T,K) + \int_0^{+\infty} y\frac{\partial^2 C_{t_0}}{\partial K^2}(T,dy)\chi_{T,y}\left(\ln\left(\frac{K}{y}\right)\right) \tag{5}$$

on  $[t_0, \infty[\times]0, \infty[$  with the initial condition:

$$\forall K > 0, \quad C_{t_0}(t_0, K) = (S_{t_0} - K)_+.$$

**Remark 1.** Recall that  $f:[t_0,\infty[\times]0,\infty[\mapsto\mathbb{R}]$  is a solution of (5) in the sense of distributions on  $[t_0,\infty[\times]0,\infty[$  if for any test function  $\varphi\in C_0^\infty([t_0,\infty[\times]0,\infty[,\mathbb{R}])$  and for any  $T>t_0$ .

$$\int_{t_0}^{T} dt \int_{0}^{\infty} dK \varphi(t, K) \left[ -\frac{\partial f}{\partial t} - r(t) K \frac{\partial f}{\partial K} + \frac{K^2 \sigma(t, K)^2}{2} \frac{\partial^2 f}{\partial K^2} + \int_{0}^{+\infty} y \frac{\partial^2 f}{\partial K^2} (t, dy) \chi_{t,y} \left( \ln \left( \frac{K}{y} \right) \right) \right] = 0,$$

where  $C_0^{\infty}([t_0, \infty[\times]0, \infty[, \mathbb{R}))$  is the set of infinitely differentiable functions with compact support in  $[t_0, \infty[\times]0, \infty[$ . This notion of generalized solution allows to separate the discussion of existence of solutions from the discussion of their regularity (which may be delicate, see [14]).

### Remark 2. The discounted asset price

$$\hat{S}_T = e^{-\int_0^T r(t)dt} S_T,$$

is the stochastic exponential of the martingale U defined by

$$U_T = \int_0^T \delta_t dW_t + \int_0^T \int (e^y - 1)\tilde{M}(dt dy).$$

Under assumption (H), we have

$$\forall T > 0, \quad \mathbb{E}\left[\exp\left(\frac{1}{2}\langle U, U \rangle_T^d + \langle U, U \rangle_T^c\right)\right] < \infty,$$

where  $\langle U, U \rangle^c$  and  $\langle U, U \rangle^d$  denote the continuous and purely discontinuous parts of [U, U]. [32, Theorem 9] implies that  $(\hat{S}_T)$  is a  $\mathbb{P}$ -martingale.

The form of the integral term in (5) may seem different from the integral term appearing in backward PIDEs [14, 25]. The following lemma expresses  $\chi_{T,y}(z)$  in a more familiar form in terms of call payoffs:

**Lemma 1.** Let  $n(t, dz, y, \omega)$  dt be a random measure on  $[0, T] \times \mathbb{R} \times \mathbb{R}^+$  verifying

$$\forall t \in [0,T], \quad \int_{-\infty}^{\infty} (e^z \wedge |z|^2) n(t,dz,y,\omega) < \infty$$
 a.s.

Then the exponential double tail  $\chi_{t,y}(z)$  of n, defined as

$$\chi_{t,y}(z) = \begin{cases} \int_{-\infty}^{z} dx \ e^{x} \int_{-\infty}^{x} n(t, du, y), & z < 0 ;\\ \int_{z}^{+\infty} dx \ e^{x} \int_{x}^{\infty} n(t, du, y), & z > 0 \end{cases}$$
 (6)

verifies

$$\int_{\mathbb{R}} [(ye^z - K)^+ - e^z(y - K)^+ - K(e^z - 1)1_{\{y > K\}}] n(t, dz, y) = y \chi_{t,y} \left( \ln \left( \frac{K}{y} \right) \right).$$

*Proof.* Let K, T > 0. Then

$$\begin{split} &\int_{\mathbb{R}} [(ye^z - K)^+ - e^z(y - K)^+ - K(e^z - 1)1_{\{y > K\}}] n(t, dz, y) \\ &= \int_{\mathbb{R}} [(ye^z - K)1_{\{z > \ln{(\frac{K}{y})}\}} - e^z(y - K)1_{\{y > K\}} - K(e^z - 1)1_{\{y > K\}}] n(t, dz, y) \\ &= \int_{\mathbb{R}} [(ye^z - K)1_{\{z > \ln{(\frac{K}{y})}\}} + (K - ye^z)1_{\{y > K\}}] n(t, dz, y). \end{split}$$

• If  $K \geq y$ , then

$$\int_{\mathbb{R}} 1_{\{K \ge y\}} [(ye^z - K) 1_{\{z > \ln(\frac{K}{y})\}} + (K - ye^z) 1_{\{y > K\}}] n(t, dz, y)$$

$$= \int_{\ln(\frac{K}{y})}^{+\infty} y(e^z - e^{\ln(\frac{K}{y})}) n(t, dz, y).$$

• If K < y, then

$$\begin{split} & \int_{\mathbb{R}} 1_{\{K < y\}} [(ye^z - K) 1_{\{z > \ln{(\frac{K}{y})}\}} + (K - ye^z) 1_{\{y > K\}}] n(t, dz, y) \\ & = \int_{\ln{(\frac{K}{y})}}^{+\infty} [(ye^z - K) + (K - ye^z)] n(t, dz, y) + \int_{-\infty}^{\ln{(\frac{K}{y})}} [K - ye^z] n(t, dz, y) \\ & = \int_{-\infty}^{\ln{(\frac{K}{y})}} y(e^{\ln{(\frac{K}{y})}} - e^z) n(t, dz, y). \end{split}$$

Using integration by parts,  $\chi_{t,y}$  can be equivalently expressed as

$$\chi_{t,y}(z) = \begin{cases} \int_{-\infty}^{z} (e^z - e^u) \, n(t, du, y), & z < 0 ;\\ \int_{z}^{\infty} (e^u - e^z) \, n(t, du, y), & z > 0. \end{cases}$$

Hence

$$\int_{\mathbb{R}} [(ye^z - K)^+ - e^z(y - K)^+ - K(e^z - 1)1_{\{y > K\}}] n(t, dz, y) = y \,\chi_{t,y} \left( \ln \left( \frac{K}{y} \right) \right).$$

## 1.2 Derivation of the forward equation

In this section we present a proof of Theorem 1 using the Tanaka-Meyer formula for semimartingales [24, Theorem 9.43] under assumption (H).

*Proof.* We first note that, by replacing  $\mathbb{P}$  by the conditional measure  $\mathbb{P}_{|\mathcal{F}_{t_0}}$  given  $\mathcal{F}_{t_0}$ , we may replace the conditional expectation in (2) by an expectation with respect to the marginal distribution  $p_T^S(dy)$  of  $S_T$  under  $\mathbb{P}_{|\mathcal{F}_{t_0}}$ . Thus, without

loss of generality, we set  $t_0 = 0$  in the sequel and consider the case where  $\mathcal{F}_0$  is the  $\sigma$ -algebra generated by all  $\mathbb{P}$ -null sets and we denote  $C_0(T, K) \equiv C(T, K)$  for simplicity. (2) can be expressed as

$$C(T,K) = e^{-\int_0^T r(t) dt} \int_{\mathbb{R}^+} (y - K)^+ \ p_T^S(dy). \tag{7}$$

By differentiating with respect to K, we obtain

$$\frac{\partial C}{\partial K}(T,K) = -e^{-\int_0^T r(t) dt} \int_K^\infty p_T^S(dy) = -e^{-\int_0^T r(t) dt} \mathbb{E}\left[1_{\{S_T > K\}}\right],$$

$$\frac{\partial^2 C}{\partial K^2}(T,dy) = e^{-\int_0^T r(t) dt} p_T^S(dy).$$
(8)

Let  $L_t^K = L_t^K(S)$  be the semimartingale local time of S at K under  $\mathbb{P}$  (see [24, Chapter 9] or [33, Ch. IV] for definitions). Applying the Tanaka-Meyer formula to  $(S_T - K)^+$ , we have

$$(S_T - K)^+ = (S_0 - K)^+ + \int_0^T 1_{\{S_{t-} > K\}} dS_t + \frac{1}{2} (L_T^K)$$

$$+ \sum_{0 < t < T} \left[ (S_t - K)^+ - (S_{t-} - K)^+ - 1_{\{S_{t-} > K\}} \Delta S_t \right].$$
(9)

As noted in Remark 2, the integrability condition (H) implies that the discounted price  $\hat{S}_t = e^{-\int_0^t r(s) \, ds} S_t = \mathcal{E}(U)_t$  is a martingale under  $\mathbb{P}$ . So (1) can be expressed as

$$dS_t = e^{\int_0^t r(s) ds} \left( r(t) S_{t-} dt + d\hat{S}_t \right)$$

and

$$\int_0^T \mathbf{1}_{\{S_{t-}>K\}} dS_t = \int_0^T e^{\int_0^t r(s) \, ds} \, \mathbf{1}_{\{S_{t-}>K\}} d\hat{S}_t + \int_0^T e^{\int_0^t r(s) \, ds} \, r(t) S_{t-} \mathbf{1}_{\{S_{t-}>K\}} dt,$$

where the first term is a martingale. Taking expectations, we obtain

$$e^{\int_0^T r(t) dt} C(T, K) - (S_0 - K)^+ = \mathbb{E} \left[ \int_0^T e^{\int_0^t r(s) ds} r(t) S_t 1_{\{S_{t-} > K\}} dt + \frac{1}{2} L_T^K \right] + \mathbb{E} \left[ \sum_{0 < t \le T} \left( (S_t - K)^+ - (S_{t-} - K)^+ - 1_{\{S_{t-} > K\}} \Delta S_t \right) \right].$$

Noting that  $S_{t-1}\{S_{t-}>K\} = (S_{t-}-K)^+ + K1\{S_{t-}>K\}$ , we obtain

$$\mathbb{E}\left[\int_{0}^{T} e^{\int_{0}^{t} r(s) ds} r(t) S_{t-1} \{S_{t-} > K\} dt\right] = \int_{0}^{T} r(t) e^{\int_{0}^{t} r(s) ds} \left[C(t, K) - K \frac{\partial C}{\partial K}(t, K)\right] dt,$$

using Fubini's theorem and (8). As for the jump term,

$$\mathbb{E}\left[\sum_{0 < t \le T} (S_t - K)^+ - (S_{t-} - K)^+ - 1_{\{S_{t-} > K\}} \Delta S_t\right]$$

$$= \mathbb{E}\left[\int_0^T dt \int m(t, dx) \left(S_{t-} e^x - K\right)^+ - \left(S_{t-} - K\right)^+ - 1_{\{S_{t-} > K\}} S_{t-} (e^x - 1)\right]$$

$$= \mathbb{E}\left[\int_0^T dt \int m(t, dx) \left(\left(S_{t-} e^x - K\right)^+ - \left(S_{t-} - K\right)^+ - \left(S_{t-} - K\right)^+ - \left(S_{t-} - K\right)^+ \left(e^x - 1\right) - K 1_{\{S_{t-} > K\}} (e^x - 1)\right)\right].$$

Applying Lemma 1 to the random measure m we obtain that

$$\int m(t,dx) \left( (S_{t-}e^x - K)^+ - e^x (S_{t-} - K)^+ - K \mathbf{1}_{\{S_{t-} > K\}} (e^x - 1) \right) = S_{t-} \psi_t \left( \ln \left( \frac{K}{S_{t-}} \right) \right)$$

holds true. One observes that for all z in  $\mathbb R$ 

$$\psi_t(z) \leq 1_{\{z<0\}} \int_{-\infty}^z e^z \, m(t, du) + 1_{\{z>0\}} \int_{-\infty}^z e^u \, m(t, du)$$
$$= 1_{\{z<0\}} e^z \int_{-\infty}^z 1. \, m(t, du) + 1_{\{z>0\}} \int_{-\infty}^z e^u \, m(t, du).$$

Using Assumption (H),

$$\mathbb{E}\left[\sum_{0 < t \le T} \left[ (S_t - K)^+ - (S_{t-} - K)^+ - 1_{\{S_{t-} > K\}} \Delta S_t \right] \right] = \mathbb{E}\left[\int_0^T dt \, S_{t-} \, \psi_t \left( \ln\left(\frac{K}{S_{t-}}\right) \right) \right] < \infty.$$

Hence applying Fubini's theorem leads to

$$\mathbb{E}\left[\sum_{0 < t \leq T} (S_t - K)^+ - (S_{t-} - K)^+ - 1_{\{S_{t-} > K\}} \Delta S_t\right] \\
= \int_0^T dt \mathbb{E}\left[\int m(t, dx) \left( (S_{t-} e^x - K)^+ - e^x (S_{t-} - K)^+ - K 1_{\{S_{t-} > K\}} (e^x - 1) \right) \right] \\
= \int_0^T dt \, \mathbb{E}\left[S_{t-} \psi_t \left(\ln\left(\frac{K}{S_{t-}}\right)\right)\right] \\
= \int_0^T dt \, \mathbb{E}\left[S_{t-} \mathbb{E}\left[\psi_t \left(\ln\left(\frac{K}{S_{t-}}\right)\right) | S_{t-}\right]\right] \\
= \int_0^T dt \, \mathbb{E}\left[S_{t-} \chi_{t,S_{t-}} \left(\ln\left(\frac{K}{S_{t-}}\right)\right)\right].$$

Let  $\varphi \in C_0^{\infty}([0,T]\times]0,\infty[)$  be an infinitely differentiable function with compact support in  $[0,T]\times]0,\infty[$ . The extended occupation time formula [34, Chap. VI, Exercise 1.15] yields

$$\int_0^{+\infty} dK \int_0^T \varphi(t, K) dL_t^K = \int_0^T \varphi(t, S_{t-}) d[S]_t^c = \int_0^T dt \, \varphi(t, S_{t-}) S_{t-}^2 \delta_t^2. \tag{10}$$

Since  $\varphi$  is bounded and has compact support, in order to apply Fubini's theorem to

$$\mathbb{E}\left[\int_0^{+\infty} dK \int_0^T \, \varphi(t,K) \, dL_t^K \right],$$

it is sufficient to show that  $\mathbb{E}\left[L_t^K\right] < \infty$  for  $t \in [0,T]$ . Rewriting equation (9) yields

$$\frac{1}{2}L_T^K = (S_T - K)^+ - (S_0 - K)^+ - \int_0^T 1_{\{S_{t-} > K\}} dS_t - \sum_{0 < t < T} \left[ (S_t - K)^+ - (S_{t-} - K)^+ - 1_{\{S_{t-} > K\}} \Delta S_t \right].$$

Since  $\hat{S}$  is a martingale,  $\mathbb{E}[S_T] < \infty$   $\mathbb{E}[(S_T - K)^+] < \mathbb{E}[S_T]$  and  $\mathbb{E}\left[\int_0^T 1_{\{S_{t-} > K\}} dS_t\right] < \infty$ . As discussed above,

$$\mathbb{E}\left[\sum_{0 < t \le T} \left( (S_t - K)^+ - (S_{t-} - K)^+ - 1_{\{S_{t-} > K\}} \Delta S_t \right) \right] < \infty,$$

yielding that  $\mathbb{E}\left[L_T^K\right] < \infty$ . Hence, one may take expectations in equation (10) to obtain

$$\mathbb{E}\left[\int_0^{+\infty} dK \int_0^T \varphi(t,K) dL_t^K\right] = \mathbb{E}\left[\int_0^T \varphi(t,S_{t-}) S_{t-}^2 \delta_t^2 dt\right] = \int_0^T dt \, \mathbb{E}\left[\varphi(t,S_{t-}) S_{t-}^2 \delta_t^2\right]$$

$$= \int_0^T dt \, \mathbb{E}\left[\mathbb{E}\left[\varphi(t,S_{t-}) S_{t-}^2 \delta_t^2 | S_{t-}\right]\right] = \mathbb{E}\left[\int_0^T dt \, \varphi(t,S_{t-}) S_{t-}^2 \sigma(t,S_{t-})^2\right]$$

$$= \int_0^\infty \int_0^T \varphi(t,K) K^2 \sigma(t,K)^2 p_t^S(dK) dt = \int_0^T dt \, e^{\int_0^t r(s) \, ds} \int_0^\infty \varphi(t,K) K^2 \sigma(t,K)^2 \frac{\partial^2 C}{\partial K^2}(t,dK),$$

where the last line is obtained by using (8). Using integration by parts,

$$\int_{0}^{\infty} dK \int_{0}^{T} dt \, \varphi(t, K) \, \frac{\partial}{\partial t} \left[ e^{\int_{0}^{t} r(s) \, ds} \, C(t, K) - (S_{0} - K)^{+} \right]$$

$$= \int_{0}^{\infty} dK \int_{0}^{T} dt \, \varphi(t, K) \, \frac{\partial}{\partial t} \left[ e^{\int_{0}^{t} r(s) \, ds} \, C(t, K) \right]$$

$$= \int_{0}^{\infty} dK \int_{0}^{T} dt \, \varphi(t, K) \, e^{\int_{0}^{t} r(s) \, ds} \left[ \frac{\partial C}{\partial t}(t, K) + r(t)C(t, K) \right]$$

$$= -\int_{0}^{\infty} dK \int_{0}^{T} dt \, \frac{\partial \varphi}{\partial t}(t, K) \, \left[ e^{\int_{0}^{t} r(s) \, ds} \, C(t, K) \right],$$

where derivatives are used in the sense of distributions. Gathering together all terms,

$$\begin{split} &\int_{0}^{\infty}dK\int_{0}^{T}dt\frac{\partial\varphi}{\partial t}(t,K)\left[e^{\int_{0}^{t}r(s)\,ds}\,C(t,K)\right]\\ &=\int_{0}^{T}dt\int_{0}^{\infty}dK\frac{\partial\varphi}{\partial t}(t,K)\int_{0}^{t}\,ds\,r(s)\,e^{\int_{0}^{s}r(u)\,du}[C(s,K)-K\frac{\partial C}{\partial K}(s,K)]\\ &+\int_{0}^{T}dt\int_{0}^{\infty}\frac{1}{2}\frac{\partial\varphi}{\partial t}(t,K)\int_{0}^{t}dL_{s}^{K}\\ &+\int_{0}^{T}dt\int_{0}^{\infty}dK\frac{\partial\varphi}{\partial t}(t,K)\int_{0}^{t}ds\,e^{\int_{0}^{s}r(u)\,du}\int_{0}^{+\infty}y\frac{\partial^{2}C}{\partial K^{2}}(s,dy)\chi_{s,y}\left(\ln\left(\frac{K}{y}\right)\right)\\ &=-\int_{0}^{T}dt\int_{0}^{\infty}dK\,\varphi(t,K)\,r(t)\,e^{\int_{0}^{t}r(s)\,ds}[C(t,K)-K\frac{\partial C}{\partial K}(t,K)]\\ &-\int_{0}^{T}dt\int_{0}^{\infty}\frac{1}{2}\,\varphi(t,K)\,dL_{t}^{K}-\int_{0}^{T}dt\int_{0}^{\infty}dK\,\varphi(t,K)\,e^{\int_{0}^{t}r(s)\,ds}\int_{0}^{+\infty}y\frac{\partial^{2}C}{\partial K^{2}}(t,dy)\chi_{t,y}\left(\ln\left(\frac{K}{y}\right)\right). \end{split}$$

So finally we have shown that for any test function  $\varphi \in C_0^{\infty}([0,T] \times ]0, \infty[,\mathbb{R}),$ 

$$\int_{0}^{\infty} dK \int_{0}^{T} dt \frac{\partial \varphi}{\partial t}(t, K) \left[ e^{\int_{0}^{t} r(s) ds} C(t, K) \right]$$

$$= -\int_{0}^{\infty} dK \int_{0}^{T} dt \, \varphi(t, K) \, e^{\int_{0}^{t} r(s) ds} \left[ \frac{\partial C}{\partial t}(t, K) + r(t)C(t, K) \right]$$

$$= -\int_{0}^{T} dt \int_{0}^{\infty} dK \, \varphi(t, K) \, r(t) \, e^{\int_{0}^{t} r(s) ds} [C(t, K) - K \frac{\partial C}{\partial K}(t, K)]$$

$$- \int_{0}^{T} dt \int_{0}^{\infty} \frac{1}{2} \, e^{\int_{0}^{t} r(s) ds} \, \varphi(t, K) \, K^{2} \sigma(t, K)^{2} \frac{\partial^{2} C}{\partial K^{2}}(t, dK)$$

$$- \int_{0}^{T} dt \int_{0}^{\infty} dK \, \varphi(t, K) \, e^{\int_{0}^{t} r(s) ds} \int_{0}^{+\infty} y \frac{\partial^{2} C}{\partial K^{2}}(t, dy) \chi_{t,y} \left( \ln \left( \frac{K}{y} \right) \right).$$

Therefore, C(.,.) is a solution of (5) in the sense of distributions.

## 1.3 Uniqueness of solutions of the forward PIDE

Theorem 1 shows that the call price  $(T, K) \mapsto C_{t_0}(T, K)$  solves the forward PIDE (5). Uniqueness of the solution of such PIDEs has been shown using analytical methods [4, 21] under various types of conditions on the coefficients. We give below a direct proof of uniqueness for (5) using a probabilistic method, under explicit conditions which cover most examples of models used in finance.

Define, for  $u \in \mathbb{R}, t \in [0, T[, z > 0 \text{ the measure } n(t, du, z) \text{ by}$ 

$$n(t, [u, \infty[, z) = -e^{-u} \frac{\partial}{\partial u} [\chi_{t,z}(u)], \quad u > 0;$$
  

$$n(t, ] - \infty, u], z) = e^{-u} \frac{\partial}{\partial u} [\chi_{t,z}(u)], \quad u < 0.$$
(11)

Throughout this section, we make the following assumption:

### Assumption 3.

$$\forall T > 0, \forall B \in \mathcal{B}(\mathbb{R}) - \{0\}, \quad (t, z) \to \sigma(t, z), \quad (t, z) \to n(t, B, z)$$

are continuous in  $z \in \mathbb{R}^+$ , uniformly in  $t \in [0,T]$ ; right-continuous in t on [0,T[ uniformly in  $z \in \mathbb{R}^+$ . and

$$\exists K_T > 0, \forall (t, z) \in [0, T] \times \mathbb{R}^+, \ |\sigma(t, z)| + \int_{\mathbb{R}} (1 \wedge |z|^2) \, n(t, du, z) \le K_T.$$
 (H')

Theorem 2. Under Assumption 3, if

either (i) 
$$\forall R > 0 \ \forall t \in [0, T[, \inf_{0 \le z \le R} \sigma(t, z) > 0,$$
  
or (ii)  $\sigma(t, z) \equiv 0$  and  $\exists \beta \in ]0, 2[, \exists C > 0, \forall R > 0, \forall (t, z) \in [0, T[ \times [0, R], \forall f \in C_0^0(\mathbb{R} - \{0\}, \mathbb{R}_+), \int \left(n(t, du, z) - \frac{C du}{|u|^{1+\beta}}\right) f(u) \ge 0,$   
 $\exists K'_{T,R} > 0, \int_{\{|u| \le 1\}} |u|^{\beta} \left(n(t, du, z) - \frac{C du}{|u|^{1+\beta}}\right) dt \le K'_{T,R},$   
and (iii)  $\lim_{R \to \infty} \int_0^T \sup_{z \in \mathbb{R}^+} n(t, \{|u| \ge R\}, z) dt = 0,$ 

then the call option price  $(T, K) \mapsto C_{t_0}(T, K)$ , as a function of maturity and strike, is the unique solution (in the sense of distributions) of the partial integrodifferential equation (5) on  $[t_0, \infty[\times]0, \infty[$  with the initial condition:  $\forall K > 0$   $C_{t_0}(t_0, K) = (S_{t_0} - K)_+$ .

The proof uses the uniqueness of the solution of the forward Kolmogorov equation associated to a certain integro-differential operator. We start by recalling the following result:

**Proposition 1.** Define, for  $t \in [0,T]$  and  $f \in C_0^{\infty}(\mathbb{R})$ , the integro-differential operator

$$L_t f(x) = r(t)x f'(x) + \frac{x^2 \sigma(t, x)^2}{2} f''(x)$$

$$+ \int_{\mathbb{D}} \left[ f(t, x e^y) - f(t, x) - x(e^y - 1) f'(x) \right] n(t, dy, x).$$
(12)

Under Assumption 3, if either conditions (i) or (ii) and (iii) of Theorem 2 hold, then for each  $x_0$  in  $\mathbb{R}^+$ , there exists a unique family  $(p_t(x_0, dy), t \geq 0)$  of bounded measures such that

$$\forall g \in \mathcal{C}_0^{\infty}(]0, \infty[, \mathbb{R}), \quad \int g(y) \frac{dp}{dt}(x_0, dy) = \int p_t(x_0, dy) L_t g(y), \quad p_0(x_0, .) = \epsilon_{x_0},$$
(13)

where  $\epsilon_{x_0}$  is the point mass at  $x_0$ . Furthermore,  $p_t(x_0, .)$  is a probability measure on  $[0, \infty[$ .

Proof. Denote by  $(X_t)_{t\in[0,T]}$  the canonical process on  $D([0,T],\mathbb{R}_+)$ . Under assumptions (i) (or (ii)) and (iii), the martingale problem for  $((L_t)_{t\in[0,T]}, \mathcal{C}_0^{\infty}(\mathbb{R}^+))$  on [0,T] is well-posed [31, Theorem 1]: for any  $x_0 \in \mathbb{R}^+, t_0 \in [0,T[$ , there exists a unique probability measure  $\mathbb{Q}_{t_0,x_0}$  on  $(D([0,T],\mathbb{R}^+),\mathcal{B}_T)$  such that  $\mathbb{Q}_{t_0,x_0}(X_{t_0}=x_0)=1$  and for any  $f\in\mathcal{C}_0^{\infty}(\mathbb{R}^+)$ ,

$$f(X_t) - f(x_0) - \int_{t_0}^t L_s f(X_s) \, ds$$

is a  $(\mathbb{Q}_{t_0,x_0},(\mathcal{B}_t)_{t\geq t_0})$ -martingale on  $[t_0,T]$ . Under  $\mathbb{Q}_{t_0,x_0}$ , X is a Markov process. Define the evolution operator  $(Q_{t_0,t})_{t\in[t_0,T]}$  by

$$\forall f \in \mathcal{C}_b^0(\mathbb{R}^+), \quad Q_{t_0,t}f(x_0) = \mathbb{E}^{\mathbb{Q}_{t_0,x_0}}[f(X_t)].$$
 (14)

Then 
$$Q_{t_0,t}f(x_0) = f(x_0) + \mathbb{E}^{\mathbb{Q}_{t_0,x_0}} \left[ \int_{t_0}^t L_s f(X_s) \, ds \right].$$

Given Assumption 3,  $t \in [0,T] \mapsto \int_{t_0}^t L_s f(X_s) ds$  is uniformly bounded on [0,T]. Given Assumption 3, since X is right continuous  $s \in [0,T[\mapsto L_s f(X_s)]$  is right-continuous up to a  $\mathbb{Q}_{t_0,x_0}$ -null set and

$$\lim_{t \downarrow t_0} \int_{t_0}^t L_s f(X_s) \, ds = 0 \quad \text{a.s.}$$

Applying the dominated convergence theorem yields

$$\lim_{t \downarrow t_0} \mathbb{E}^{\mathbb{Q}_{t_0, x_0}} \left[ \int_{t_0}^t L_s f(X_s) \, ds \right] = 0, \quad \text{so} \quad \lim_{t \downarrow t_0} Q_{t_0, t} f(x_0) = f(x_0),$$

implying that  $t \in [0, T[ \mapsto Q_{t_0,t} f(x_0)]$  is right-continuous at  $t_0$  for each  $x_0 \in \mathbb{R}^+$ . Hence the evolution operator  $(Q_{t_0,t})_{t \in [t_0,T]}$  verifies the following continuity property:

$$\forall f \in \mathcal{C}_b^0(\mathbb{R}^+), \forall x \in \mathbb{R}^+, \quad \lim_{t \downarrow t_0} Q_{t_0,t} f(x) = f(x). \tag{15}$$

In particular, denoting  $q_t(dy)$  the marginal distribution of  $X_t$ , the map

$$t \in [0, T[ \mapsto \int_{\mathbb{R}^+} q_t(dy) f(y)$$
 (16)

is right-continuous, for any  $f \in \mathcal{C}_b^0(\mathbb{R}^+), x_0 \in \mathbb{R}^+$ . The martingale property implies that  $q_t(x_0, dy)$  satisfies

$$\forall g \in \mathcal{C}_0^{\infty}(\mathbb{R}^+), \quad \int_{\mathbb{R}^+} q_t(x_0, dy) g(y) = g(x_0) + \int_0^t \int_{\mathbb{R}^+} q_s(x_0, dy) L_s g(y) \, ds. \tag{17}$$

Given Assumption 3,  $q_t$  is a solution of (13) with initial condition  $q_0(dy) = \epsilon_{x_0}$ . In particular, the measure  $q_t$  has mass 1. To show uniqueness of solutions of (13), we will rewrite (13) as the forward Kolmogorov equation associated with a homogeneous operator on space-time domain and use uniqueness results for the corresponding homogeneous equation. Let  $\mathcal{C}^1([0,T]) \otimes \mathcal{C}_0^{\infty}(\mathbb{R}^+)$  be the tensor product of  $\mathcal{C}^1([0,T])$  and  $\mathcal{C}_0^{\infty}(\mathbb{R}^+)$ .  $L_t$  can be extended to a (homogeneous) linear operator A on  $\mathcal{C}^1([0,T]) \otimes \mathcal{C}_0^{\infty}(\mathbb{R}^+)$  defined via

$$\forall f \in C_0^{\infty}(\mathbb{R}^+), \quad \forall \gamma \in \mathcal{C}^1([0,T]), \quad A(f\gamma)(t,x) = \gamma(t)L_t f(x) + f(x)\gamma'(t).$$
 (18)

[19, Theorem 7.1, Chapter 4] implies that for any  $x_0 \in \mathbb{R}^+$ , if  $(X, \mathbb{Q}_{t_0, x_0})$  is a solution of the martingale problem for L, then the law of  $\eta_t = (t, X_t)$  under  $\mathbb{Q}_{t_0, x_0}$  is a solution of the martingale problem for A: in particular for any  $f \in \mathcal{C}_0^{\infty}(\mathbb{R}^+)$  and  $\gamma \in \mathcal{C}([0, T])$ ,

$$\int q_t(x_0, dy) f(y) \gamma(t) = f(x_0) \gamma(0) + \int_0^t \int q_s(x_0, dy) A(f\gamma)(s, y) \, ds.$$
 (19)

[19, Theorem 7.1, Chapter 4] implies also that if the law of  $\eta_t = (t, X_t)$  is a solution of the martingale problem for A then the law of X is also a solution of the martingale problem for L, namely: uniqueness holds for the martingale problem associated to the operator L on  $C_0^{\infty}(\mathbb{R}^+)$  if and only if uniqueness holds for the martingale problem associated to the martingale problem for A on  $C^1([0,T]) \otimes C_0^{\infty}(\mathbb{R}^+)$ . Define, for  $t \in [0,T]$  and  $h \in C_0^h([0,T] \times \mathbb{R}^+)$ ,

$$\forall (s,x) \in [0,T] \times \mathbb{R}^+, \quad \mathcal{Q}_t h(s,x) = \int_{\mathbb{R}^+} q_t(x,dy) h(t,y). \tag{20}$$

which extends  $Q_{0,t}$  to a 'homogeneous' operator on  $\mathcal{C}_b^0([0,T]\times\mathbb{R}^+)$ . Using (17), we have, for  $\epsilon>0$ ,

$$\forall (f,\gamma) \in \mathcal{C}^{1}([0,T]) \times \mathcal{C}^{\infty}_{0}(\mathbb{R}^{+}), \quad \mathcal{Q}_{t}(f\gamma)(s,x_{0}) - \mathcal{Q}_{\epsilon}(f\gamma)(s,x_{0}) =$$

$$\int_{\epsilon}^{t} \int_{\mathbb{R}^{+}} q_{u}(x_{0},dy)A(f\gamma)(u,y) du = \int_{\epsilon}^{t} \mathcal{Q}_{u}(A(f\gamma))(s,x_{0}) du.$$
 (21)

By linearity, for any  $h \in \mathcal{C}^1([0,T]) \otimes \mathcal{C}_0^{\infty}(\mathbb{R}^+)$  we have

$$Q_t h(s, x_0) - Q_{\epsilon} h(s, x_0) = \int_{\epsilon}^{t} \int_{\mathbb{R}^+} q_u(x_0, dy) Ah(u, y) du = \int_{\epsilon}^{t} Q_u Ah(s, x_0) du.$$
(22)

Consider now a family  $p_t(x_0, dy)$  of positive measures solution of (13) such that  $p_0(x_0, dy) = \epsilon_{x_0}(dy)$ . Then  $p_t$  is also a solution of (17). An integration by parts implies that, for  $(f, \gamma) \in \mathcal{C}^1([0, T]) \times \mathcal{C}^\infty_0(\mathbb{R}^+)$ ,

$$\int_{\mathbb{R}^+} p_t(x_0, dy) f(y) \gamma(t) = f(x_0) \gamma(0) + \int_0^t \int_{\mathbb{R}^+} p_s(x_0, dy) A(f\gamma)(s, y) \, ds. \quad (23)$$

Define, for t in  $[0,T], h \in \mathcal{C}_h^0([0,T] \times \mathbb{R}^+)$ ,

$$\forall (s, x_0) \in [0, T] \times \mathbb{R}^+, \qquad \mathcal{P}_t h(s, x_0) = \int_{\mathbb{R}^+} p_t(x_0, dy) h(t, y).$$

 $(\mathcal{P}_t)_{t\geq 0}$  is then a homogeneous semigroup. Using (23), for  $(f,\gamma)\in\mathcal{C}^1([0,T])\times\mathcal{C}^\infty_0(\mathbb{R}^+)$ ,

$$\forall \epsilon > 0, \quad \mathcal{P}_t(f\gamma) - \mathcal{P}_\epsilon(f\gamma) = \int_{\epsilon}^t \int_{\mathbb{R}^+} p_u(dy) A(f\gamma)(u,y) \, du = \int_{\epsilon}^t \mathcal{P}_u(A(f\gamma)) \, du. \tag{24}$$

which is identical to (21). Multiplying by  $e^{-\lambda t}$  and integrating with respect to t we obtain

$$\lambda \int_0^\infty e^{-\lambda t} \, \mathcal{P}_t(f\gamma)(s, x_0) \, dt = f(x_0)\gamma(0) + \lambda \int_0^\infty e^{-\lambda t} \int_0^t \mathcal{P}_u(A(f\gamma))(s, x_0) \, du \, dt$$

$$= f(x_0)\gamma(0) + \lambda \int_0^\infty \left( \int_u^\infty e^{-\lambda t} \, dt \right) \, \mathcal{P}_u(A(f\gamma))(s, x_0) \, du$$

$$= f(x_0)\gamma(0) + \int_0^\infty e^{-\lambda u} \, \mathcal{P}_u(A(f\gamma))(s, x_0) \, du.$$

for any  $\lambda > 0$ . Similarly, from (21) we obtain for any  $\lambda > 0$ ,

$$\lambda \int_0^\infty e^{-\lambda t} \mathcal{Q}_t(f\gamma)(s,x_0) dt = f(x_0)\gamma(0) + \int_0^\infty e^{-\lambda u} \mathcal{Q}_u(A(f\gamma))(s,x_0) du.$$

Hence for  $(f, \gamma) \in \mathcal{C}^1([0, T]) \times \mathcal{C}_0^{\infty}(\mathbb{R}^+)$  we have

$$\int_0^\infty e^{-\lambda t} \mathcal{Q}_t(\lambda - A)(f\gamma)(s, x_0) dt = f(x_0)\gamma(0) = \int_0^\infty e^{-\lambda t} \mathcal{P}_t(\lambda - A)(f\gamma) dt.$$
(25)

By linearity, for any  $h \in \mathcal{C}^1([0,T]) \otimes \mathcal{C}_0^{\infty}(\mathbb{R}^+)$  we have

$$\int_0^\infty e^{-\lambda t} \mathcal{Q}_t(\lambda - A) h(s, x_0) dt = h(0, x_0) = \int_0^\infty e^{-\lambda t} \mathcal{P}_t(\lambda - A) h dt.$$
 (26)

From [19, Proposition 2.1, Chapter 1], for all  $\lambda > 0$ 

$$Im(\lambda - A) = C_b^0([0, T] \times \mathbb{R}^+)$$

where  $Im(\lambda - A)$  denotes the image of  $\mathcal{C}^1([0,T]) \otimes \mathcal{C}_0^{\infty}(\mathbb{R}^+)$  by the mapping  $(\lambda - A)$ . Hence, since (26) holds

$$\forall h_i n \mathcal{C}_b^0([0,T] \times \mathbb{R}^+), \quad \int_0^\infty e^{-\lambda t} \, \mathcal{Q}_t h\left(s, x_0\right) dt = \int_0^\infty e^{-\lambda t} \, \mathcal{P}_t h(s, x_0) \, dt, \quad (27)$$

so the Laplace transform of  $t \mapsto \mathcal{Q}_t h\left(s, x_0\right)$  is uniquely determined. Using (24),

$$\forall \epsilon > 0, \forall h \in \mathcal{C}^1([0,T]) \otimes \mathcal{C}_0^{\infty}(\mathbb{R}^+),$$

$$\mathcal{P}_t h - \mathcal{P}_\epsilon h = \int_{\epsilon}^t \int_{\mathbb{R}^+} p_u(dy) Ah(u,y) \, du = \int_{\epsilon}^t \mathcal{P}_u(Ah) \, du$$
(28)

by linearity, which allows to show that, for any  $h \in \mathcal{C}^1([0,T]) \otimes \mathcal{C}_0^{\infty}(\mathbb{R}^+)$ ,  $t \mapsto \mathcal{P}_t h(s,x_0)$  is right-continuous:

$$\forall h \in \mathcal{C}^1([0,T]) \otimes \mathcal{C}_0^{\infty}(\mathbb{R}^+), \qquad \lim_{t' \downarrow t} \mathcal{P}_{t'}h(s,x_0) = \mathcal{P}_th(s,x_0).$$

An identical argument using (24) shows that  $t \mapsto \mathcal{Q}_t h(s, x_0)$  is right-continuous. These two right-continuous functions have the same Laplace transform by (27), so they are equal. Thus we have shown that

$$\forall h \in \mathcal{C}^1([0,T]) \otimes \mathcal{C}_0^{\infty}(\mathbb{R}^+), \quad \int h(t,y)q_t(x_0,dy) = \int h(t,y)p_t(x_0,dy). \quad (29)$$

[19, Proposition 4.4, Chapter 3] implies that  $C^1([0,T]) \otimes C_0^{\infty}(\mathbb{R}^+)$  is separating, so (29) allows to conclude that  $p_t(x_0, dy) = q_t(x_0, dy)$ .

We can now study the uniqueness of the forward PIDE (5) and prove Theorem 2.

*Proof.* of Theorem 2. We start by decomposing  $L_t$  as  $L_t = A_t + B_t$  where

$$A_t f(y) = r(t)y f'(y) + \frac{y^2 \sigma(t, y)^2}{2} f''(y), \text{ and}$$

$$B_t f(y) = \int_{\mathbb{R}} [f(ye^z) - f(y) - y(e^z - 1)f'(y)] n(t, dz, y).$$

Then using the fact that  $y \frac{\partial}{\partial y} (y-x)^+ = x \mathbb{1}_{\{y>x\}} + (y-x)_+ = y \mathbb{1}_{\{y>x\}}$  and  $\frac{\partial^2}{\partial y^2} (y-x)^+ = \epsilon_x(y)$  where  $\epsilon_x$  is a unit mass at x, we obtain

$$A_t(y-x)^+ = r(t)y \, 1_{\{y>x\}} + \frac{y^2 \sigma(t,y)^2}{2} \epsilon_x(y) \quad \text{and}$$
$$(y-x)^+ = \int \left[ (ye^z - x)^+ - (y-x)^+ - (e^z - 1) \left( x 1_{\{y>x\}} + (y-x)^+ \right) \right]$$

$$B_T(y-x)^+ = \int_{\mathbb{R}} [(ye^z - x)^+ - (y-x)^+ - (e^z - 1) (x \mathbb{1}_{\{y>x\}} + (y-x)^+)] n(t, dz, y)$$
$$= \int_{\mathbb{R}} [(ye^z - x)^+ - e^z (y-x)^+ - x (e^z - 1) \mathbb{1}_{\{y>x\}}] n(t, dz, y).$$

Using Lemma 1 for the random measure n(t, dz, y) and  $\psi_{t,y}$  its exponential double tail,

$$B_t(y-x)^+ = y\psi_{t,y}\left(\ln\left(\frac{x}{y}\right)\right)$$

Hence, the following identity holds

$$L_t(y-x)^+ = r(t) \left( x \mathbb{1}_{\{y>x\}} + (y-x)_+ \right) + \frac{y^2 \sigma(t,y)^2}{2} \epsilon_x(y) + y \psi_{t,y} \left( \ln \left( \frac{x}{y} \right) \right).$$
(30)

Let  $f: [t_0, \infty[\times]0, \infty[\mapsto \mathbb{R}]$  be a solution in the sense of distributions of (5) with the initial condition:  $f(0, x) = (S_0 - x)^+$ . Integration by parts yields

$$\int_{0}^{\infty} \frac{\partial^{2} f}{\partial x^{2}}(t, dy) L_{t}(y - x)^{+}$$

$$= \int_{0}^{\infty} \frac{\partial^{2} f}{\partial x^{2}}(t, dy) \left( r(t)(x \mathbf{1}_{\{y > x\}} + (y - x)_{+}) + \frac{y^{2} \sigma(t, y)^{2}}{2} \epsilon_{x}(y) + y \psi_{t, y} \left( \ln \left( \frac{x}{y} \right) \right) \right)$$

$$= -r(t) x \int_{0}^{\infty} \frac{\partial^{2} f}{\partial x^{2}}(t, dy) \mathbf{1}_{\{y > x\}} + r(t) \int_{0}^{\infty} \frac{\partial^{2} f}{\partial x^{2}}(t, dy)(y - x)^{+}$$

$$+ \frac{x^{2} \sigma(t, x)^{2}}{2} \frac{\partial^{2} f}{\partial x^{2}} + \int_{0}^{\infty} \frac{\partial^{2} f}{\partial x^{2}}(t, dy) y \psi_{t, y} \left( \ln \left( \frac{x}{y} \right) \right)$$

$$= -r(t) x \frac{\partial f}{\partial x} + r(t) f(t, x) + \frac{x^{2} \sigma(t, x)^{2}}{2} \frac{\partial^{2} f}{\partial x^{2}} + \int_{0}^{\infty} \frac{\partial^{2} f}{\partial x^{2}}(t, dy) y \psi_{t, y} \left( \ln \left( \frac{x}{y} \right) \right).$$

Hence given (5), the following equality holds

$$\frac{\partial f}{\partial t}(t,x) = -r(t)f(t,x) + \int_0^\infty \frac{\partial^2 f}{\partial x^2}(t,dy)L_t(y-x)^+ , \qquad (31)$$

or, equivalently, after integration with respect to time t

$$e^{\int_0^t r(s) \, ds} f(t, x) - f(0, x) = \int_0^\infty e^{\int_0^t r(s) \, ds} \frac{\partial^2 f}{\partial x^2} (t, dy) L_t(y - x)^+. \tag{32}$$

Integration by parts shows that

$$f(t,x) = \int_0^\infty \frac{\partial^2 f}{\partial x^2}(t, dy)(y-x)^+. \tag{33}$$

Hence (31) may be rewritten as

$$\int_{0}^{\infty} e^{\int_{0}^{t} r(s) ds} \frac{\partial^{2} f}{\partial x^{2}}(t, dy)(y-x)^{+} - (S_{0} - x)^{+} = \int_{0}^{t} \int_{0}^{\infty} e^{\int_{0}^{s} r(u) du} \frac{\partial^{2} f}{\partial x^{2}}(s, dy) L_{s}(y-x)^{+} ds.$$
(34)

Define  $q_t(dy) \equiv e^{\int_0^t r(s) ds} \frac{\partial^2 f}{\partial x^2}(t, dy)$ , we have  $q_0(dy) = \epsilon_{S_0}(dy) = p_0(S_0, dy)$ . For  $g \in \mathcal{C}_0^{\infty}(]0, \infty[, \mathbb{R})$ , integration by parts yields

$$g(y) = \int_0^\infty g''(z)(y-z)^+ dz.$$
 (35)

Replacing the above expression in  $\int_{\mathbb{R}} g(y)q_t(dy)$  and using (34) we obtain

$$\int_{0}^{\infty} g(y)q_{t}(dy) = \int_{0}^{\infty} g(y) e^{\int_{0}^{t} r(s) ds} \frac{\partial^{2} f}{\partial x^{2}}(t, dy)$$

$$= \int_{0}^{\infty} g''(z) \int_{0}^{\infty} e^{\int_{0}^{t} r(s) ds} \frac{\partial^{2} f}{\partial x^{2}}(t, dy)(y - z)^{+} dz$$

$$= \int_{0}^{\infty} g''(z)(S_{0} - z)^{+} dz + \int_{0}^{\infty} g''(z) \int_{0}^{t} \int_{0}^{\infty} e^{\int_{0}^{s} r(u) du} \frac{\partial^{2} f}{\partial x^{2}}(s, dy) L_{s}(y - z)^{+} dz$$

$$= g(S_{0}) + \int_{0}^{t} \int_{0}^{\infty} e^{\int_{0}^{s} r(u) du} \frac{\partial^{2} f}{\partial x^{2}}(s, dy) L_{s}[\int_{0}^{\infty} g''(z)(y - z)^{+} dz]$$

$$= g(S_{0}) + \int_{0}^{t} \int_{0}^{\infty} q_{s}(dy) L_{s}g(y) ds.$$

This is none other than equation (13). By uniqueness of the solution  $p_t(S_0, dy)$  of (13) in Proposition 1),

$$e^{\int_0^t r(s) ds} \frac{\partial^2 f}{\partial x^2}(t, dy) = p_t(S_0, dy).$$

One may rewrite equation (32) as

$$f(t,x) = e^{-\int_0^t r(s) ds} \left( f(0,x) + \int_0^\infty p_t(S_0, dy) L_t(y-x)^+ \right),$$

showing that the solution of (5) with initial condition  $f(0,x) = (S_0 - x)^+$  is unique.

# 2 Examples

We now give various examples of pricing models for which Theorem 1 allows to retrieve or generalize previously known forms of forward pricing equations.

## 2.1 Itô processes

When  $(S_t)$  is an Itô process i.e. when the jump part is absent, the forward equation (5) reduces to the Dupire equation [16]. In this case our result reduces to the following:

**Proposition 2** (Dupire equation). Consider the price process  $(S_t)$  whose dynamics under the pricing measure  $\mathbb{P}$  is given by

$$S_T = S_0 + \int_0^T r(t)S_t dt + \int_0^T S_t \delta_t dW_t.$$

Assume there exists a measurable function  $\sigma: [t_0, T] \times \mathbb{R}^+ - \{0\} \mapsto \mathbb{R}^+$  such that

$$\forall t \in t \in [t_0, T], \qquad \sigma(t, S_{t-}) = \sqrt{\mathbb{E}\left[\delta_t^2 | S_{t-}\right]}. \tag{36}$$

If

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T \delta_t^2 dt\right)\right] < \infty \qquad a.s. \tag{37}$$

the call option price (2) is a solution (in the sense of distributions) of the partial differential equation

$$\frac{\partial C_{t_0}}{\partial T}(T,K) = -r(T)K\frac{\partial C_{t_0}}{\partial K}(T,K) + \frac{K^2\sigma(T,K)^2}{2}\frac{\partial^2 C_{t_0}}{\partial K^2}(T,K)$$
(38)

on  $[t_0, \infty[\times]0, \infty[$  with the initial condition:

$$\forall K > 0, \quad C_{t_0}(t_0, K) = (S_{t_0} - K)_+.$$

Notice in particular that this result does not require a non-degeneracy condition on the diffusion term.

*Proof.* It is sufficient to take  $\mu \equiv 0$  in (1) then equivalently in (5). We leave the end of the proof to the reader.

## 2.2 Markovian jump-diffusion models

Another important particular case in the literature is the case of a Markov jump-diffusion driven by a Poisson random measure. Andersen and Andreasen [2] derived a forward PIDE in the situation where the jumps are driven by a compound Poisson process with time-homogeneous Gaussian jumps. We will now show here that Theorem 1 implies the PIDE derived in [2], given here in a more general context allowing for a time- and state-dependent Lévy measure, as well as infinite number of jumps per unit time ("infinite jump activity").

**Proposition 3** (Forward PIDE for jump diffusion model). Consider the price process S whose dynamics under the pricing measure  $\mathbb{P}$  is given by

$$S_{t} = S_{0} + \int_{0}^{T} r(t)S_{t-}dt + \int_{0}^{T} S_{t-}\sigma(t, S_{t-})dB_{t} + \int_{0}^{T} \int_{-\infty}^{+\infty} S_{t-}(e^{y} - 1)\tilde{N}(dtdy)$$
(39)

where  $B_t$  is a Brownian motion and N a Poisson random measure on  $[0,T] \times \mathbb{R}$  with compensator  $\nu(dz) dt$ ,  $\tilde{N}$  the associated compensated random measure. Assume that

$$\sigma(.,.)$$
 is bounded and  $\int_{\{|y|>1\}} e^{2y} \nu(dy) < \infty.$  (40)

Then the call option price

$$C_{t_0}(T, K) = e^{-\int_{t_0}^T r(t) dt} E^{\mathbb{P}}[\max(S_T - K, 0) | \mathcal{F}_{t_0}]$$

is a solution (in the sense of distributions) of the PIDE

$$\frac{\partial C_{t_0}}{\partial T}(T,K) = -r(T)K\frac{\partial C_{t_0}}{\partial K}(T,K) + \frac{K^2\sigma(T,K)^2}{2}\frac{\partial^2 C_{t_0}}{\partial K^2}(T,K) 
+ \int_{\mathbb{R}} \nu(dz) e^z \left[ C_{t_0}(T,Ke^{-z}) - C_{t_0}(T,K) - K(e^{-z} - 1)\frac{\partial C_{t_0}}{\partial K} \right]$$
(41)

on  $[t_0, \infty[\times]0, \infty[$  with the initial condition:

$$\forall K > 0, \quad C_{t_0}(t_0, K) = (S_{t_0} - K)_+.$$

*Proof.* As in the proof of Theorem 1, by replacing  $\mathbb{P}$  by the conditional measure  $\mathbb{P}_{\mathcal{F}_{t_0}}$  given  $\mathcal{F}_{t_0}$ , we may replace the conditional expectation in (2) by an expectation with respect to the marginal distribution  $p_T^S(dy)$  of  $S_T$  under  $\mathbb{P}_{|\mathcal{F}_{t_0}}$ . Thus, without loss of generality, we put  $t_0 = 0$  in the sequel, consider the case where  $\mathcal{F}_0$  is the  $\sigma$ -algebra generated by all  $\mathbb{P}$ -null sets and we denote  $C_0(T,K) \equiv C(T,K)$  for simplicity.

Differentiating (2) in the sense of distributions with respect to K, we obtain:

$$\frac{\partial C}{\partial K}(T,K) = -e^{-\int_0^T r(t) dt} \int_K^\infty p_T^S(dy), \qquad \frac{\partial^2 C}{\partial K^2}(T,dy) = e^{-\int_0^T r(t) dt} p_T^S(dy).$$

In this particular case,  $m(t, dz) dt \equiv \nu(dz) dt$  and  $\psi_t$  are simply given by:

$$\psi_t(z) \equiv \psi(z) = \begin{cases} \int_{-\infty}^z dx \ e^x \int_{-\infty}^x \nu(du) & z < 0\\ \int_z^{+\infty} dx \ e^x \int_x^\infty \nu(du) & z > 0 \end{cases}$$

Then (4) yields

$$\chi_{t,S_{t-}}(z) = \mathbb{E}\left[\psi_{t}\left(z\right)|S_{t-}\right] = \psi(z).$$

Let us now focus on the term

$$\int_0^{+\infty} y \frac{\partial^2 C}{\partial K^2}(T,dy) \, \chi \left( \ln \left( \frac{K}{y} \right) \right)$$

in (5). Applying Lemma 1 yields

$$\int_{0}^{+\infty} y \frac{\partial^{2} C}{\partial K^{2}}(T, dy) \chi \left( \ln \left( \frac{K}{y} \right) \right) 
= \int_{0}^{\infty} e^{-\int_{0}^{T} r(t) dt} p_{T}^{S}(dy) \int_{\mathbb{R}} [(ye^{z} - K)^{+} - e^{z}(y - K)^{+} - K(e^{z} - 1)1_{\{y > K\}}] \nu(dz) 
= \int_{\mathbb{R}} e^{z} \int_{0}^{\infty} e^{-\int_{0}^{T} r(t) dt} p_{T}^{S}(dy) [(y - Ke^{-z})^{+} - (y - K)^{+} - K(1 - e^{-z})1_{\{y > K\}}] \nu(dz) 
= \int_{\mathbb{R}} e^{z} \left[ C(T, Ke^{-z}) - C(T, K) - K(e^{-z} - 1) \frac{\partial C}{\partial K} \right] \nu(dz).$$
(42)

This ends the proof.

## 2.3 Pure jump processes

For price processes with no Brownian component, Assumption (H) reduces to

$$\forall T > 0, \quad \mathbb{E}\left[\exp\left(\int_0^T dt \int (e^y - 1)^2 m(t, dy)\right)\right] < \infty.$$

Assume there exists a measurable function  $\chi:[t_0,T]\times\mathbb{R}^+-\{0\}\mapsto\mathbb{R}^+$  such that for all  $t\in[t_0,T]$  and for all  $z\in\mathbb{R}$ :

$$\chi_{t,S_{t-}}(z) = \mathbb{E}\left[\psi_{t}\left(z\right)|S_{t-}\right],\tag{43}$$

with

$$\psi_T(z) = \begin{cases} \int_{-\infty}^z dx \ e^x \int_{-\infty}^x m(T, du), & z < 0 ;\\ \int_z^{+\infty} dx \ e^x \int_x^\infty m(T, du), & z > 0, \end{cases}$$

then, the forward equation for call option becomes

$$\frac{\partial C}{\partial T} + r(T)K\frac{\partial C}{\partial K} = \int_0^{+\infty} y \frac{\partial^2 C}{\partial K^2}(T, dy) \,\chi_{T,y}\left(\ln\left(\frac{K}{y}\right)\right). \tag{44}$$

It is convenient to use the change of variable:  $v = \ln y, k = \ln K$ . Define  $c(k,T) = C(e^k,T)$ . Then one can write this PIDE as

$$\frac{\partial c}{\partial T} + r(T)\frac{\partial c}{\partial k} = \int_{-\infty}^{+\infty} e^{2(v-k)} \left(\frac{\partial^2 c}{\partial k^2} - \frac{\partial c}{\partial k}\right) (T, dv) \chi_{T,v}(k-v). \tag{45}$$

In the case, considered in [9], where the Lévy density  $m_Y$  has a deterministic separable form

$$m_Y(t, dz, y) dt = \alpha(y, t) k(z) dz dt, \tag{46}$$

Equation (45) allows us to recover<sup>1</sup> equation (14) in [9]

$$\frac{\partial c}{\partial T} + r(T)\frac{\partial c}{\partial k} = \int_{-\infty}^{+\infty} \kappa(k - v)e^{2(v - k)}\alpha(e^v, T) \left(\frac{\partial^2 c}{\partial k^2} - \frac{\partial c}{\partial k}\right) (T, dv)$$

where  $\kappa$  is defined as the exponential double tail of k(u) du, i.e.

$$\kappa(z) = \begin{cases} \int_{-\infty}^{z} dx \ e^{x} \int_{-\infty}^{x} k(u) du & z < 0 ; \\ \int_{z}^{+\infty} dx \ e^{x} \int_{x}^{\infty} k(u) du & z > 0. \end{cases}$$

The right hand side can be written as a convolution of distributions:

$$\frac{\partial c}{\partial T} + r(T)\frac{\partial c}{\partial k} = \left[a_T(.) \left(\frac{\partial^2 c}{\partial k^2} - \frac{\partial c}{\partial k}\right)\right] * g \quad \text{where}$$
 (47)

$$g(u) = e^{-2u}\kappa(u) \qquad a_T(u) = \alpha(e^u, T). \tag{48}$$

<sup>&</sup>lt;sup>1</sup>Note however that the equation given in [9] does not seem to be correct: it involves the double tail of k(z) dz instead of the exponential double tail.

Therefore, knowing c(.,.) and given  $\kappa(.)$  we can recover  $a_T$  hence  $\alpha(.,.)$ . As noted by Carr et al. [9], this equation is analogous to the Dupire formula for diffusions: it enables to "invert" the structure of the jumps–represented by  $\alpha$ –from the cross-section of option prices. Note that, like the Dupire formula, this inversion involves a double deconvolution/differentiation of c which illustrates the ill-posedness of the inverse problem.

## 2.4 Time changed Lévy processes

Time changed Lévy processes were proposed in [8] in the context of option pricing. Consider the price process S whose dynamics under the pricing measure  $\mathbb{P}$  is given by

$$S_t \equiv e^{\int_0^t r(u) \, du} \, X_t \qquad X_t = \exp\left(L_{\Theta_t}\right) \qquad \Theta_t = \int_0^t \theta_s ds \tag{49}$$

where  $L_t$  is a Lévy process with characteristic triplet  $(b, \sigma^2, \nu)$ , N its jump measure and  $(\theta_t)$  is a locally bounded positive semimartingale. X is a  $\mathbb{P}$ -martingale if

$$b + \frac{1}{2}\sigma^2 + \int_{\mathbb{R}} (e^z - 1 - z \, 1_{\{|z| \le 1\}}) \nu(dy) = 0.$$
 (50)

Define the value  $C_{t_0}(T, K)$  at  $t_0$  of the call option with expiry  $T > t_0$  and strike K > 0 as

$$C_{t_0}(T, K) = e^{-\int_0^T r(t) dt} E^{\mathbb{P}}[\max(S_T - K, 0) | \mathcal{F}_{t_0}].$$
 (51)

**Proposition 4.** Assume there exists a measurable function  $\alpha : [0,T] \times \mathbb{R} \mapsto \mathbb{R}$  such that

$$\alpha(t, X_{t-}) = E[\theta_t | X_{t-}], \tag{52}$$

and let  $\chi$  be the exponential double tail of  $\nu$ , defined as

$$\chi(z) = \begin{cases} \int_{-\infty}^{z} dx \ e^{x} \int_{-\infty}^{x} \nu(du), & z < 0 ;\\ \int_{z}^{+\infty} dx \ e^{x} \int_{x}^{\infty} \nu(du), & z > 0. \end{cases}$$
 (53)

If  $\beta = \frac{1}{2}\sigma^2 + \int_{\mathbb{R}} (e^y - 1)^2 \nu(dy) < \infty$  and

$$\mathbb{E}\left[\exp\left(\beta\Theta_T\right)\right] < \infty,\tag{54}$$

then the call option price  $C_{t_0}: (T,K) \mapsto C_{t_0}(T,K)$  at date  $t_0$ , as a function of maturity and strike, is a solution (in the sense of distributions) of the partial integro-differential equation

$$\frac{\partial C}{\partial T}(T,K) = -r\alpha(T,K)K\frac{\partial C}{\partial K}(T,K) + \frac{K^2\alpha(T,K)\sigma^2}{2}\frac{\partial^2 C}{\partial K^2}(T,K) + \int_0^{+\infty} y\frac{\partial^2 C}{\partial K^2}(T,dy)\alpha(T,y)\chi\left(\ln\left(\frac{K}{y}\right)\right) \tag{55}$$

on  $[t, \infty[\times]0, \infty[$  with the initial condition:  $\forall K > 0$   $C_{t_0}(t_0, K) = (S_{t_0} - K)_+$ .

*Proof.* Using Lemma [5, Lemma 2],  $(L_{\Theta_t})$  writes

$$L_{\Theta_t} = L_0 + \int_0^t \sigma \sqrt{\theta_s} dB_s + \int_0^t b\theta_s ds$$
$$+ \int_0^t \theta_s \int_{|z| \le 1} z \tilde{N}(ds dz) + \int_0^t \int_{\{|z| > 1\}} z N(ds dz)$$

where N is an integer-valued random measure with compensator  $\theta_t \nu(dz) dt$ ,  $\tilde{N}$  its compensated random measure. Applying the Itô formula yields

$$X_{t} = X_{0} + \int_{0}^{t} X_{s-} dL_{T_{s}} + \frac{1}{2} \int_{0}^{t} X_{s-} \sigma^{2} \theta_{s} ds + \sum_{s \leq t} (X_{s} - X_{s-} - X_{s-} \Delta L_{T_{s}})$$

$$= X_{0} + \int_{0}^{t} X_{s-} \left[ b\theta_{s} + \frac{1}{2} \sigma^{2} \theta_{s} \right] ds + \int_{0}^{t} X_{s-} \sigma \sqrt{\theta_{s}} dB_{s}$$

$$+ \int_{0}^{t} X_{s-} \theta_{s} \int_{\{|z| \leq 1\}} z \tilde{N}(ds dz) + \int_{0}^{t} X_{s-} \theta_{s} \int_{\{|z| > 1\}} z N(ds dz)$$

$$+ \int_{0}^{t} \int_{\mathbb{R}} X_{s-}(e^{z} - 1 - z) N(ds dz)$$

Under our assumptions,  $\int (e^z - 1 - z \, 1_{\{|z| \le 1\}}) \nu(dz) < \infty$ , hence:

$$\begin{split} X_t &= X_0 + \int_0^t X_{s-} \left[ b\theta_s + \frac{1}{2} \sigma^2 \theta_s + \int_{\mathbb{R}} (e^z - 1 - z \, \mathbf{1}_{\{|z| \le 1\}}) \theta_s \nu(dz) \right] \, ds + \int_0^t X_{s-} \sigma \sqrt{\theta} \, dB_s \\ &+ \int_0^t \int_{\mathbb{R}} X_{s-} \theta_s(e^z - 1) \tilde{N}(ds \, dz) \\ &= X_0 + \int_0^t X_{s-} \sigma \sqrt{\theta_s} \, dB_s + \int_0^t \int_{\mathbb{R}} X_{s-}(e^z - 1) \tilde{N}(ds \, dz) \end{split}$$

and  $(S_t)$  may be expressed as

$$S_t = S_0 + \int_0^t S_{s-} r(s) \, ds + \int_0^t S_{s-} \sigma \sqrt{\theta_s} \, dB_s + \int_0^t \int_{\mathbb{R}} S_{s-} (e^z - 1) \tilde{N}(ds \, dz).$$

Assumption (54) implies that S fulfills Assumption (H) of Theorem 1 and ( $S_t$ ) is now in the suitable form (1) to apply Theorem 1, which yields the result.  $\square$ 

## 2.5 Index options in a multivariate jump-diffusion model

Consider a multivariate model with d assets

$$S_T^i = S_0^i + \int_0^T r(t)S_{t-}^i dt + \int_0^T S_{t-}\delta_t^i dW_t^i + \int_0^T \int_{\mathbb{R}^d} S_{t-}^i (e^{y_i} - 1)\tilde{N}(dt \, dy)$$

where  $\delta^i$  is an adapted process taking values in  $\mathbb{R}$  representing the volatility of asset i, W is a d-dimensional Wiener process, N is a Poisson random measure

on  $[0,T] \times \mathbb{R}^d$  with compensator  $\nu(dy) dt$ ,  $\tilde{N}$  denotes its compensated random measure. The Wiener processes  $W^i$  are correlated

$$\forall 1 \leq (i,j) \leq d, \langle W^i, W^j \rangle_t = \rho_{i,j}t,$$

with  $\rho_{ij} > 0$  and  $\rho_{ii} = 1$ . An index is defined as a weighted sum of asset prices

$$I_t = \sum_{i=1}^d w_i S_t^i \quad w_i > 0 \quad \sum_{1}^d w_i = 1.$$

The value  $C_{t_0}(T, K)$  at time  $t_0$  of an index call option with expiry  $T > t_0$  and strike K > 0 is given by

$$C_{t_0}(T, K) = e^{-\int_{t_0}^T r(t) dt} E^{\mathbb{P}}[\max(I_T - K, 0) | \mathcal{F}_{t_0}].$$
 (56)

The following result is a generalization of the forward PIDE studied by Avellaneda et al. [3] for the diffusion case:

Theorem 3. Forward PIDE for index options. Assume

$$\begin{cases}
\forall T > 0 \quad \mathbb{E}\left[\exp\left(\frac{1}{2}\int_{0}^{T}\|\delta_{t}\|^{2}dt\right)\right] < \infty \\
\int_{\mathbb{R}^{d}}(1 \wedge \|y\|) \nu(dy) < \infty \quad a.s.
\end{cases} (57)$$

$$\int_{\{\|y\|>1\}} e^{2\|y\|} \nu(dy) < \infty \quad a.s.$$

Define

$$\eta_{t}(z) = \begin{cases} \int_{-\infty}^{z} dx \ e^{x} \int_{\mathbb{R}^{d}} 1_{\ln\left(\frac{\sum_{1 \leq i \leq d-1} w_{i} S_{t-}^{i} e^{y_{i}}}{I_{t-}}\right) \leq x} \nu(dy) & z < 0 \\ \int_{z}^{\infty} dx \ e^{x} \int_{\mathbb{R}^{d}} 1_{\ln\left(\frac{\sum_{1 \leq i \leq d-1} w_{i} S_{t-}^{i} e^{y_{i}}}{I_{t-}}\right) \geq x} \nu(dy) & z > 0 \end{cases}$$
(58)

and assume there exists measurable functions  $\sigma: [t_0, T] \times \mathbb{R}^+ - \{0\} \mapsto \mathbb{R}^+,$  $\chi: [t_0, T] \times \mathbb{R}^+ - \{0\} \mapsto \mathbb{R}^+$  such that for all  $t \in [t_0, T]$  and for all  $z \in \mathbb{R}$ :

$$\begin{cases}
\sigma(t, I_{t-}) &= \frac{1}{z} \sqrt{\mathbb{E}\left[\left(\sum_{i,j=1}^{d} w_{i} w_{j} \rho_{ij} \, \delta_{t}^{i} \delta_{t}^{j} \, S_{t-}^{i} S_{t-}^{j}\right) | I_{t-}\right]} \quad a.s., \\
\chi_{t, I_{t-}}(z) &= \mathbb{E}\left[\eta_{t}(z) | I_{t-}\right] \quad a.s.
\end{cases}$$
(59)

Then the index call price  $(T,K) \mapsto C_{t_0}(T,K)$ , as a function of maturity and strike, is a solution (in the sense of distributions) of the partial integro-differential equation

$$\frac{\partial C_{t_0}}{\partial T} = -r(T)K\frac{\partial Ct_0}{\partial K} + \frac{\sigma(T,K)^2}{2}\frac{\partial^2 Ct_0}{\partial K^2} + \int_0^{+\infty} y \frac{\partial^2 Ct_0}{\partial K^2}(T,dy) \chi_{T,y} \left(\ln\left(\frac{K}{y}\right)\right)$$
(60)

on  $[t_0, \infty[\times]0, \infty[$  with the initial condition:

$$\forall K > 0, \quad C_{t_0}(t_0, K) = (I_{t_0} - K)_+.$$

*Proof.*  $(B_t)_{t\geq 0}$  defined by

$$dB_{t} = \frac{\sum_{i=1}^{d} w_{i} S_{t-}^{i} \delta_{t}^{i} dW_{t}^{i}}{\left(\sum_{i,j=1}^{d} w_{i} w_{j} \rho_{ij} \delta_{t}^{i} \delta_{t}^{j} S_{t-}^{i} S_{t-}^{j}\right)^{1/2}}$$

is a continuous local martingale with quadratic variation t: by Lévy's theorem, B is a Brownian motion. Hence I may be decomposed as

$$I_{T} = \sum_{i=1}^{d} w_{i} S_{0}^{i} + \int_{0}^{T} r(t) I_{t-} dt + \int_{0}^{T} \left( \sum_{i,j=1}^{d} w_{i} w_{j} \rho_{ij} \, \delta_{t}^{i} \delta_{t}^{j} \, S_{t-}^{i} S_{t-}^{j} \right)^{\frac{1}{2}} dB_{t}$$

$$+ \int_{0}^{T} \int_{\mathbb{R}^{d}} \sum_{i=1}^{d} w_{i} S_{t-}^{i} (e^{y_{i}} - 1) \tilde{N}(dt \, dy)$$

$$(61)$$

The essential part of the proof consists in rewriting  $(I_t)$  in the suitable form (1) to apply Theorem 1. Applying the Itô formula to  $\ln(I_T)$  yields

$$\ln \frac{I_T}{I_0} = \int_0^T \left[ r(t) - \frac{1}{2I_{t-}^2} \sum_{i,j=1}^d w_i w_j \rho_{ij} \, \delta_t^i \delta_t^j \, S_{t-}^i S_{t-}^j \right. \\ \left. - \int \left( \frac{\sum_{1 \le i \le d} w_i S_{t-}^i e^{y_i}}{I_{t-}} - 1 - \ln \left( \frac{\sum_{1 \le i \le d} w_i S_{t-}^i e^{y_i}}{I_{t-}} \right) \right) \nu(dy) \right] dt \\ + \int_0^T \frac{1}{I_{t-}} \left( \sum_{i,j=1}^d w_i w_j \rho_{ij} \, \delta_t^i \delta_t^j \, S_{t-}^i S_{t-}^j \right)^{\frac{1}{2}} dB_t + \int_0^T \int \ln \left( \frac{\sum_{1 \le i \le d} w_i S_{t-}^i e^{y_i}}{I_{t-}} \right) \tilde{N}(dt \, dy).$$

Using the convexity property of the logarithm,

$$\ln\left(\frac{\sum_{1 \leq i \leq d} w_i S_{t-}^i e^{y_i}}{I_{t-}}\right) \geq \frac{\sum_{1 \leq i \leq d} w_i S_{t-}^i y_i}{I_{t-}} \geq -\|y\|, \quad \text{and} \quad \ln\left(\frac{\sum_{1 \leq i \leq d} w_i S_{t-}^i e^{y_i}}{I_{t-}}\right) \leq \ln\left(\max_{1 \leq i \leq d} e^{y_i}\right),$$

implying that

$$\left| \ln \left( \frac{\sum_{1 \le i \le d} w_i S_{t-}^i e^{y_i}}{I_{t-}} \right) \right| \le \left| \sum_{1 \le i \le d} \frac{w_i S_{t-}^i}{I_{t-}} y_i \right| \le \sum_{1 \le i \le d} |y_i| \le ||y||,$$

so the functions  $y \to \ln\left(\frac{\sum_{1 \le i \le d} w_i S_{t-}^i e^{y_i}}{I_{t-}}\right)$  and  $y \to \frac{\sum_{1 \le i \le d} w_i S_{t-}^i e^{y_i}}{I_{t-}}$  are integrable with respect to  $\nu(dy)$  under the assumptions (57). We furthermore observe that

$$\int 1 \wedge \left| \ln \left( \frac{\sum_{1 \leq i \leq d} w_i S_{t-}^i e^{y_i}}{I_{t-}} \right) \right| \nu(dy) < \infty \quad a.s.$$

$$\int_0^T \int_{\{\|y\| > 1\}} e^{2 \left| \ln \left( \frac{\sum_{1 \leq i \leq d} w_i S_{t-}^i e^{y_i}}{I_{t-}} \right) \right|} \nu(dy) dt < \infty \quad a.s.$$
(62)

Similarly, (57) implies that  $\int (e^{y_i} - 1 - 1_{\{|y_i| \le 1\}} y_i) \nu(dy) < \infty$  so  $\ln(S_T^i)$  may be expressed as

$$\ln(S_T^i) = \ln(S_0^i) + \int_0^T \left( r(t) - \frac{1}{2} (\delta_t^i)^2 - \int \left( e^{y_i} - 1 - 1_{\{|y_i| \le 1\}} y_i \right) \nu(dy) \right) dt + \int_0^T \delta_t^i dW_t^i + \int_0^T \int y_i \, \tilde{N}(dt \, dy)$$

Define the d-dimensional martingale  $W_t = (W_t^1, \dots, W_t^{d-1}, B_t)$ . For  $1 \le i, j \le d-1$  we have

$$\langle W^i, W^j \rangle_t = \rho_{i,j} t \quad \text{and} \quad \langle W^i, B \rangle_t = \frac{\sum_{j=1}^d w_j \rho_{ij} S^j_{t-} \delta^j_t}{\left(\sum_{i,j=1}^d w_i w_j \rho_{ij} \, \delta^i_t \delta^j_t \, S^i_{t-} S^j_{t-}\right)^{1/2}} \, t.$$

Define

$$\Theta_{t} = \begin{pmatrix} 1 & \cdots & \rho_{1,d-1} & \frac{\sum_{j=1}^{d} w_{j} \rho_{1j} S_{t-}^{j} \delta_{t}^{j}}{\left(\sum_{i,j=1}^{d} w_{i} w_{j} \rho_{ij} \delta_{i}^{i} \delta_{t}^{j} S_{t-}^{j} S_{t-}^{j}\right)^{1/2}} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ \rho_{d-1,1} & \cdots & 1 & \frac{\sum_{j=1}^{d} w_{j} \rho_{d-1,j} S_{t-}^{j} \delta_{t}^{j}}{\left(\sum_{i,j=1}^{d} w_{i} w_{j} \rho_{ij} \delta_{i}^{i} \delta_{t}^{j} S_{t-}^{i} S_{t-}^{j}\right)^{1/2}} \\ \frac{\sum_{j=1}^{d} w_{j} \rho_{1,j} S_{t-}^{j} \delta_{t}^{j}}{\left(\sum_{i,j=1}^{d} w_{i} w_{j} \rho_{ij} \delta_{i}^{i} \delta_{t}^{j} S_{t-}^{i} S_{t-}^{j}\right)^{1/2}} & \cdots & \frac{\sum_{j=1}^{d} w_{j} \rho_{d-1,j} S_{t-}^{j} \delta_{t}^{j}}{\left(\sum_{i,j=1}^{d} w_{i} w_{j} \rho_{ij} \delta_{i}^{i} \delta_{t}^{j} S_{t-}^{i} S_{t-}^{j}\right)^{1/2}} & 1 \end{pmatrix}$$

There exists a standard Brownian motion  $(Z_t)$  such that  $W_t = AZ_t$  where A is a  $d \times d$  matrix verifying  $\Theta = {}^t A A$ . Define  $X_T \equiv (\ln(S_T^1), \dots, \ln(S_T^{d-1}), \ln(I_T))$ ;

$$\delta = \begin{pmatrix} \delta_t^1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & \delta_t^{d-1} & 0 \\ 0 & \cdots & 0 & \frac{1}{I_{t-}} \left( \sum_{i,j=1}^d w_i w_j \rho_{ij} \, \delta_t^i \delta_t^j \, S_{t-}^i S_{t-}^j \right)^{\frac{1}{2}} \end{pmatrix},$$

$$\beta_t = \begin{pmatrix} r(t) - \frac{1}{2} (\delta_t^1)^2 - \int (e^{y_1} - 1 - y_1) \, \nu(dy) \\ \vdots \\ r(t) - \frac{1}{2} (\delta_t^{d-1})^2 - \int (e^{y_{d-1}} - 1 - y_{d-1}) \, \nu(dy) \\ r(t) - \frac{1}{2I_{t-}^2} \sum_{i,j=1}^d w_i w_j \rho_{ij} \, \delta_t^i \delta_t^j \, S_{t-}^i S_{t-}^j - \int \left( \frac{\sum_{1 \le i \le d} w_i S_{t-}^i e^{y_i}}{I_{t-}} - 1 - \ln \left( \frac{\sum_{1 \le i \le d} w_i S_{t-}^i e^{y_i}}{I_{t-}} \right) \right) \nu(dy) \end{pmatrix},$$
and 
$$\psi_t(y) = \begin{pmatrix} y_1 \\ \vdots \\ y_{d-1} \\ \ln \left( \frac{\sum_{1 \le i \le d} w_i S_{t-}^i e^{y_i}}{I_{t-}} \right) \end{pmatrix}.$$

Then  $X_T$  may be expressed as

$$X_{T} = X_{0} + \int_{0}^{T} \beta_{t} dt + \int_{0}^{T} \delta_{t} A dZ_{t} + \int_{0}^{T} \int_{\mathbb{R}^{d}} \psi_{t}(y) \,\tilde{N}(dt \, dy)$$
 (63)

The predictable function  $\phi_t$  defined, for  $t \in [0,T], y \in \psi_t(\mathbb{R}^d)$ , by

$$\phi_t(y) = \left(y_1, \dots, y_{d-1}, \ln\left(\frac{e^{y_d} I_{t-} - \sum_{1 \le i \le d-1} w_i S_{t-}^i e^{y_i}}{w_d S_{t-}^d}\right)\right)$$

is the left inverse of  $\psi_t$ :  $\phi_t(\omega, \psi_t(\omega, y)) = y$ . Observe that  $\psi_t(., 0) = 0$ ,  $\phi$  is predictable, and  $\phi_t(\omega, .)$  is differentiable on  $Im(\psi_t)$  with Jacobian matrix  $\nabla_y \phi_t(y)$  given by

$$(\nabla_y \phi_t(y)) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \dots & 1 & 0 \\ \frac{-e^{y_1} w_1 S_{t-}^1}{e^{y_d} I_{t-} - \sum_{1 \le i \le d-1} w_i S_{t-}^i e^{y_i}} & \dots & \frac{-e^{y_{d-1}} w_{d-1} S_{t-}^{d-1}}{e^{y_d} I_{t-} - \sum_{1 \le i \le d-1} w_i S_{t-}^i e^{y_i}} & \frac{e^{y_d} I_{t-}}{e^{y_d} I_{t-} - \sum_{1 \le i \le d-1} w_i S_{t-}^i e^{y_i}} \end{pmatrix}$$

so  $(\psi, \nu)$  satisfies the assumptions of [5, Lemma 2]: using Assumption  $(A_{2b})$ , for all  $T \ge t \ge 0$ ,

$$\mathbb{E}\left[\int_{0}^{T} \int_{\mathbb{R}^{d}} (1 \wedge \|\psi_{t}(.,y)\|^{2}) \nu(dy) dt\right]$$

$$= \int_{0}^{T} \int_{\mathbb{R}^{d}} 1 \wedge \left(y_{1}^{2} + \dots + y_{d-1}^{2} + \ln\left(\frac{\sum_{1 \leq i \leq d} w_{i} S_{t-}^{i} e^{y_{i}}}{I_{t-}}\right)^{2}\right) \nu(dy) dt$$

$$\leq \int_{0}^{T} \int_{\mathbb{R}^{d}} 1 \wedge (2\|y\|^{2}) \nu(dy) dt < \infty.$$

Define  $\nu_{\phi}$ , the image of  $\nu$  by  $\phi$  by

$$\nu_{\phi}(\omega, t, B) = \nu(\phi_t(\omega, B)) \quad \text{for} \quad B \subset \psi_t(\mathbb{R}^d). \tag{64}$$

Applying [5, Lemma 2],  $X_T$  may be expressed as

$$X_T = X_0 + \int_0^T \beta_t dt + \int_0^T \delta_t A dZ_t + \int_0^T \int y \, \tilde{M}(dt \, dy)$$

where M is an integer-valued random measure (resp.  $\tilde{M}$  its compensated random measure) with compensator

$$\mu(\omega; dt \ dy) = m(t, dy; \omega) dt,$$

defined via its density

$$\frac{d\mu}{d\nu_{\phi}}(\omega, t, y) = 1_{\{\psi_{t}(\mathbb{R}^{d})\}}(y) \left| \det \nabla_{y} \phi_{t} \right|(y) = 1_{\{\psi_{t}(\mathbb{R}^{d})\}}(y) \left| \frac{e^{y_{d}} I_{t-}}{e^{y_{d}} I_{t-} - \sum_{1 \leq i \leq d-1} w_{i} S_{t-}^{i} e^{y_{i}}} \right|$$

with respect to  $\nu_{\phi}$ . Considering now the d-th component of  $X_T$ , one obtains the semimartingale decomposition of  $\ln(I_t)$ :

$$\ln (I_T) - \ln (I_0) 
= \int_0^T \left( r(t) - \frac{1}{2I_{t-}^2} \left( \sum_{i,j=1}^d w_i w_j \rho_{ij} \, \delta_t^i \delta_t^j \, S_{t-}^i S_{t-}^j \right) 
- \int \left( \frac{\sum_{1 \le i \le d} w_i S_{t-}^i e^{y_i}}{I_{t-}} - 1 - \ln \left( \frac{\sum_{1 \le i \le d} w_i S_{t-}^i e^{y_i}}{I_{t-}} \right) \right) \nu(dy) \right) dt 
+ \int_0^T \frac{1}{I_{t-}} \left( \sum_{i,j=1}^d w_i w_j \rho_{ij} \, \delta_t^i \delta_t^j \, S_{t-}^i S_{t-}^j \right)^{\frac{1}{2}} dB_t + \int_0^T \int y \, \tilde{K}(dt \, dy)$$

where K is an integer-valued random measure on  $[0,T] \times \mathbb{R}$  with compensator k(t,dy) dt where

$$k(t,B) = \int_{\mathbb{R}^{d-1} \times B} \mu(t,dy) = \int_{\mathbb{R}^{d-1} \times B} 1_{\{\psi_t(\mathbb{R}^d)\}}(y) |\det \nabla_y \phi_t|(y) \nu_{\phi}(t,dy)$$

$$= \int_{\mathbb{R}^{d-1} \times B \cap \psi_t(\mathbb{R}^d)} |\det \nabla_y \phi_t|(\psi_t(y)) \nu(dy)$$

$$= \int_{\{y \in \mathbb{R}^d - \{0\}, \ln\left(\frac{\sum_{1 \le i \le d-1} w_i S_{t-}^i e^{y_i}}{I_{t-}}\right) \in B\}} \nu(dy) \text{ for } B \in \mathcal{B}(\mathbb{R} - \{0\}).$$

In particular, the exponential double tail of k(t, dy) which we denote  $\eta_t(z)$ 

$$\eta_t(z) = \begin{cases} \int_{-\infty}^z dx \ e^x k(t, ] - \infty, x]), & z < 0 ;\\ \int_z^{+\infty} dx \ e^x k(t, [x, \infty[), z > 0, ]) \end{cases}$$

is given by (58). So finally  $I_T$  may be expressed as

$$I_{T} = I_{0} + \int_{0}^{T} r(t)I_{t-} dt + \int_{0}^{T} \left( \sum_{i,j=1}^{d} w_{i}w_{j}\rho_{ij} \,\delta_{t}^{i}\delta_{t}^{j} \,S_{t-}^{i} S_{t-}^{j} \right)^{\frac{1}{2}} dB_{t}$$

$$+ \int_{0}^{T} \int_{\mathbb{R}^{d}} (e^{y} - 1) \,I_{t-} \tilde{K}(dt \, dy).$$

The normalized volatility of  $I_t$  satisfies, for  $t \in [0, T]$ ,

$$\left| \frac{\sum_{i,j=1}^{d} w_i w_j \rho_{ij} \, \delta_t^i \delta_t^j \, S_{t-}^i S_{t-}^j}{I_{t-}^2} \leq \sum_{i,j=1}^{d} \rho_{ij} \, \delta_t^i \delta_t^j, \quad \text{and} \quad \left| \ln \left( \frac{\sum_{1 \leq i \leq d} w_i S_{t-}^i e^{y_i}}{I_{t-}} \right) \right| \leq \|y\|.$$

Hence

$$\frac{1}{2} \int_{0}^{T} \frac{\sum_{i,j=1}^{d} w_{i} w_{j} \rho_{ij} \, \delta_{t}^{i} \delta_{t}^{j} \, S_{t-}^{i} S_{t-}^{j}}{I_{t-}^{2}} \, dt + \int_{0}^{T} \int (e^{y} - 1)^{2} k(t, \, dy) \, dt$$

$$= \frac{1}{2} \int_{0}^{T} \frac{\sum_{i,j=1}^{d} w_{i} w_{j} \rho_{ij} \, \delta_{t}^{i} \delta_{t}^{j} \, S_{t-}^{i} S_{t-}^{j}}{I_{t-}^{2}} \, dt$$

$$+ \int_{0}^{T} \int_{\mathbb{R}^{d}} \left( \frac{\sum_{1 \leq i \leq d-1} w_{i} S_{t-}^{i} e^{y_{i}} + w_{d} S_{t-}^{d} e^{y}}{I_{t-}} - 1 \right)^{2} \nu(dy_{1}, \dots, dy_{d-1}, dy) \, dt$$

$$\leq \frac{1}{2} \sum_{i,j=1}^{d} \rho_{ij} \, \delta_{t}^{i} \delta_{t}^{j} + \int_{0}^{T} \int_{\mathbb{R}^{d}} (e^{||y||} - 1)^{2} \nu(dy_{1}, \dots, dy_{d-1}, dy) \, dt.$$

Using assumptions (57), the last inequality implies that  $I_t$  satisfies (H). Hence Theorem 1 can now be applied to I, which yields the result.

## 2.6 Forward equations for CDO pricing

Portfolio credit derivatives such as CDOs or index default swaps are derivatives whose payoff depends on the total loss  $L_t$  due to defaults in a reference portfolio of obligors. Reduced-form top-down models of portfolio default risk [20, 22, 35, 11, 36] represent the default losses of a portfolio as a marked point process  $(L_t)_{t\geq 0}$  where the jump times represents credit events in the portfolio and the jump sizes  $\Delta L_t$  represent the portfolio loss upon a default event. Marked point processes with random intensities are increasingly used as ingredients in such models [20, 22, 28, 35, 36]. In all such models the loss process (represented as a fraction of the portfolio notional) may be represented as

$$L_t = \int_0^t \int_0^1 x M(ds \, dx),$$

where M(dt dx) is an integer-valued random measure with compensator

$$\mu(dt dx; \omega) = m(t, dx; \omega) dt.$$

If furthermore

$$\int_0^1 x \, m(t, dx) < \infty,\tag{65}$$

then  $L_t$  may be expressed in the form

$$L_t = \int_0^t \int_0^1 x \, \left( m(s, dx) \, ds + \tilde{M}(ds \, dx) \right),$$

where

$$\int_0^t \int_0^1 x \, \tilde{M}(ds \, dx),$$

is a  $\mathbb{P}$ -martingale. The point process  $N_t = M([0,t] \times [0,1])$  represents the number of defaults and

 $\lambda_t(\omega) = \int_0^1 m(t, dx; \omega)$ 

represents the default intensity. Denote by  $T_1 \leq T_2 \leq ...$  the jump times of N. The cumulative loss process L may also be represented as

$$L_t = \sum_{k=1}^{N_t} Z_k,$$

where the "mark"  $Z_k$ , with values in [0,1], is distributed according to

$$F_t(dx;\omega) = \frac{m_X(t,dx;\omega)}{\lambda_t(\omega)}.$$

Note that the percentage loss  $L_t$  belongs to [0,1], so  $\Delta L_t \in [0,1-L_{t-}]$ . For the equity tranche [0,K], we define the expected tranche notional at maturity T as

$$C_{t_0}(T, K) = \mathbb{E}[(K - L_T)_+ | \mathcal{F}_{t_0}].$$
 (66)

As noted in [11], the prices of portfolio credit derivatives such as CDO tranches only depend on the loss process through the expected tranche notionals. Therefore, if one is able to compute  $C_{t_0}(T,K)$  then one is able to compute the values of all CDO tranches at date  $t_0$ . In the case of a loss process with constant loss increment, Cont and Savescu [12] derived a forward equation for the expected tranche notional. The following result generalizes the forward equation derived by Cont and Savescu [12] to a more general setting which allows for random, dependent loss sizes and possible dependence between the loss given default and the default intensity:

**Proposition 5** (Forward equation for expected tranche notionals). Assume there exists a measurable function  $m_Y : [0,T] \times [0,1] \mapsto \mathcal{R}([0,1])$  such that for all  $t \in [t_0,T]$  and for all  $A \in \mathcal{B}([0,1)]$ ,

$$m_Y(t, A, L_{t-}) = E[m_X(t, A, .)|L_{t-}],$$
(67)

and denote  $M_Y(dt dy)$  the integer-valued random measure with compensator  $m_Y(t, dy, z) dt$ . Define the effective default intensity

$$\lambda^{Y}(t,z) = \int_{0}^{1-z} m_{Y}(t,dy,z). \tag{68}$$

Then the expected tranche notional  $(T, K) \mapsto C_{t_0}(T, K)$ , as a function of maturity and strike, is a solution of the partial integro-differential equation

$$\frac{\partial C_{t_0}}{\partial T}(T,K) = -\int_0^K \frac{\partial^2 C_{t_0}}{\partial K^2}(T,dy) \left[ \int_0^{K-y} (K-y-z) \, m_Y(T,dz,y) - (K-y)\lambda^Y(T,y) \right],$$
(60)

on  $[t_0, \infty[\times]0, 1[$  with the initial condition:  $\forall K \in [0, 1], C_{t_0}(t_0, K) = (K - L_{t_0})_+$ .

*Proof.* By replacing  $\mathbb{P}$  by the conditional measure  $\mathbb{P}_{|\mathcal{F}_0}$  given  $\mathcal{F}_0$ , we may replace the conditional expectation in (66) by an expectation with respect to the marginal distribution  $p_T(dy)$  of  $L_T$  under  $\mathbb{P}_{|\mathcal{F}_{t_0}}$ . Thus, without loss of generality, we put  $t_0 = 0$  in the sequel and consider the case where  $\mathcal{F}_0$  is the  $\sigma$ -algebra generated by all  $\mathbb{P}$ -null sets. (66) can be expressed as

$$C(T,K) = \int_{\mathbb{R}^+} (K - y)^+ \ p_T(dy). \tag{70}$$

Differentiating with respect to K, we get

$$\frac{\partial C}{\partial K} = \int_0^K p_T(dy) = \mathbb{E}\left[1_{\{L_{t-} \le K\}}\right], \qquad \frac{\partial^2 C}{\partial K^2}(T, dy) = p_T(dy). \tag{71}$$

For h > 0 applying the Tanaka-Meyer formula to  $(K - L_t)^+$  between T and T + h, we have

$$(K - L_{T+h})^{+} = (K - L_{T})^{+} - \int_{T}^{T+h} 1_{\{L_{t-} \le K\}} dL_{t}$$

$$+ \sum_{T < t \le T+h} \left[ (K - L_{t})^{+} - (K - L_{t-})^{+} + 1_{\{L_{t-} \le K\}} \Delta L_{t} \right].$$
(72)

Taking expectations, we get

$$C(T+h,K) - C(T,K) = \mathbb{E}\left[\int_{T}^{T+h} dt \, 1_{\{L_{t-} \le K\}} \int_{0}^{1-L_{t-}} x \, m(t,dx)\right]$$

$$+ \mathbb{E}\left[\sum_{T < t \le T+h} (K-L_{t})^{+} - (K-L_{t-})^{+} + 1_{\{L_{t-} \le K\}} \Delta L_{t}\right].$$

The first term may be computed as

$$\mathbb{E}\left[\int_{T}^{T+h} dt \, 1_{\{L_{t-} \leq K\}} \int_{0}^{1-L_{t-}} x \, m(t, dx)\right] = \int_{T}^{T+h} dt \, \mathbb{E}\left[1_{\{L_{t-} \leq K\}} \int_{0}^{1-L_{t-}} x \, m(t, dx)\right]$$

$$= \int_{T}^{T+h} dt \, \mathbb{E}\left[\mathbb{E}\left[1_{\{L_{t-} \leq K\}} \int_{0}^{1-L_{t-}} x \, m(t, dx) \Big| L_{t-}\right]\right]$$

$$= \int_{T}^{T+h} dt \, \mathbb{E}\left[1_{\{L_{t-} \leq K\}} \int_{0}^{1-L_{t-}} x \, m_{Y}(t, dx, L_{t-})\right]$$

$$= \int_{T}^{T+h} dt \, \int_{0}^{K} p_{T}(dy) \left(\int_{0}^{1-y} x \, m_{Y}(t, dx, y)\right).$$

As for the jump term,

$$\mathbb{E}\left[\sum_{T < t \leq T+h} (K - L_{t})^{+} - (K - L_{t-})^{+} + 1_{\{L_{t-} \leq K\}} \Delta L_{t}\right] \\
= \mathbb{E}\left[\int_{T}^{T+h} dt \int_{0}^{1-L_{t-}} m(t, dx) \left((K - L_{t-} - x)^{+} - (K - L_{t-})^{+} + 1_{\{L_{t-} \leq K\}} x\right)\right] \\
= \int_{T}^{T+h} dt \,\mathbb{E}\left[\int_{0}^{1-L_{t-}} m(t, dx) \left((K - L_{t-} - x)^{+} - (K - L_{t-})^{+} + 1_{\{L_{t-} \leq K\}} x\right)\right] \\
= \int_{T}^{T+h} dt \,\mathbb{E}\left[\mathbb{E}\left[\int_{0}^{1-L_{t-}} m(t, dx) \left((K - L_{t-} - x)^{+} - (K - L_{t-})^{+} + 1_{\{L_{t-} \leq K\}} x\right) \Big| L_{t-}\right]\right] \\
= \int_{T}^{T+h} dt \,\mathbb{E}\left[\int_{0}^{1-L_{t-}} m_{Y}(t, dx, L_{t-}) \left((K - L_{t-} - x)^{+} - (K - L_{t-})^{+} + 1_{\{L_{t-} \leq K\}} x\right)\right] \\
= \int_{T}^{T+h} dt \,\mathbb{E}\left[\int_{0}^{1-L_{t-}} m_{Y}(t, dx, L_{t-}) \left((K - L_{t-} - x)^{+} - (K - L_{t-})^{+} + 1_{\{L_{t-} \leq K\}} x\right)\right] \\
= \int_{T}^{T+h} dt \,\mathbb{E}\left[\int_{0}^{1-L_{t-}} m_{Y}(t, dx, L_{t-}) \left((K - L_{t-} - x)^{+} - (K - L_{t-})^{+} + 1_{\{L_{t-} \leq K\}} x\right)\right] \\
= \int_{T}^{T+h} dt \,\mathbb{E}\left[\int_{0}^{1-L_{t-}} m_{Y}(t, dx, L_{t-}) \left((K - L_{t-} - x)^{+} - (K - L_{t-})^{+} + 1_{\{L_{t-} \leq K\}} x\right)\right] \\
= \int_{T}^{T+h} dt \,\mathbb{E}\left[\int_{0}^{1-L_{t-}} m_{Y}(t, dx, L_{t-}) \left((K - L_{t-} - x)^{+} - (K - L_{t-})^{+} + 1_{\{L_{t-} \leq K\}} x\right)\right] \\
= \int_{T}^{T+h} dt \,\mathbb{E}\left[\int_{0}^{1-L_{t-}} m_{Y}(t, dx, L_{t-}) \left((K - L_{t-} - x)^{+} - (K - L_{t-})^{+} + 1_{\{L_{t-} \leq K\}} x\right)\right] \\
= \int_{T}^{T+h} dt \,\mathbb{E}\left[\int_{0}^{1-L_{t-}} m_{Y}(t, dx, L_{t-}) \left((K - L_{t-} - x)^{+} - (K - L_{t-})^{+} + 1_{\{L_{t-} \leq K\}} x\right)\right] \\
= \int_{T}^{T+h} dt \,\mathbb{E}\left[\int_{0}^{1-L_{t-}} m_{Y}(t, dx, L_{t-}) \left((K - L_{t-} - x)^{+} - (K - L_{t-})^{+} + 1_{\{L_{t-} \leq K\}} x\right)\right] \\
= \int_{T}^{T+h} dt \,\mathbb{E}\left[\int_{0}^{1-L_{t-}} m_{Y}(t, dx, L_{t-}) \left((K - L_{t-} - x)^{+} - (K - L_{t-})^{+} + 1_{\{L_{t-} \leq K\}} x\right)\right] \\
= \int_{T}^{T+h} dt \,\mathbb{E}\left[\int_{0}^{1-L_{t-}} m_{Y}(t, dx, L_{t-}) \left((K - L_{t-} - x)^{+} - (K - L_{t-})^{+} + 1_{\{L_{t-} \leq K\}} x\right)\right] \\
= \int_{T}^{T+h} dt \,\mathbb{E}\left[\int_{0}^{1-L_{t-}} m_{Y}(t, dx, L_{t-}) \left((K - L_{t-} - x)^{+} - (K - L_{t-})^{+} + 1_{\{L_{t-} \leq K\}} x\right)\right] \\
= \int_{T}^{T+h} dt \,\mathbb{E}\left[\int_{0}^{1-L_{t-}} m_{Y}(t, dx, L_{t-}) \left((K - L_{t-} - x)^{+} - (K - L_{t-})^{+} + 1_{\{L_{t-} \leq K\}} x\right)\right] \\
= \int_{T}^{$$

where the inner integrals may be computed as

$$\int_{0}^{1} p_{T}(dy) \int_{0}^{1-y} m_{Y}(t, dx, y) \left( (K - y - x)^{+} - (K - y)^{+} + 1_{\{y \le K\}} x \right)$$

$$= \int_{0}^{K} p_{T}(dy) \int_{0}^{1-y} m_{Y}(t, dx, y) \left( (K - y - x) 1_{\{K - y > x\}} - (K - y - x) \right)$$

$$= \int_{0}^{K} p_{T}(dy) \int_{K-y}^{1-y} m_{Y}(t, dx, y) (K - y - x).$$

Gathering together all the terms, we obtain

$$C(T+h,K) - C(T,K) = \int_{T}^{T+h} dt \int_{0}^{K} p_{T}(dy) \left( \int_{0}^{1-y} x \, m_{Y}(t,dx,y) \right) + \int_{T}^{T+h} dt \int_{0}^{K} p_{T}(dy) \left( \int_{K-y}^{1-y} m_{Y}(t,dx,y)(K-y-x) \right)$$

$$= \int_{T}^{T+h} dt \int_{0}^{K} p_{T}(dy) \left( -\int_{0}^{K-y} m_{Y}(t,dx,y)(K-y-x) + (K-y)\lambda^{Y}(T,y) \right).$$

Dividing by h and taking the limit  $h \to 0$  yields

$$\frac{\partial C}{\partial T} = -\int_0^K p_T(dy) \left[ \int_0^{K-y} (K - y - x) m_Y(T, dx, y) - (K - y) \lambda^Y(T, y) \right] 
= -\int_0^K \frac{\partial^2 C}{\partial K^2} (T, dy) \left[ \int_0^{K-y} (K - y - x) m_Y(T, dx, y) - (K - y) \lambda^Y(T, y) \right].$$

In [12], loss given default (i.e. the jump size of L) is assumed constant  $\delta = (1 - R)/n$ : then  $Z_k = \delta$ , so  $L_t = \delta N_t$  and one can compute C(T, K) using the law of  $N_t$ . Setting  $t_0 = 0$  and assuming as above that  $\mathcal{F}_{t_0}$  is generated by null sets, we have

$$C(T, K) = \mathbb{E}[(K - L_T)^+] = \mathbb{E}[(k \delta - L_T)^+] = \delta \mathbb{E}[(k - N_T)^+] \equiv \delta C_k(T).$$
 (73)

The compensator of  $L_t$  is  $\lambda_t \epsilon_{\delta}(dz) dt$ , where  $\epsilon_{\delta}(dz)$  is the point mass at the point  $\delta$ . The effective compensator becomes

$$m_Y(t, dz, y) = E[\lambda_t | L_{t-} = y] \epsilon_{\delta}(dz) dt = \lambda^Y(t, y) \epsilon_{\delta}(dz),$$

and the effective default intensity is  $\lambda^{Y}(t,y) = E[\lambda_t | L_{t-} = y]$ . Using the notations in [12], if we set  $y = j\delta$  then

$$\lambda^{Y}(t, j\delta) = E[\lambda_t | L_{t-} = j\delta] = E[\lambda_t | N_{t-} = j] = a_j(t)$$

and  $p_t(dy) = \sum_{j=0}^n q_j(t)\epsilon_{j\delta}(dy)$ . Let us focus on (69) in this case. We recall from the proof of Proposition 5 that

$$\frac{\partial C}{\partial T}(T, k\delta) = \int_0^1 p_T(dy) H_T \cdot (k\delta - y)^+ 
= \int_0^1 p_T(dy) \int_0^{1-y} [(k\delta - y - z)^+ - (k\delta - y)^+] \lambda^Y(T, y) \epsilon_{\delta}(dz) 
= \int_0^1 p_T(dy) \lambda^Y(T, y) [(k\delta - y - \delta)^+ - (k\delta - y)^+] 1_{\{\delta < 1 - y\}} 
= -\delta \sum_{j=0}^n q_j(T) a_j(T) 1_{\{j \le k-1\}}.$$

This expression can be simplified as in [12, Proposition 2], leading to the forward equation

$$\frac{\partial C_k(T)}{\partial T} = a_k(T)C_{k-1}(T) - a_{k-1}(T)C_k(T) - \sum_{j=1}^{k-2} C_j(T)[a_{j+1}(T) - 2a_j(T) + a_{j-1}(T)]$$

$$= [a_k(T) - a_{k-1}(T)]C_{k-1}(T) - \sum_{j=1}^{k-2} (\nabla^2 a)_j C_j(T) - a_{k-1}(T)[C_k(T) - C_{k-1}(T)].$$

Hence we recover [12, Proposition 2] as a special case of Proposition 5.

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